Problem 1. Consider convex functions $f_0, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ and a vector of parameters $y = [y_1, \dots, y_m]^\top \in \mathbb{R}^m$. Prove that the extended real-valued function $g(y) : \mathbb{R}^m \to \overline{\mathbb{R}}$

$$g(y) := \inf_x f_0(x) \text{ such that } f_1(x) \le y_1, \dots, f_m(x) \le y_m$$

is a convex function of y.

Problem 2. We call a function $p : \mathbb{R} \to \mathbb{R}$ a univariate polynomial of degree d if $p(x) = c^\top m^d(x)$, where $m^d(x) := [1, x, \dots, x^d]^\top$ is the vector of monomials of degree d and $c := [c_0, \dots, c_d]^\top$ is the vector of coefficients of p (in other words, $p(x) = \sum_{i=0}^d c_i x^i$).

- Prove that the functions $f : \mathbb{R} \to \mathbb{R}$ that are equal to sums of squares of univariate polynomials, i.e., $f(x) = \sum_{j=1}^m (p_j(x))^2$ for some polynomials $p_1, \dots, p_m$, form a convex cone.
- Show that polynomial f of degree $2d$ is a sum of squares of univariate polynomials if and only if f can be written as $f(x) = m^d(x)^\top C m^d(x)$, where the matrix C is positive semidefinite.
- Assume you are given a univariate polynomial function g of degree $2d$ and you suspect that it is non-negative on $\mathbb{R}$. Write an optimization problem which, if its optimal value is equal to zero, will confirm the non-negativity of g. Hint: two polynomials are equal on $\mathbb{R}$ if and only if their vectors of coefficients are equal.
- Is the problem you obtained above convex? Explain why.