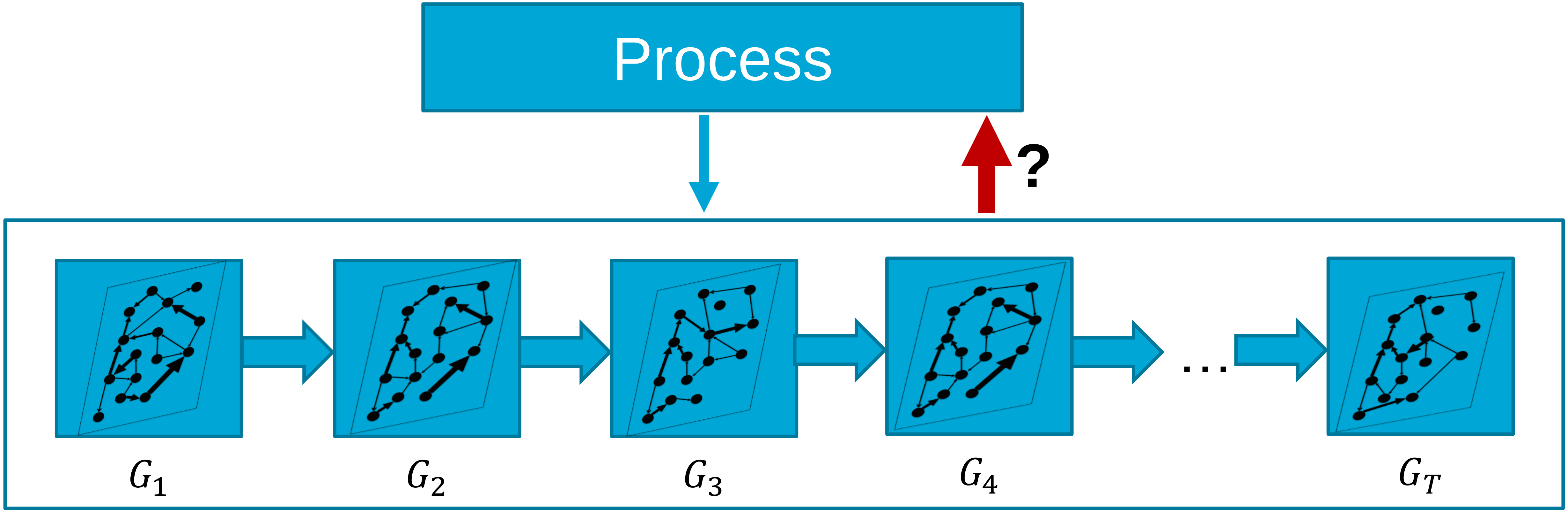
“System Identification for Temporal Networks”

Sergey Shvydun & Piet Van Mieghem (TU Delft)

Outline

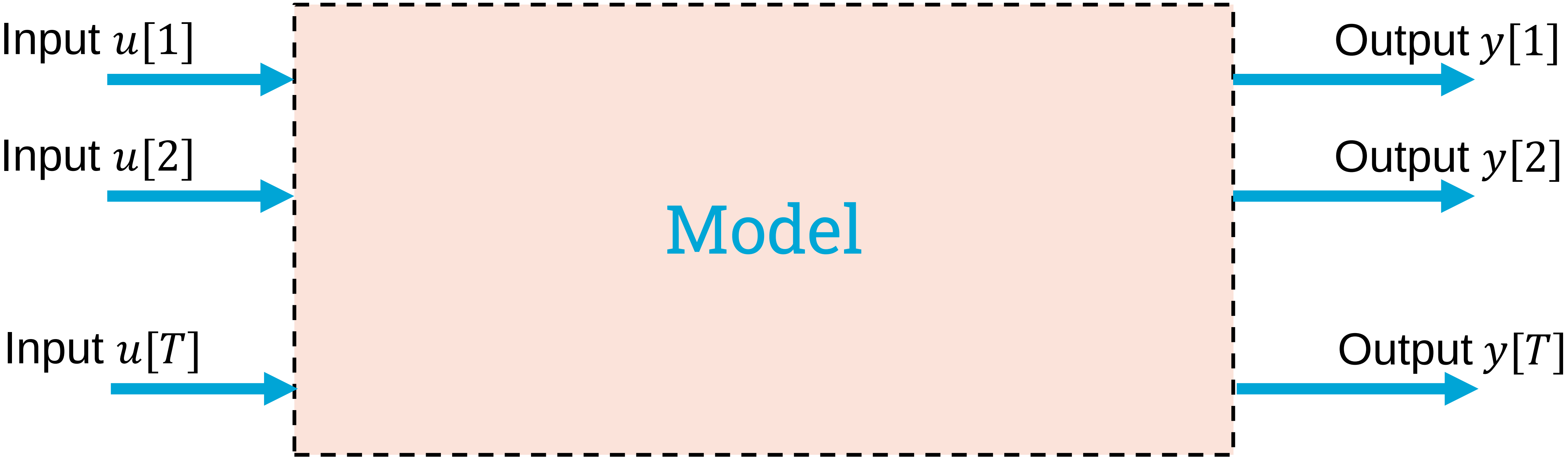
1) Problem Statement;
2) Linear Time-Invariant (LTI) State-Space Model;
3) Application of the LTI Model to Time-Evolving Networks;
    - Periodic Graph Dynamics;
    - Non-Periodic Graph Dynamics.

Problem Statement



Time k

Linear Time-Invariant (LTI) State Space Model



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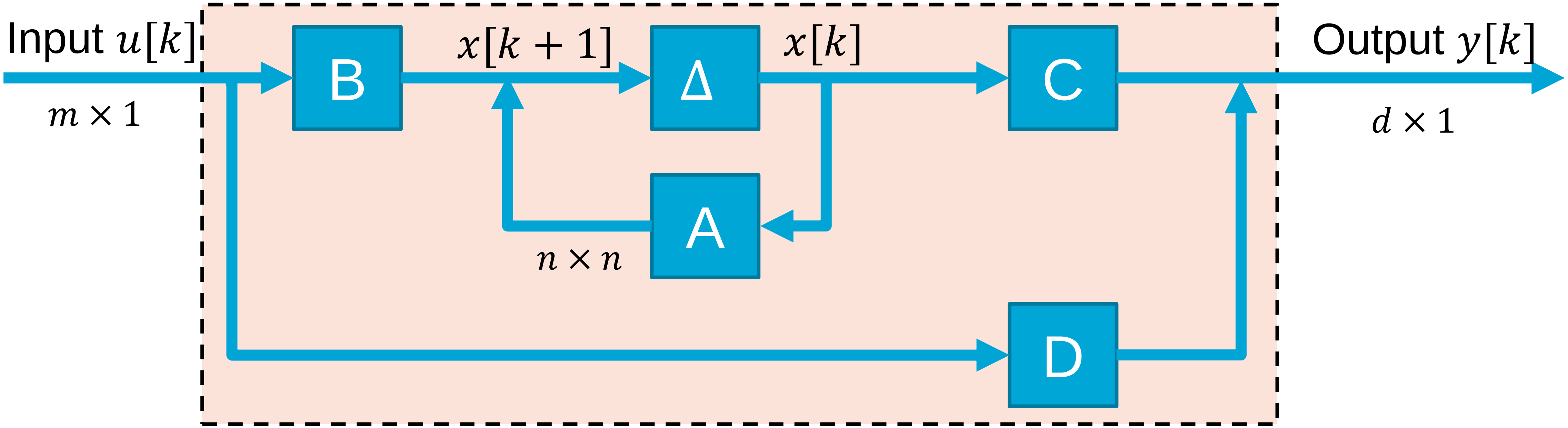
$u[1], \ldots, u[T]$ are $m \times 1$ vectors;
$y[1], \ldots, y[T]$ are $d \times 1$ vectors.

Linear Time-Invariant (LTI) State Space Model



- $x[k]$ – state vector at discrete time k;
- Δ – delay;

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$$\begin{bmatrix} x[k+1] \\ y[k] \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot \begin{bmatrix} x[k] \\ u[k] \end{bmatrix} = Q \cdot \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

Subspace Method for System Identification

Compact form of the equation is

$$Y_{1,s,N} = \Gamma_s X_{1,N} + H_s U_{1,s,N}$$

where $Y_{i,s,N}, U_{i,s,N}$ - block Hankel matrices:

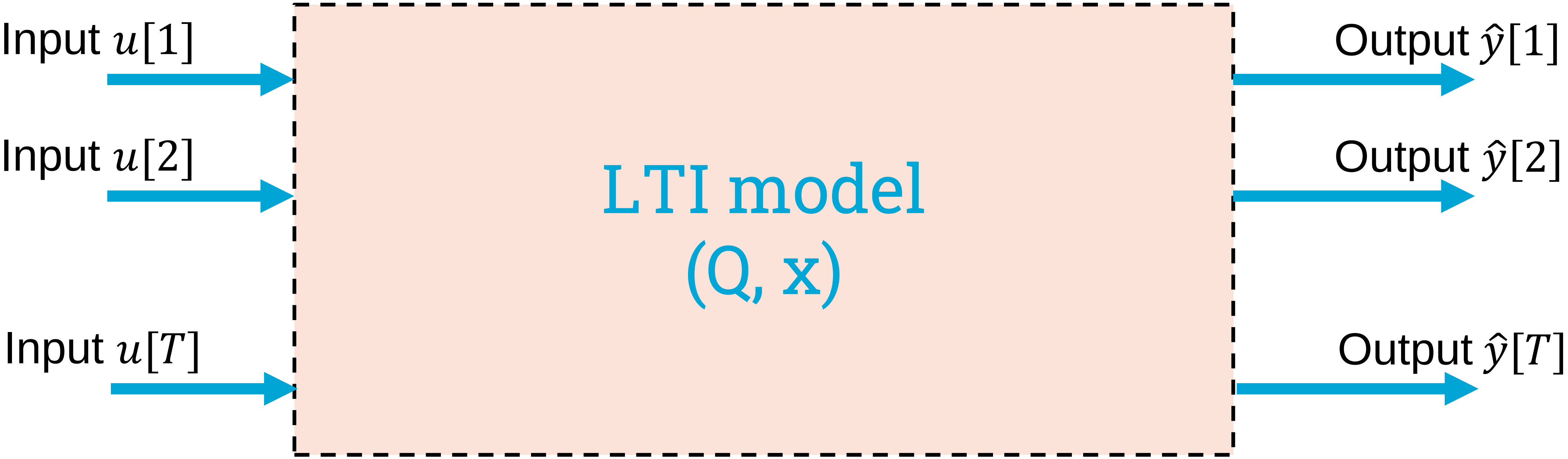
$$Y_{1,s,N} = \begin{pmatrix} y[1] & y[2] & \cdots & y[N] \\ y[2] & y[3] & \cdots & y[N+1] \\ \vdots & \vdots & \ddots & \vdots \\ y[s] & y[s+1] & \cdots & y[N+s] \end{pmatrix}, X_{1,N} = (x[1] \quad \dots \quad x[N]), U_{1,s,N} = \begin{pmatrix} u[1] & u[2] & \cdots & y[N] \\ u[2] & u[3] & \cdots & y[N+1] \\ \vdots & \vdots & \ddots & \vdots \\ u[s] & y[s+1] & \cdots & y[N+s] \end{pmatrix},$$

s, N – parameters and Γ_s, H_s - observability and controllability matrices:

$$\Gamma_s = \begin{bmatrix} C \\ CA \\ \cdots \\ CA^{s-1} \end{bmatrix}, \quad H_s = \begin{bmatrix} D & 0 & \cdots & 0 \\ CB & D & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ CA^{s-2}B & CA^{s-1}B & \cdots & D] \end{bmatrix}.$$

<u>N4SID Algorithm</u>: a **N**umerical algorithm for **s**ubspace **s**tate **s**pace **s**ystem **id**entification, developed by Van Overschee & De Moor (1994).

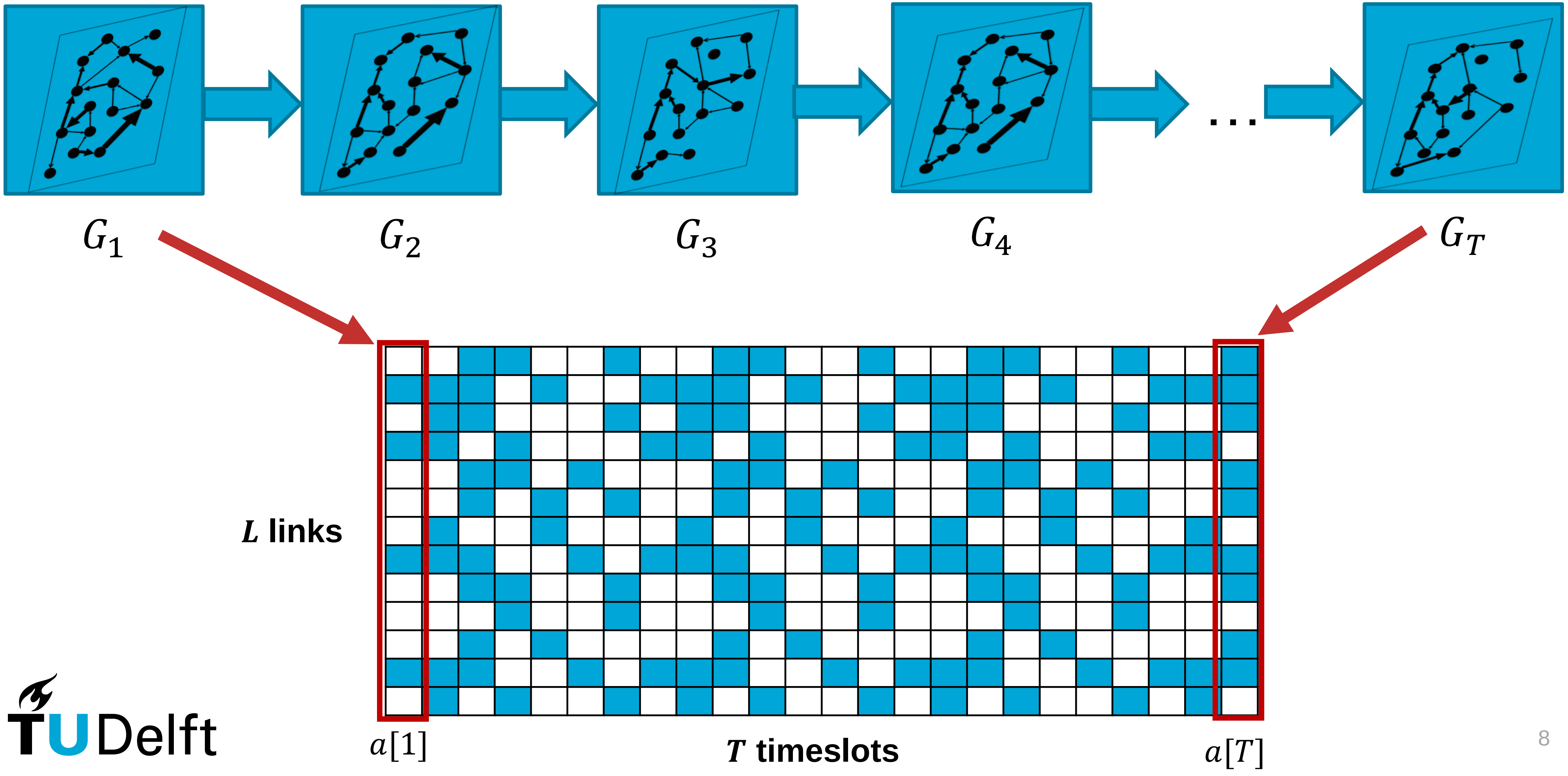
Linear Time-Invariant (LTI) State Space Model



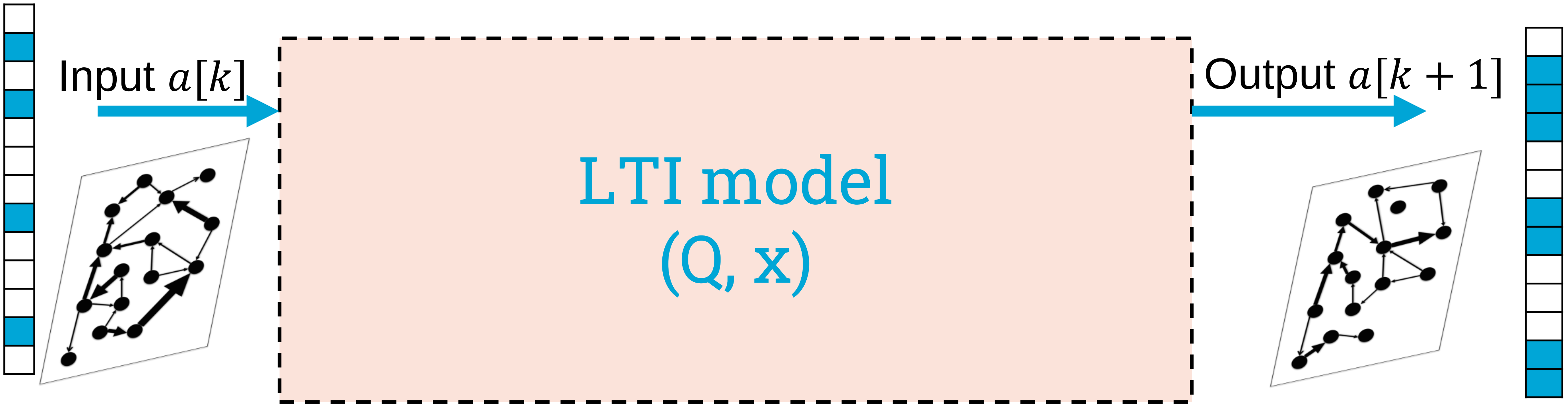
$$\begin{bmatrix} x[k+1] \\ \hat{y}[k] \end{bmatrix} = Q \cdot \begin{bmatrix} x[k] \\ u[k] \end{bmatrix}$$

TU Delft **What is the input/output for graphs?**

Graph Representation



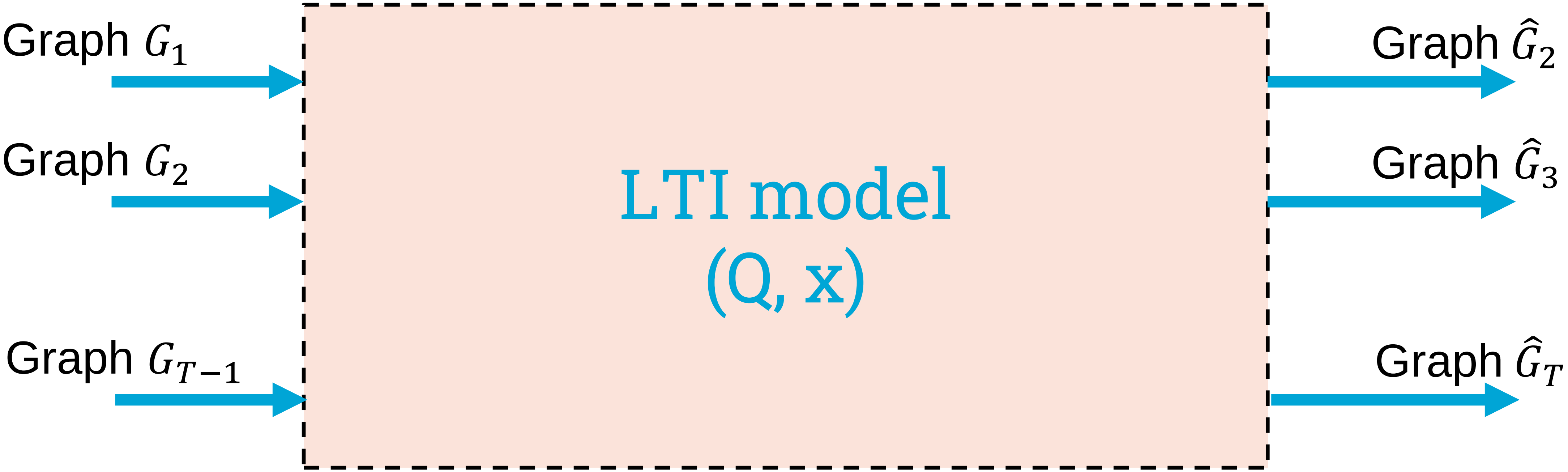
Application to Time-Evolving Graphs



Input: Graph G_k, $L \times 1$ binary vector $a[k]$.

Output: Graph G_{k+1}, $L \times 1$ binary vector $a[k+1]$.

Linear Time-Invariant (LTI) State Space Model



Input: Graph G_k, $L \times 1$ binary vector.

Output: Graph $\hat{G}_{k+1}$, $L \times 1$ binary vector.

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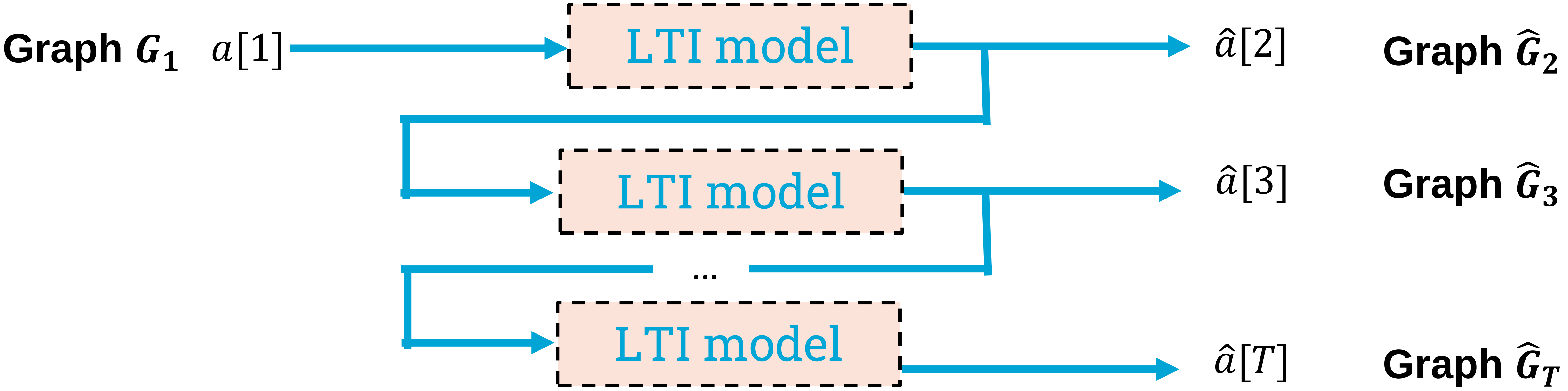
How to generate more than 1 graph?

Application to Time-Evolving Graphs

Subspace Graph Generator (SG-Gen)



k-th output:

$$\begin{bmatrix} x[k+1] \\ \hat{a}[k+1] \end{bmatrix} = Q^k \begin{bmatrix} x[1] \\ a[1] \end{bmatrix}$$

Performance of the Model

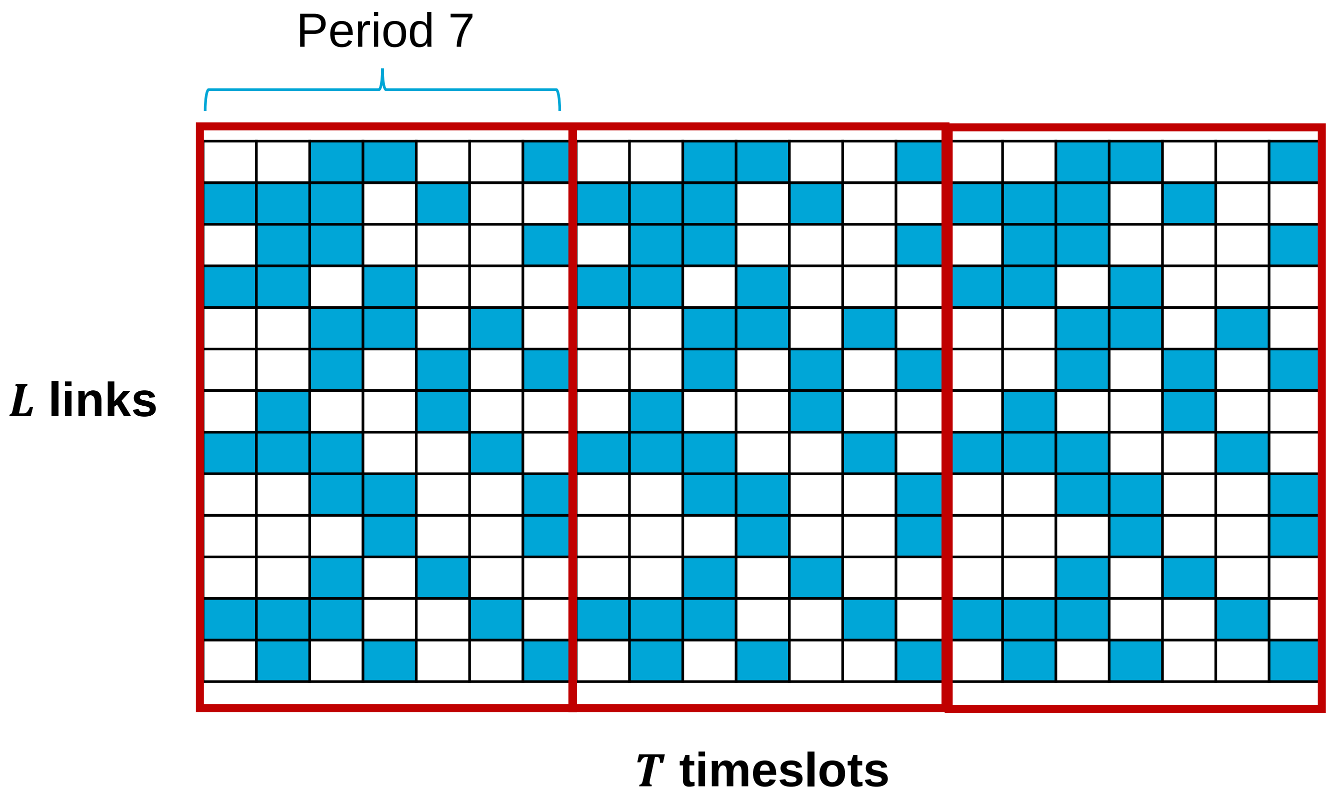
Mean square error (MSE):

$$MSE(a, \hat{a}) = \frac{1}{T-1} \sum_{k=1}^{T-1} \sum_{i=1}^{L} (a_i[k] - \hat{a}_i[k])^2 \,,$$

- $a[k]$ – real vector corresponding to the graph G_{k+1};
- $\hat{a}[k]$ – estimated vector corresponding to the graph G_{k+1}.
- T – number of graphs, L – dimension of $a[k]$ and $\hat{a}[k]$.

Periodic Graph Dynamics

Periodic Graph Dynamics



Application to Periodic Graph Dynamics

- **Observation**: ANY periodic graph dynamic can be modelled accurately by SG-Gen:

$$\begin{bmatrix} x[k+1] \\ a[k+1] \end{bmatrix} = Q^k \begin{bmatrix} x[1] \\ a[1] \end{bmatrix}$$

Intuition:

Let $v[k] = \begin{bmatrix} \boxed{x[k]} \\ \boxed{a[k]} \end{bmatrix}$.

?

Vector is fixed (G_k)

If $v[k]$ is periodic with period p, the SG-gen model can be rewritten as

$$\boxed{[v[2]\ \ v[3]\ \ \ldots v[p]\ v[1]] = Q \cdot [v[1]\ \ v[2]\ \ \ldots v[p-1]\ \ v[p]].}$$

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Application to Periodic Graph Dynamics

$$[v[2]\ \ v[3]\ \ \dots v[p]\ v[1]] = Q \cdot [v[1]\ \ v[2]\ \ \dots v[p-1]\ \ v[p]].$$

Lemma. If $v[1],\ \dots, v[p]$ are linearly independent in $\mathbb{R}^p$, then any periodic graph sequence can be modelled by the LTI model with

$$Q = \left[v[1]\ \ v[2]\ \ \dots v[p-1]\ \ v[p]\right]^{-1} \cdot [v[2]\ \ v[3]\ \ \dots v[p]\ v[1]]$$

Application to Periodic Graph Dynamics

$$\boxed{[v[2]\ \ v[3]\ \ \ldots v[p]\ v[1]] = Q \cdot [v[1]\ \ v[2]\ \ \ldots v[p-1]\ \ v[p]].}$$

Is it possible to map vectors $\boldsymbol{v}[\mathbf{1}],\ \ldots, \boldsymbol{v}[\boldsymbol{p}]$ onto $\mathbb{R}^{\boldsymbol{r}}$ with $\boldsymbol{r} < \boldsymbol{p}$?

- **YES (**for some graph dynamics).

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Application to Periodic Graph Dynamics

Theorem. Let $v[1], \ldots, v[p]$ be an $r \times 1$ vectors with $r \leq p$. Then the minimal order r of the system matrix Q is defined as

$$r = rank \begin{bmatrix} a[1] & a[2] & \cdots & a[p-1] & a[p] \\ a[2] & a[3] & & a[p] & a[1] \\ \vdots & & \ddots & & \vdots \\ a[p] & a[1] & \cdots & a[p-2] & a[p-1] \end{bmatrix}_{p \cdot L \times p},$$

where $a[1]$, …, $a[p]$ - input vectors corresponding to $G_1, \ldots, G_p$.

Algorithm: Linear Periodic Graph Generator (LPG-gen)

Non-Periodic Graph Dynamics

Non-Periodic Graph Dynamics

What if the graph sequence $G_1, \dots, G_T$ is not periodic?

- **Linear Graph Generator (LG-gen)**.

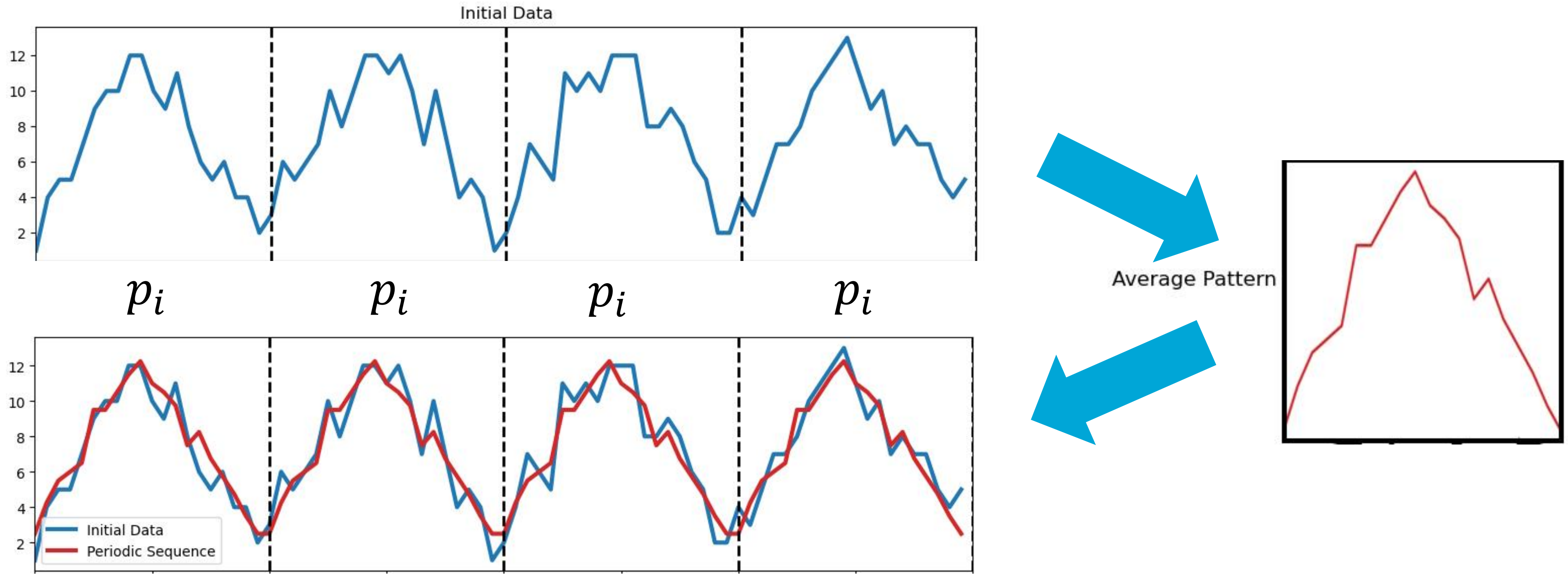
Idea:

- $G_1^{(i)}, G_2^{(i)}, \dots, G_T^{(i)}$ a *i*-th periodic graph sequence with period p_i;
- *Construct* l periodic graph sequences such that for $\forall k = 1, \dots, T$

$$G_k \approx G_k^{(1)} + G_k^{(2)} + \cdots + G_k^{(l)}.$$

- Each periodic graph sequence is modelled by LPG-gen.

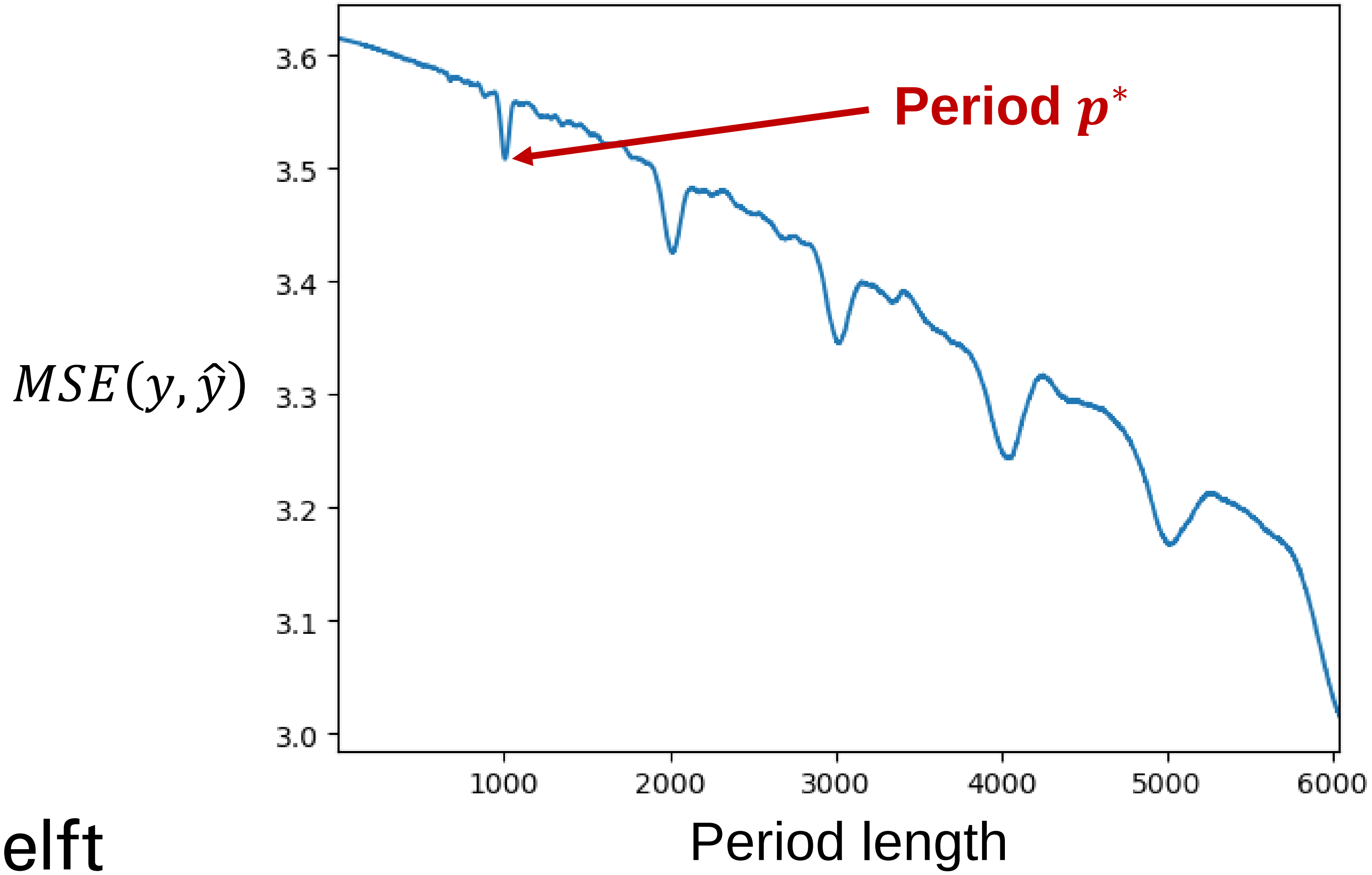
LG-gen: Periodic Transform



$$\boxed{x_i = x_{i-1} - \pi(x_{i-1}, \mathcal{P}(x_{i-1}, \mathcal{P}_{p_i}),}$$

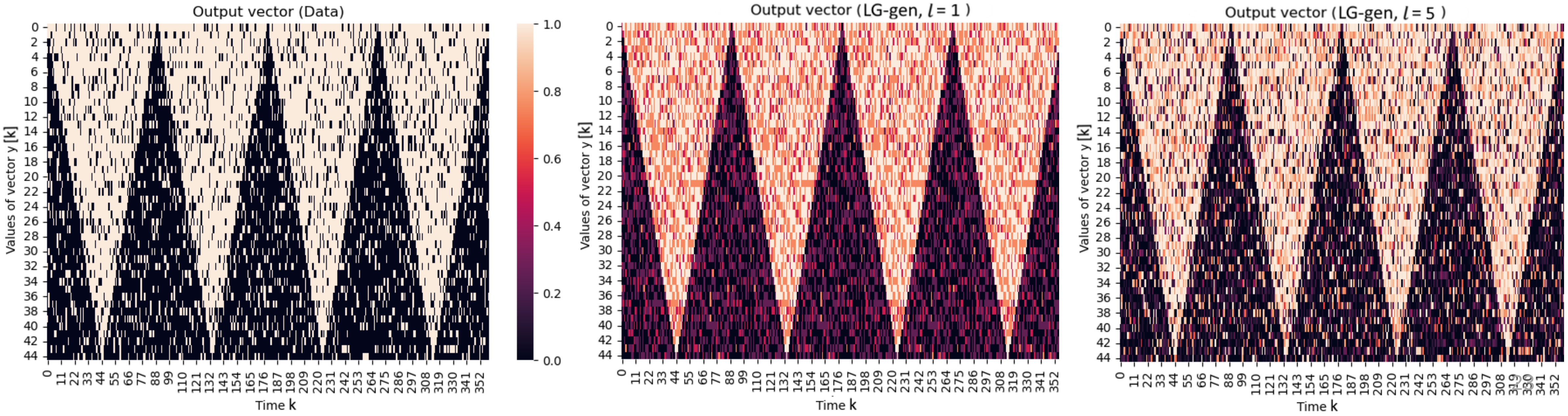
- $\pi(x_{i-1}, \mathcal{P}(x_{i-1}, \mathcal{P}_{p_i})$ – projection of x_{i-1} onto a p_i-periodic subspace $\mathcal{P}_{p_i}$;
- $x_0 = [a[1], a[2], \dots, a[T]]$ initial sequence.

LG-gen: Identification of Periodic Sequences



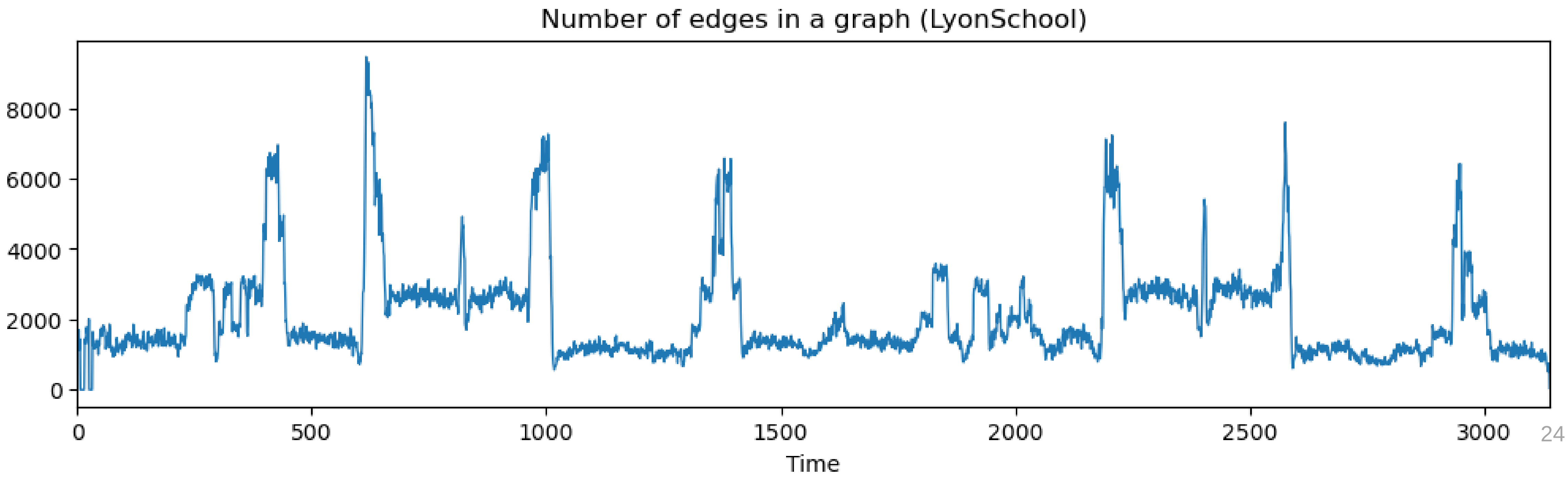
Artificial Graph Dynamic: Results of LG-Gen

- Periodic dynamic with r random changes per time slot.
- LG-gen provides a good performance;
- 7 periodic graph sequences are sufficient to provide an ideal performance of the model, i.e., $G_k = round\left(\sum_{i=1}^{7} G_k^{(i)}\right)$.



Graph Dynamic: LyonSchool

- Contact events between 242 individuals (232 children and 10 teachers) during two days in October 2009.
- The total number of contacts is 6,594,492 (29,161 unique links).

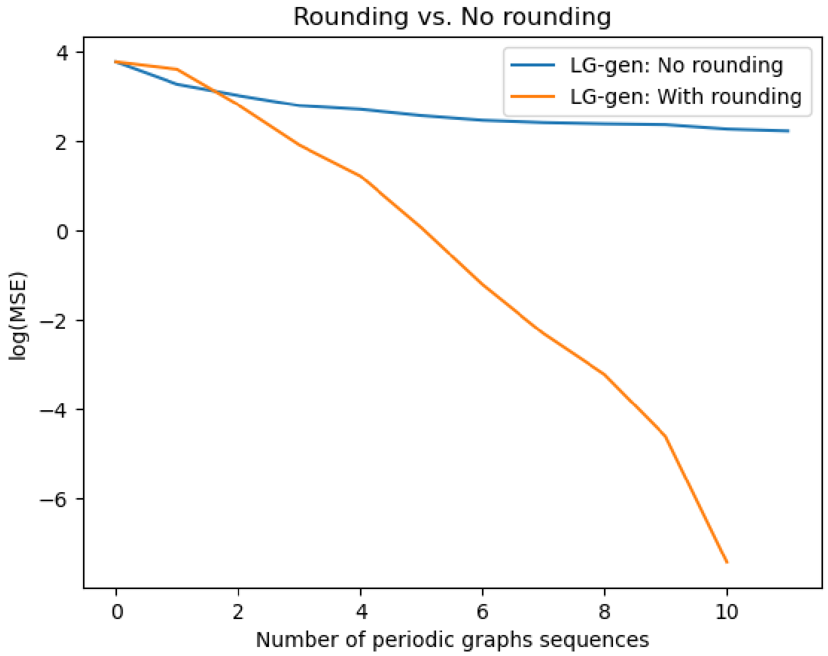


Graph Dynamic 4: Results of LG-Gen

Type of model	Number of periodic graphs sequences l										
	1	2	3	4	5	6	7	8	9	10	11
No rounding	26.5	20.6	16.4	15.1	13.1	11.8	11.2	10.9	10.7	9.7	9.3
With rounding	37	16.7	6.8	3.4	1.07	0.3	0.1	0.04	0.01	0.0006	0

No rounding: $G_k \approx G_k^{(1)} + G_k^{(2)} + \cdots + G_k^{(l)}$

With rounding: $G_k \approx round\left(G_k^{(1)} + G_k^{(2)} + \cdots + G_k^{(l)}\right)$



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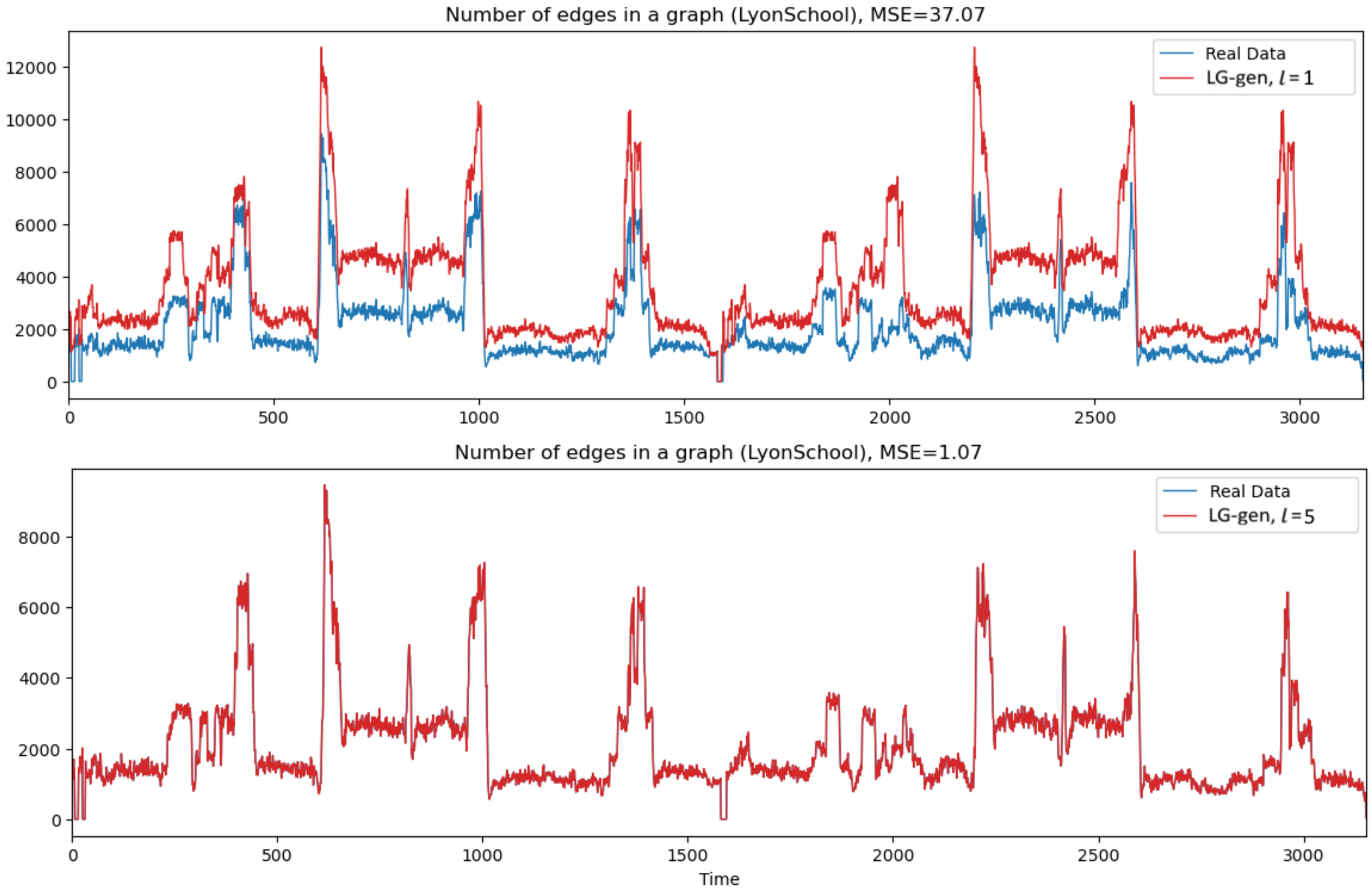
Graph Dynamic 4: Results of LG-Gen

- For $l = 5$, the MSE is 1.07 (≈1 incorrect contact per timestamp).
- For $l = 11$, the LG-gen generates the LyonSchool network exactly.



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Results

- We modeled the graph dynamics as a *linear process*.
- Any periodic graph sequence can be modelled accurately by the LTI model.
- We proposed two algorithms, called LPG-gen and LG-gen, to model periodic and non-periodic graph sequences and discuss their efficiency.
- LPG-gen and LG-gen algorithms provided a good performance on various artificial and real graph sequences.

Thank you!

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