Convex Analysis for Optimization - Exercise set 2

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Important note: you can certainly find the solutions to some of the below exercises. However, if you do so right at the start, you will not learn. So, try as much as you can on your own before searching for an existing solution.

Problem 1. Prove or give a counterexample:

$$A\mathrm{ri}(S) = \mathrm{ri}(AS)$$

where $A : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation and $S \subset \mathbb{R}^n$ is a nonempty convex set.

Problem 2. Consider a nonempty convex set $S \subset \mathbb{R}^n$. Recall the definition of relative boundary $\mathrm{rb}(S)$ of S being $S \setminus \mathrm{ri}(S)$. Is it true that if we select $x_0 \in \mathrm{ri}(S)$ and a point $x \notin S$, then the straight line segment connecting x_0 and x must contain a point on $\mathrm{rb}(S)$? Motivate your answer.

Problem 3. Prove or give a counterexample:

$$\mathrm{ri}(A) + \mathrm{ri}(B) = \mathrm{ri}(A + B).$$

where A, B are convex sets. How does the statement change if the sets are not necessarily convex?

Problem 4. Prove that for any $S \subseteq \mathbb{R}^n$, the convex hull $\mathrm{conv}(S)$ is the smallest convex set containing S, i.e., the intersection of all convex sets containing S. Analogously, prove that the conic hull $\mathrm{cone}(S)$ is the smallest convex cone containing S.

Problem 5. Show that each extreme point of a closed convex set $S \subseteq \mathbb{R}^n$ is a relative boundary point (i.e., belongs to $S \setminus \mathrm{ri}(S)$).

Problem 6. Provide an example of a non-empty compact convex set S in $\mathbb{R}^2$ whose $\mathrm{cone}(S)$ is not closed. Which additional assumption do you need to ensure closedness of $\mathrm{cone}(S)$ for a non-empty compact convex set S and why?

Problem 7. Prove or provide a counterexample: any convex set S contains a nonzero direction of recession if and only if S is unbounded.