CIEM5110-2: FEM, lecture 6.2

Finite element analysis for dynamics of solids and structures

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Agenda for today

1. Derivation of semi-discretized form
2. Time stepping schemes
    - Central difference (explicit)
    - Newmark (implicit)
3. Frequency analysis

CIEM5110-2 workshops and lectures

	(Theory)	BarModel (MUDE)	SolidModel (1.2)	TimoshenkoModel (2.1)	FrameModel (4.1)
SolverModule	(1.2)	2.2	2.2	3.2	3.2
NonlinModule	(3.1)		6.1		4.1 + 4.2 + 5.1
LinBuckModule	(4.1)				4.1 + 5.1
ModeShapeModule	**(6.2)**		7.1		7.1 + 8.1
ExplicitTimeModule	**(6.2)**				7.2 + 8.1
NewmarkModule	**(6.2)**				7.2 + 8.1

Derivation of semi-discrete form

Focus is on continuum mechanics. We add an inertia term to the PDE

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Following the steps of strong–weak–discretized form gives

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}$$

with

$$\mathbf{M} = \int_\Omega \mathbf{N}^T \rho \mathbf{N} \, \mathrm{d}\Omega$$

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega$$

$$\mathbf{f} = \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

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\end{aligned}$$

For other problems (`models`), the semi-discrete equation is the same, but the definition of $\mathbf{M}$ may be different

TU Delft

Damping

No similar derivation exists for damping, but a damped system of equations should have this form

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}$$

For instance with Rayleigh damping

$$\mathbf{C} = \tau\mathbf{M} + \phi\mathbf{K}$$

Time discretization

We will solve the time dependent problem using time steps

Find $[\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_{n-1}, \mathbf{a}_n, \mathbf{a}_{n+1}, \ldots, \mathbf{a}_{nt}]$ that satisfies

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}$$

Require: Stiffness matrix $\mathbf{K}$; mass matrix $\mathbf{M}$; damping matrix $\mathbf{C}$; external force $\mathbf{f}(t)$; time step size Δt

1: Initialize $n = 0$. Set $\mathbf{a}_0, \dot{\mathbf{a}}_0$ from initial conditions
2: **while** $n <$ number of time steps **do**
3: Compute solution $\mathbf{a}_{n+1}$ with selected **time stepping scheme**
4: Update velocity $\dot{\mathbf{a}}_{n+1}$ and acceleration $\ddot{\mathbf{a}}_{n+1}$
5: Go to the next time step $n = n + 1$
6: **end while**

Central difference scheme

Semi-discretized form ($\mathbf{a}$ is still a continuous function of t):

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{C}\dot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}$$

Using central difference approximations for the time derivatives at t_n, this can be discretized as:

$$\mathbf{M}\frac{\mathbf{a}_{n-1} - 2\mathbf{a}_n + \mathbf{a}_{n+1}}{\Delta t^2} + \mathbf{C}\frac{\mathbf{a}_{n+1} - \mathbf{a}_{n-1}}{2\Delta t} + \mathbf{K}\mathbf{a}_n = \mathbf{f}_n$$

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And then reorganized as:

$$\hat{\mathbf{M}}\mathbf{a}_{n+1} = \hat{\mathbf{f}}_n$$

with

$$\hat{\mathbf{M}} = \frac{1}{\Delta t^2}\mathbf{M} + \frac{1}{2\Delta t}\mathbf{C} \qquad \text{and} \qquad \hat{\mathbf{f}} = \frac{1}{\Delta t^2}\mathbf{M}\left(2\mathbf{a}_n - \mathbf{a}_{n-1}\right) + \frac{1}{2\Delta t}\mathbf{C}\mathbf{a}_{n-1} - \mathbf{K}\mathbf{a}_n + \mathbf{f}_n$$

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Diagonalizing $\hat{\mathbf{M}}$ is beneficial for efficiency of solving the system of equation

Central difference scheme: discussion

The central difference scheme

- Single solve per time step, even with nonlinear $\mathbf{f}_{\mathrm{int}}(\mathbf{a})$
- Very efficient after diagonalizationg ('lumping')
- Conditionally stable

For linear elements: $\Delta t \leq 2/\omega^h$ with $\omega^h \approx 2c/L^{\mathrm{e}}$ and $c = \sqrt{E/\rho}$

Newmark time integration

Introduce the following approximations (cf. trapezoidal integration)

$$\mathbf{a}_{n+1} = \mathbf{a}_n + \Delta t \dot{\mathbf{a}}_n + \tfrac{1}{2}\Delta t^2 \left((1-2\beta)\ddot{\mathbf{a}}_n + 2\beta\ddot{\mathbf{a}}_{n+1}\right)$$
$$\dot{\mathbf{a}}_{n+1} = \dot{\mathbf{a}}_n + \Delta t \left((1-\gamma)\ddot{\mathbf{a}}_n + \gamma\ddot{\mathbf{a}}_{n+1}\right)$$

Rearranging and substituting the first into the second equation gives:

$$\ddot{\mathbf{a}}_{n+1} = \frac{1}{\beta\Delta t^2}\left(\mathbf{a}_{n+1} - \mathbf{a}_n\right) - \frac{1}{\beta\Delta t}\dot{\mathbf{a}}_n + \left(1 - \frac{1}{2\beta}\right)\ddot{\mathbf{a}}_n$$
$$\dot{\mathbf{a}}_{n+1} = \dot{\mathbf{a}}_n + \Delta t\left((1-\gamma)\ddot{\mathbf{a}}_n + \gamma\left(\frac{1}{\beta\Delta t^2}\left(\mathbf{a}_{n+1} - \mathbf{a}_n\right) - \frac{1}{\beta\Delta t}\dot{\mathbf{a}}_n + \left(1 - \frac{1}{2\beta}\right)\ddot{\mathbf{a}}_n\right)\right)$$

Then we can express the solution at t_{n+1} in terms of $\mathbf{a}_{n+1}$ (ignoring damping for brevity):

$$\mathbf{M}\ddot{\mathbf{a}}_{n+1} + \mathbf{K}\mathbf{a}_{n+1} = \mathbf{f}_{n+1} \quad \Rightarrow \quad \hat{\mathbf{K}}\mathbf{a}_{n+1} = \hat{\mathbf{f}}_{n+1}$$

with

$$\hat{\mathbf{K}} = \frac{1}{\beta\Delta t^2}\mathbf{M} + \mathbf{K} \quad \text{and} \quad \hat{\mathbf{f}} = \mathbf{M}\left(\frac{1}{\beta\Delta t^2}\mathbf{a}_n + \frac{1}{\beta\Delta t}\dot{\mathbf{a}}_n + \frac{1}{2\beta}\dot{\mathbf{a}}_n - \ddot{\mathbf{a}}_n\right) + \mathbf{f}_n$$

Newmark time integration scheme: discussion

The Newmark scheme is unconditionally stable for $2\beta \geq \gamma \geq \frac{1}{2}$

Newmark is equivalent to the central difference scheme with $\beta = 0$ and $\gamma = \frac{1}{2}$

Even without damping, we need to compute $\dot{\mathbf{a}}$

With trapezoidal integration ($\beta = \frac{1}{4}, \gamma = \frac{1}{2}$), the scheme is second order accurate

Numerical damping is obtained with $\gamma > \frac{1}{2}$, but the scheme becomes first order accurate

$\hat{\mathbf{K}}$ contains $\mathbf{K}$ which cannot be diagonalized

For nonlinear problems $\mathbf{f}_{\text{int}}(\mathbf{a}_{n+1})$ replaces $\mathbf{K}\mathbf{a}_{n+1}$ and iterations are needed

Frequency analysis

Back to the undamped semi-discretized system of equations

$$\mathbf{M}\ddot{\mathbf{a}} + \mathbf{K}\mathbf{a} = \mathbf{f}$$

We can find natural frequencies with a generalized eigenvalue problem

$$\det\left(\mathbf{K} - \omega^2\mathbf{M}\right) = 0$$

Eigenmodes are the modes of vibration

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Compare with linear buckling analysis, where we solved

$$\det\left(\mathbf{K}_M + \lambda\mathbf{K}_G\right) = 0$$

To compute the critical buckling load and buckling mode

FE analysis of dynamics of solids and structures: discussion

Explicit time-dependent analysis (central difference scheme)

- Individual time steps are very efficient (especially with mass lumping)
- Need small time steps, related to the mesh size
- Sometimes this is used/abused for quasi-static analysis (mass scaling)

Implicit time-dependent analysis (Newmark scheme)

- Stable but more costly per time step
- Time steps can be much larger, related to the time scale of the problem at hand
- Numerical damping comes at a price in accuracy

Frequency analysis

- Provides natural frequencies and vibration modes
- If structure (bridge/building/...) cannot be modeled as a prismatic beam
- Gives information on vibration mode, could proceed with forced modal analysis