Homework 7 - CAO 2024

5. Let $h \in \Gamma_0$ prove that prox_h is $\frac{1}{2}$-averaged.

6. **Proximal gradient of quadratic optimization**. Let $m < n$. Let H be an $n \times n$ positive semidefinite matrix. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank m (i.e. A has independent rows). And let $b \in \mathbb{R}^m$. We are interested on using proximal gradient to solve the *quadratic optimization problem*:

$$\min_{x : Ax \leq b} \frac{1}{2} x^T H x. \tag{QP}$$

Let $X := \{x : Ax \leq b\} \neq \emptyset$. Notice that problem (QP) can be written as $\min_x g(x) + \delta_X$, where $g : \mathbb{R}^n \to \mathbb{R}$ is defined by $g(x) = \frac{1}{2} x^T H x$.

In the next problems, express your solution (in simplified form) in terms of H, A and b.

(a) Compute ∇g and prox_{δ_X}.

In addition, for the following problems, consider separately (if necessary) the case when H is positive definite and the case when H is positive semidefinite but not positive definite.

(b) Write down the proximal gradient operator $F_{PG} : \mathbb{R}^n \to \mathbb{R}^n$, where η_k the step size is fixed (i.e. $\eta_k = \eta$ for all k).

(c) Find the fixed-points of F_{PG}

(d) Let $\lambda_{\max}$ be the largest eigenvalue of H. Show that if $\eta < \lambda_{\max}$ then F_{PG} is averaged. (This implies the proximal gradient method converges with rate $O(1/\sqrt{k})$.)

(e) Give some (natural) condition(s) under which F_{PG} satisfies the error bound condition. This implies linear convergence rate for the proximal gradient. Give (a bound on) the rate of convergence.