What do you get when you look at supersymmetric extremal black holes for a long time?

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A Black hole Farey tail

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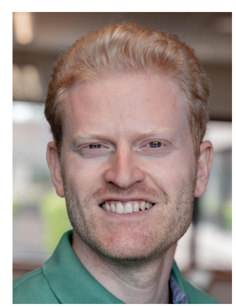


Useful discussions with Joaquin Turiaci and Vladimir Narovlansky

We, as a field, have been looking at supersymmetric extremal black holes for a long time…

Their entropies match beautifully with index computations…

Strominger-Vafa, 1996.

…

Dabholkar, Gomis, Murthy

What more could we say?

There are many questions that remain

There is more to a black hole than its entropy!

What about the details of is AdS_2 near horizon geometry?

Is there an AdS_2/CFT_1 ?

There is more information in this AdS_2 theory:

Its correlation functions

Our main technical results involve computing some of them.

Haven’t people been computing AdS correlators “for ever” ?

Yes, in AdS_D , D>2

But, in two dimensions the situation is a bit harder. And it has been understood only within the last few years.

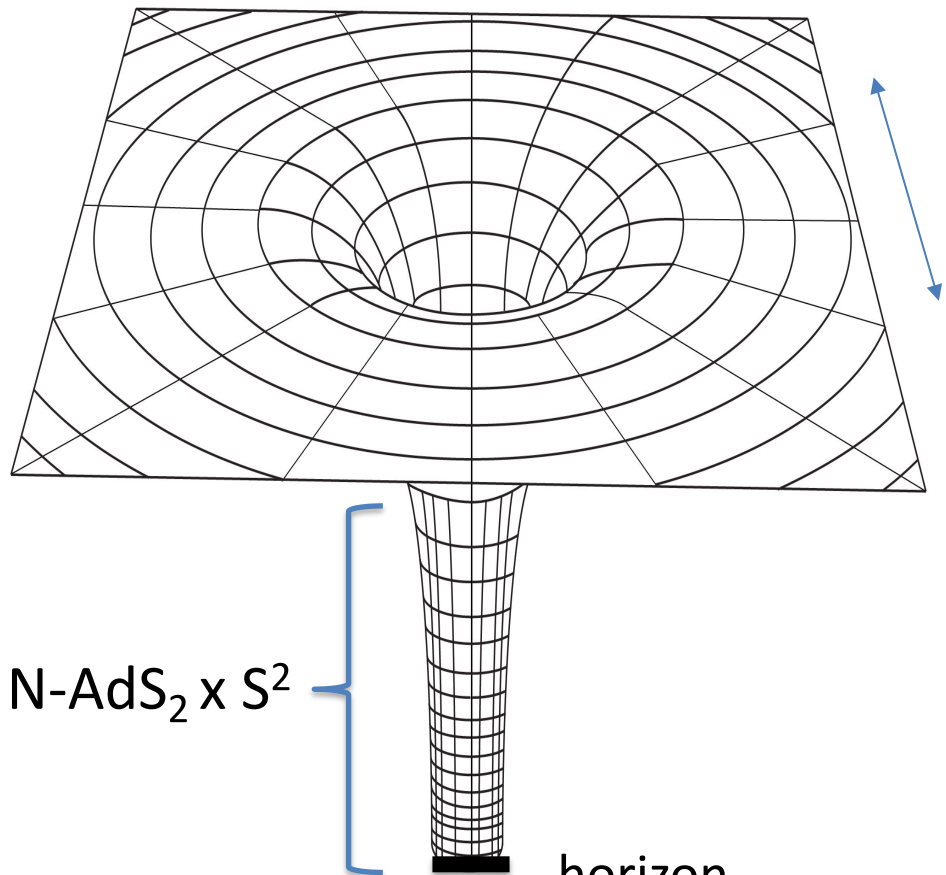
The new feature is that there is a gravitational mode that becomes strongly coupled at low energies. Even for large N (or large Q).

Let’s talk about charged extremal black holes

Towards extremality

$$M \geq Q$$

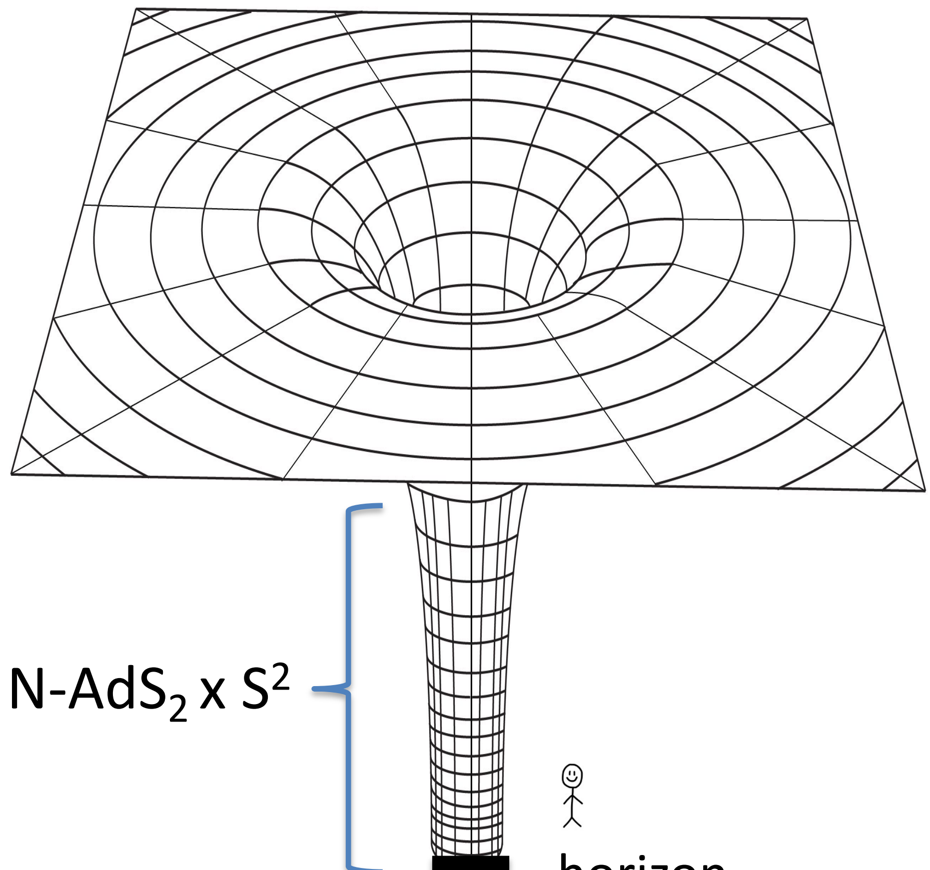
$$M \sim Q$$



The throat becomes deeper → larger redshift factor → lower temperature

First hint of a problem

$$M \geq Q$$
$$M \sim Q$$



r_e UV scale.

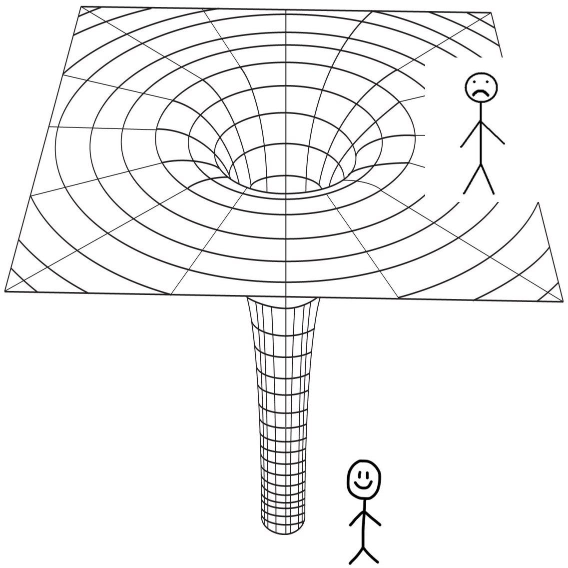
$$E - E_e \propto T^2$$
$$S - S_e \propto T$$

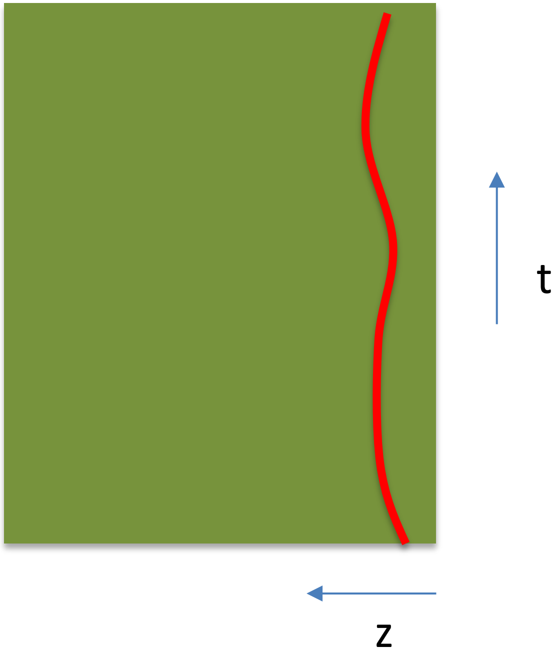
A problem arises when the length of the throat is $\beta_0 = \frac{1}{T_0} \sim S_e r_e$.

The specific heat becomes of order one → backreaction becomes large

Preskill, Schwarz, Shapere, Trivedi, Wilczek.

What goes wrong with the classical approximation ?





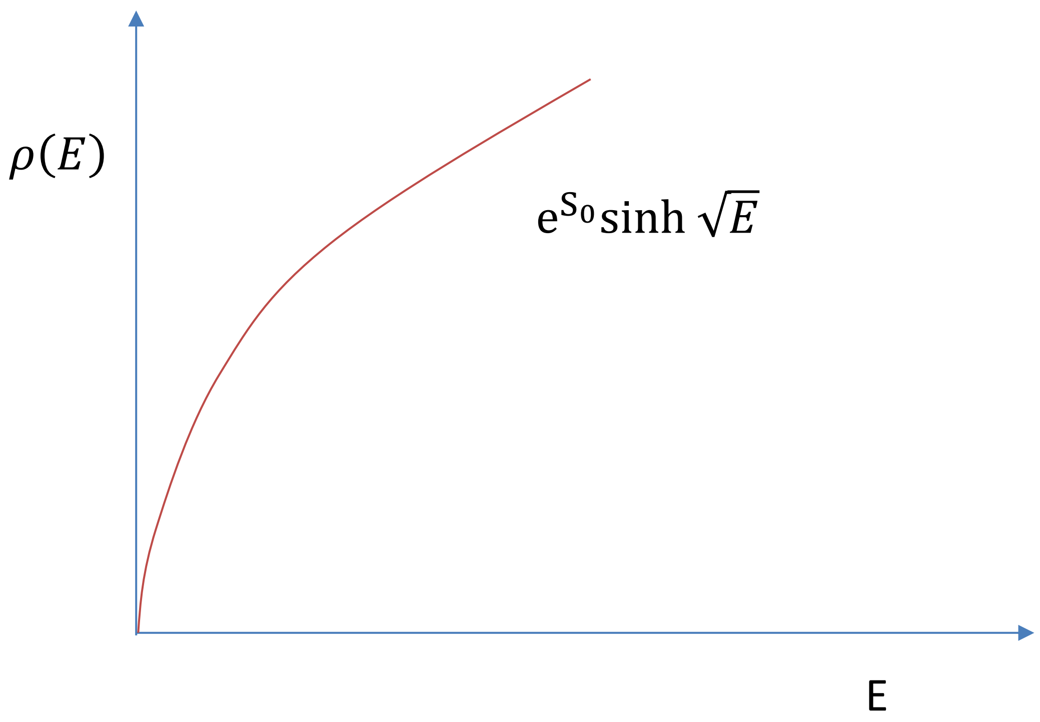
- Seemingly, nothing with the bulk geometry.
- There is a gravity mode telling us how we connect the throat region to the exterior.
- From the throat point of view, it is a ``boundary gravity mode''.
- This boundary mode becomes highly quantum mechanical → we need to treat it exactly.

Results from quantizing the boundary mode and computing the partition function

The results depend on whether we have pure gravity or supergravity (supersymmetry)

Non-SUSY

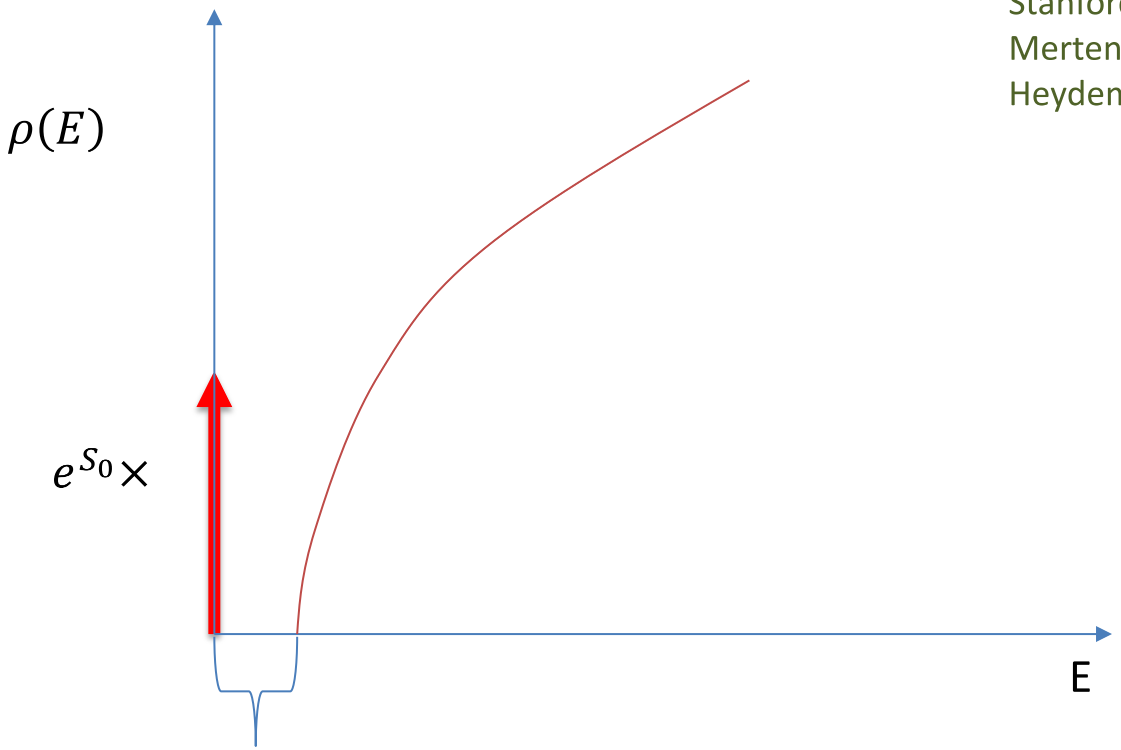
Bagrets, Altland, Kamenev
Stanford, Witten
Kitaev, Suh
Mertens Turiaci Verlinde



Entropy goes to zero at low energies.

$\mathcal{N}$=2, 4 SUSY

Stanford, Witten
Mertens Turiaci Verlinde
Heydeman, Iliesiu, Turiaci, Zhao



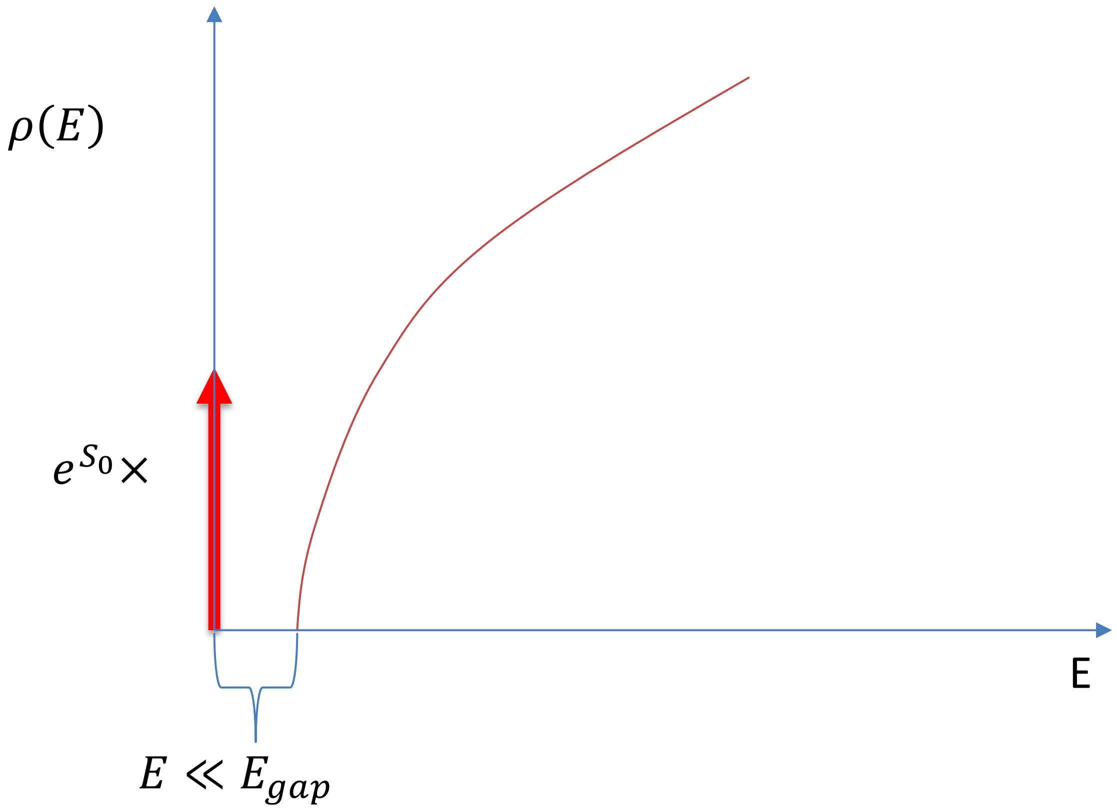
$$E_{gap} \sim \frac{E_{UV}}{S_e} \sim \frac{1}{r_e\, S_e} \sim \frac{1}{\phi_r} \sim T_0$$

$$S = -\phi_r \int du\{t(u), u\} + \text{partners}$$

Conclusion

- Nothing goes ``wrong'' at this scale, T_0 .
- We simply need to take into account the quantum mechanics of the Schwarzian mode.
- We transition from a classical physics regime to a quantum gravity regime → just the quantum mechanics of the boundary mode.
- It is surprising that this quantum mechanics predicts a gap for some cases (with SUSY)
- It is surprising because one would have though that the gap would arise from some feature of the geometry, such as an end of the world brane.

A decoupling low energy limit



Look at the system at low energies, smaller than this gap. → Only ground states survive.

If you look at supersymmetric black holes for a long (Euclidean) time

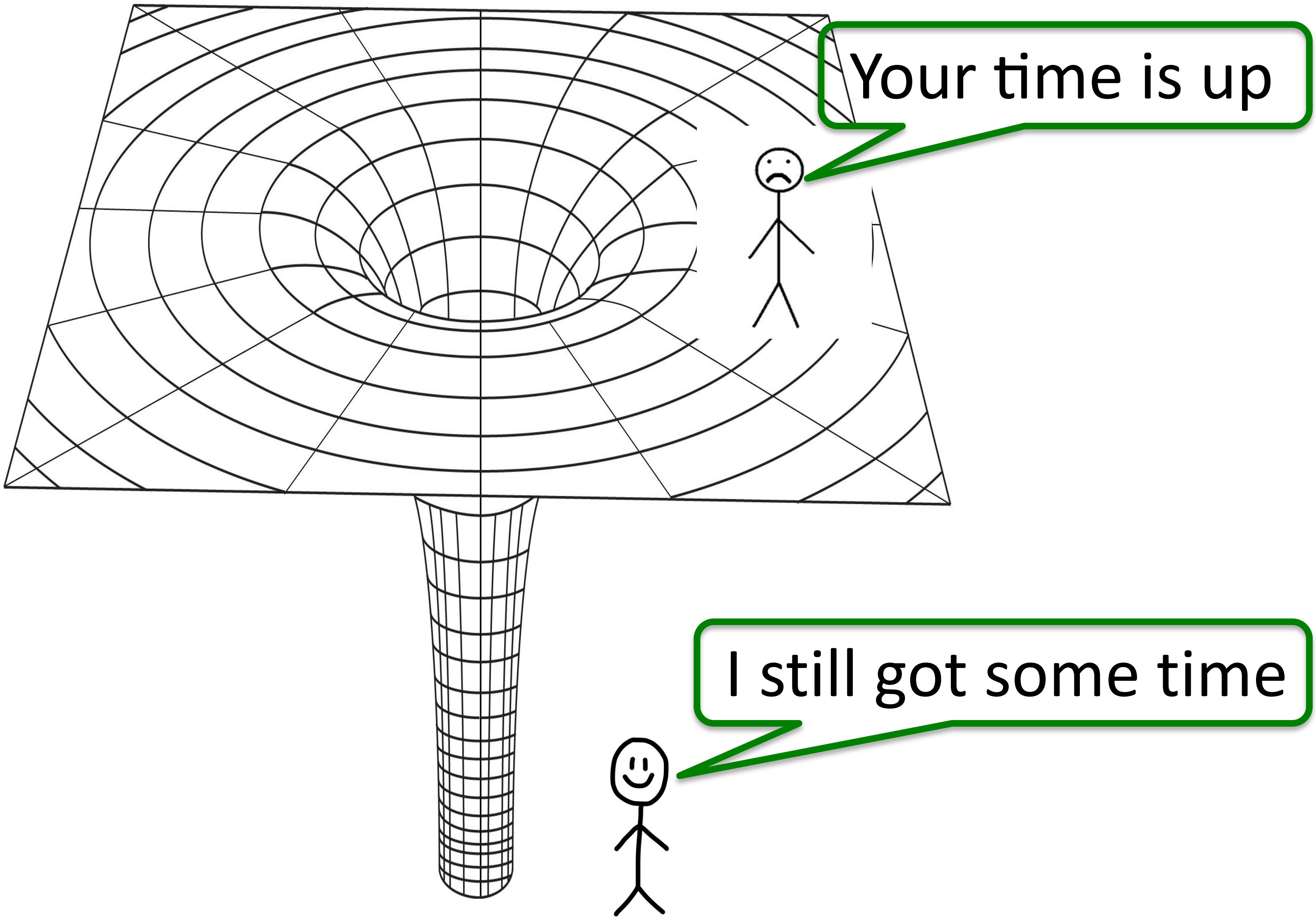
→

Only the ground states survive.

There is no more boundary time. H=0.

What happens with the bulk?

In this SUSY case, we still have large N, or Q, and a large number of quantum states.


Your time is up
I still got some time

Why do we study supersymmetric extremal black holes?

- This limit exists only with supersymmetry.
- No dynamics, H=0. No boundary time.
- What is the connection between the geometry and the microstates? (fuzzballs?)

Supersymmetric black hole examples

- 4 or 5 dimensional black holes in supergravity have N=4 supersymmetry.
- The 1/16 BPS black hole in $AdS_5 x S^5$ has N=2 supersymmetry.

Boruch, Heydeman, Iliesiu, Turiaci

- At low energies, we can Kaluza-Klein reduce to AdS_2 . And the dominant gravitational mode is what we have discussed above. (If we fix other possible charges). All these other KK modes can be viewed as living in a rigid AdS_2

The boundary mode dynamics

- To understand it, it is important to understand the asymptotic symmetries of AdS_2
- SL(2) → full time reparametrizations $f(\tau)$
- These symmetries are spontaneously broken.
- They are also explicitly broken by the boundary conditions

$$I = -\phi_r \int dt \{ f(t), t \}$$

- Governs the boundary mode dynamics.

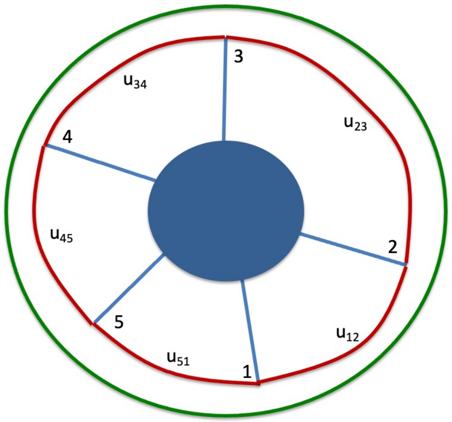
The boundary mode dynamics with SUSY

- OSp(2|2) → full time super-reparametrizations.

$$I = -\phi_r \int dt \{f(t), t\} + \text{partners}$$

- At very long times → the symmetries are restored!.
- The extremely low energy theory has H=0, and is topological, no time dependence.
- Restoration of the symmetry by quantum effects.

Correlation functions



- They are computed by ``dressing'' the bulk field theory correlators
- At long times, they become time independent → topological theory.

- AdS_2/TFT_1

- We will neglect the sum over geometries for this talk, but it is also interesting to include it. (We've considered the cylinder).

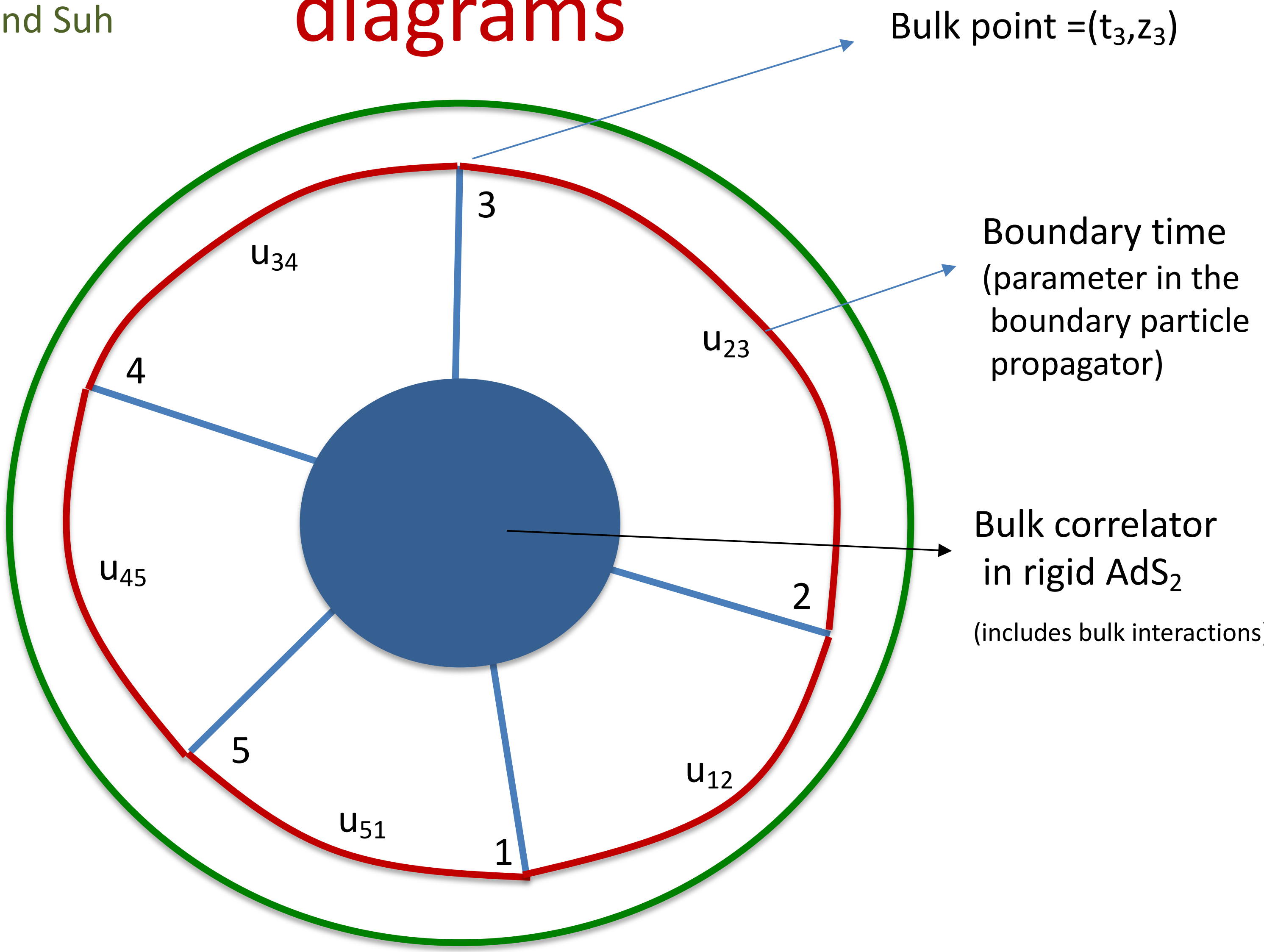
Review of Nearly-AdS_2 gravity correlators

Nearly AdS_2 gravity

- Matter fields moving in a rigid AdS_2 spacetime.
- The boundary becomes dynamical and behaves as a particle moving in AdS_2
- The quantum mechanics of this boundary particle can be exactly solved. Z. Yang, Kitaev and Suh
- It behaves as a non-relativistic particle moving in AdS_2 with an ``electric field''.

Quantum gravity from Witten-like diagrams

Z. Yang, Kitaev and Suh



Z. Yang, Kitaev and Suh

$$\langle O(u_1)\cdots O(u_n)\rangle = (\text{Boundary Particle})(\text{Correlator in AdS}_2)$$

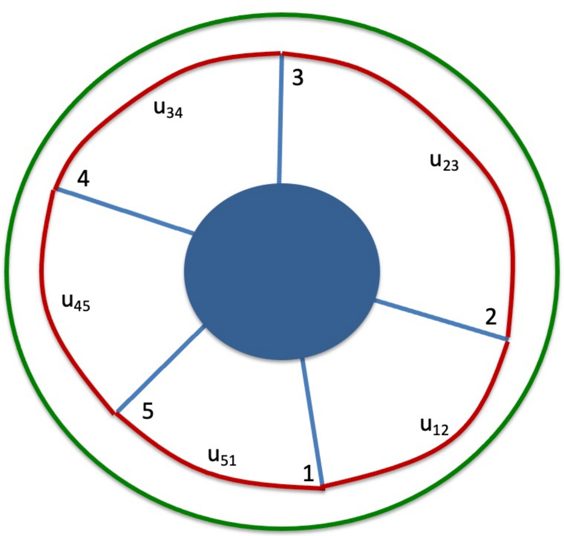
$$\langle O(u_1)\cdots O(u_n)\rangle = \int \frac{\prod_i dx_i dz_i/z_i^2}{\text{Vol}(SL(2))} P(\vec{x}_i,\vec{x}_{i+1};u_{i,i+1}) \prod_i z_i^{\Delta_i} \langle O(x_1)\cdots O(x_n)\rangle$$

QFT in AdS_2 correlators

They simplify because we are near the boundary

Boundary particle propagator.

(We will review later how they are computed)



The N=2 case

- We have a similar expression.
- At low energies, or $u_{ij} \gg \beta_0$
- The propagator becomes independent of u.
- Only the zero energy states contribute.
- The correlator becomes topological.

$$\langle O(u_1)\cdots O(u_n)\rangle = \int \frac{\prod_i dx_i dz_i d^2\theta_i / z_i}{\mathrm{Vol}(SU(1,1|1))} P_0(\vec{x}_i, \vec{x}_{i+1}) \prod_i z_i^{\Delta_i} \langle O(x_1,\theta_1,\bar{\theta}_1)\cdots O(x_n,\theta_n,\bar{\theta}_n)\rangle$$

Independent of the u_i . Depends on the order.

$$\langle O(u_1)\cdots O(u_n)\rangle = \mathrm{number} = F(\Delta_i, g_i)$$

How can we have a theory with no Hamiltonian?

- The structure is in the form of the observables (simple operators).
- Ground states + some simple operators.

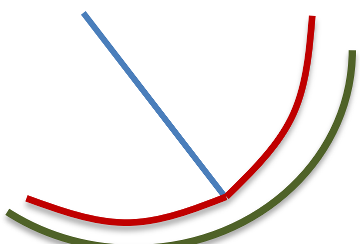
Infrared operators

$$\hat{O} = P_0 O P_0 \sim \lim_{u\to\infty} e^{-uH} O e^{-uH}$$

Projector on to the microstates.

O is not BPS.
But $\hat{O}$ is BPS.

O's are simple in the UV theory. But $\hat{O}$ is complicated due to the projector P_0, which depends on the flow and characterizes how the ground states are embedded in the full Hilbert space.



= bulk picture

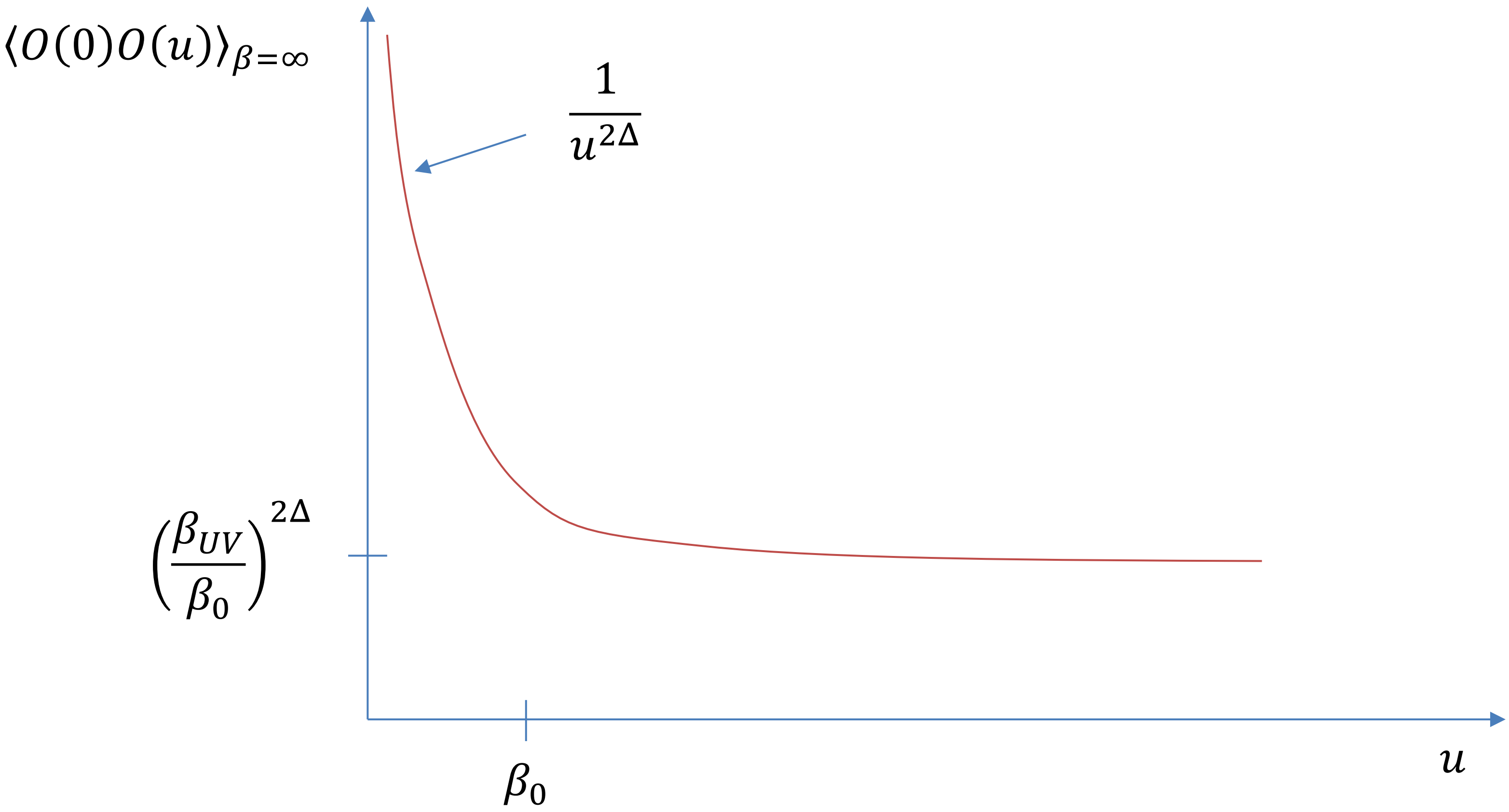
The two point function in the N=2 susy theory

- We can compute the two point function using a variety of methods.
  - The chord diagram technique in N=2 SYK.

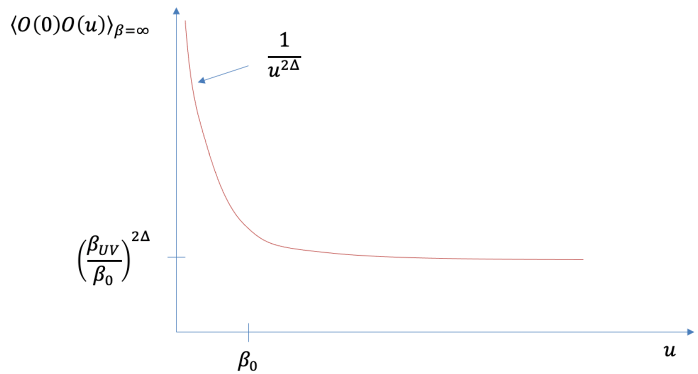
    Berkooz, Brukner, Narovlansky, Raz
  - The super-Liouville approach. As in Mertens, Turiaci, Verlinde
  - Using the boundary propagators.

We first discuss some qualitative features of the answer

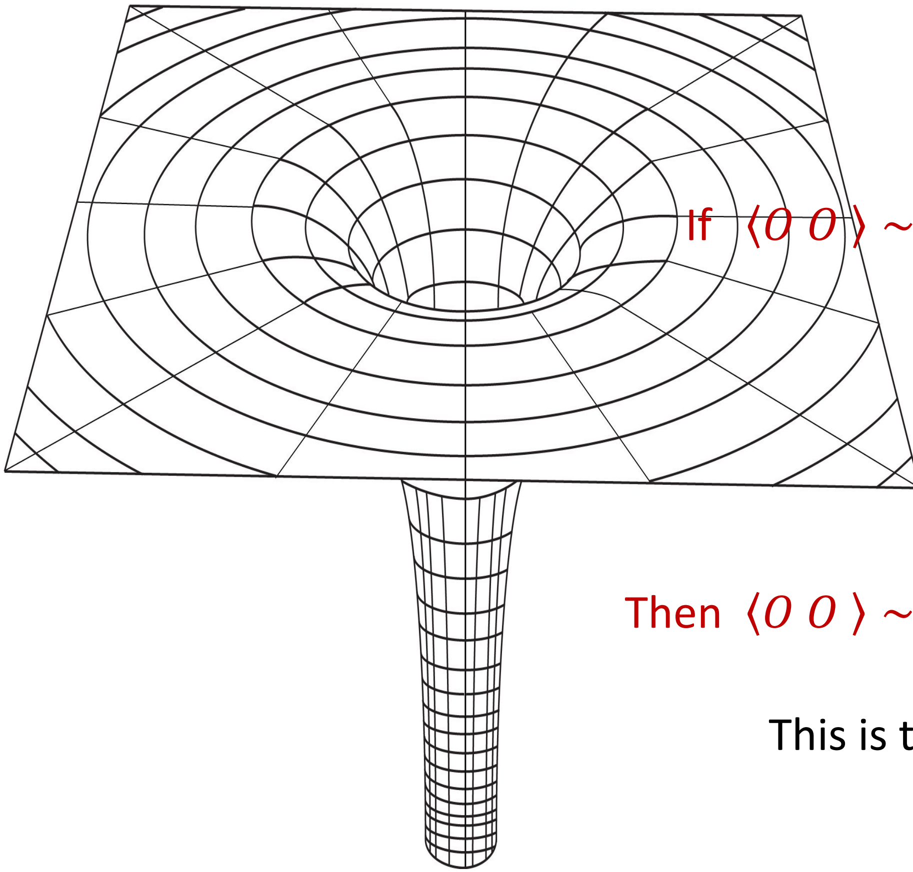
The two point function at zero temperature



$$\beta_{UV} \sim r_e \sim \frac{1}{J}$$



- It connects the shorter distance limit to the long distance, exactly AdS_2 regime.
- It is non-zero.
- This non-zero value has a power law suppression (power of the entropy) relative to its natural UV value.



If $\langle O\ O \rangle \sim 1$

Operator normalized so that its two point function is one in the region outside the black hole.

Then $\langle O\ O \rangle \sim \left(\frac{r_e}{\beta_0}\right)^{2\Delta} \sim \frac{1}{S_e^{2\Delta}}$

This is the value at very long times.

Typical values of matrix elements

The two point function is telling us information about the average value of the matrix elements of the operator in a microstate

Raju, Shrivastava

$$\langle OO\rangle_{IR} = e^{-S_0} Tr[\hat{O}\hat{O}] = e^{-S_0} \sum_{ij} |O_{ij}|^2$$

Typical values of eigenvalues

We could diagonalize the (Hermitian) operator O, $O_{\alpha,\beta} \sim o_\alpha \delta_{\alpha,\beta}$

$$\langle OO \rangle_{IR} = e^{-S_0} Tr[\hat{O}\hat{O}] = e^{-S_0} \sum_\alpha o_\alpha^2$$

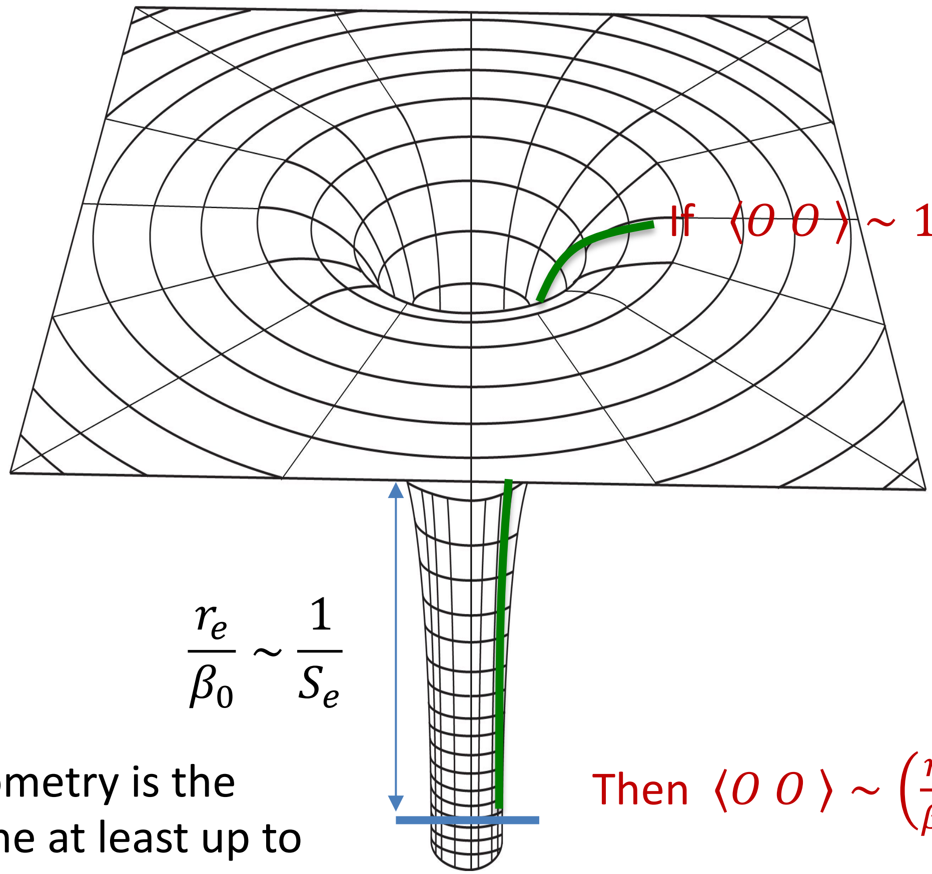
The two point function is giving us size of the typical eigenvalues of the operator.

This is the typical value of O in the basis that diagonalizes O.

This is larger than the typical value of the one point function of O on a random quantum states

$$\int d\psi \left[\langle \psi | O | \psi \rangle\right]^2 = e^{-S_0} \langle OO \rangle$$

This gives an interesting implication for where the geometry can start differing for various microstates in this basis



Universal factor coming from the propagation of the particle in AdS_2

$$\frac{r_e}{\beta_0} \sim \frac{1}{S_e}$$

Then $\langle O\, O \rangle \sim \left(\frac{r_e}{\beta_0}\right)^{2\Delta} \sim \frac{1}{S_e^{2\Delta}}$

The geometry is the usual one at least up to this point

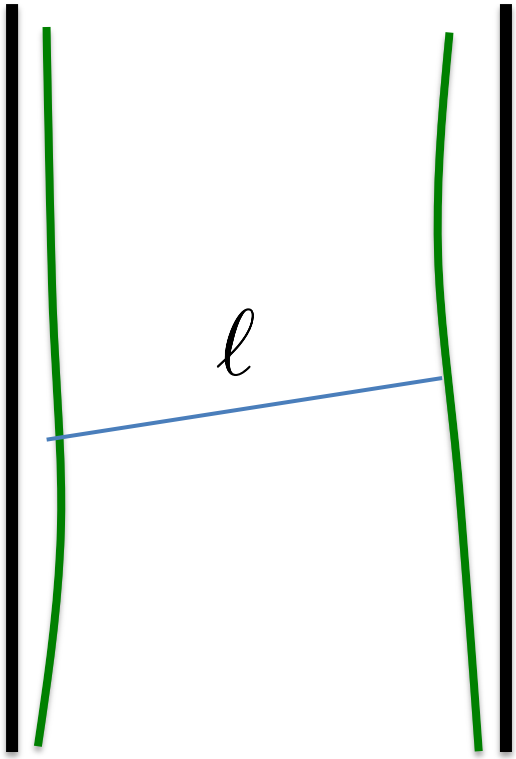
The geometry might be different from this point forwards for the different microstates in this basis where we diagonalize O.

This gives us a constraint on what we should expect for individual microstates.

Another implication…

Some details on the computation of the two point function using the Liouville method

Mertens, Turiaci, Verlinde



Basic variable of the two sided problem:
the distance.

This is a gauge invariant coordinate for the wormhole.
Kuchar

It turns out that its action is a Liouville like action

$$\int du[\dot{\ell}^2 + e^{-\ell}]$$

Harlow-Jafferis, Lin

With supersymmetry → Super Liouville theory.

Naively we would try to consider an N=2 superLiouville theory. However, we need an N=4 one because we have 2 SUSYs on the left and 2 SUSYs on the right.

Some more details on the computation of the 2pt function

- With the super-Liouville method.
- N=2 Superliouville theory in 2d → N=2 Schwarzian in 1d.

Mertens, Turiaci, Verlinde

- Liouville quantum mechanics.

$$S = \int du \left[\frac{1}{4} \dot{\ell}^2 + \dot{a}^2 + \bar{\psi}_\pm \dot{\psi}_\pm + e^{-\ell/2 - ia} \psi_+ \psi_- + e^{-\ell/2 + ia} \bar{\psi}_+ \bar{\psi}_- + e^{-\ell} \right]$$

- Find eigenfunctions.
- Build the Hartle-Hawking state (use input from the disk partition function)
- Compute $\langle e^{-\Delta \ell} \rangle$, or $\langle e^{-\Delta(\ell + 2\, i\, a)} \rangle$

The full answer is a bit long…

$$Tr\left[e^{-u H} O\, e^{-u' H} O\right] = \langle\psi| e^{-\Delta \ell} |\psi\rangle =$$

$$= \sum_r e^{-\frac{(r-1)^2}{4}(u+u')} \frac{2^{2+2\Delta}}{\pi^2 \hat{q}^2 \Gamma(2\Delta)} \int_0^\infty ds \int_0^\infty ds' e^{-4s^2 u - 4s'^2 u'}$$

$$\times \frac{s \sinh(2\pi s)}{E} \frac{s' \sinh(2\pi s')}{E'} 2\left(E + E' + 4\Delta^2\right) \Gamma(\Delta \pm is \pm is')$$

$$+ 2 \sum_r e^{-\frac{(r-1)^2}{4}u - \frac{(r-3)^2}{4}u'} \frac{2^{4+2\Delta}}{\pi^2 \hat{q}^2 \Gamma(2\Delta)} \int_0^\infty ds \int_0^\infty ds' e^{-4s^2 u - 4s'^2 u'}$$

$$\times \frac{s \sinh(2\pi s)}{\left(4s^2 + \frac{(r-1)^2}{4}\right)} \frac{s' \sinh(2\pi s')}{\left(4s'^2 + \frac{(r-3)^2}{4}\right)} \Gamma(\Delta + \frac{1}{2} \pm is \pm is')$$

$$+ 2 \sum_{|r|<1} e^{-\frac{(r-1)^2}{4}u} \frac{2^{2\Delta}}{\pi \hat{q}^2 \Gamma(2\Delta)} \cos(\frac{\pi r}{2}) \int_0^\infty ds e^{-4s^2 u}$$

$$\times \frac{s \sinh(2\pi s)}{\left(4s^2 + \frac{(r-1)^2}{4}\right)} \left(4s^2 + \frac{(r-1)^2}{4} + 2\Delta(r + 2\Delta - 1)\right) \Gamma(\Delta \pm \frac{(r-1)}{4} \pm is)$$

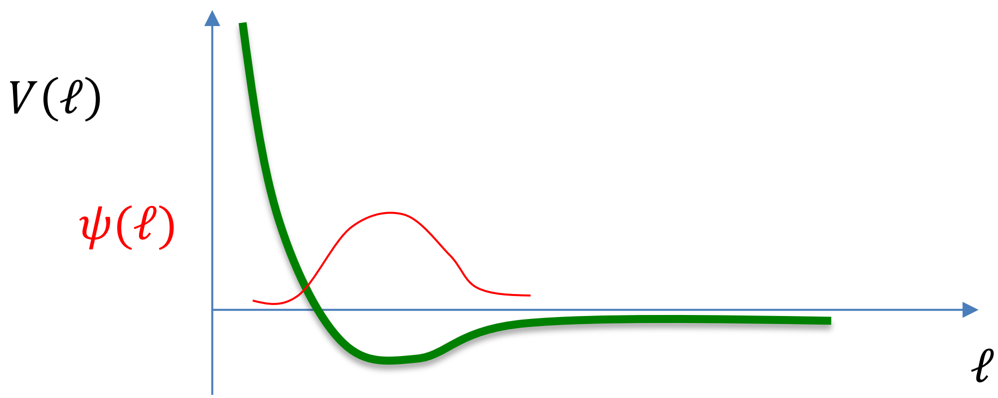
$$+ 2 \sum_{|r|<1} e^{-\frac{(r+1)^2}{4}u'} \frac{2^{2+2\Delta}}{\pi \hat{q}^2 \Gamma(2\Delta)} \cos(\frac{\pi r}{2}) \int_0^\infty ds e^{-4s^2 u'} \frac{s \sinh(2\pi s)}{\left(4s^2 + \frac{(r+1)^2}{4}\right)} \Gamma(\Delta + \frac{1}{2} \pm \frac{(r-1)}{4} \pm is)$$

$$+ \sum_{|r|<1} \frac{2^{2\Delta}\Delta}{\hat{q}^2} \cos(\frac{\pi r}{2})^2 \frac{\Gamma(\Delta)^2 \Gamma(\Delta + \frac{1}{2} \pm \frac{r}{2})}{\Gamma(2\Delta)}$$

$$S = \int du \left[\frac{1}{4}\dot{\ell}^2 + \dot{a}^2 + \bar{\psi}_\pm \dot{\psi}_\pm + e^{-\ell/2 - ia} \psi_+ \psi_- + e^{-\ell/2 + ia} \bar{\psi}_+ \bar{\psi}_- + e^{-\ell} \right]$$

$$S = \int du \left[\frac{1}{4}\dot{\ell}^2 - e^{-\ell/2} + e^{-\ell} \right]$$

There is a zero energy normalizable ground state



For this state the length of the wormhole is bounded, and time independent.

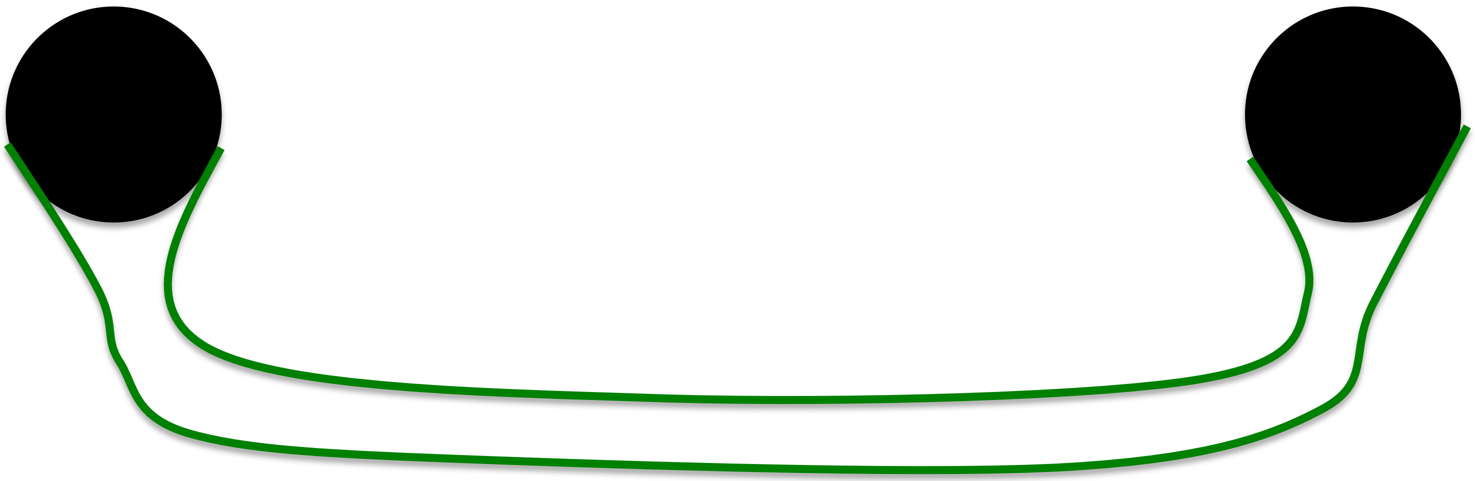
This is different from the naïve classical picture of an infinitely long throat, or

$$ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2}$$

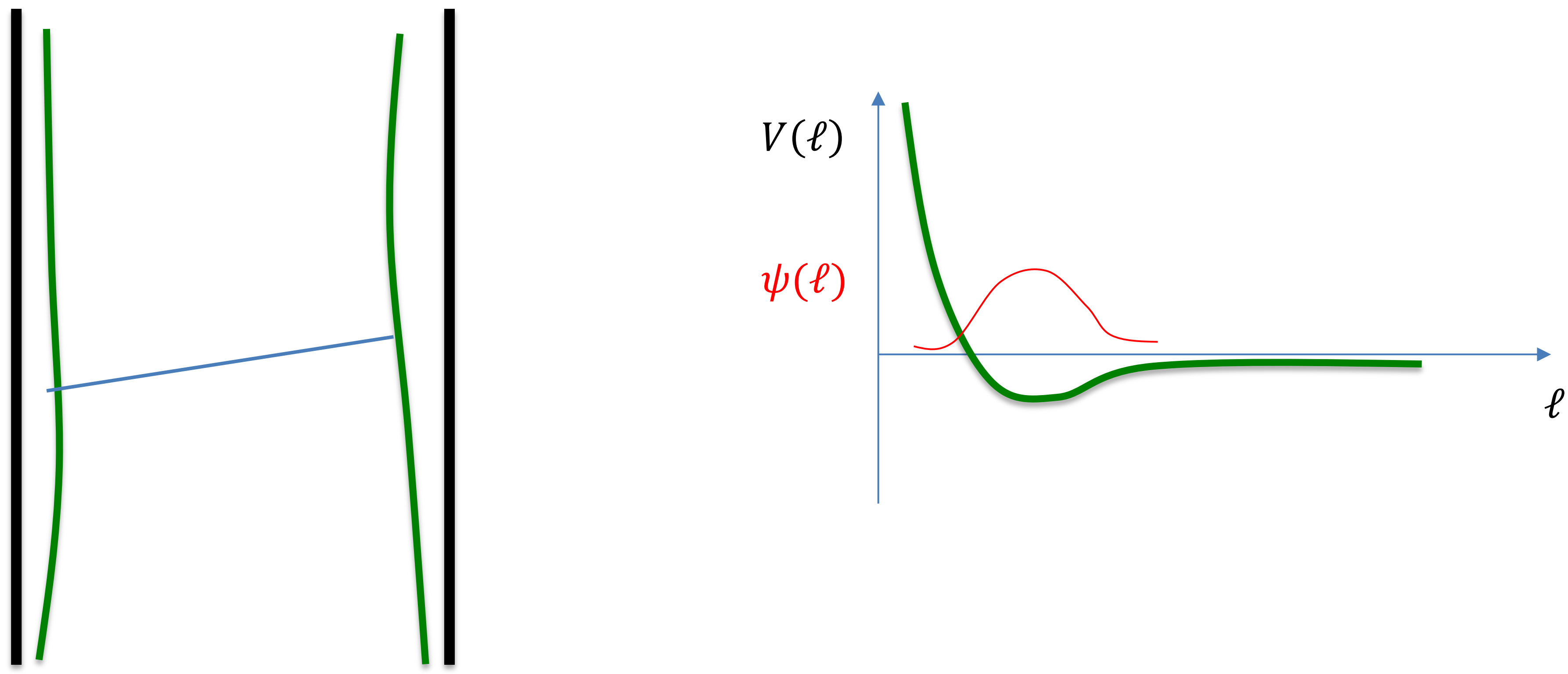
Let us emphasize the last point by asking:

Are there supersymmetric wormhole configurations?

SUSY ER= EPR ?



From the previous comments → Yes!



$V(\ell)$
$\psi(\ell)$
ℓ

Now we turn to the boundary particle propagator

We need it to construct more general correlators

Z. Yang, Kitaev Suh

The boundary particle formalism

- Boundary = Particle moving in AdS_2

Z. Yang, Kitaev Suh

The propagator, no SUSY

- AdS_2 = SL(2)/U(1)
- $g \sim g\, h\,,\;\; g \in SL(2),\; h \in U(1)$
- Wavefunctions over the group manifold that have a specified charge under U(1). (charged particle in AdS_2)
- $g = e^{x\, E_-} e^{\phi D} e^{\gamma_+ E_+}$,

- U(1): shifts of γ_+
- Quotient: consider states with fixed momentum along γ_+

Z. Yang, Kitaev Suh

The coset description, no SUSY

- Propagator:
- $P(1,2;u) = e^{i\,q\left(\gamma_+^1-\gamma_+^2+\varphi(1,2)\right)}$ F(distance; u) $= \langle 1|e^{-u\,H}|2\rangle$
- 1 = (x_1,ϕ_1), 2= (x_2,ϕ_2)
- $i\partial_u P = H_1 P \to F = \int_0^\infty dE\ \rho(E)e^{-u\,E}K_E(distance)$
- H_1 = Laplacian on the group = J^2
- Composition law fixes $\rho(E)$:
- $\int d\,\vec{x}_2\, P(1,2;u_{12})P(2,3;u_{23}) = P(1,3;u_{13})$

The boundary super-particle formalism

- Boundary = Particle moving in AdS_2
- Symmetries:
  - Full symmetry under the SU(1,1|1)= OSp(2|2) supergroup. This is a gauge symmetry.
  - N=2 worldline supersymmetry (Poincare)
  - Physical U(1) R symmetry.

(N=1 case: Fan, Mertens)

With N=2 SUSY

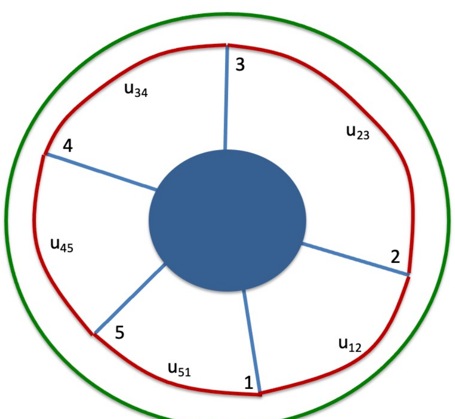
- AdS_2 + superspace variables = SU(1,1|1)/U(1)
- Add several Grassmann coordinates.
- $g = e^{x\, E_-}\, e^{\theta_- F_- + \overline{\theta_-}\, \overline{F_-}}\, e^{\phi D}\, e^{\theta_+ F_+ + \overline{\theta_+}\, \overline{F_+}}\, e^{\gamma_+ E_+} e^{\sigma J}$,

Z. Yang, Kitaev Suh

Propagator for zero energy states

- Propagator in the u → infinity limit .
- $Q_1 P_0 = \overline{Q_1} P_0 = 0 \rightarrow H_1 P_0 = 0$
- This propagator will enables us to compute any correlator in AdS_2 at zero energy.

$$P_0 = \theta(x_{12}) \frac{(z_1 z_2)^{1/4}}{\sqrt{x_{12}}} \exp\left(- \frac{(\sqrt{z_1} + \sqrt{z_2})^2}{x_{12}} \right) (+\text{fermions})$$



$$\langle O_1 \cdots O_n \rangle \sim e^{-S_0} Tr \left[\hat{O}_1 \cdots \hat{O}_n \right]$$

We checked that it does obeys the composition law.

Explicit form

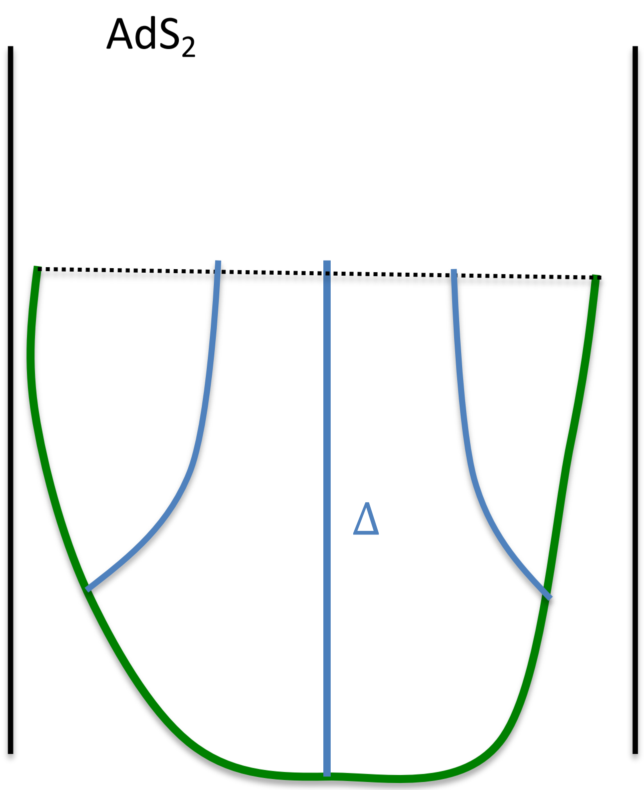
$$P_0 = e^{-q\tilde{\varphi}} \left(\tilde{f}_2 \Sigma^{1/2} + \tilde{f}_1 \Sigma^{-1/2} \right) \frac{e^{-\frac{2q}{w}}}{\sqrt{w}}$$

$$\begin{aligned}
\tilde{f}_1 &= \chi_1 + i\sqrt{2} e^{\sigma_1} e^{\phi_2} \frac{(\bar{\theta}_{-1} - \bar{\theta}_{-2})}{w} \\
\tilde{f}_2 &= \chi_2 + i\sqrt{2} e^{\sigma_2} e^{\phi_1} \frac{(\bar{\theta}_{-1} - \bar{\theta}_{-2})}{w} \\
\tilde{\varphi} &= \gamma_{+1} - \gamma_{+2} + \frac{2\cosh(\phi_1 - \phi_2)}{w} + \frac{i\sqrt{2}}{w} (-e^{-\sigma_1} e^{\phi_2} \chi_1 + e^{-\sigma_2} e^{\phi_1} \chi_2)(\theta_{-1} - \theta_{-2}) + \\
&\quad + (e^{2\phi_1} - e^{2\phi_2})(\theta_{-1} - \theta_{-2})(\bar{\theta}_{-1} - \bar{\theta}_{-2}) \frac{1}{w^2}
\end{aligned}$$

$$\begin{aligned}
\Sigma &= e^{\sigma_1 - \sigma_2} \left(1 - \frac{(\theta_{-1} - \theta_{-2})(\bar{\theta}_{-1} - \bar{\theta}_{-2})}{(x_1 - x_2)} \right) \\
w &= e^{\phi_1 + \phi_2} \left[x_1 - x_2 + \theta_{-1} \bar{\theta}_{-2} + \bar{\theta}_{-1} \theta_{-2} \right]
\end{aligned}$$

For zero R charge.
Similar expression but with a Bessel K function with non-zero R charge

Filling the inside



The distance increases, but remains finite.

We could insert many particles.

The distance depends only on the total dimension.

The entanglement entropy between the two sides decreases.

$S = S_0$ - (finite)

Type II_1 algebra.

Similarities with dS: H=0, type II_1 in the semiclassical limit.

Chandrasekharan, Long, Penington, Witten

Chaos in operators

- Since H=0, no chaos from energy levels.
- One can argue that the IR operators, $\hat{O}$=POP, are random matrices, probably with some evidence for eigenvalue repulsion.
- → Chaos in operators, or their eigenvalues.

We can compare the super-Schwarzian answers against those of the $\mathcal{N}$=2 SYK model

$\mathcal{N}$=2 SYK model

Fu, Gaiotto, Sachdev, JM

- Similar to the SYK model.
- N complex fermions ψ^i
- Supercharge involves a product of three fermions with random couplings.
- $Q = \sum_{ijk} \psi^i \psi^j \psi^k$
- $H = \{Q, Q^\dagger\}$

$\mathcal{N}$=2 SYK model

- We can compute the number of ground states analytically and numerically. Fu, Gaiotto, Sachdev, JM
- Now, we can compute correlators.
- They indeed go to constants at long times, both for R-charged operators such as ψ^i as well as for neutral operators such as $\psi^{i\dagger}\psi^j$
- At zero energies TOC and OTOC correlators are similar. In the particular case of ψ^i operators → actually the same!. Argument using SUSY.

Numerical computation of the energy gap

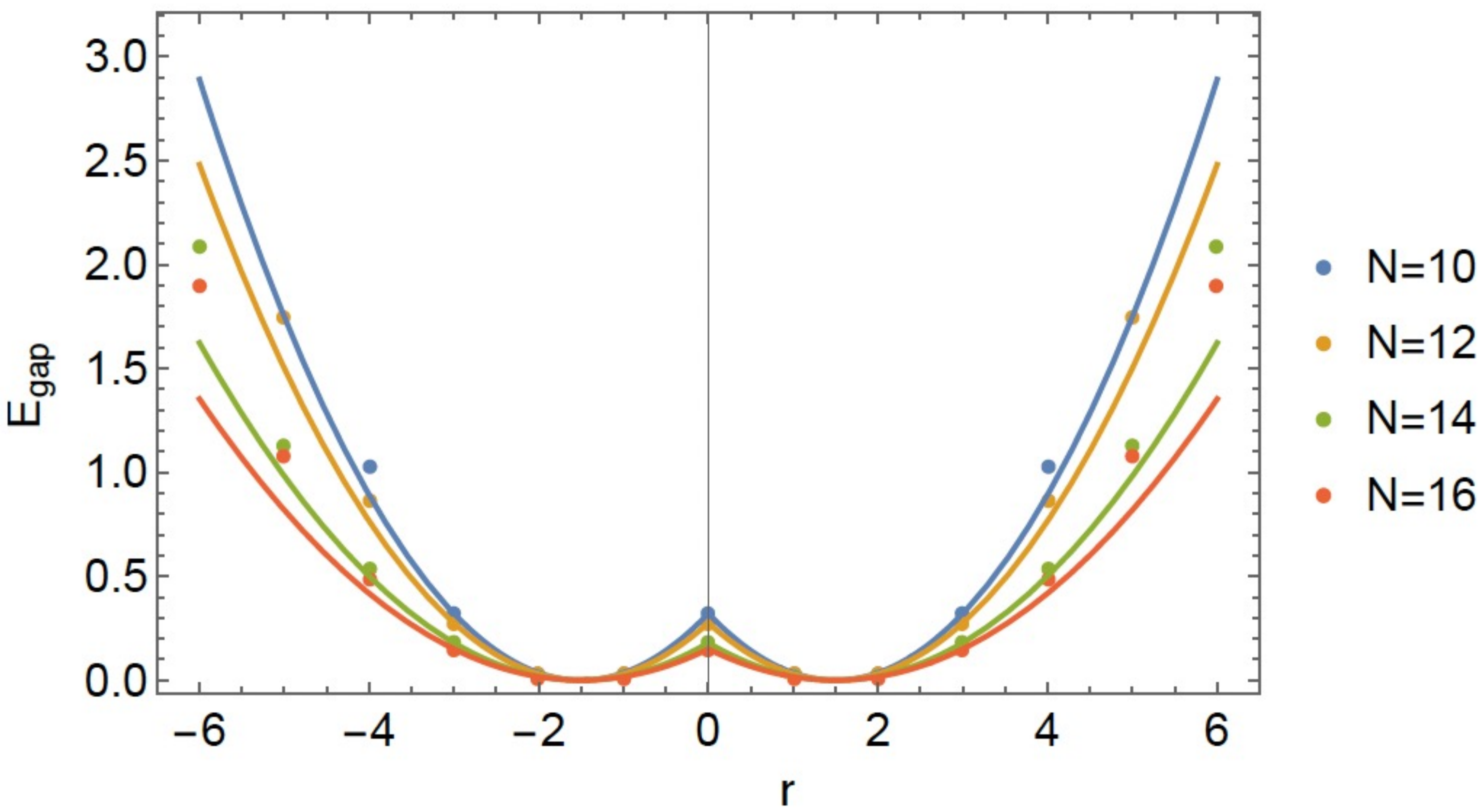


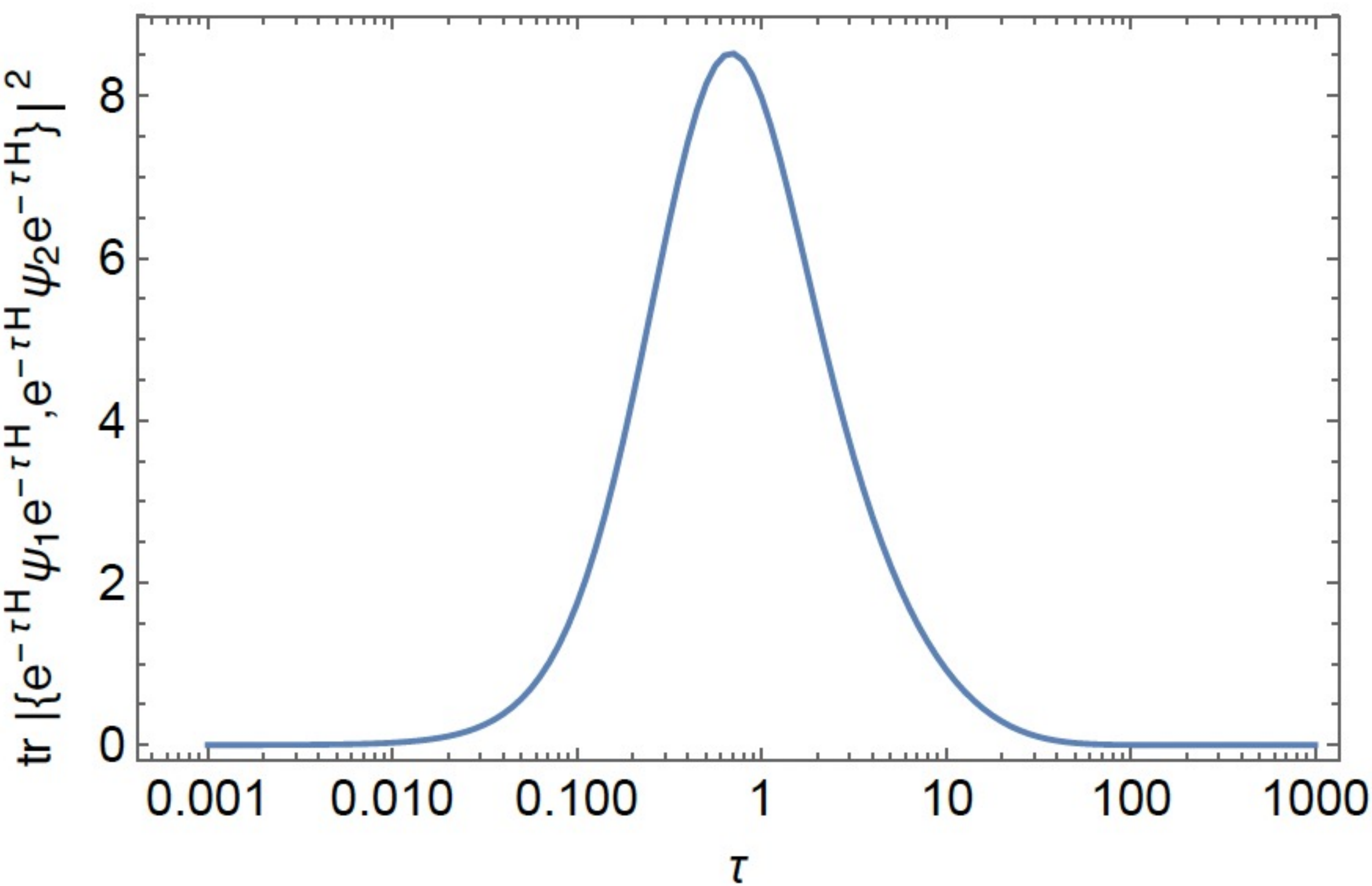
Figure 1: Gap as a function of R-charge for various values of N.

$$E_{gap} \propto \frac{1}{\phi_r}\left(\frac{|r|}{3}-\frac{1}{2}\right)^2 \propto \frac{J}{N}\left(\frac{|r|}{3}-\frac{1}{2}\right)^2$$

Constant two point function at long Euclidean times

Operator	R-charge	Schwarzian prediction	Numerical answer (N=16)
ψ_i	0	0.103	0.110 ± 0.005
	$-1/3$	0.103	0.110 ± 0.005
$\psi_i \psi_j$	$-1/3$	0.0213	0.024 ± 0.003
$\bar{\psi}_i \psi_j$	$-1/3$	0.0243	0.027 ± 0.001
	0	0.0754	0.079 ± 0.001
	$+1/3$	0.0243	0.027 ± 0.001

TOC = OTOC



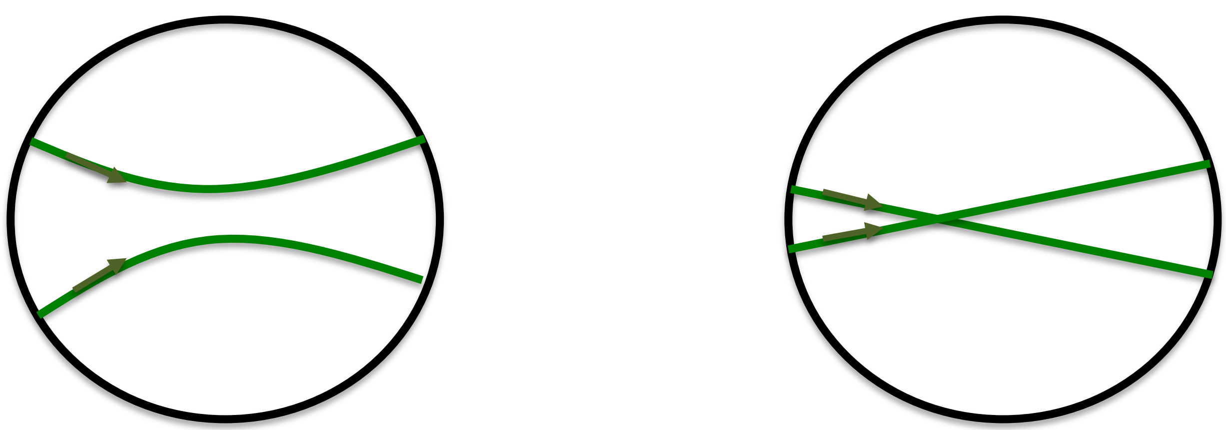
tr $|\{e^{-\tau H}\psi_1 e^{-\tau H}, e^{-\tau H}\psi_2 e^{-\tau H}\}|^2$
0
2
4
6
8
0.001
0.010
0.100
1
10
100
1000
τ

TOC = OTOC at long times

$$\cdots e^{-\infty H} \psi^i e^{-tH} \psi^j e^{-\infty H} \cdots = \langle 0 | \psi^i e^{-tH} \psi^j | 0' \rangle$$

$$[Q, \psi^i] = 0 \ , \qquad Q|0'\rangle = 0 = \langle 0 | Q$$

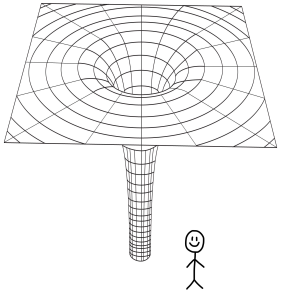
$$\partial_t . \longrightarrow \qquad \langle 0 | \psi^i (Q Q^\dagger + Q^\dagger Q) e^{-Ht} \psi^j | 0' \rangle$$



Conclusions

- AdS_2/TFT_1
- There is a bulk time but no boundary time. H=0.
- There are other interesting observables for SUSY AdS_2 : the correlators.
- We computed the two point function.
- We computed the zero energy propagator→ any correlator.
- These put constraints on how different various microstates can be.
- Good match to numerical SYK answers.

Future



- What is bulk time in this limit ?
- Connections with the explicit solutions people are constructing as part of the fuzzball program?

Happy birthday

Herman and Erik

Extra slides

N=2 Propagator

Propagator:

- $P(1,2; u_{12}, \kappa_1, \overline{\kappa_1}, \kappa_2, \overline{\kappa_2}) = \langle 1| e^{\kappa_1 Q + \overline{\kappa_1}\,\bar{Q}}\, e^{\,u_{12} H} e^{\kappa_2 Q + \overline{\kappa_2}\,\bar{Q}} |2\rangle$

  $= e^{i\,q\left(\gamma_+^1 - \gamma_+^2 + \varphi(1,2)\right)}\, F(invariants, u_{12}, \kappa_1, \overline{\kappa_1}, \kappa_2, \overline{\kappa_2})$
- $i\partial_u P = H_1 P$
- $iD_{\kappa_1} P = Q_1 P, \qquad i\overline{D_{\overline{\kappa_1}}}\,P = \overline{Q_1}\,P$
- Q_1 is a Grassman odd differential operator. It is invariant under the left symmetries. We had to guess its form.