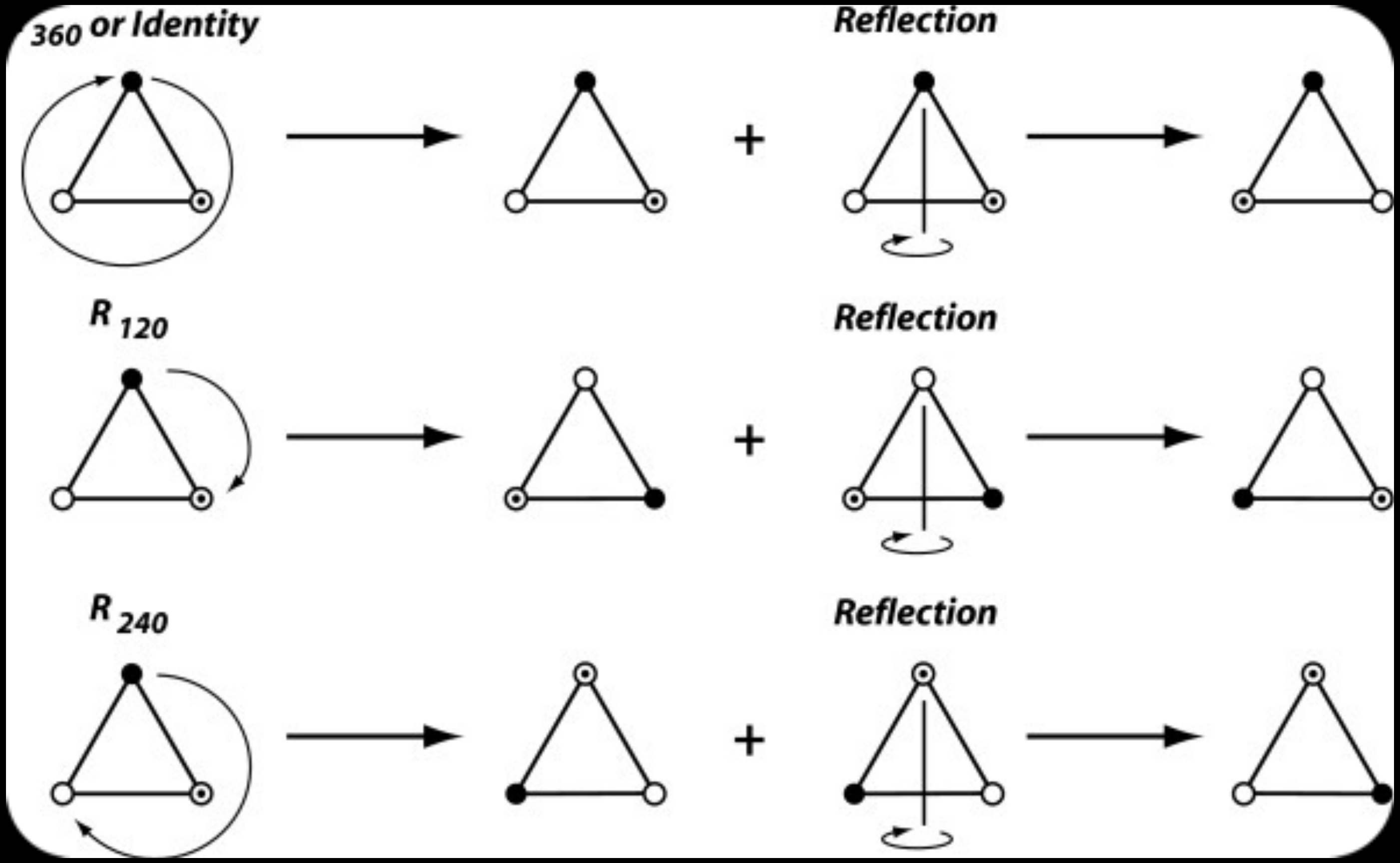
Symmetries and conservation laws



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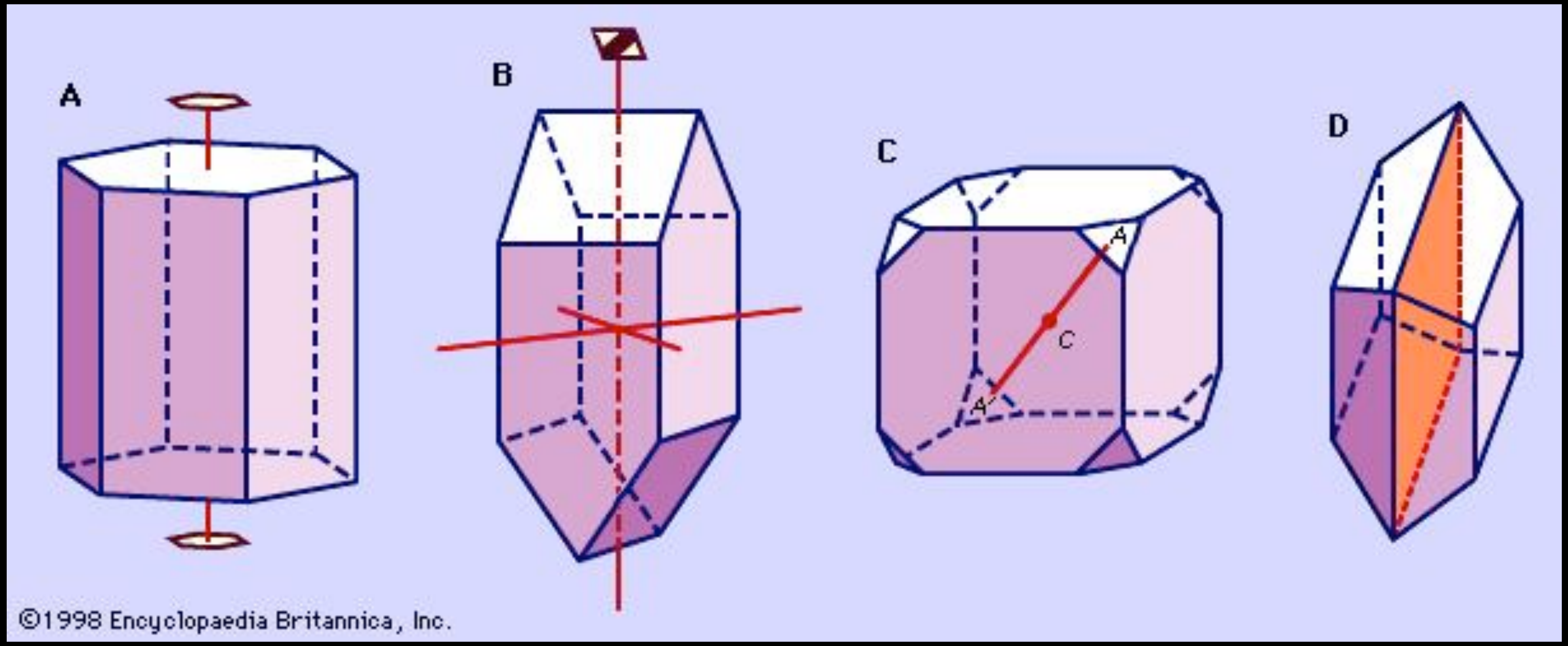
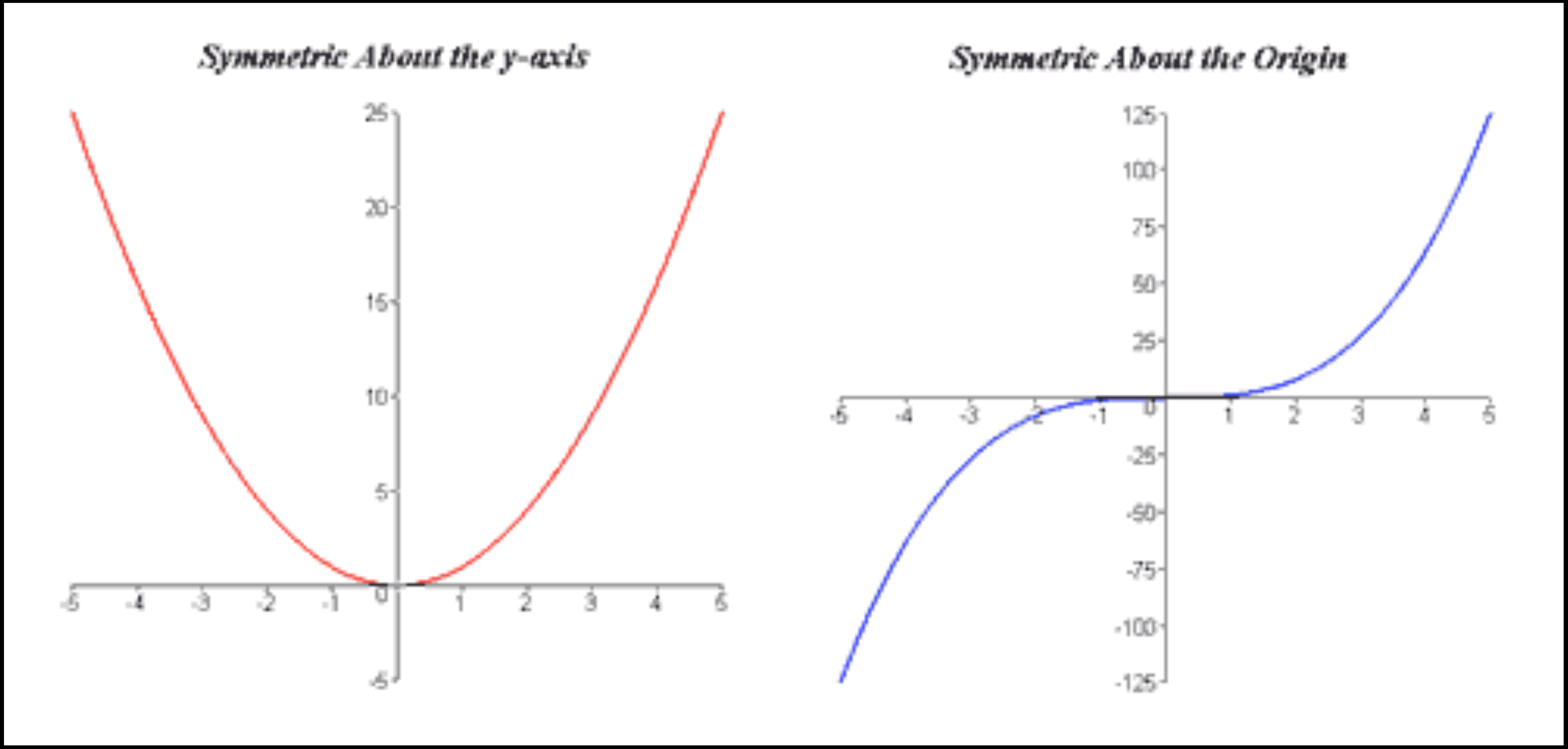
About symmetries

- Symmetries and invariances are important notions in physics
  - They describe how a system remains unaltered under a given transformation
- We will focus on dynamical symmetries of motion and not on static symmetries e.g. as in crystals



Odd/even functions



Lagrangian formalism

- In classical dynamics, the motion of a system is described in terms of forces using the second law of Newton

$$\vec{F} = m\vec{a} = m\ddot{\vec{x}}$$

- The same equation of motion can be obtained from the Lagrangian
  - T: the part that describes the kinetic energy of a system
  - V: the part that describes the potential

$$\mathcal{L} = T - V$$

- The Lagrangian is a function of generalised coordinates $(q_i, \dot{q}_i)$
- The equation of motion of a system is given by the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i}\right) - \frac{\partial \mathcal{L}}{\partial q_i} = 0$$

Parenthesis: Dirac's bra-ket notation

- A state can be represented by a wave function ψ or by a column of complex numbers denoted by the ket $|\psi\rangle$
- To every ket we associate a bra-vector $\langle\psi|$
  - This is represented by the conjugate transpose of the relevant matrix
    - transpose transforms the column to a row
    - each entry becomes the complex conjugate of the initial entry
- The operation that transforms a bra to a ket and vice-versa is called the Hermitian conjugation $|\psi\rangle^{\dagger}=\langle\psi|$
- An operator for which $P=P^{\dagger}$ is called self-adjoint or Hermitian
- Observables are always quantities that are represented by Hermitian operators.
  - Their expectation value is real

$$\langle a|P|a\rangle=\langle a|P^{\dagger}|a\rangle=\langle a|P|a\rangle^{*}$$

When is an observable conserved?

- Observables are always quantities that are represented by Hermitian operators.

$$F = F^{\dagger}$$

- The expectation value is

$$\langle F \rangle = \langle \psi | F | \psi \rangle$$

$$\langle F \rangle^{*} = \langle \psi | F^{\dagger} | \psi \rangle$$

- An observable constant of motion F is Hermitian and commutes with the Hamiltonian

How can conserved quantities be found?

- We can write down the Hamiltonian of a system and test the commutation criterium [H,F]=0, for all possible observables
  - Not really practical or feasible
- The Hamiltonian of a system does not need to be fully known
  - we need to establish the invariance of H under a transformation
    - this leads to a conserved quantity
- Introducing a transformation operator U that transforms a wave function ψ into a ψ'
  - the normalisation relation of the wave function does not change between ψ and ψ'
  - this leads to the fact that the transformation operator must be unitary

When is an operator called a symmetry operator?

- Consider a transformation U that takes a state ψ and transforms it into a ψ'
  - The wave functions are normalised:
  - The transformation needs to be unitary

$$\langle\psi'|\psi'\rangle=\langle\psi|\psi\rangle=1$$

- We call U a symmetry operator if the new state ψ' obeys the same Schrodinger equation as the initial wave function ψ

- A symmetry operator is unitary and commutes with the Hamiltonian

$$[H,U]=0$$

Symmetries and conservation laws

- A system is normally described by its Lagrangian
  - The Lagrangian can be found from first principles or
  - can be deduced through the conservation laws of the system
- Noether's theorem connects symmetries with conservation laws
  - "Every symmetry in nature yields a conservation law and inversely every conservation law reveals an underlying symmetry
  - Momentum conservation: invariance under a translation in space
  - Angular momentum conservation: invariance under rotation in space

Emmy Noether (1882 - 1935)

Categories of symmetries

- Symmetries can be categorised as local or global
  - Global symmetries hold at all points of space-time (e.g. translation)
  - Local symmetries are only valid in certain regions of space-time (i.e. local domains - important for this topic)
- Symmetries can also be characterised as continuous or discrete:
  - Continuous symmetries are viewed as “motions” (e.g. rotations)
  - Discrete symmetries describe non-continuous symmetries in a system (e.g. parity & charge conjugation)

Continuous transformations

- They are unitary by definition but not necessarily Hermitian
- They rely on one or more continuous parameters so that $|\psi'\rangle = U(a)\,|\psi\rangle$
  - rotation by an angle α
- These transformations can be written as a succession of infinitesimal deviations from the identity

$$U(a) = \lim_{n\to\infty}\left(I + \frac{ia}{n}F\right)^n = e^{iaF}$$

- F is called the generator of U
- The generator of a unitary operator is Hermitian
- The generator of a symmetry operator commutes with the Hamiltonian
  - If U is a symmetry operator that commutes with H, then its generator is a Hermitian operator that also commutes with H

Identifying the generators of a transformation

- Let us consider a group of transformations defined by $x_i' = f_i(x_1,\ldots,x_n; a_1,\ldots,a_n)$
  - x_i are the coordinates on which the transformation acts
  - a_i are the elements (i.e. real numbers) of the transformation
- By convention the identity element is a=0 such that

$$x_i' = f_i(x;0)$$

- A transformation in the neighbour of the identity reads

$$dx_i = \sum_{v=1}^{r} \frac{\partial f_i}{\partial a_v} da_v$$

- The generators of the transformation are found by considering a change in a function f(x) and are given by

$$\hat{X}_v = i \sum_{i=1}^{N} \frac{\partial f_i}{\partial a_v} \frac{\partial}{\partial x_i}$$

Invariance under spatial translations

- A system described by a quantum state ψ and is moving, thus changing its spatial coordinates
- For infinitesimally small translations of (ε_1, ε_2, ε_3) we can write

$$X' = X + \varepsilon \Rightarrow \begin{pmatrix} x'_1 \\ x'_2 \\ x'_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}$$

- The generators of the transformation are the elements of momentum

$$\hat{X}_i = -\frac{1}{\hbar}\hat{P}_i$$

- Invariance under spatial translation leads to conservation of momentum

Invariance under spatial rotations

- A system described by a quantum state ψ and is rotating in 3D
- For infinitesimally small rotations of $\delta R(\varepsilon) \sim (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ we can write

$$R = I + \delta R$$

- This operator is unitary $RR^\dagger = I \Rightarrow (I+\delta R)(I+\delta R^\dagger) = I \Rightarrow \delta R = -\delta R^\dagger$
  - δR is represented by an antisymmetric matrix of the form

$$\delta R = \begin{pmatrix} 0 & \varepsilon_3 & -\varepsilon_2 \\ -\varepsilon_3 & 0 & \varepsilon_1 \\ \varepsilon_2 & -\varepsilon_1 & 0 \end{pmatrix}$$

- The generators of the transformation are the elements of angular momentum

$$\hat{X} = \hat{x} \times (-i)\,\hat{\nabla} \sim \hat{L}$$

- Invariance under spatial rotations leads to conservation of angular momentum

Introduction to groups

- A group G is a collection of elements or operators $a_1, a_2, \ldots, a_n$
  - The have defined laws describing how one can combine any of the two elements with an operator e.g. "x" fulfilling the following conditions
  - **Closure**: For each two elements of G, their product is also an element of G

$$a_i \times a_j = a_k$$

  - **Associativity**: Combining two elements is associative

$$(a_i \times a_j) \times a_k = a_i \times (a_j \times a_k)$$

  - **Identity element**: Every group has an identity element e such that for all elements of the group

$$a_i \times e = e \times a_i = a_i$$

  - **Inverse element**: For all elements in G there is a unique element such that

$$a_i \times (a_i)^{-1} = (a_i)^{-1} \times a_i = e$$

- When a group consists of elements that any of two commute, then the group is called **Abelian**, otherwise non-Abelian
- When a group contains finite number of elements n, then it is called finite group of order n

Charge conservation

- All particles have electric charge in multiples of the charge of the quarks q = Nx(1/3)
- In any interaction of this form charge is conserved

$$a+b \rightarrow c+d+e \qquad q_a+q_b=q_c+q_d+q_e$$

- Every conservation law is connected to a symmetry principle
- Assume that ψ is the wave function of a state with charge q, then it obeys the Schrodinger equation

$$i\hbar \frac{\partial|\psi\rangle}{\partial t}=\mathrm{H}|\psi\rangle$$

- If Q is the charge operator then <Q> is conserved
  - the operator commutes with the Hamiltonian of the system [H,Q] = 0
  - The eigenfunction ψ can be an eigenfunction of Q as well

$$\hat{Q}\,|\psi\rangle=q\,|\psi\rangle$$

- Let's identify the observable which is conserved as the electric charge if we apply a global gauge transformation of the form

$$|\psi'\rangle=e^{i\varepsilon Q}|\psi\rangle$$

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Discrete symmetries

- Charge conjugation (C):
  - Converts a particle to its anti-particle (i.e. sign-flip) $Q \rightarrow -Q$

$$C|\pi^+\rangle \rightarrow |\pi^-\rangle \neq \pm|\pi^+\rangle$$

$$C|\pi^0\rangle = \pm 1|\pi^0\rangle \qquad \pi^0 \rightarrow 2\gamma \qquad C|\pi^0\rangle = +1|\pi^0\rangle$$

- Parity (P):
  - Converts right-handed systems to left-handed ones

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \rightarrow \begin{pmatrix} -x \\ -y \\ -z \end{pmatrix}$$

  - Vectors change sign but axial vectors remain unchanged

$$\vec{x} \rightarrow -\vec{x}, \ \vec{p} \rightarrow -\vec{p}, \ \vec{L} = \vec{x} \times \vec{p} \rightarrow \vec{L}$$

- Time reversal T
  - Reverses the direction of motion of particles

Symmetries are part of the building blocks of particle physics. However their validity rests on **experimental verification**!!!

Baryon number conservation

- The much hypothesised decay of protons has been studied by measuring possible decays in a large quantity of water which contains many protons and with large counters shielded from cosmic rays by being positioned underground
  - The reaction $p \rightarrow e^+\pi^0$ has not been detected
  - A lower limit was set to the decay of proton to 1.6×10^{33} years
- The fact that such decay does not seem to take place, led to the introduction of a new quantum number for baryons and antibaryons: the **baryon number (BN)**
- Each baryon is assigned a +1 BN
- Each antibaryon is assigned a -1 BN
- The conservation of BN is not connected to an exact symmetry
  - More of an empirical conservation law than a fundamental one
- There are theories beyond the Standard Model where such a BN violation is allowed

p	+1	$\overline{p}$	-1
n	+1	$\overline{n}$	-1
Λ	+1	$\overline{\Lambda}$	-1
Ξ^-, Ξ^0	+1	$\overline{\Xi^0}$ Ξ^+	-1
Ω^-	+1	Ω^+	-1

Lepton number conservation

- Evidence for lepton number conservation comes mainly from neutrino reactions

$$\bar{\nu}_e + p \rightarrow e^+ + n \qquad n \rightarrow p + e^- + \bar{\nu}_e$$

- Reaction such as the ones below have not been observed, indicating that lepton number violation is not natural

$$\bar{\nu}_e + n \rightarrow p + e^- \qquad \nu_e + p \rightarrow n + e^+$$

- Neutrinos and antineutrinos have different characteristics
  - helicity shows if the spin of a particle points in the direction of or opposite to the momentum
  - Experiments on β-decay have shown that neutrinos are left-handed while antineutrinos are right-handed

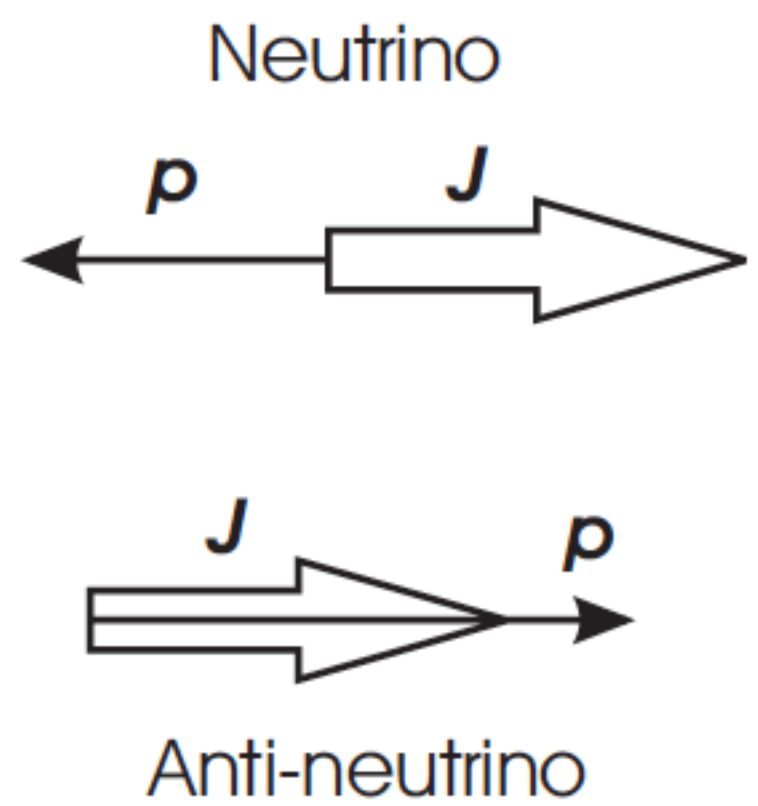


Figure 7.2: Neutrino and antineutrino are always *polarized* if we neglect their very small masses. The neutrino has its spin always opposite to its momentum; the antineutrino has parallel spin and momentum.

Lepton number conservation (cont.)

- The following decays of muons are allowed and are quite common
  - They involve two types of neutrinos, one for the electron and the other for the muon

$$\mu^+ \rightarrow e^+ + \nu_e + \overline{\nu}_\mu \qquad \mu^- \rightarrow e^- + \overline{\nu}_e + \nu_\mu$$

- On the other hand, this decay channel has not been observed
  - The branching ratio is less than 1.2×10^{-11}

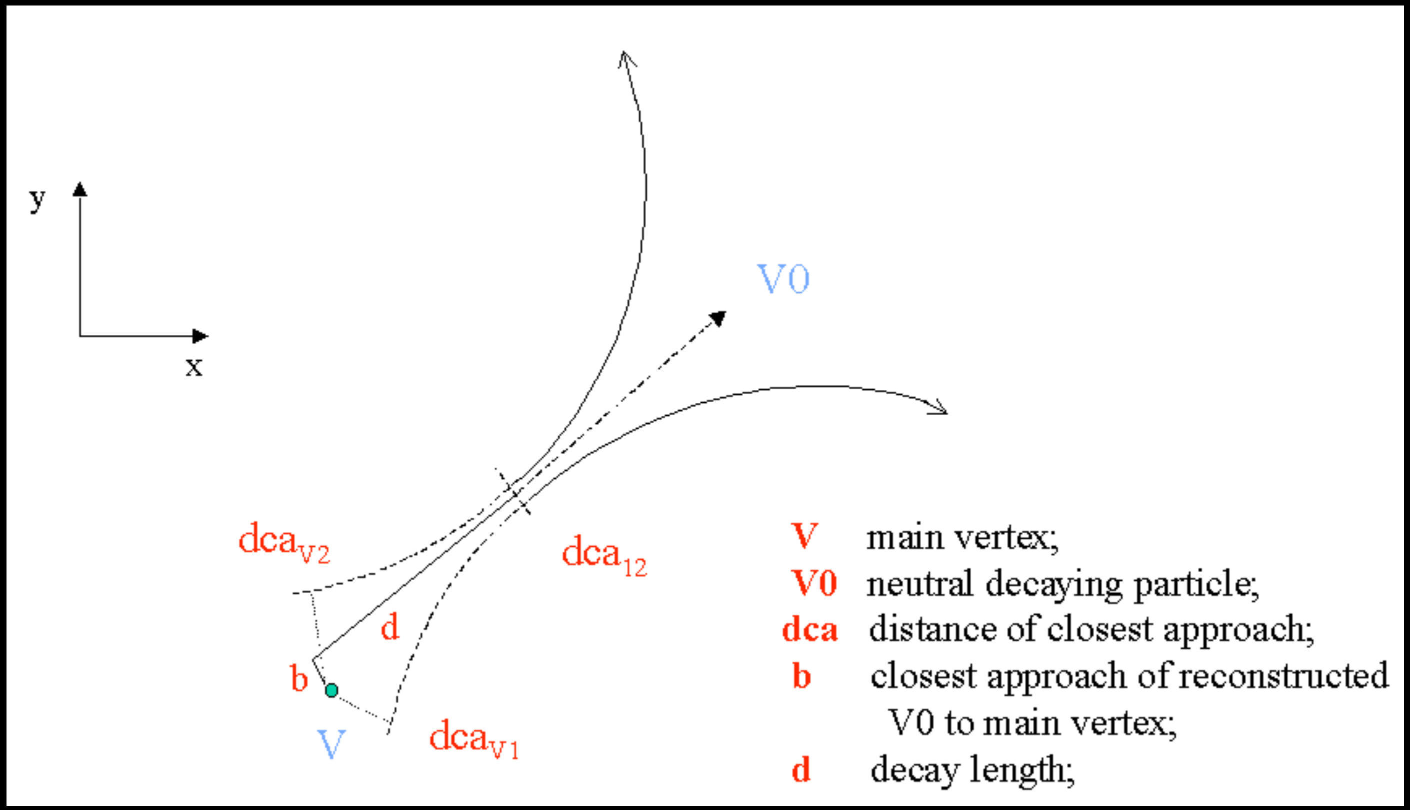
$$\mu^\pm \rightarrow e^\pm + \gamma$$

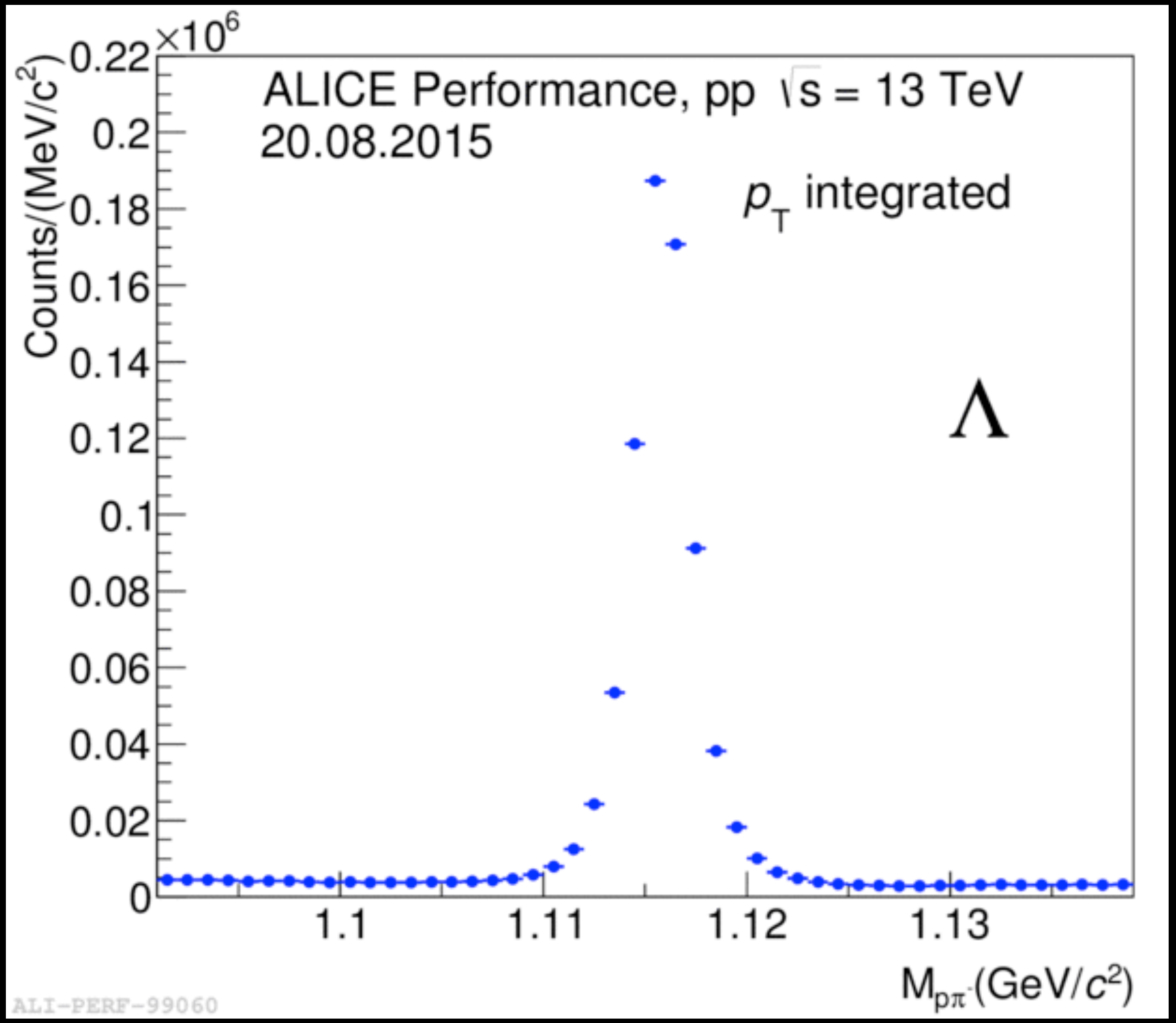
- This indicates that there should be another conservation law, the one of **lepton flavour** (e,μ,τ)

Particle	L_e	L_μ	L_τ	Antiparticle	L_e	L_μ	L_τ
e^-	+1	0	0	e^+	-1	0	0
ν_e	+1	0	0	$\overline{\nu}_e$	-1	0	0
μ^-	0	+1	0	μ^+	0	-1	0
ν_μ	0	+1	0	$\overline{\nu}_\mu$	0	-1	0
τ^-	0	0	+1	τ^+	0	0	-1
ν_τ	0	0	+1	$\overline{\nu}_\tau$	0	0	-1

Strangeness conservation

- Observation of decay of particles with a characteristic V-shape decay topology
  - These decays were rather slow, with a lifetime of 10^{-10}s, typical lifetime of a weak decay
- Gell-Mann and Nishijima introduced a new quantum numbers, called **strangeness**
- Strangeness is conserved in the strong and electromagnetic interactions, but it is violated in the weak interactions





Hypercharge

- We can define hypercharge as the sum of the baryon number and strangeness (and all additional quantum numbers for heavier quark states)

Hypercharge

$$Y = S + C + B + T + BN$$

Table 7.1: Baryon Number A, Strangeness S, Hypercharge Y, and Average Value of the Charge Number $N_q = q/e$.

Particle		A	S	Y	$\langle N_q \rangle$
Photon	γ	0	0	0	0
Pion	$\pi^+\pi^0\pi^-$	0	0	0	0
Kaon	K^+K^0	0	1	1	$\frac{1}{2}$
Nucleon	pn	1	0	1	$\frac{1}{2}$
Lambda	Λ^0	1	-1	0	0
Sigma	$\Sigma^+\Sigma^0\Sigma^-$	1	-1	0	0
Cascade	$\Xi^-\Xi^0$	1	-2	-1	$-\frac{1}{2}$
Omega	Ω^-	1	-3	-2	-1

Quantum numbers of quarks

Table 7.2: Quantum Number Assignments for the Six Quarks.

	Quantum Number					
Quark	A	S	C	B	T	Y_{gen}
d	1/3	0	0	0	0	1/3
u	1/3	0	0	0	0	1/3
s	1/3	−1	0	0	0	−2/3
c	1/3	0	1	0	0	4/3
b	1/3	0	0	−1	0	−2/3
t	1/3	0	0	0	1	4/3

- ✓ For quarks we can introduce the generalised hypercharge, defined as

$$Y_{gen} = A + S + C + B + T = 2\langle q/e \rangle$$

Isospin

- Neutron and proton are quite similar apart from their charge
- Heisenberg proposed that they are regarded as the two states of the same particle
    - the nucleon
- Similar to the notation related to spin we can write p and n with a two component column matrix
- By direct analogy to spin we introduce isospin with coordinates in the isospin space:
    - I_1, I_2, I_3
- Strong interactions are invariant under rotations in isospin space
    - Isospin is conserved
- Group theory wording:
    - Strong interactions are invariant under an internal symmetry of SU(2)