Homework 5 CAO 2024

1. **Distance and projection into a set**. Let $S \subset \mathbb{R}^n$ be a closed set. Consider *the distance from* $u \in \mathbb{R}^n$ *to* S

$$\operatorname{dist}_S(u) := \min_{x \in S} \|u - x\|. \tag{Prj}$$

   (a) Show that (Prj) always has (at least) one solution for each u.

   (b) Show that if S is convex then, for each given u, (Prj) has exactly one optimal solution.

   (c) Show that $\operatorname{dist}_S(u)$ is a convex function of u if and only if S is convex.

   (d) Assume S is convex. Find $\partial \operatorname{dist}_S(u)$.

   Now, assume S is convex and closed. Denote the unique solution to (Prj) (defined in exercise 1) by $\Pi_S(u)$, the *projection of* u *on* S.

   (e) Assume $u \notin S$. Show that for all $x \in S$ we have

$$(u - \Pi_S(u))^T (x - \Pi_S(u)) \leq 0.$$

   (f) Show that for all $x \in S$ and all $u \in \mathbb{R}^n$ we have

$$\|u - x\|^2 \geq \|u - \Pi_S(u)\|^2 + \|\Pi_S(u) - x\|^2.$$

2. **Fenchel dual of quadratic optimization**. Let $m < n$. Let H be an $n \times n$ positive semidefinite matrix. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank m (i.e. A has independent rows). And let $b \in \mathbb{R}^m$. We are interested on computing the fenchel dual of the *quadratic optimization problem*:

$$\min_{x : Ax \leq b} \tfrac{1}{2} x^T H x. \tag{QP}$$

   Notice that problem (QP) can be written as $\min_x f(x) + g(Ax)$, where $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x) = \frac{1}{2} x^T H x$ and $g : \mathbb{R}^m \to \overline{R}$ is defined by $g(y) = \begin{cases} 0 & \text{if } y \leq b \\ +\infty & \text{otherwise.} \end{cases}$.

   In the next problems, express your solution (in simplified form) in terms of H, A and b.

   (a) Compute g^*

   In addition, for the following problems, consider separately the case when H is positive definite and the case when H is positive semidefinite but not positive definite.

   (b) Compute f^*.

   (c) Compute the Fenchel dual of (QP).