CIEM1110-1: Numerical modeling, lecture 3.1

Nonlinear FEM: solution procedure

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Agenda for today

1. Characteristics of nonlinear problems
2. Virtual work interpretation of weak form
3. Sources of nonlinearity
4. General formulation for the nonlinear system of equations
5. Incremental-iterative solution procedure

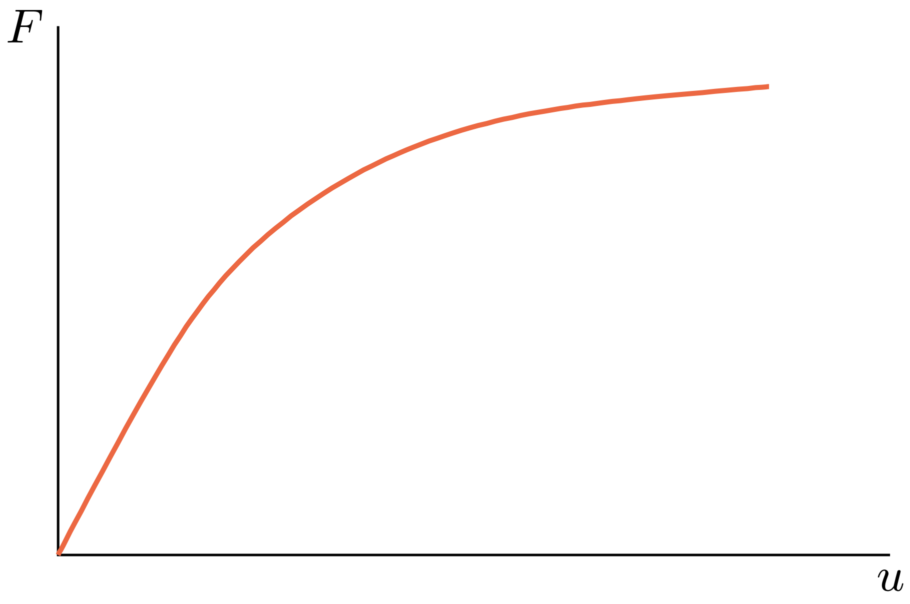
Characteristics of nonlinear problems

In nonlinear simulations, we simulate a process

Often this is quasi-static $\rightarrow$ no actual time, but still 'time steps' or increments

Even if we are only interested in a final state, a number of increments can be needed to get there

The classical output of a nonlinear finite element simulation is a force-displacement curve



Remember: this is a 1D representation of an n_{dof}-dimensional solution

Interpreting the weak formulation as virtual work equation (continuum mechanics)

Weak form (before assuming linear elasticity):

$$-\int_\Omega \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\sigma}\,\mathrm{d}\Omega + \int_\Omega \mathbf{w}\cdot\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{w}\cdot\mathbf{t}\,\mathrm{d}\Gamma = \mathbf{0}, \quad \forall \quad \mathbf{w}$$

Let $\mathbf{w} \leftarrow \delta\mathbf{u}$ (just a change of symbol):

$$-\int_\Omega \nabla^{\mathrm{s}}\delta\mathbf{u} : \boldsymbol{\sigma}\,\mathrm{d}\Omega + \int_\Omega \delta\mathbf{u}\cdot\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \delta\mathbf{u}\cdot\mathbf{t}\,\mathrm{d}\Gamma = \mathbf{0}, \quad \forall \quad \delta\mathbf{u}$$

With $\nabla^{\mathrm{s}}\delta\mathbf{u} = \delta\boldsymbol{\varepsilon}$ we can give a physical interpretation to the weak form:

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After discretization (with $\delta\mathbf{u} = \mathbf{N}\delta\mathbf{a}$ and $\delta\boldsymbol{\varepsilon} = \mathbf{B}\delta\mathbf{a}$):

$$\delta\mathbf{a}^T \int_\Omega \mathbf{B}^T\boldsymbol{\sigma}\,\mathrm{d}\Omega = \delta\mathbf{a}^T \left(\int_\Omega \mathbf{N}^T\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T\mathbf{t}\,\mathrm{d}\Gamma \right) \qquad \Rightarrow \qquad \int_\Omega \mathbf{B}^T\boldsymbol{\sigma}\,\mathrm{d}\Omega = \int_\Omega \mathbf{N}^T\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T\mathbf{t}\,\mathrm{d}\Gamma$$

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Back to the linear case

This is the general discretized equilibrium equation:

$$\underbrace{\int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \,\mathrm{d}\Omega}_{\mathbf{f}_\text{int}} = \underbrace{\int_\Omega \mathbf{N}^T \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \,\mathrm{d}\Gamma}_{\mathbf{f}_\text{ext}}$$

Assuming linear elasticity, we could substitute $\boldsymbol{\sigma} = \mathbf{D}\mathbf{B}\mathbf{a}$ to get

$$\int_\Omega \mathbf{B}^T \mathbf{D}\mathbf{B} \,\mathrm{d}\Omega \,\mathbf{a} = \int_\Omega \mathbf{N}^T \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \,\mathrm{d}\Gamma \qquad \Rightarrow \qquad \mathbf{K}\mathbf{a} = \mathbf{f}_\text{ext}$$

Linearity is assumed twice there

$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}$ (kinematic relation)

and

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ (constitutive relation)

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Sources of nonlinearity

This remains the general discretized equilibrium equation:

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For large displacements, we can have a nonlinear kinematic relation:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{a}) \quad \text{with} \quad \mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}}$$

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For instance, so-called *true strain*, which can in 1D be defined as

$$\varepsilon = \int_{l_0}^{l} \frac{dl}{l} = \ln \frac{l}{l_0} = \ln(1 + \nabla u)$$

Note: for $\nabla u \ll 1$, we have $\varepsilon \approx \nabla u$

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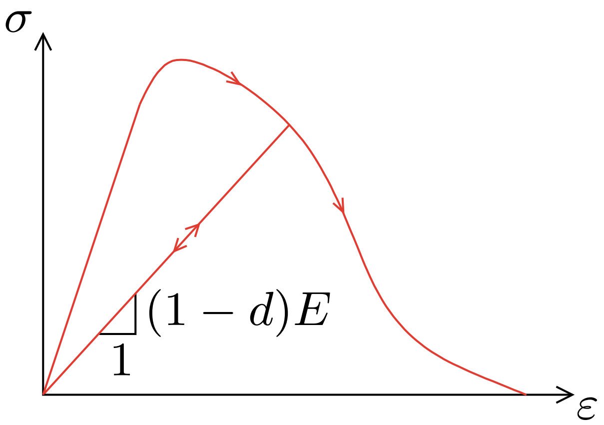
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\mathbf{a}) \quad \text{with} \quad \mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}}$$

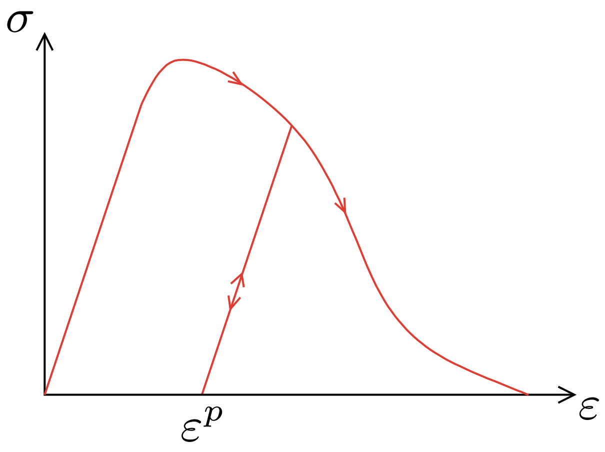
and for modeling material behavior a nonlinear constitutive relation:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \text{history}) \quad \text{with} \quad \mathbf{D} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$$

For instance damage



or plasticity



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We want to solve a nonlinear system of equations:

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- For given $\mathbf{a}$ we can compute $\mathbf{f}_{\text{int}} = \int \mathbf{B}^T \boldsymbol{\sigma}\, \mathrm{d}\Omega$

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- External force changes in increments
- At every increment $t = t^n$, $\mathbf{f}_{\text{ext}}$ is known
- Possibly $\mathbf{f}_{\text{ext}} = 0$ and Dirichlet boundary conditions change

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$\rightarrow$ For every increment, we will need to iterate

Incremental-iterative solution algorithm

In every time-step we solve a nonlinear system of equations with Newton-Raphson (or Newton's) method

Require: Solution from previous time step $\mathbf{a}^n$

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$

Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$

Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$

Evaluate residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$

repeat

  Solve linear system of equations: $\mathbf{K}^{n+1}\Delta\mathbf{a} = \mathbf{r}$

  Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta\mathbf{a}$

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until $|\mathbf{r}| < \text{tolerance}$

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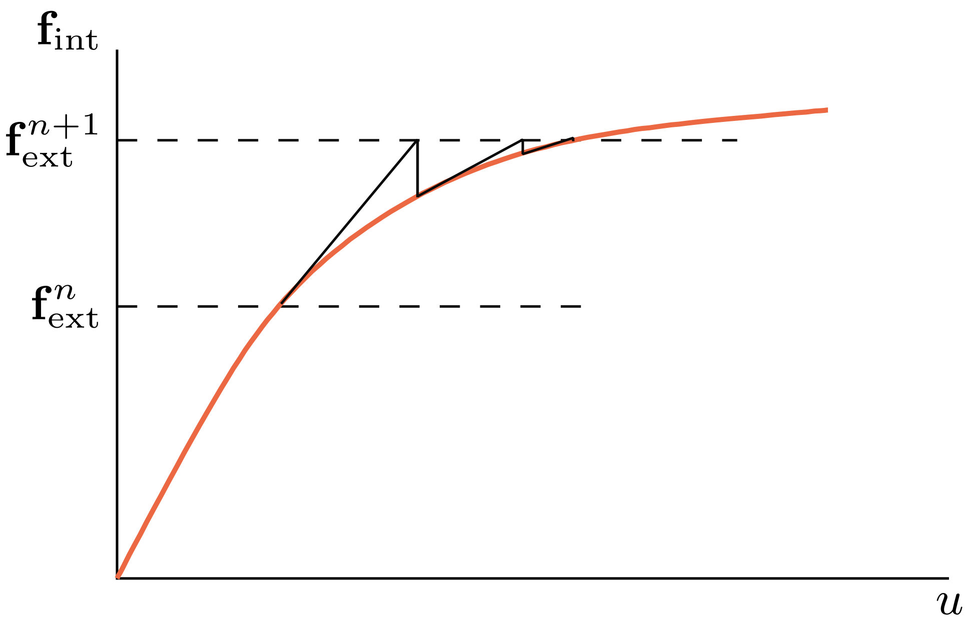

Incremental-iterative solution algorithm, including time step loop

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps **do**
    Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
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while $n <$ number of time steps **do**

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- Neumann boundary conditions
- Point loads also go here
- Possibly increasing step by step

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- Dirichlet boundary conditions
- Enforced by manipulating system of eqs.
- $\Delta\mathbf{u}_c$ contains increments in first iteration
- $\Delta\mathbf{u}_c = 0$ in other iterations

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Convergence

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

```
Initialize n = 0, a^0 = 0
while n < number of time steps do
    Get new external force vector: f_ext^{n+1}
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    Compute internal force and stiffness: f_int^{n+1}(a^{n+1}), K^{n+1}(a^{n+1})
    Evaluate residual: r = f_ext^{n+1} - f_int^{n+1}
    repeat
        Solve linear system of equations: K^{n+1} Δa = r
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```

Convergence

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

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- Different norms are possible
- Additional criterion: max # of iterations
- Convergence is not always guaranteed
- Non-converged solutions should not be kept
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- Linearization is crucial

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Linearization

In the algorithm we have $\mathbf{K}$ as the derivative of $\mathbf{f}_{\text{int}}$ to $\mathbf{a}$ with :

$$\mathbf{f}_{\text{int}} = \int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega$$

Applying the product rule and chain rule of differentation:

$$\mathbf{K} = \int_\Omega \frac{\partial \mathbf{B}^T}{\partial \mathbf{a}} \boldsymbol{\sigma} + \mathbf{B}^T \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}} \, \mathrm{d}\Omega$$

We already had $\dfrac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \mathbf{D}$ and $\dfrac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}} = \mathbf{B}$, so we get:

$$\mathbf{K} = \int_\Omega \frac{\partial \mathbf{B}^T}{\partial \mathbf{a}} \boldsymbol{\sigma} + \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega$$

For the geometrically linear situation, we get:

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega$$

Very similar to the matrix for linear FEM, but $\mathbf{D}$ should be the consistent linearization of $\boldsymbol{\sigma}(\boldsymbol{\varepsilon})$

Linearization and convergence

Theoretically, consistent linearization offers quadratic convergence

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- sufficiently close initial guess

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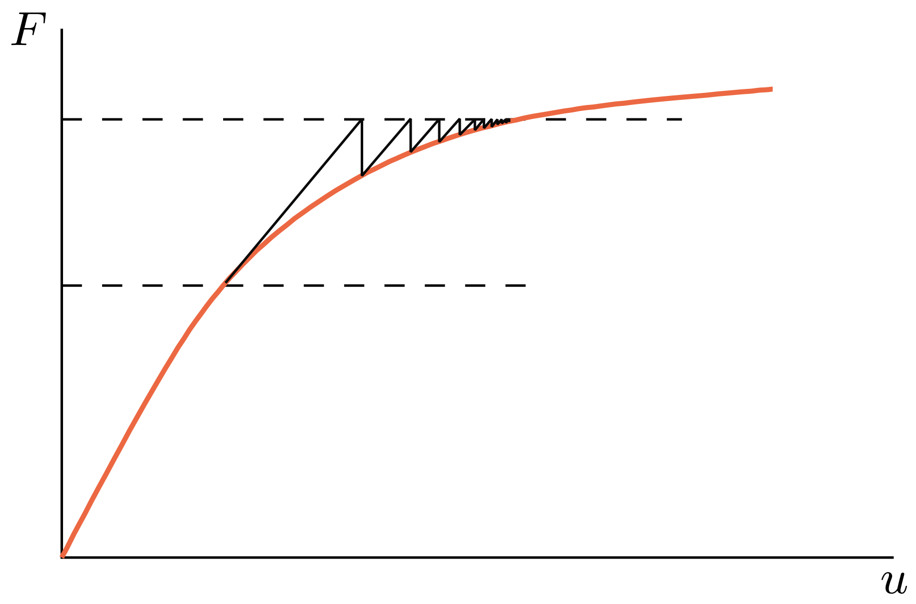

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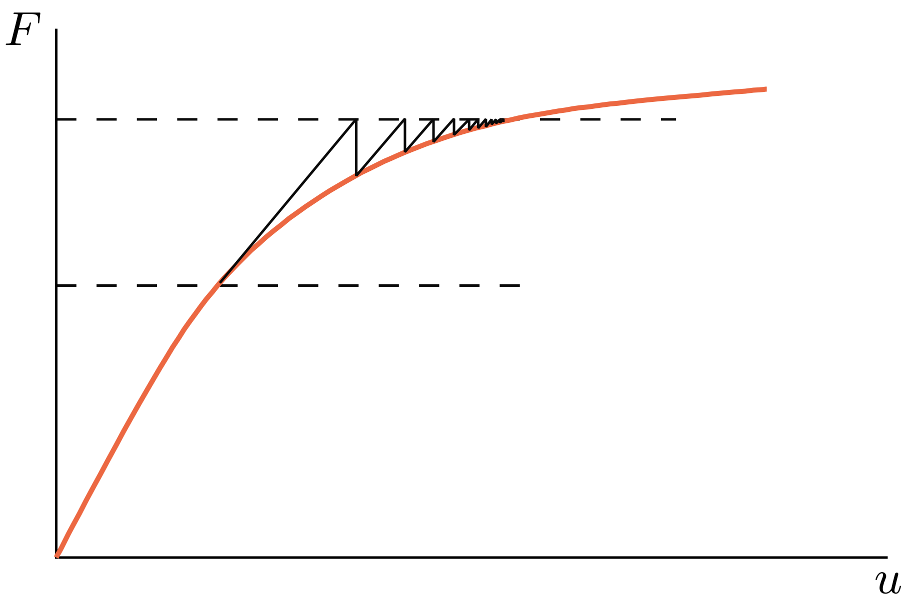

Although this requires many more iterations

```
iter = 1, scaled residual = 2.3269e-02
iter = 2, scaled residual = 2.2279e-02
iter = 3, scaled residual = 1.9872e-02
iter = 4, scaled residual = 1.6512e-02
iter = 5, scaled residual = 1.3107e-02
iter = 6, scaled residual = 1.0113e-02
iter = 7, scaled residual = 7.6675e-03
iter = 8, scaled residual = 5.7517e-03
iter = 9, scaled residual = 4.2868e-03
iter = 10, scaled residual = 3.1826e-03
iter = 11, scaled residual = 2.3574e-03
iter = 12, scaled residual = 1.7438e-03
iter = 13, scaled residual = 1.2890e-03
iter = 14, scaled residual = 9.5234e-04
iter = 15, scaled residual = 7.0348e-04
iter = 16, scaled residual = 5.1959e-04
iter = 17, scaled residual = 3.8374e-04
iter = 18, scaled residual = 2.8341e-04
iter = 19, scaled residual = 2.0931e-04
iter = 20, scaled residual = 1.5459e-04
iter = 21, scaled residual = 1.1417e-04
iter = 22, scaled residual = 8.4326e-05
```

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Modified Newton-Raphson

The algorithm remains the same but $\mathbf{K}$ is updated once per time step

- Convergence will be slower
- Reduced change of divergence or oscillatory behavior

Alternatives:

- Use incomplete linearization for $\mathbf{D}$ (secant matrix)
- Use initial elastic stiffness matrix $\mathbf{K}^0$
- $\ldots$

Recap of agenda for today

1. Characteristics of nonlinear problems
2. Virtual work interpretation of weak form
3. Sources of nonlinearity
4. General formulation for the nonlinear system of equations
5. Incremental-iterative solution procedure