CIEM5110-2: FEM, lecture 3.1

Nonlinear FEM: solution procedure

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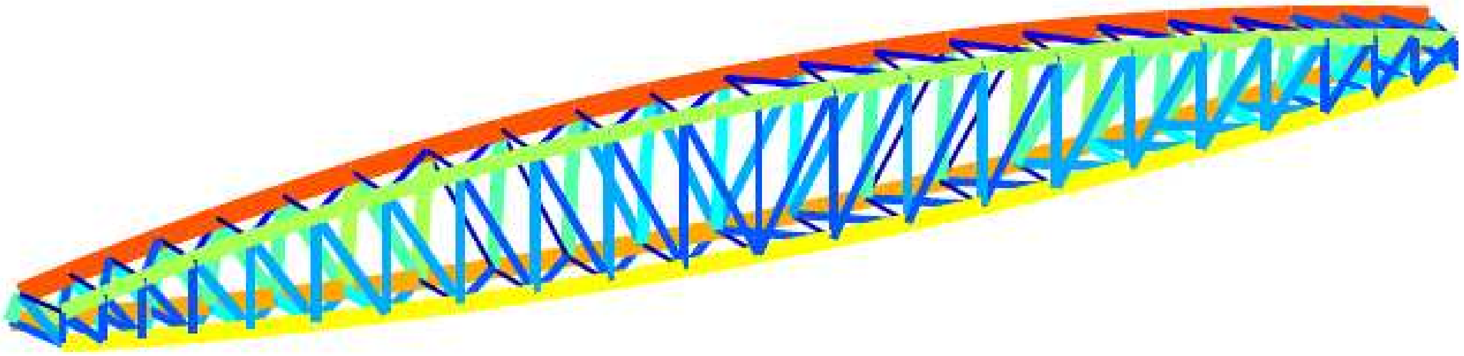
Agenda for today

1. Characteristics of nonlinear problems
2. Virtual work interpretation of weak form
3. Sources of nonlinearity
4. General formulation for the nonlinear system of equations
5. Incremental-iterative solution procedure

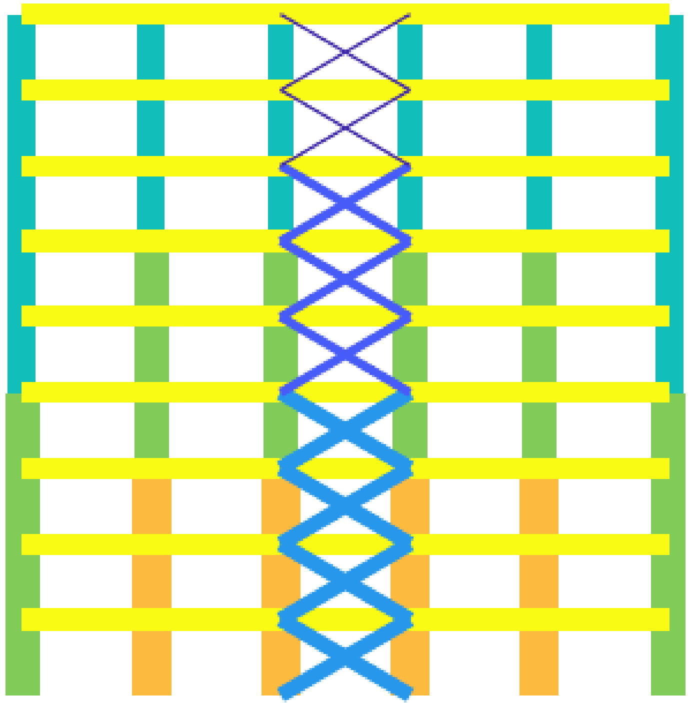
First some illustrations of linear vs nonlinear FEM

Example: Steel structures, Tom van Woudenberg (2020)

Objective: design optimization limiting the number of different steel profiles



Truss structure

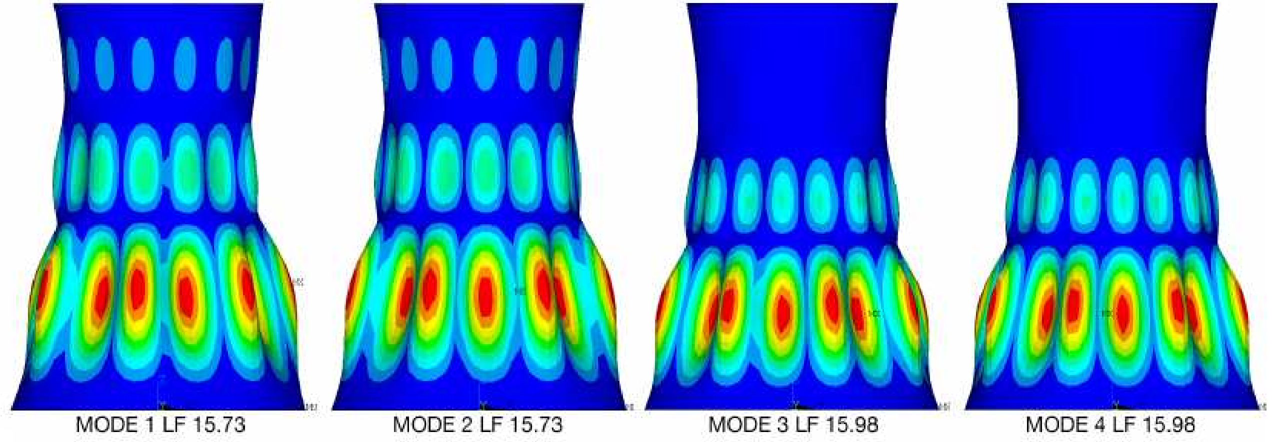


Frame structure

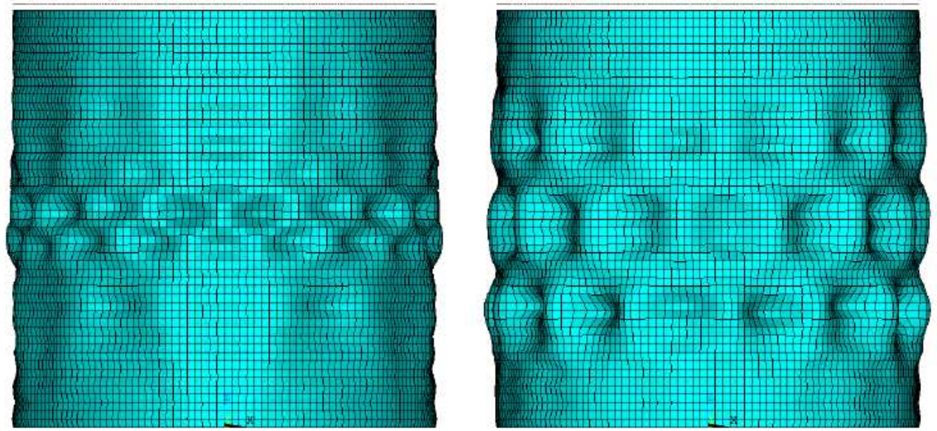
Analysis type: **linear elastic analysis**

Example: Shell buckling, Tim Chen (2014)

Objective: investigate the influence of imperfections on shell buckling



Buckling modes of a cooling tower

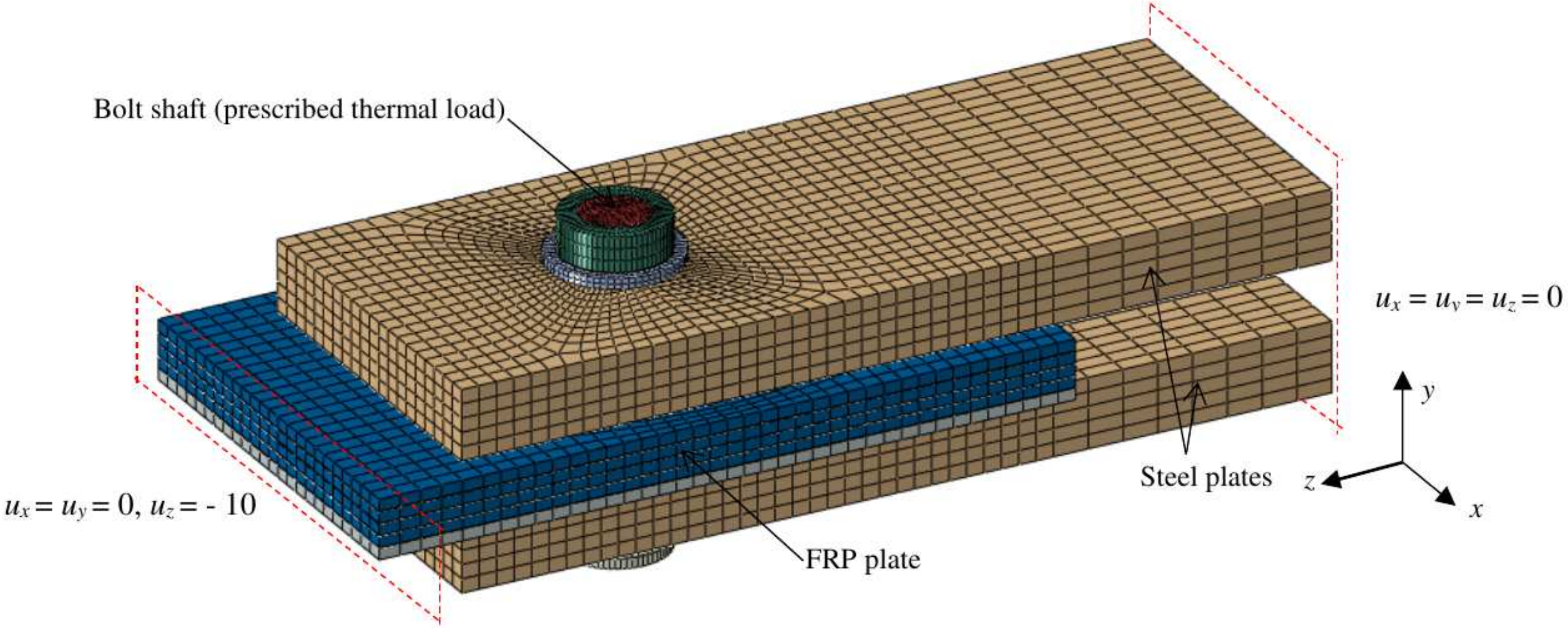


Post-buckling deformations of a cylinder

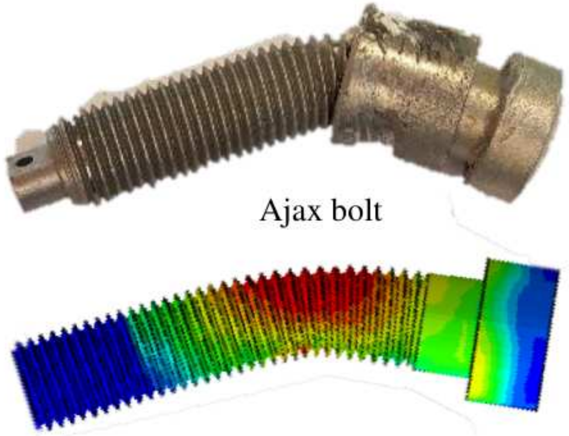
Analysis types: linear buckling analysis, **geometrically nonlinear analysis**

Example: Bolted joints, Fruzsina Csillag (2018)

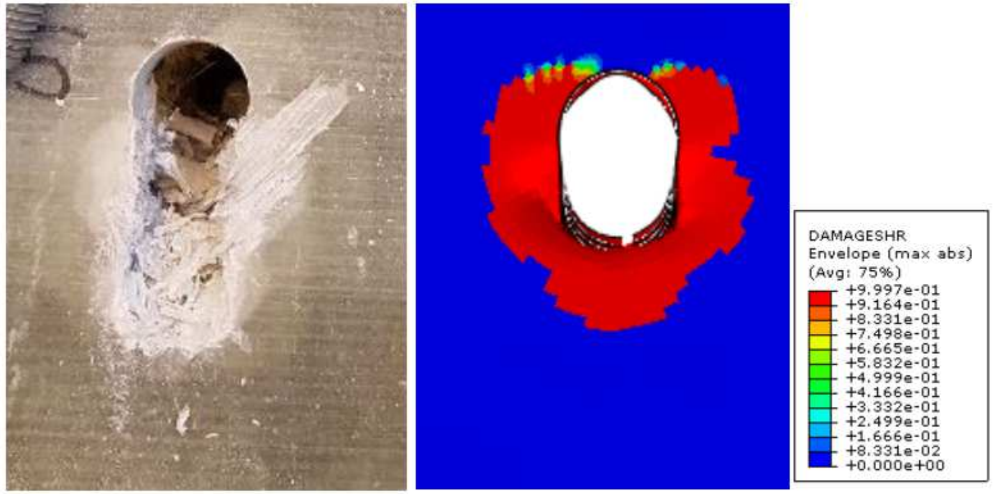
Objective: investigate the behavior of FRP-steel bolted connections



Setup



Deformation of the bolt



Damage of the FRP plate

Analysis type: **nonlinear analysis (material nonlinearity)**

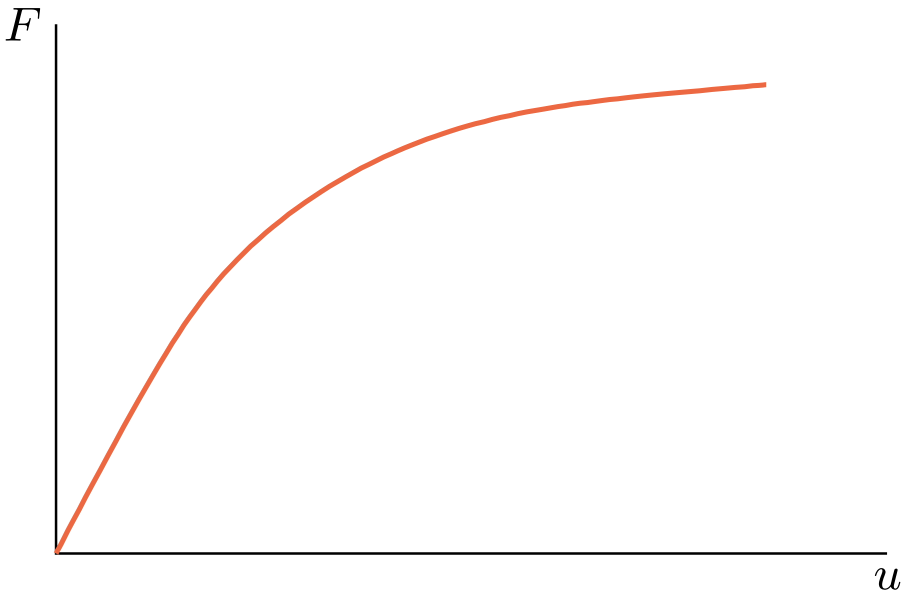
Characteristics of nonlinear problems

In nonlinear simulations, we simulate a process

Often this is quasi-static → no actual time, but still 'time steps' or increments

Even if we are only interested in a final state, a number of increments can be needed to get there

The classical output of a nonlinear finite element simulation is a force-displacement curve



Remember: this is a 1D representation of an n_{dof}-dimensional solution

Interpreting the weak formulation as virtual work equation (continuum mechanics)

Weak form (before assuming linear elasticity):

$$-\int_{\Omega} \nabla^{\mathrm{s}} \mathbf{w} : \boldsymbol{\sigma} \, \mathrm{d}\Omega + \int_{\Omega} \mathbf{w} \cdot \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{w} \cdot \mathbf{t} \, \mathrm{d}\Gamma = \mathbf{0}, \quad \forall \quad \mathbf{w}$$

Let $\mathbf{w} \leftarrow \delta\mathbf{u}$ (just a change of symbol):

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After discretization (with $\delta\mathbf{u} = \mathbf{N}\delta\mathbf{a}$ and $\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{a}$):

$$\delta\mathbf{a}^T \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \delta\mathbf{a}^T \left(\int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma \right) \qquad \Rightarrow \qquad \int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

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Interpreting the weak formulation as virtual work equation (extensible Timoshenko elements)

Recall the alternative extensible Timoshenko formulation

$$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ w_{,x} - \phi \\ \phi_{,x} \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}}$$

with

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_u & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_w & -\mathbf{N}_\phi \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_\phi \end{bmatrix}$$

The stiffness matrix takes a familiar form

$$\mathbf{K}^{\mathrm{e}} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \,\mathrm{d}\Omega$$

with

$$\mathbf{D} = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{bmatrix}$$

Collect stress-like quantities in a vector

$$\boldsymbol{\sigma} = \begin{Bmatrix} N \\ V \\ M \end{Bmatrix}$$

Then we can here too write

$$\int_\Omega \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \,\mathrm{d}\Omega = \delta\mathbf{a}^T \mathbf{f}_{\mathrm{ext}}$$

With the kinematic relation

$$\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{a}$$

we arrive at

$$\mathbf{f}_{\mathrm{int}} = \mathbf{f}_{\mathrm{ext}}$$

with

$$\mathbf{f}_{\mathrm{int}} = \int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \,\mathrm{d}\Omega$$

Back to the linear case

This is the general discretized equilibrium equation:

$$\underbrace{\int_\Omega \mathbf{B}^T \boldsymbol{\sigma}\,\mathrm{d}\Omega}_{\mathbf{f}_\text{int}} = \underbrace{\int_\Omega \mathbf{N}^T \mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t}\,\mathrm{d}\Gamma}_{\mathbf{f}_\text{ext}}$$

Assuming linear elasticity, we could substitute $\boldsymbol{\sigma} = \mathbf{DBa}$ to get

$$\int_\Omega \mathbf{B}^T \mathbf{D}\mathbf{B}\,\mathrm{d}\Omega\,\mathbf{a} = \int_\Omega \mathbf{N}^T \mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t}\,\mathrm{d}\Gamma \qquad \Rightarrow \qquad \mathbf{Ka} = \mathbf{f}_\text{ext}$$

Linearity is assumed twice there

$\boldsymbol{\varepsilon} = \mathbf{Ba}$ (kinematic relation)

and

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ (constitutive relation)

Sources of nonlinearity

This remains the general discretized equilibrium equation:

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For large displacements, we can have a nonlinear kinematic relation:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{a}) \quad \text{with} \quad \mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}}$$

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For instance, so-called *true strain*, which can in 1D be defined as

$$\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln \frac{l}{l_0} = \ln(1 + \nabla u)$$

Note: for $\nabla u \ll 1$, we have $\varepsilon \approx \nabla u$

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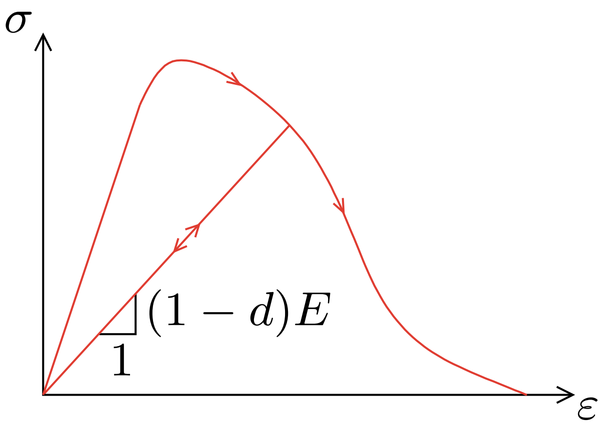
For large displacements, we can have a nonlinear kinematic relation:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{a}) \quad \text{with} \quad \mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}}$$

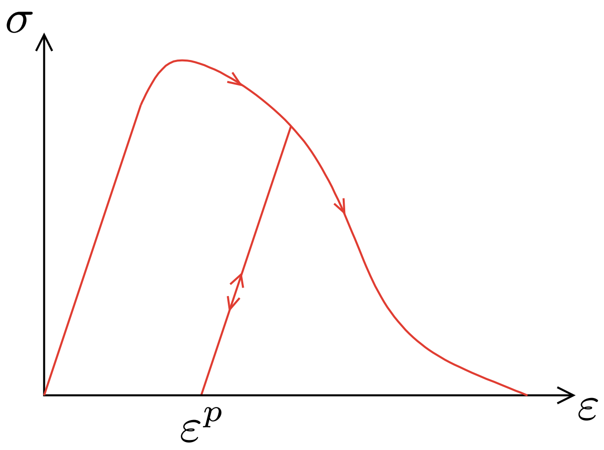
and for modeling material behavior a nonlinear constitutive relation:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \text{history}) \quad \text{with} \quad \mathbf{D} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$$

For instance damage



or plasticity



Problem statement

We want to solve a nonlinear system of equations:

$$\underbrace{\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \,\mathrm{d}\Omega}_{\mathbf{f}_{\text{int}}(\mathbf{a})} = \underbrace{\int_{\Omega} \mathbf{N}^T \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \,\mathrm{d}\Gamma}_{\mathbf{f}_{\text{ext}}(t)}$$

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- Internal force is a nonlinear function of $\mathbf{a}$
- For given $\mathbf{a}$ we can compute $\mathbf{f}_{\text{int}} = \int \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega$

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- External force changes in increments
- At every increment $t = t^n$, $\mathbf{f}_{\mathrm{ext}}$ is known
- Possibly $\mathbf{f}_{\mathrm{ext}} = 0$ and Dirichlet boundary conditions change

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For linear $\mathbf{f}_\mathrm{int}(\mathbf{a})$ we get a linear system of equations for every increment: $\mathbf{K}\mathbf{a}^n = \mathbf{f}^n_\mathrm{ext}$

$\rightarrow$ But what about a nonlinear $\mathbf{f}_\mathrm{int}(\mathbf{a})$?

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→ But what about a nonlinear $\mathbf{f}_{\mathrm{int}}(\mathbf{a})$?

→ For every increment, we will need to iterate

Incremental-iterative solution algorithm

In every time-step we solve a nonlinear system of equations with Newton-Raphson (or Newton's) method

```
Require: Solution from previous time step $\mathbf{a}^n$
Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$
Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$
Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
Evaluate residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
repeat
    Solve linear system of equations: $\mathbf{K}^{n+1}\Delta\mathbf{a} = \mathbf{r}$
    Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta\mathbf{a}$
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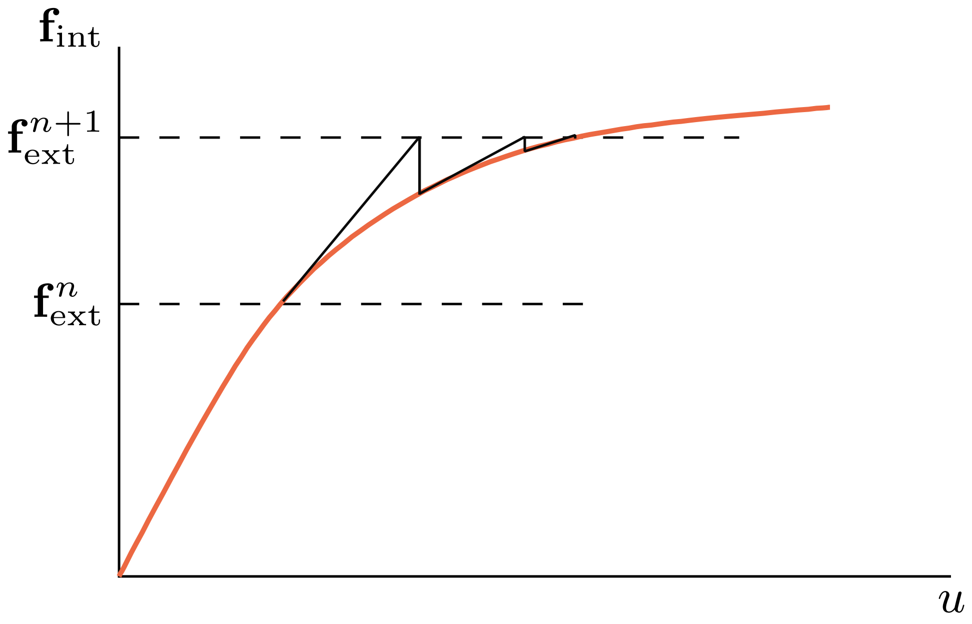

Incremental-iterative solution algorithm, including time step loop

```
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Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps do
    Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
    Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$
    Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
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    $n = n + 1$
end while
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What about boundary conditions?

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Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps do
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- Neumann boundary conditions
- Point loads also go here
- Possibly increasing step by step

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- Dirichlet boundary conditions
- Enforced by manipulating system of eqs.
- $\Delta \mathbf{u}_c$ contains increments in first iteration
- $\Delta \mathbf{u}_c = 0$ in other iterations

Convergence

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- Different norms are possible
- Additional criterion: max # of iterations
- Convergence is not always guaranteed
- Non-converged solutions should not be kept
- Adaptive step size may be needed
- Linearization is crucial

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Linearization

In the algorithm we have $\mathbf{K}$ as the derivative of $\mathbf{f}_{\mathrm{int}}$ to $\mathbf{a}$ with :

$$\mathbf{f}_{\mathrm{int}} = \int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega$$

Applying the product rule and chain rule of differentation:

$$\mathbf{K} = \int_\Omega \frac{\partial \mathbf{B}^T}{\partial \mathbf{a}} \boldsymbol{\sigma} + \mathbf{B}^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}} \, \mathrm{d}\Omega$$

We already had $\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \mathbf{D}$ and $\frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}} = \mathbf{B}$, so we get:

$$\mathbf{K} = \int_\Omega \frac{\partial \mathbf{B}^T}{\partial \mathbf{a}} \boldsymbol{\sigma} + \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega$$

For the geometrically linear situation, we get:

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega$$

Very similar to the matrix for linear FEM, but $\mathbf{D}$ should be the consistent linearization of $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$

Linearization and convergence

Theoretically, consistent linearization offers quadratic convergence

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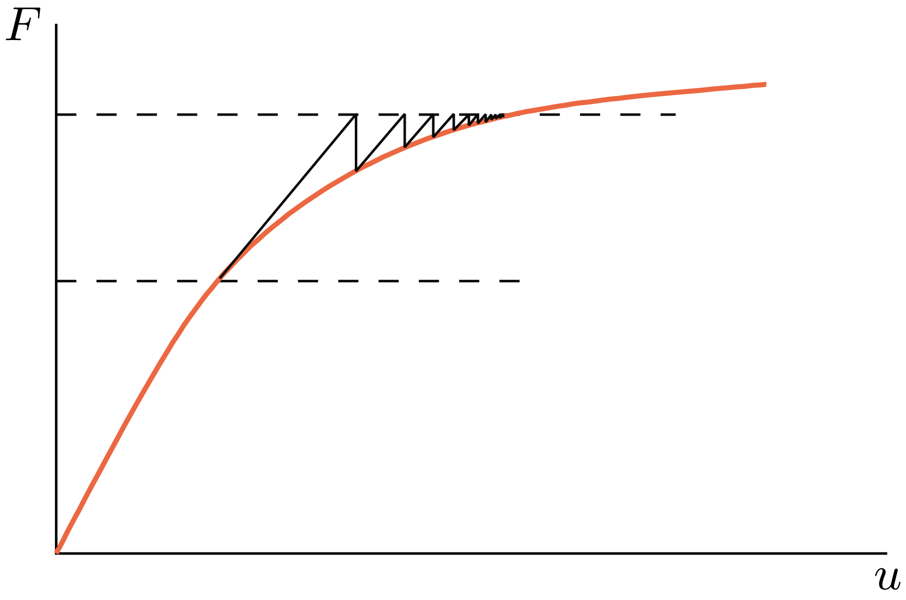

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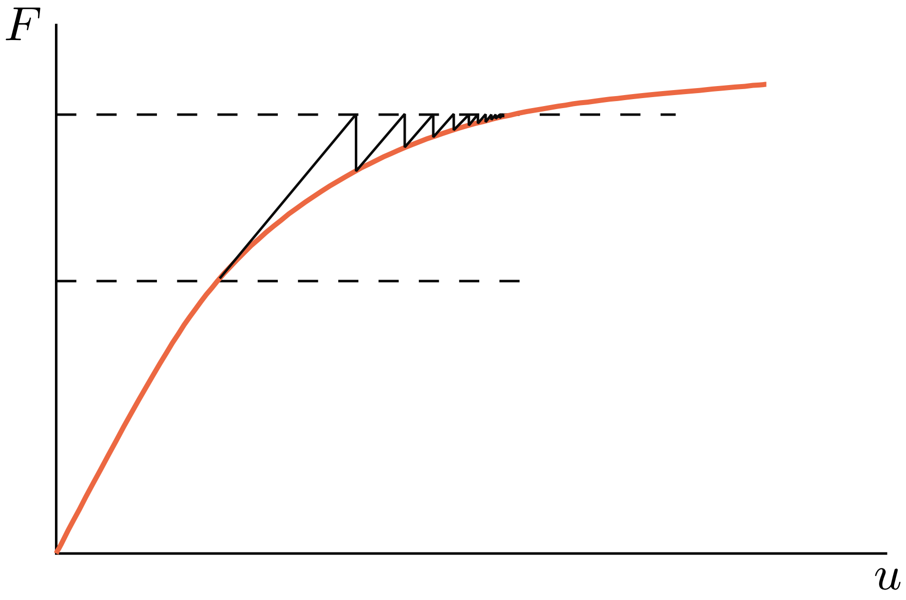

Although this requires many more iterations

```
iter = 1, scaled residual = 2.3269e-02
iter = 2, scaled residual = 2.2279e-02
iter = 3, scaled residual = 1.9872e-02
iter = 4, scaled residual = 1.6512e-02
iter = 5, scaled residual = 1.3107e-02
iter = 6, scaled residual = 1.0113e-02
iter = 7, scaled residual = 7.6675e-03
iter = 8, scaled residual = 5.7517e-03
iter = 9, scaled residual = 4.2868e-03
iter = 10, scaled residual = 3.1826e-03
iter = 11, scaled residual = 2.3574e-03
iter = 12, scaled residual = 1.7438e-03
iter = 13, scaled residual = 1.2890e-03
iter = 14, scaled residual = 9.5234e-04
iter = 15, scaled residual = 7.0348e-04
iter = 16, scaled residual = 5.1959e-04
iter = 17, scaled residual = 3.8374e-04
iter = 18, scaled residual = 2.8341e-04
iter = 19, scaled residual = 2.0931e-04
iter = 20, scaled residual = 1.5459e-04
iter = 21, scaled residual = 1.1417e-04
iter = 22, scaled residual = 8.4326e-05
```

Modified Newton-Raphson

The algorithm remains the same but $\mathbf{K}$ is updated once per time step

- Convergence will be slower
- Reduced change of divergence or oscillatory behavior

Alternatives:

- Use incomplete linearization for $\mathbf{D}$ (secant matrix)
- Use initial elastic stiffness matrix $\mathbf{K}^0$
- $\ldots$

Recap of agenda for today

1. Characteristics of nonlinear problems
2. Virtual work interpretation of weak form
3. Sources of nonlinearity
4. General formulation for the nonlinear system of equations
5. Incremental-iterative solution procedure