Stability Properties of Proportional Fairness and MaxWeight

Neil Walton (Manchester)

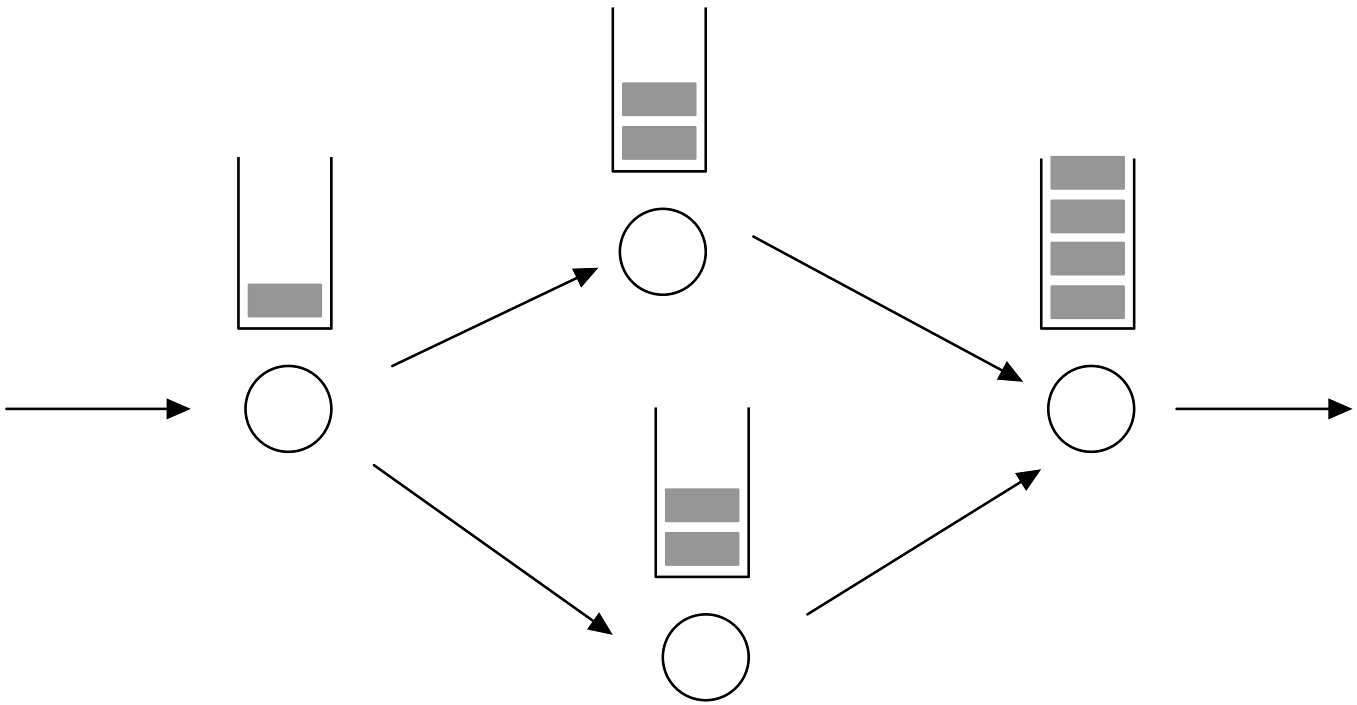
Joint with Maury Bramson, Bernardo D'Auria

Outline

- Background
- Switch Queueing Model
- Stability and Instability of MaxWeight
- Stability of Proportional Fairness
- Fixing Instability of MaxWeight

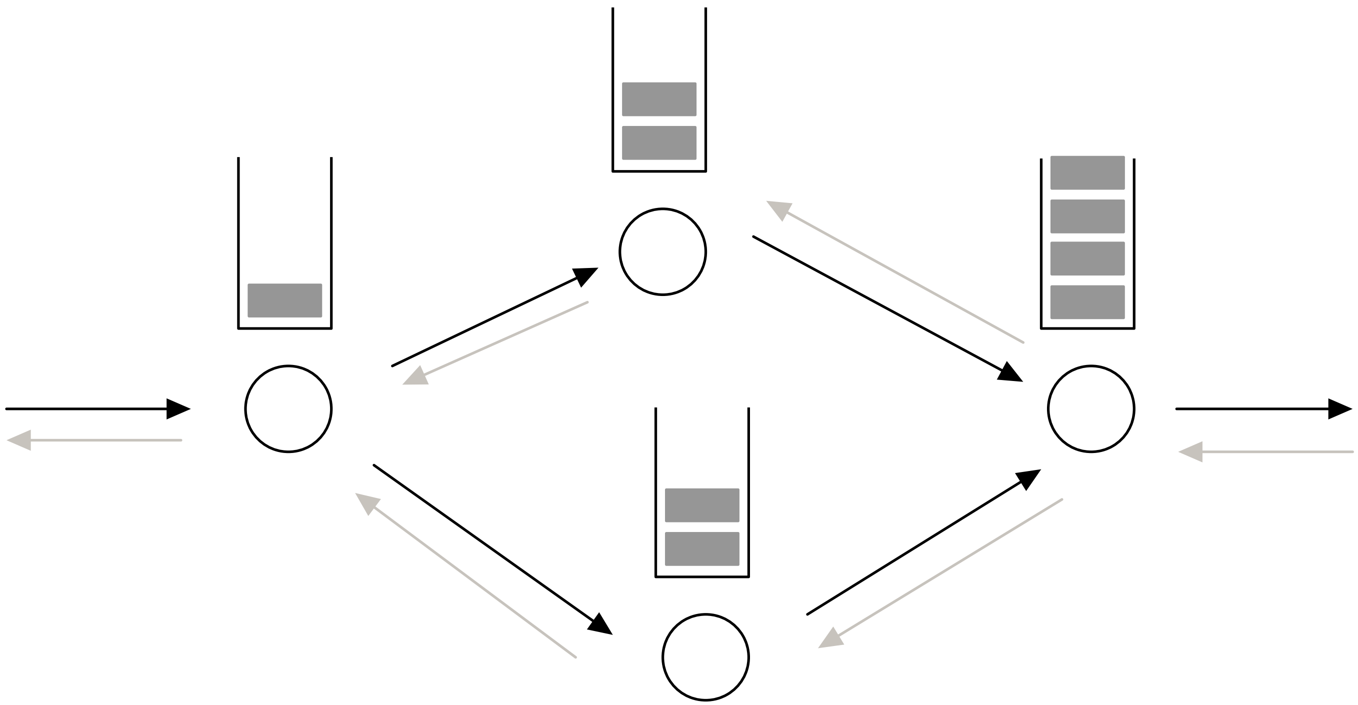
1. Background

Classical Queueing



$$\text{if} \qquad \rho_j < C_j \quad \forall j$$

Classical Queueing



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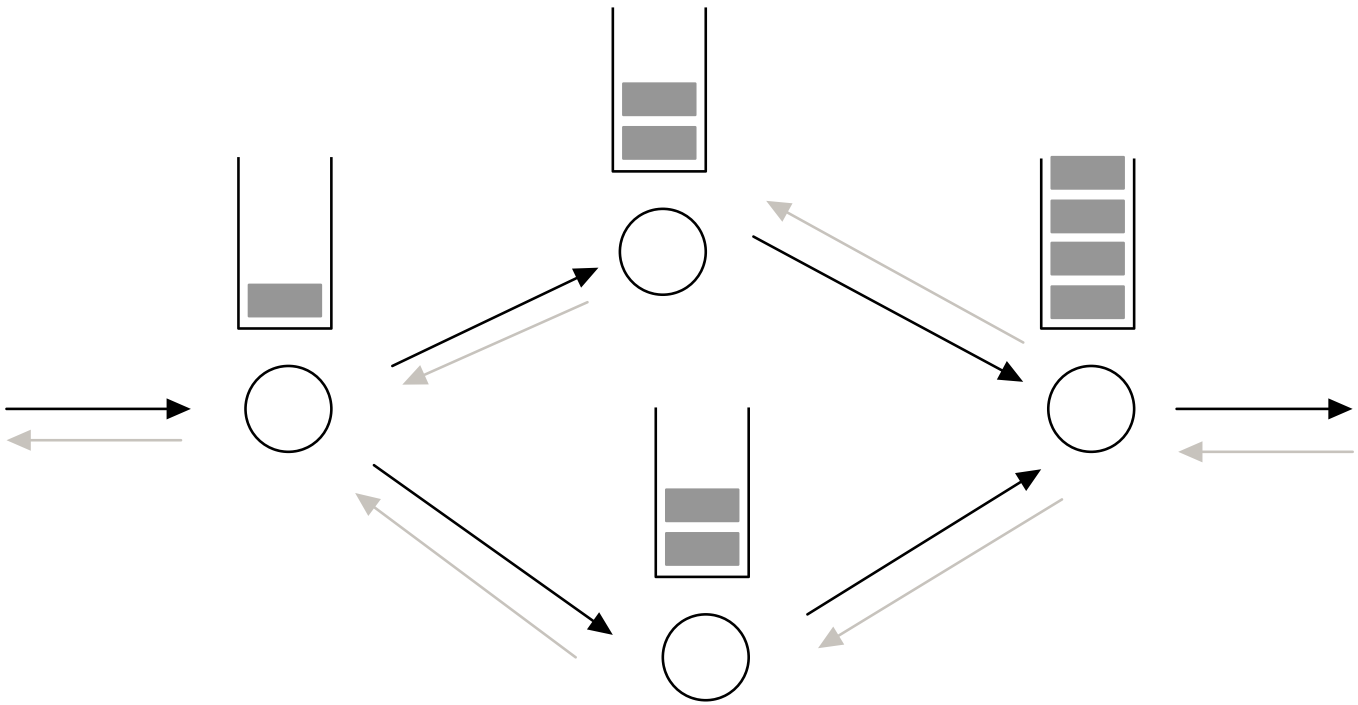
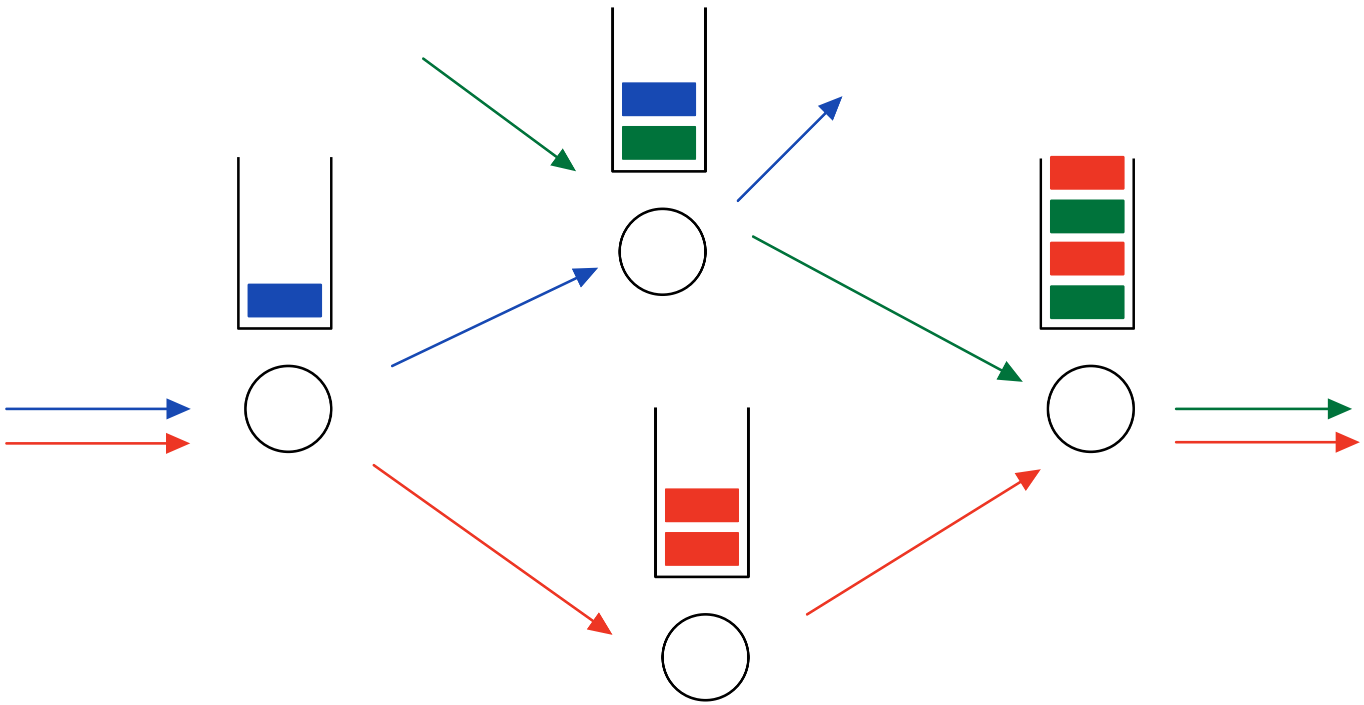


Reversing time establishes independence :

$$\pi(q) = \prod_{j \in J} \left(\frac{\rho_j}{C_j} \right)^{q_j} \qquad \text{if} \qquad \rho_j < C_j \quad \forall j$$

Classical Queueing



Quasi-reversibility establishes independence :

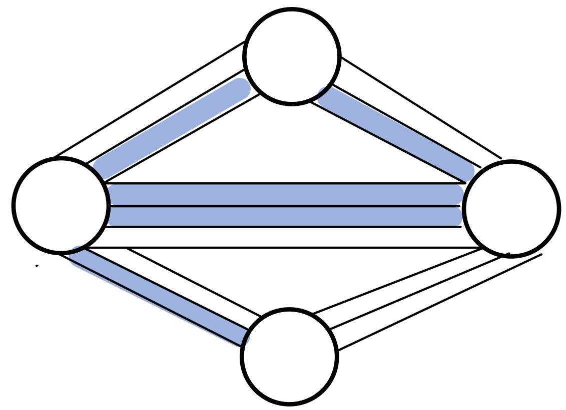
$$\pi(q) = \prod_{j \in J} \binom{q_i}{q_{ij} : i \in j} \prod_{i \in j} \left(\frac{\rho_i}{C_j} \right)^{q_{ij}} \qquad \text{if} \quad \sum_{i \in j} \rho_{ij} < C_j \quad \forall j$$

√GOOD

Asymptotic Analysis

Studying Limit behaviour provides insights :

Example 1 (Loss network)



As the number of links gets large

$$\pi(nq) \xrightarrow[n\to\infty]{} f(q)$$

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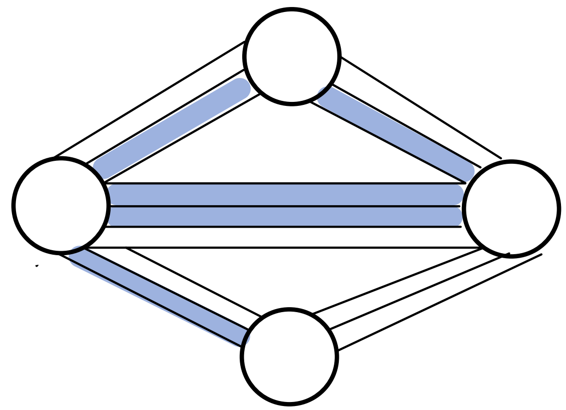


As the number of links gets large

$$\pi(nq) \xrightarrow[n\to\infty]{} f(q)$$

Primal (most likely state)

$$\max_q \quad f(q) \quad \text{subject to} \quad \sum_{i\in j} q_{ij} \leq C_j$$

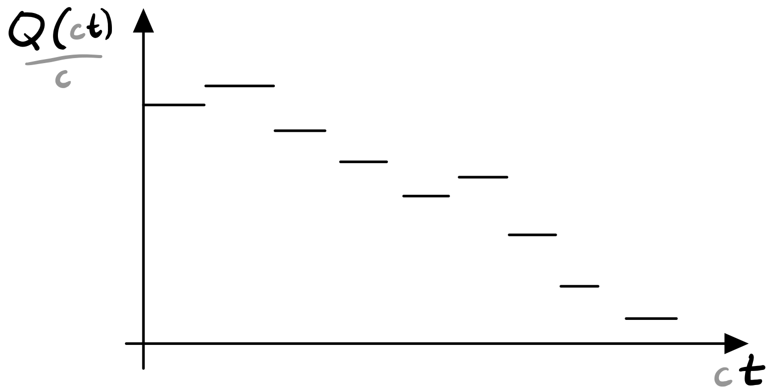
Dual (blocking probabilities)

$$\min_B \quad \delta(B) \quad \text{subject to} \quad 0 \leq B_j \leq 1$$

Asymptotic Analysis

Studying Limit behaviour provides insights :

Example 2 (Fluid Limits and Stability)



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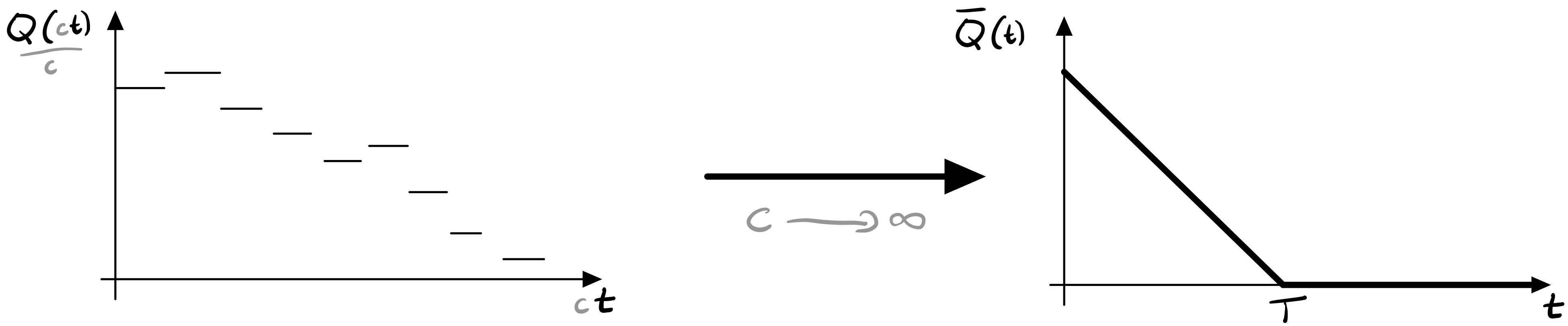

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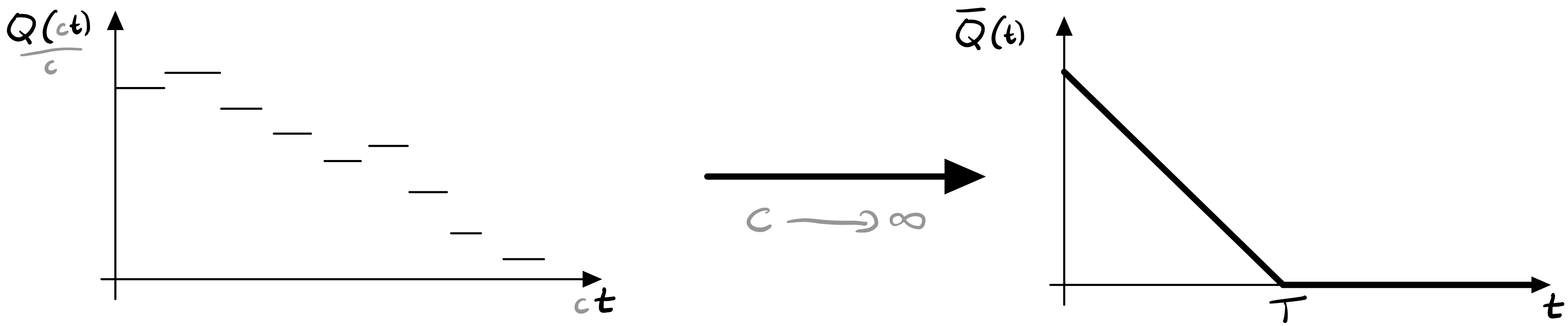


Theorem (Dai): Under appropriate conditions

If $\overline{Q}(t)$ is stable then $Q(t)$ is stable.

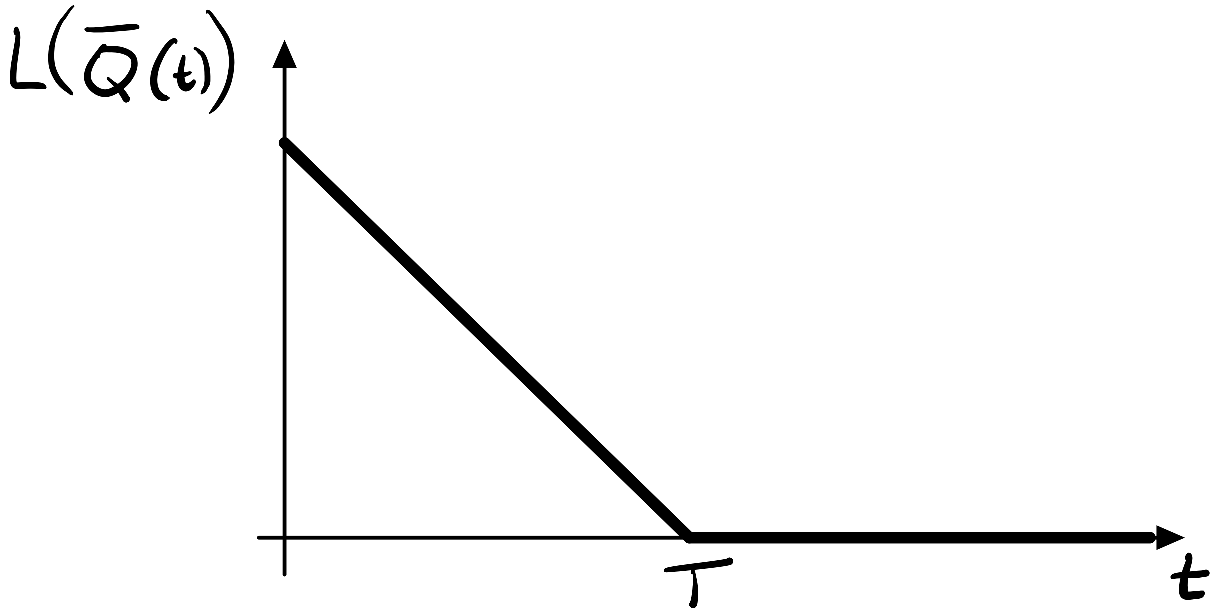
$\overline{Q}(t) = 0 \ \forall t \geq T$

POSITIVE RECURRENT MARKOV CHAIN.

Stability

Lyapunov Function:

$$L : \mathbb{R}^q \rightarrow \mathbb{R}_+$$



Such that

$$\frac{dL\left(\overline{Q}\left(t\right)\right)}{dt} < -\varepsilon \quad \text{when} \quad \bar{Q}\left(t\right) \neq 0$$

Stability

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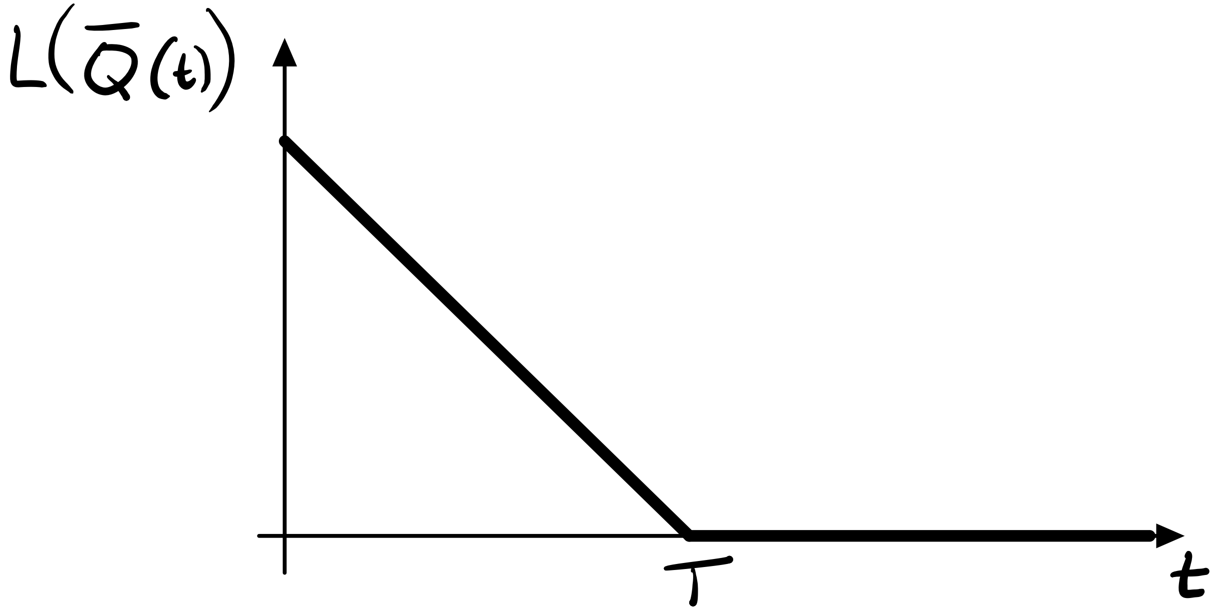

Such that

$$\frac{dL\left(\overline{Q}\left(t\right)\right)}{dt} < -\varepsilon \qquad \text{when} \quad \bar{Q}\left(t\right) \neq 0$$

This implies

$$\overline{Q}\left(t\right) = 0 \qquad \forall t > T$$

Instability

Is necessary for stability :

$$\sum_{i \in j} \rho_{ij} < C_j \quad \forall j$$

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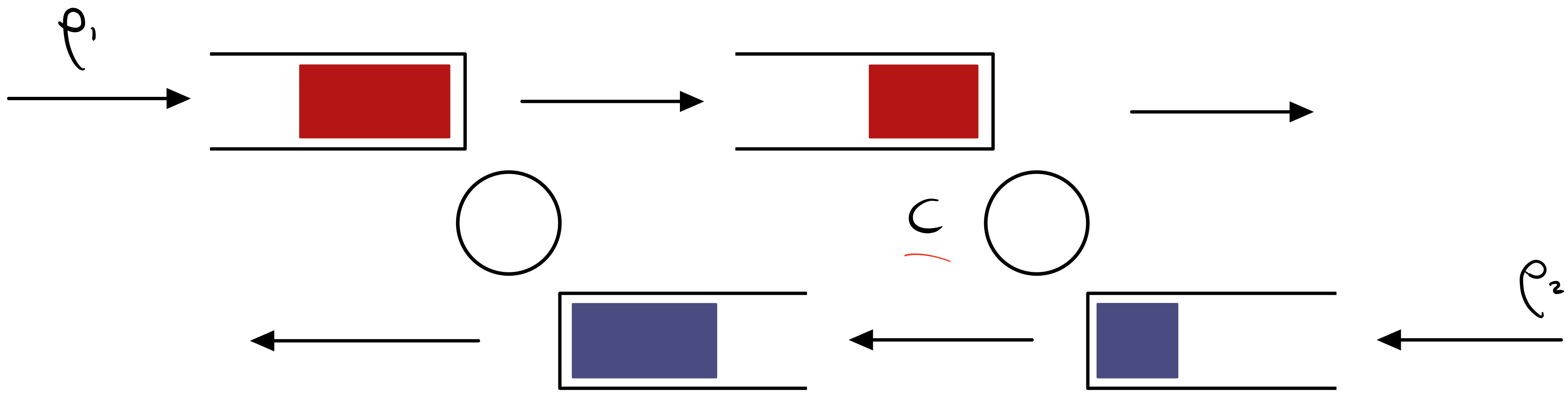
$$\sum_{i \in j} \rho_{ij} < C_j \quad \forall j$$

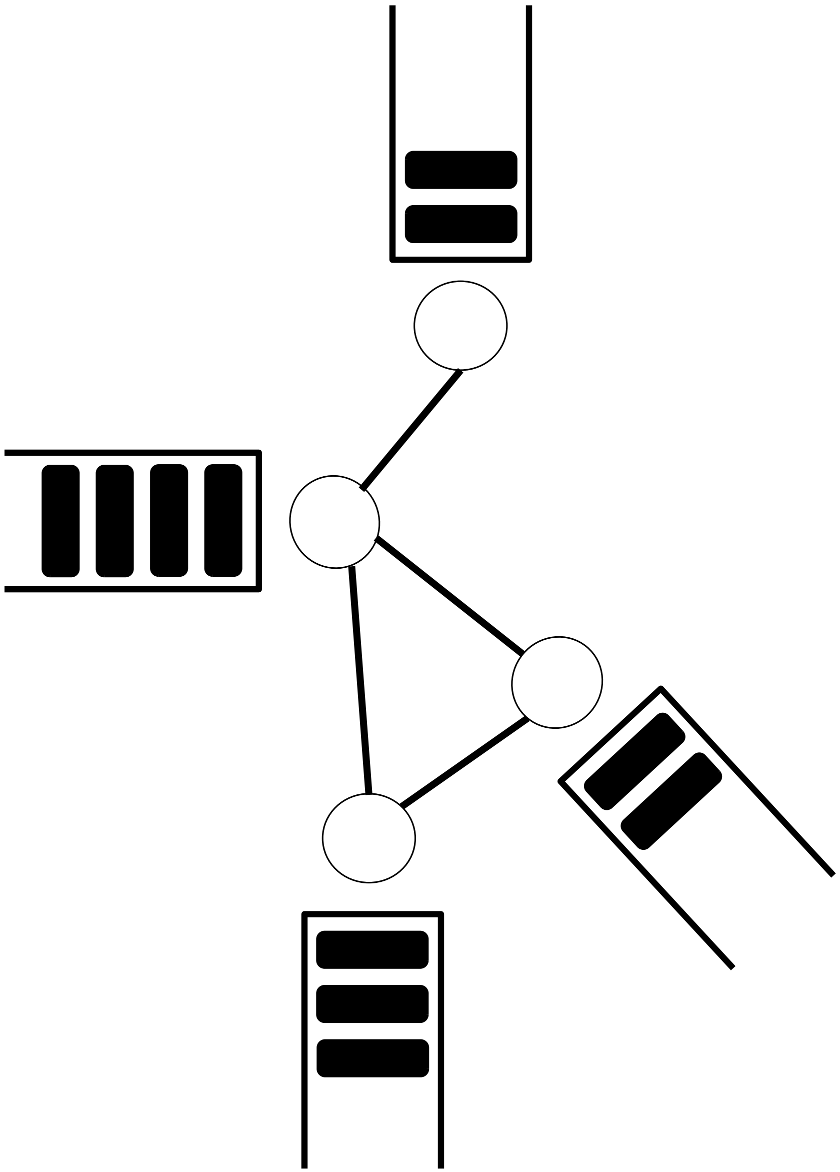
But is it sufficient? No:

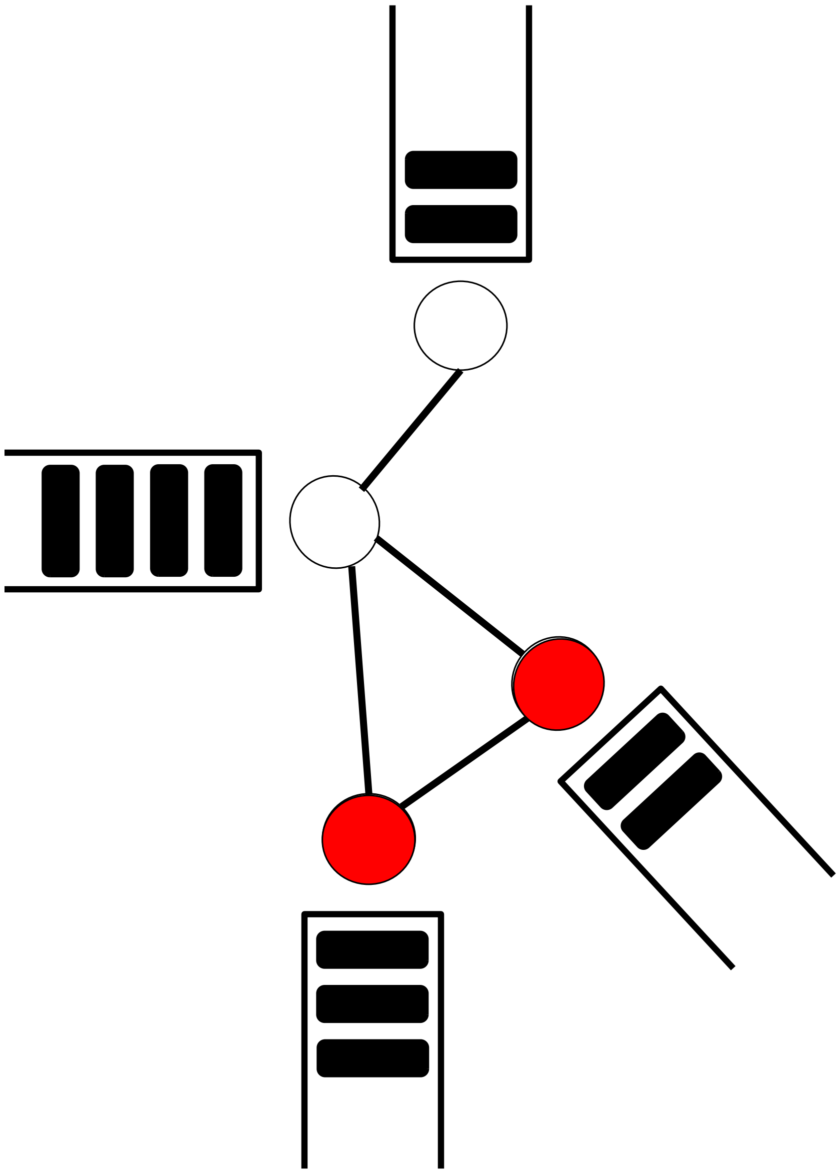


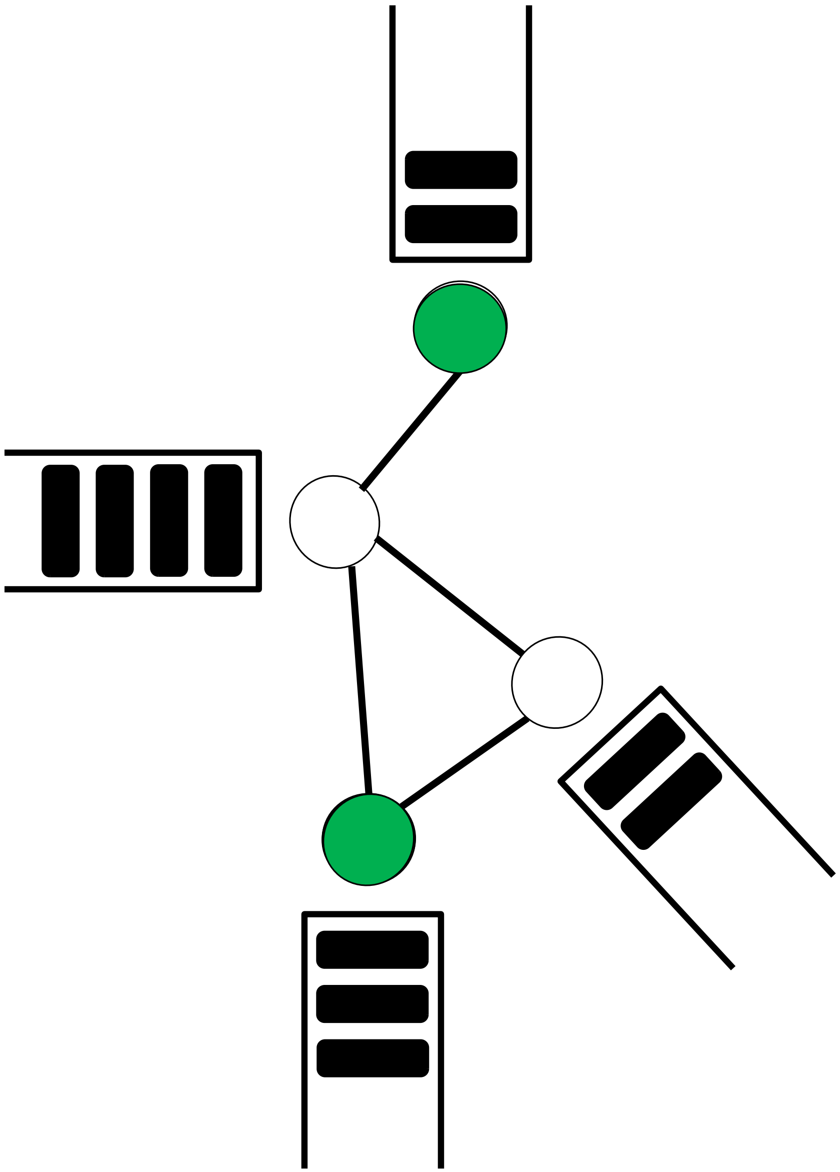
$\rho_1 + \rho_2 \leq C$

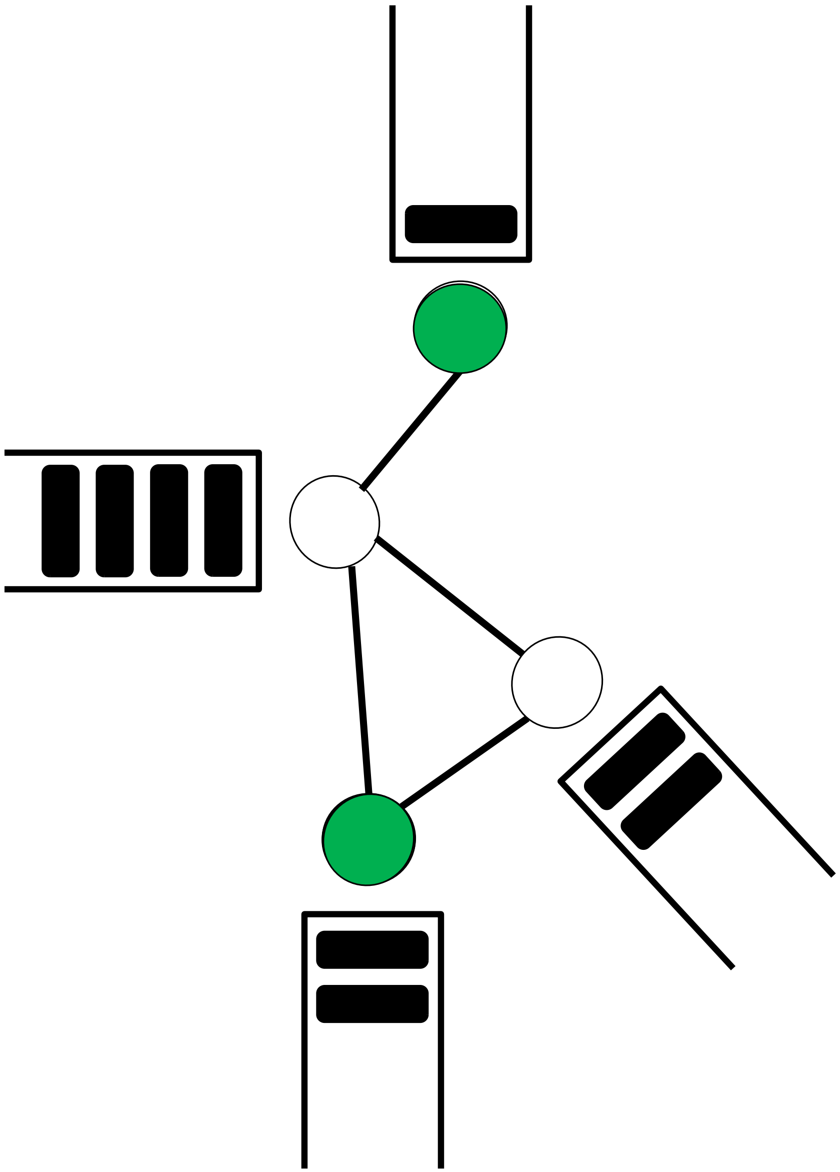
Rybko-Stolyar

2. Switched Queueing Networks

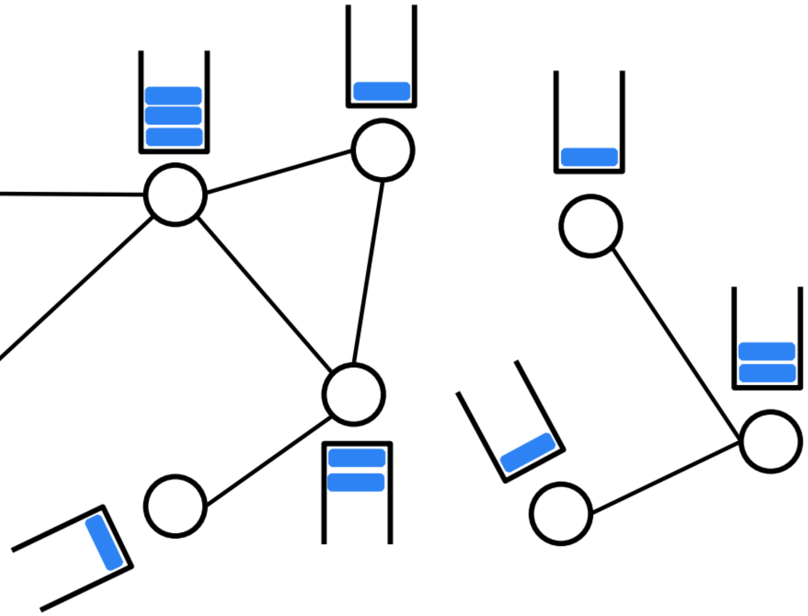




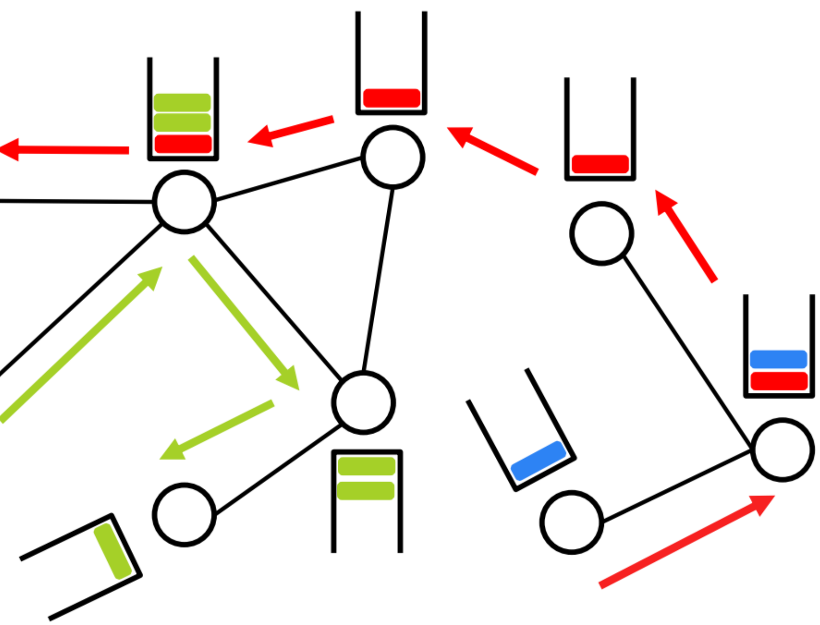




Single-hop Switched Networks

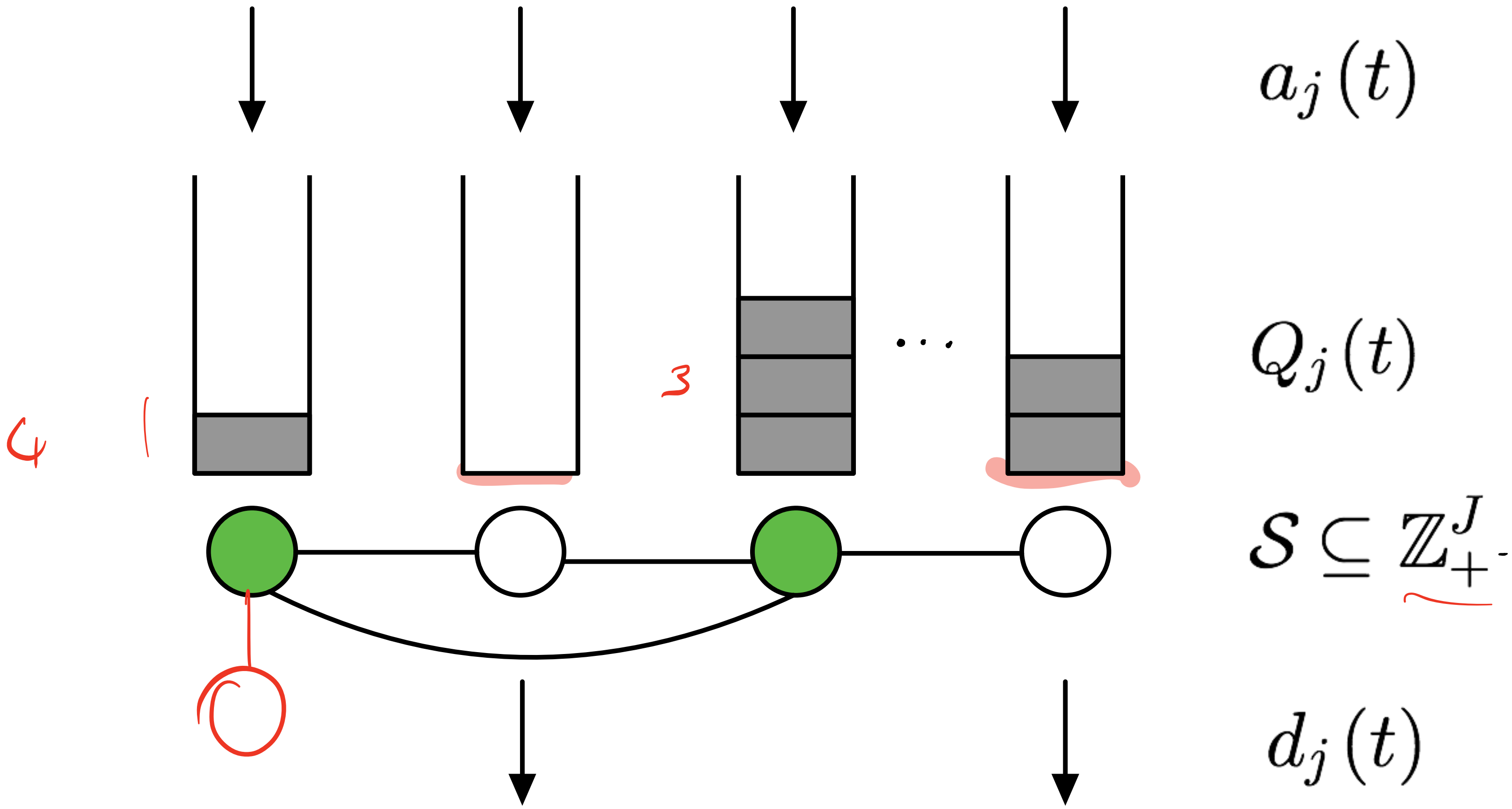


Multihop Switched Networks



3. Stability and Instability of MaxWeight

MaxWeight

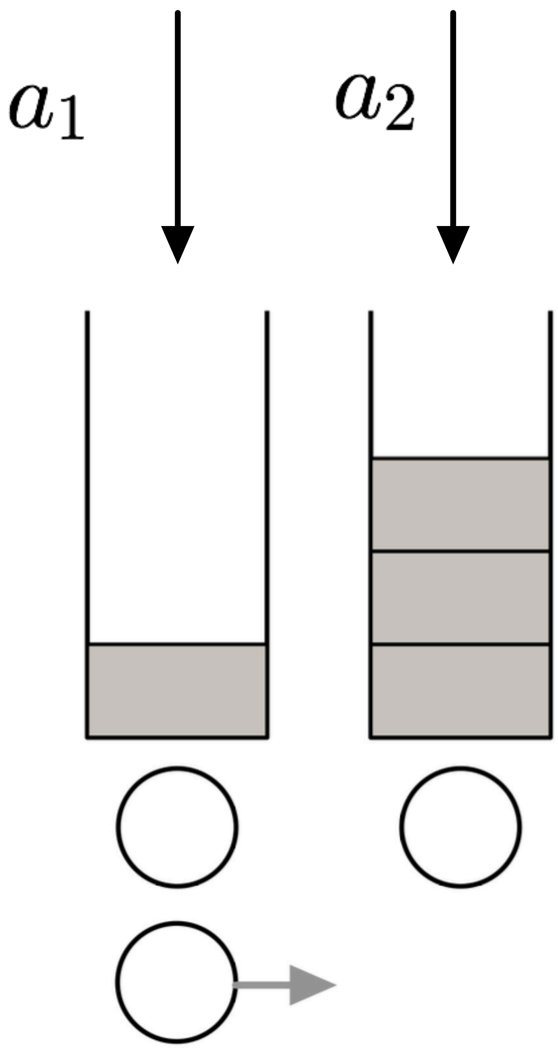


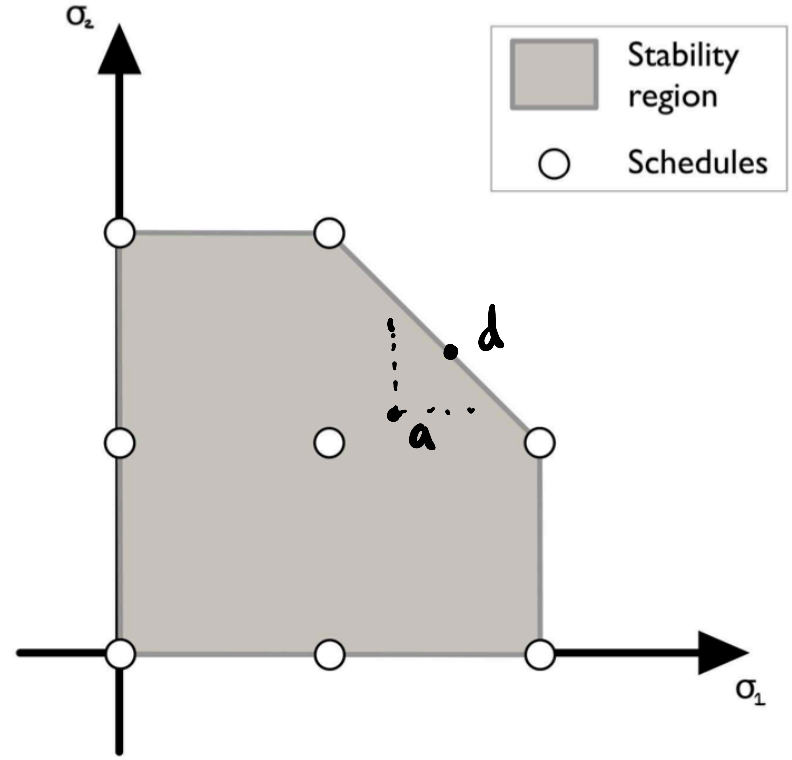
MAX-WEIGHT:

$$d(t) \in \operatorname*{argmax}_{d \in \mathcal{S}} \sum_j d_j Q_j(t)$$

When can we stabilise?

MAXIMAL STABILITY REGION IS CONVEX CLOSURE





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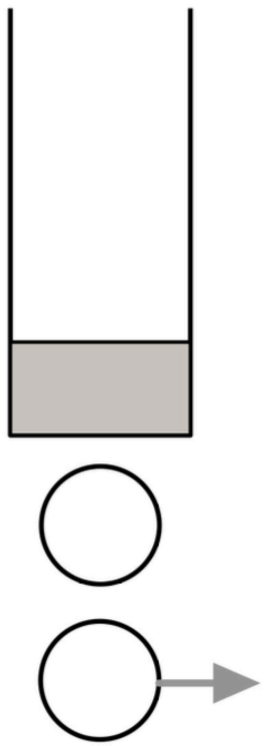
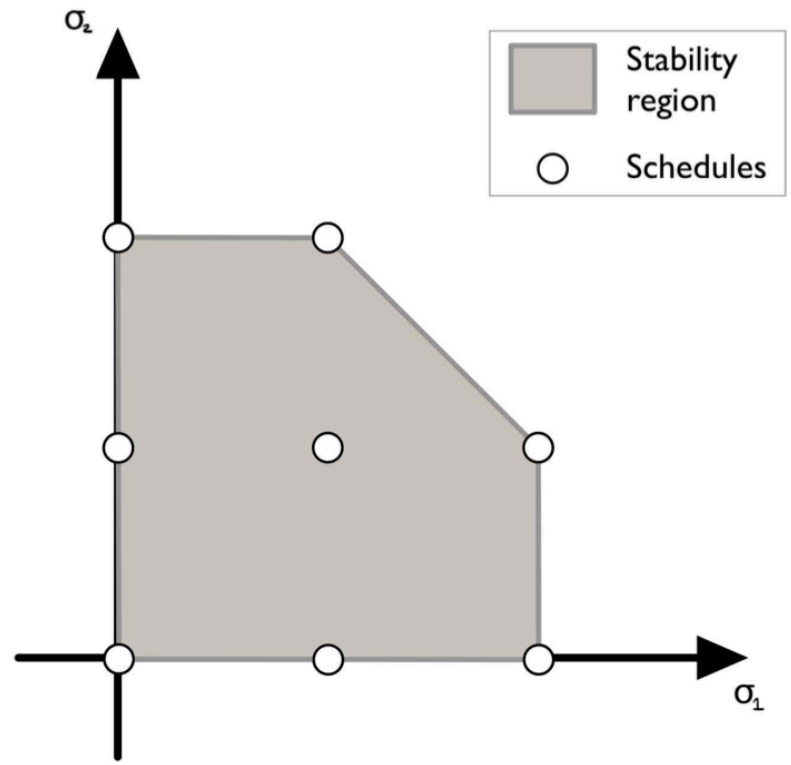




THEOREM: (Tassiulas & Ephremedes)

MAXWEIGHT IS STABLE FOR ALL RATES IN $\langle S \rangle$

Sketch Proof

$$L(Q) = \sum_j \frac{Q_j^2}{2} \qquad \& \qquad \frac{dQ_j}{dt} = a_j - d_j(t) + d_w$$

$$\therefore \quad \frac{dL(Q(t))}{dt} = \sum_j \frac{\partial L}{\partial Q_j} \frac{dQ_j}{dt}$$

$$= \sum_j Q_j \left(a_j - d_j(t)\right)$$

$$= \sum_j Q_j a_j - \sum_j Q_j d_j(t) < 0$$

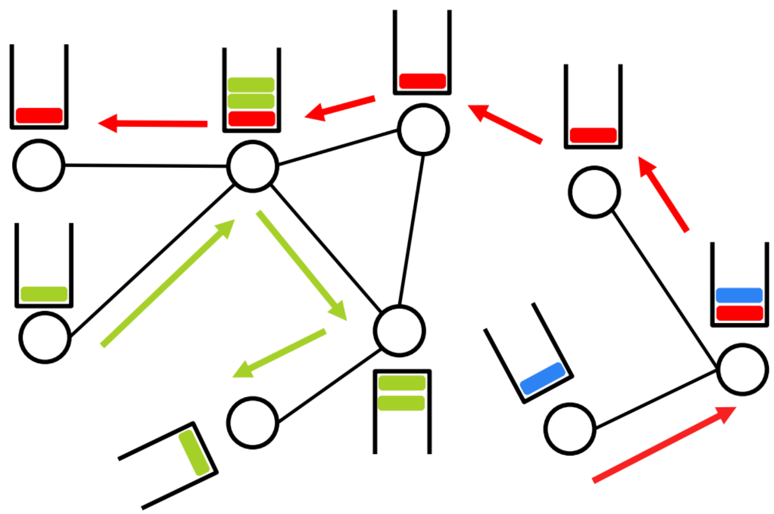
MaxWeight Maximizes This.

$\square$

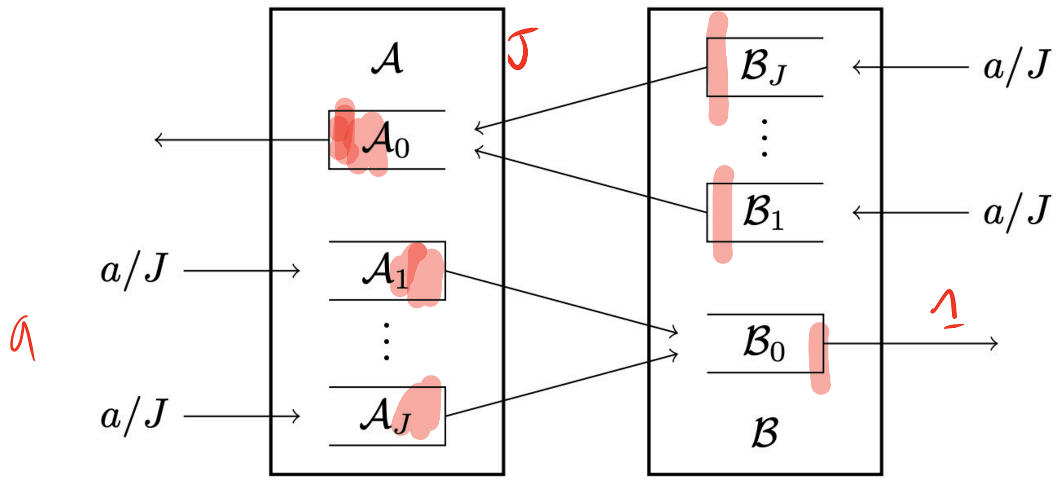
Because of these stability results MaxWeight has been studied extensively in the context of communication systems.

However packets leave after service.
So there is no communication.

<u>Question</u>: Is MaxWeight Stable when there is communication?



Answer: No


$\mathcal{A}$
$\mathcal{A}_0$
a/J
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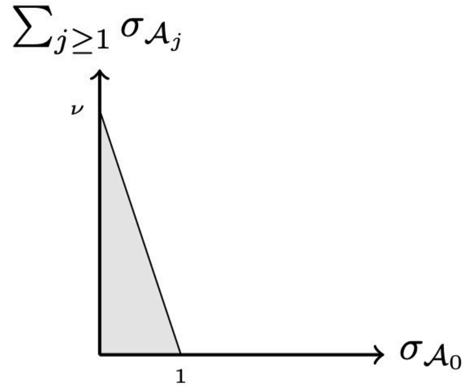
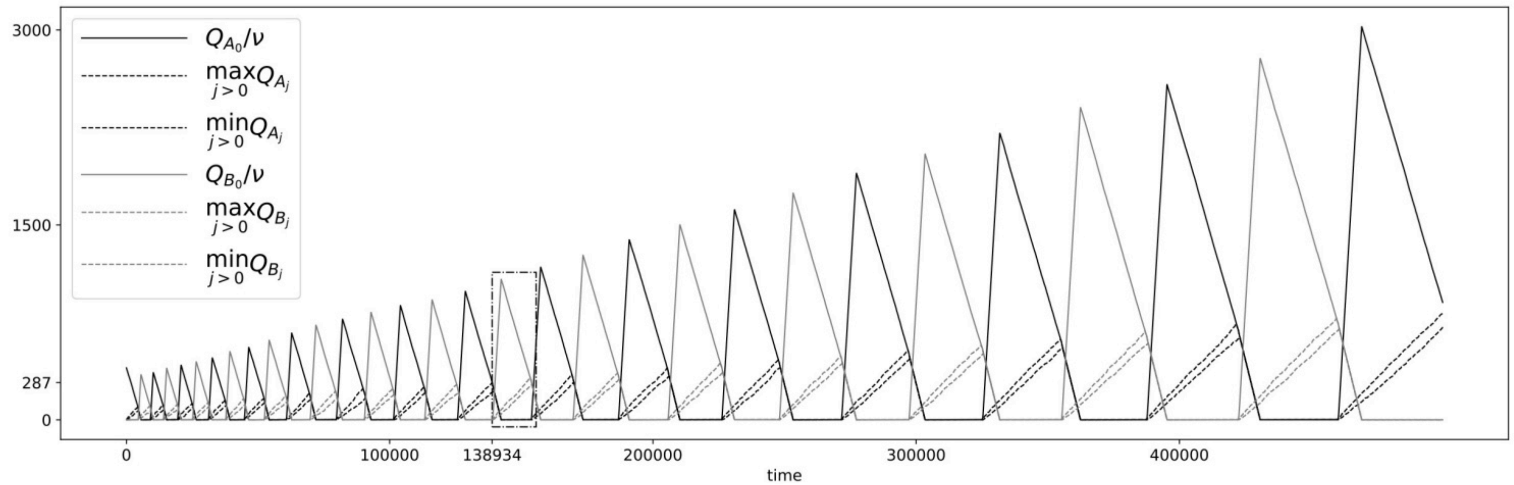
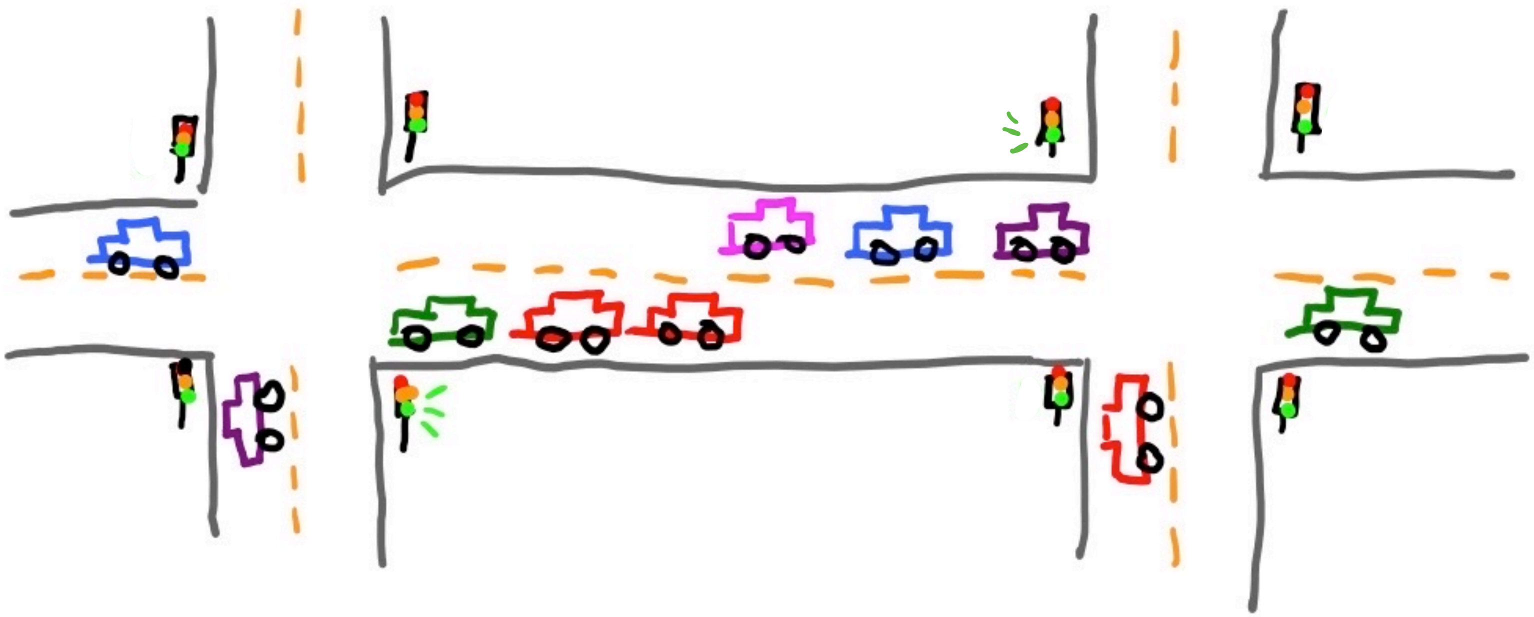
$\mathcal{B}_J$
a/J
$\mathcal{B}_1$
a/J
$\mathcal{B}_0$
$\mathcal{B}$


$\sum_{j\geq 1} \sigma_{\mathcal{A}_j}$
ν
1
$\sigma_{\mathcal{A}_0}$


Q_{A_0}/ν
$\max_{j>0} Q_{A_j}$
$\min_{j>0} Q_{A_j}$
Q_{B_0}/ν
$\max_{j>0} Q_{B_j}$
$\min_{j>0} Q_{B_j}$
3000
1500
287
0
0
100000
138934
200000
300000
400000
time

Answer: No



SURPRISINGLY, THE MOST OBVIOUS POLICY:

SERVE THE LONGEST QUEUE

IS A BAD IDEA.

4. Stability of Proportional fairness

Aim: To find a maximally stable policy “similar” to MaxWeight.

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One Solution: Extend the quadratic Lyapunov argument used for MaxWeight.

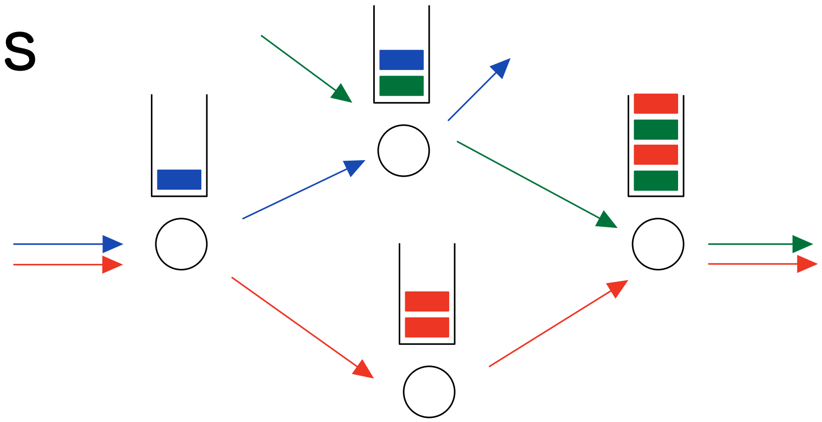
↖ THERE ARE SOME ISSUES WITH THIS

Another Solution: Analyse a different policy with good stability properties.

↖ WE PURSUE THIS APPROACH.

Classical queueing network has good stability properties:

$$\sum_{i \in j} \rho_{ij} < C_j \quad \forall j$$



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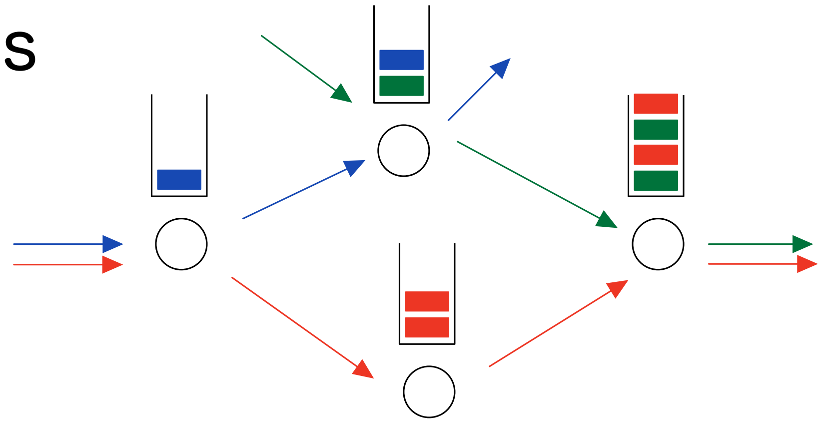

$$\sum_{i \in j} \rho_{ij} < C_j \quad \forall j$$

$$\frac{1}{n} \log \pi (nq) \xrightarrow[n \to \infty]{} - \sum_j D_j \left(q \middle\| \frac{\rho}{c} \right)$$

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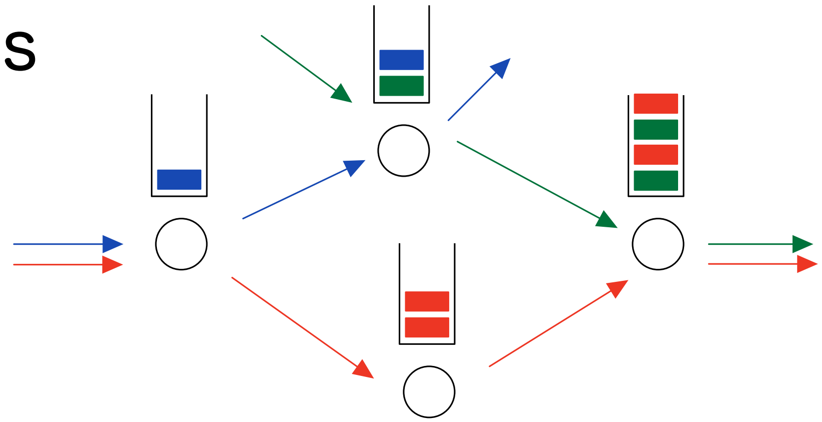

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Primal:

$$\min_q \sum_j D_i \left(q \middle\| \frac{\rho}{C} \right) \quad s.t. \quad \sum_j q_{ij} = Q_i$$

Dual:

$$\max_\Lambda \sum_i Q_i \log \Lambda_i \quad s.t. \quad \sum_{i : j \in i} \Lambda_i \leq C_j$$

↑ PROPORTIONAL FAIRNESS

Proportional Fairness

$$\max_{\Lambda} \sum_i Q_i \log \Lambda_i \quad s.t. \quad \sum_{i:j \in i} \Lambda_i \leq C_j$$

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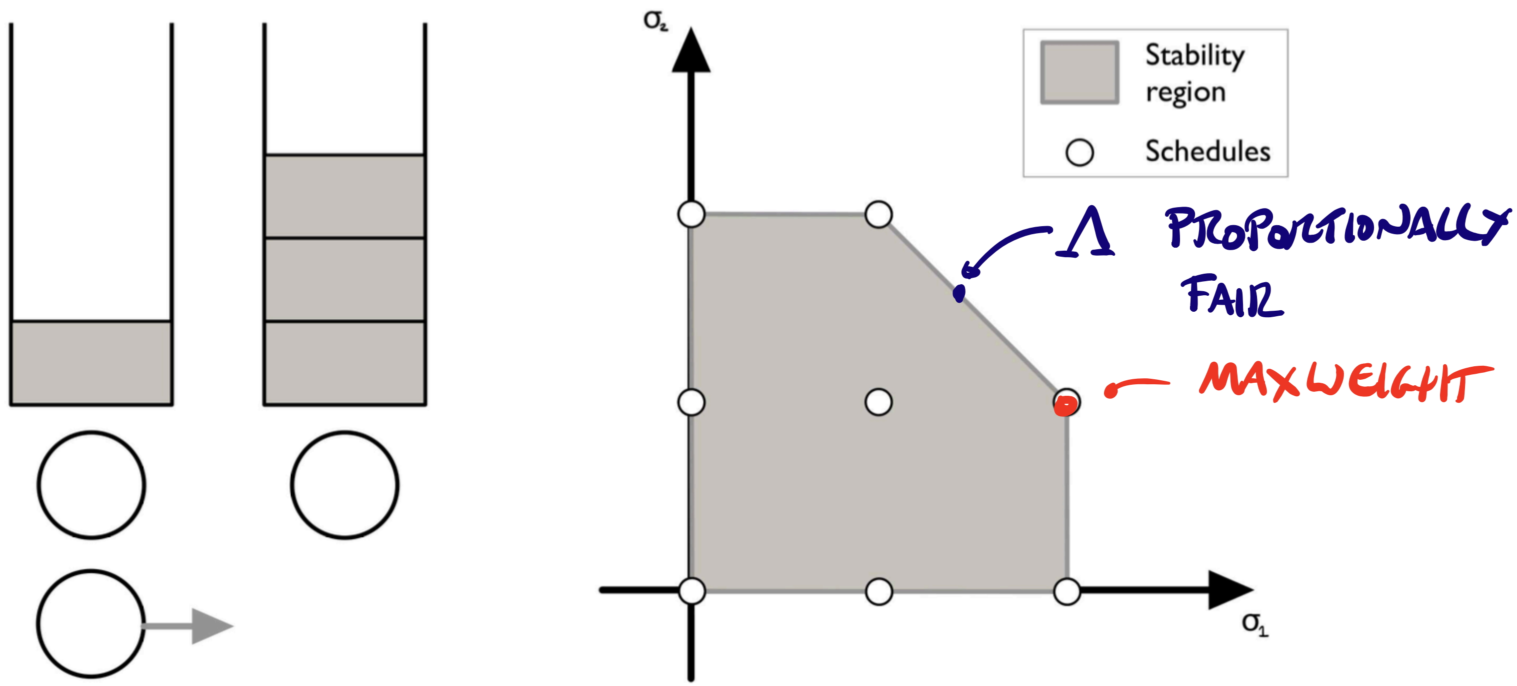


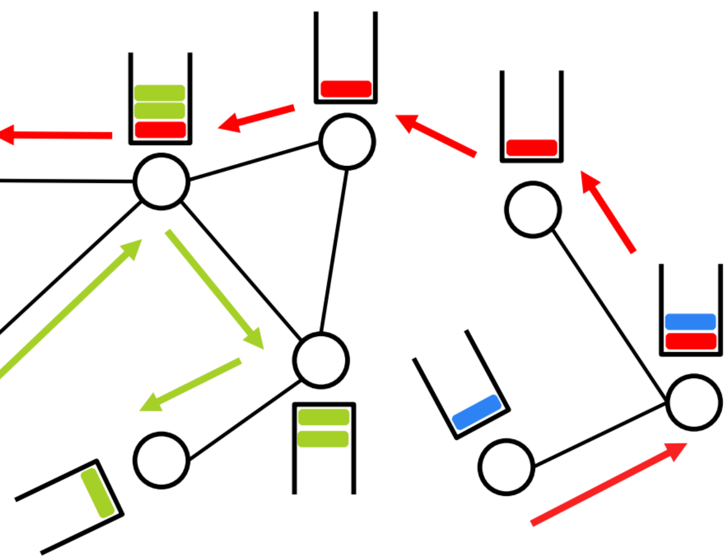
Is used for data scheduling on 4G mobile

Convergence and Stability

Theorem (Fluid Stability)

For Proportionally fair fluid model, when $a \in < S >^{\circ}$ there exists

T>0 such that $Q(t)=0$, for all $t > T$.



FIFO Model

1. $Q_j(t) = Q_j(0) + A_j(t) - D_j(t)$

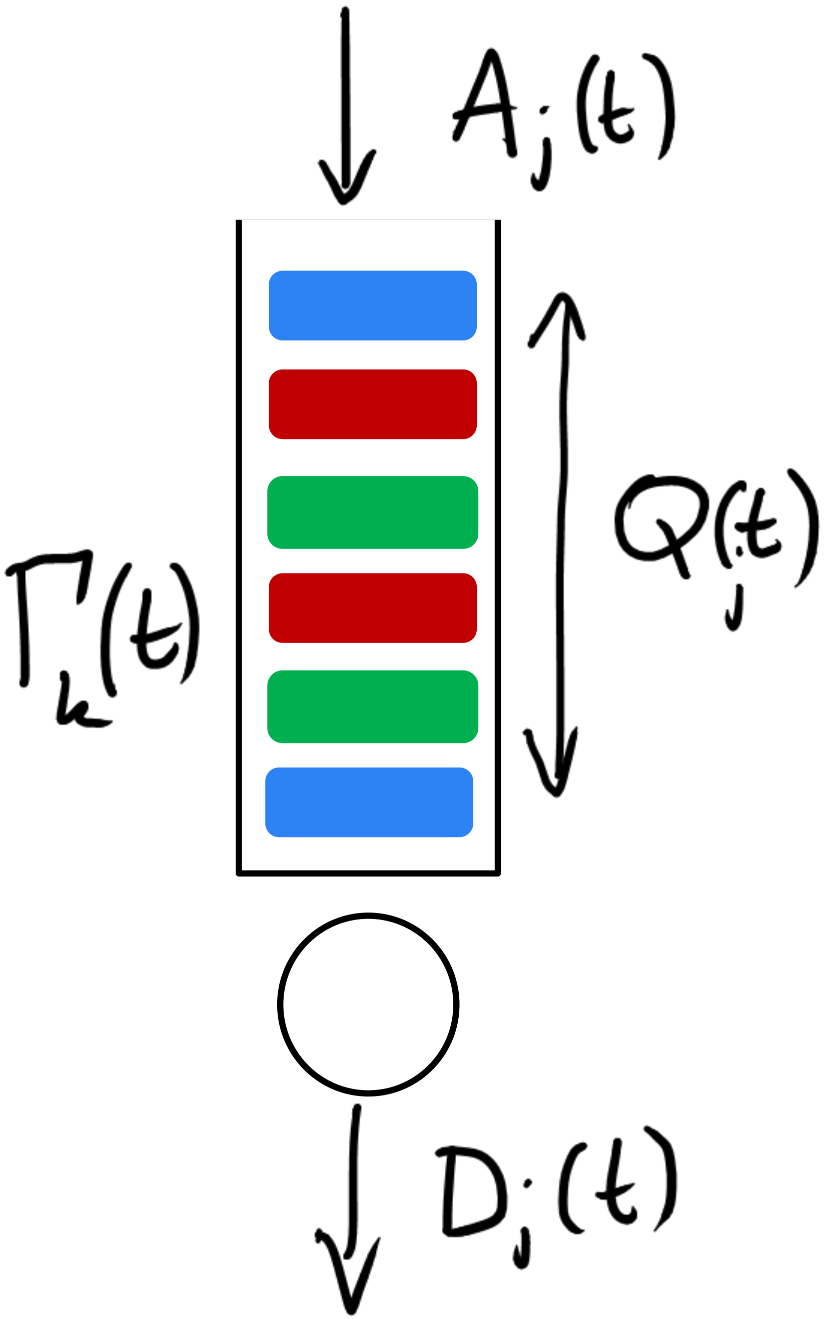
2. $\sum_{k \in j} \Gamma_k(s) = s$

3. $\left(\frac{D_j(t) - D_j(s)}{t-s} : j \in J\right) \in < S >$

4. $A_k(t) = \Gamma_k\left(A_j(t)\right)$

5. $D_k(t) = \Gamma_k\left(D_j(t)\right)$

6. $A_{n(k)}(t) = D_k(t)$



Fluid Limit

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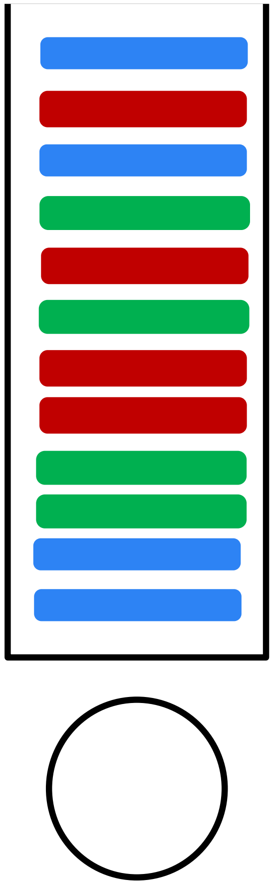

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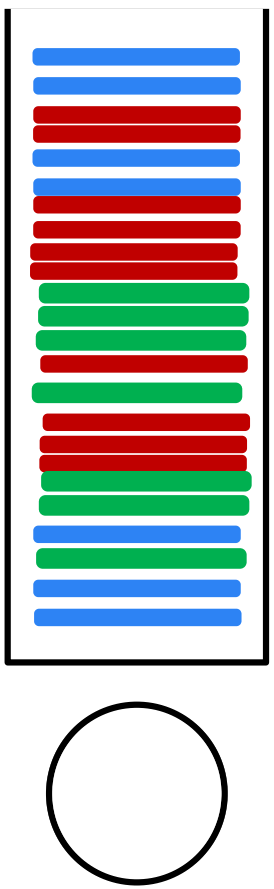

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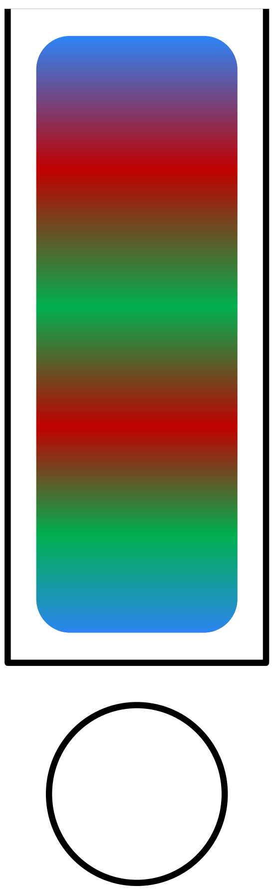


Lyapunov Function

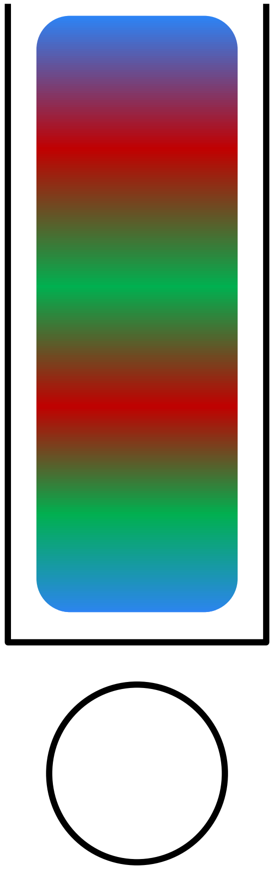
Idea: Under SLLN scaling, what is the Large Deviations rate function for the equilibrium network

1.

$$\lim_{c\to\infty}\frac{1}{c}\log P(Q) = -\max_{\sigma\in<S>}\sum_{j\in J} Q_j \log\sigma_j =: -L$$

2. (Sanov's Theorem)

$$\lim_{c\to\infty}\frac{1}{c}\log P(\Gamma|Q) = -\sum_{j\in J}\sum_{k\in j}\int_{D_j}^{A_j}\Gamma_k'(s)\log\frac{\Gamma_k'(s)}{a_k}ds =: M$$



Entropy Lyapunov function:

H(t):=L(t)+M(t)

Derivative of Lyapunov Function

Proposition:

$$H'(t) = -\sum_{r\in R} D_r'(t) \log \frac{D_r'(t)}{A_r'(t)} - \sum_{j\in J} A_j'(t) \log \frac{A_j'(t)}{D_j'(t)} < 0$$

✓ PROOF OF THEOREM COMPLETE □

5. Fixing the instability of MaxWeight

It is possible to adjust the MaxWeight policy to achieve stability.

In particular, we consider the following weighted Max Weight policy:

$$\text{maximize} \quad \sum_{j \in \mathcal{J}} Q_j \frac{\sigma_j}{\rho_j} \quad \text{over} \quad \boldsymbol{\sigma} \in \mathcal{S}\,.$$

Theorem. Given $\rho \in \langle S \rangle$, the weighted MaxWeight policy is positive recurrent.

Proof

$$L(\boldsymbol{q}(t)) := \max_{\boldsymbol{\sigma}\in\mathcal{S}} \sum_{j\in\mathcal{J}} q_j(t)\left(\frac{\sigma_j}{\rho_j} - 1\right)$$

$$\frac{dL}{dt} = -\frac{1}{2}\sum_{j,j'\in\mathcal{J}} \lambda_j P_{jj'}\left(\frac{d'_j(t)}{\lambda_j} - \frac{d'_{j'}(t)}{\lambda_{j'}}\right)^2 - \frac{1}{2}\sum_{j\in\mathcal{J}} a_j\left(\frac{d'_j(t)}{\lambda_j} - 1\right)^2 - \frac{1}{2}\sum_{j\in\mathcal{J}} \lambda_j P_{j\omega}\left(\frac{d'_j(t)}{\lambda_j} - 1\right)^2.$$

SIMILAR TO P.F.

< 0

□

Finally, Two Practical Extensions:

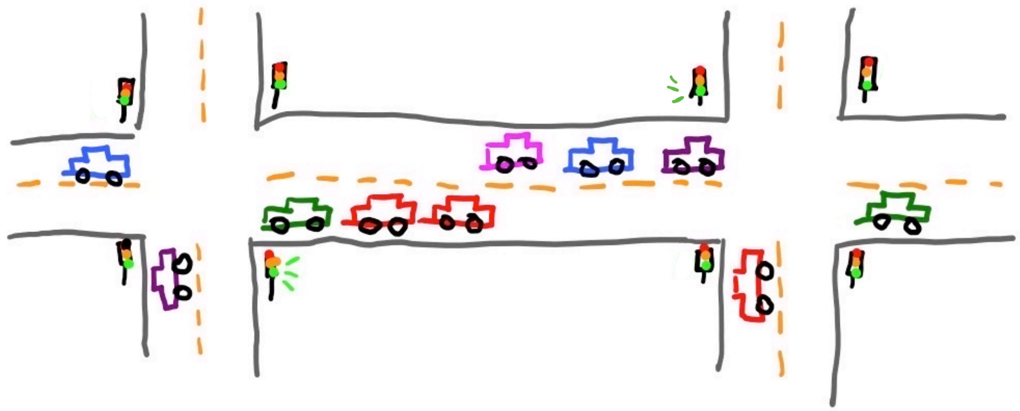
- Each job has an impact of ρ

$$W_j = w_j \sum_{k=1}^{Q_j} \sigma(\boldsymbol{\theta}_j^\top \boldsymbol{x}^{(k)})$$

& schedule according to

$$\text{maximize} \quad \sum_{j \in \mathcal{J}} W_j s_j \quad \text{over} \quad \boldsymbol{s} \in \mathcal{S}.$$

Use in road traffic applications

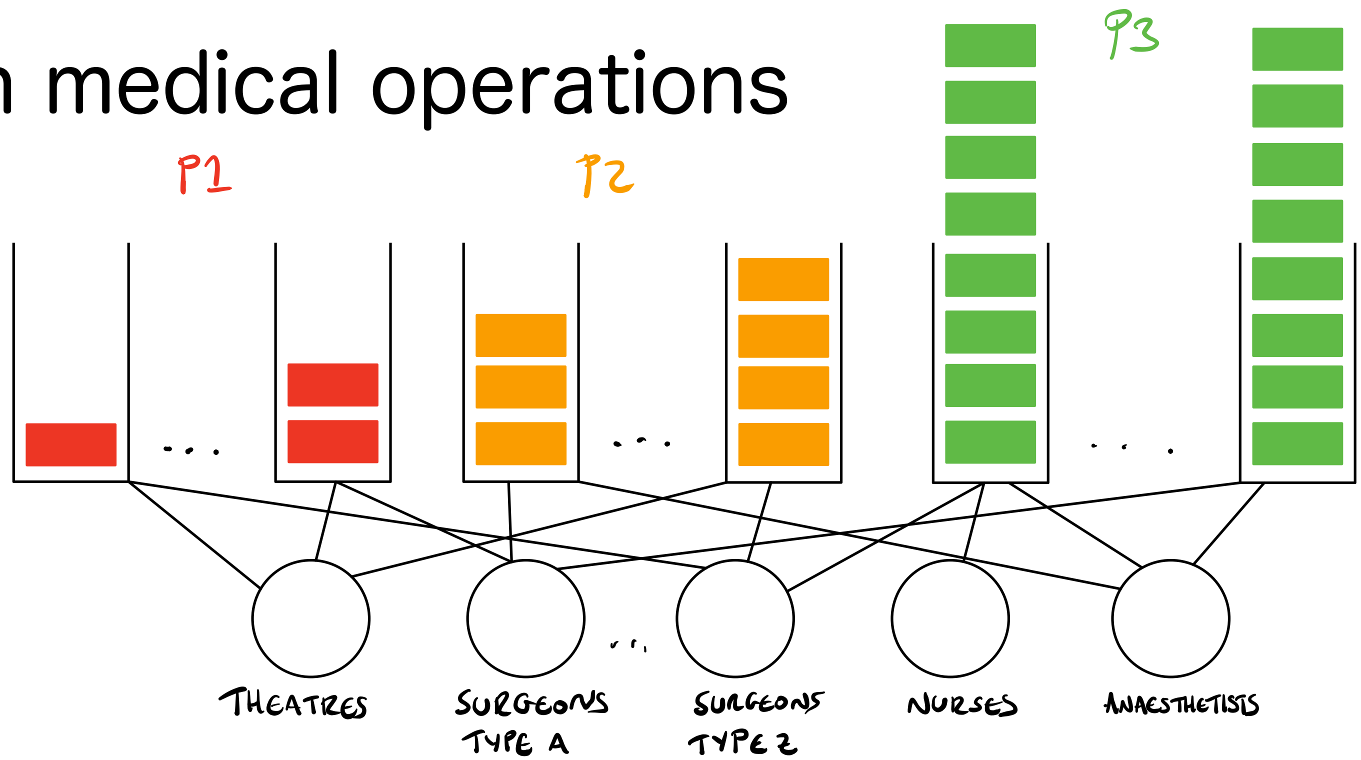


Two Practical Extensions:

- Weighted Delay

$$\max \quad \sum_j \frac{D_j(t)}{w_j}\sigma_j \quad s.t. \quad \sigma_j \in S$$

Use in medical operations



Summary

・ Rich set of techniques are employed to analyse queueing systems.

・ Max Weight is a fantastic policy but has some shortcomings

・ Its analysis shows stability of queueing systems is highly non-trivial.

・ Classical theory and asymptotics can help fix issues.

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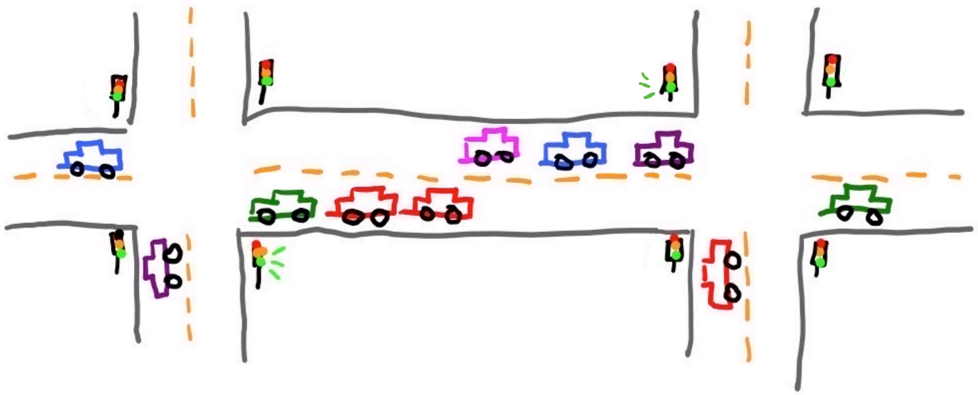

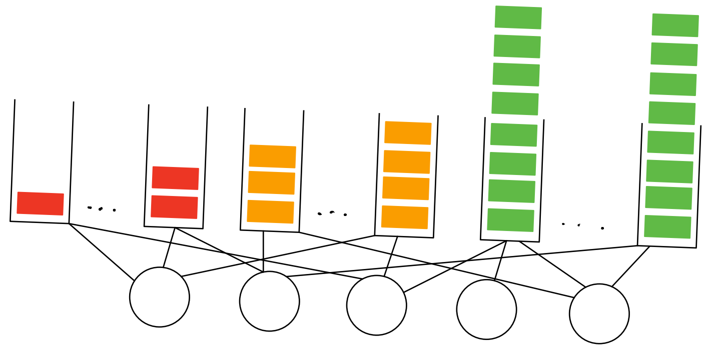

Thank yoo for listening!