CIEM5110

FEM lecture 4.1: Geometrically nonlinear frame analysis

Frans van der Meer

TU Delft

Agenda

This lecture is about geometric nonlinearity in finite elements

- From beam to frame element
- Geometrically nonlinear analysis
- Linear buckling analysis

CIEM5110-2 workshops and lectures

	(Theory)	BarModel (MUDE)	SolidModel (1.2)	TimoshenkoModel (2.1)	FrameModel **(4.1)**
SolverModule	(1.2)	2.2	2.2	3.2	3.2
NonlinModule	(3.1)		6.1		**4.1** + 4.2 + 5.1
LinBuckModule	**(4.1)**				**4.1** + 5.1
ModeShapeModule	(6.2)		7.1		8.2
ExplicitTimeModule	(6.2)				7.2 + 8.2

TU Delft

Nonlinear finite element analysis

Nonlinear system of equations

$$\mathbf{f}_{\text{int}}(\mathbf{a}) = \mathbf{f}_{\text{ext}}$$

with

$$\mathbf{f}_{\text{int}} = \int \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega$$

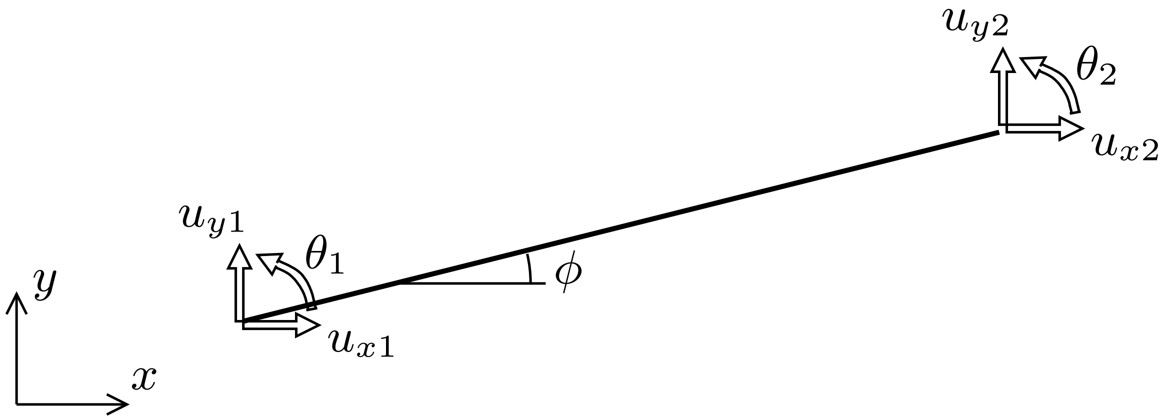
- Material nonlinearity ($\boldsymbol{\sigma}(\mathbf{a})$) and/or geometric nonlinearity ($\mathbf{B}(\mathbf{a})$)
- Incremental-iterative analysis: “time stepping”
- Displacement control or load control
- Newton-Raphson scheme to solve nonlinear system of equations
- Arclength method for snapback and proportional load

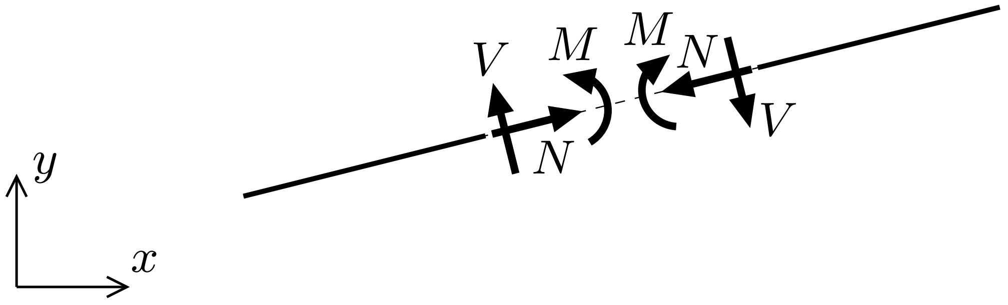
2-Node beam element

Extensible Timoshenko beam element (3 DOFs per node)

In the lecture on Timoshenko beams we used u and w for the displacement dofs

Here we work towards arbitrarily oriented elements and use u_x and u_y

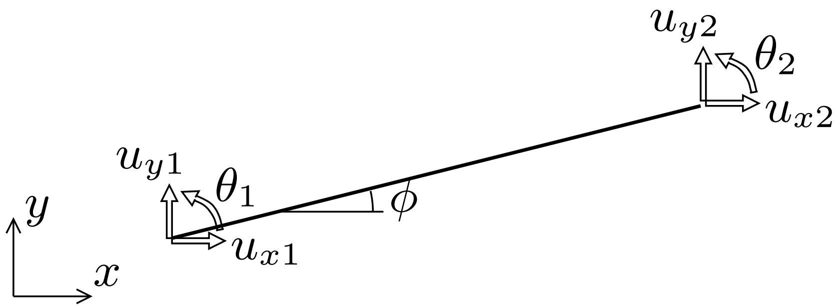


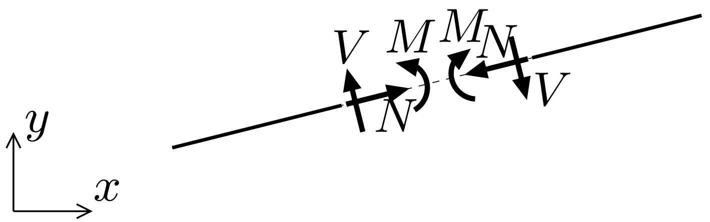


2-Node beam element

Extensible Timoshenko beam element (3 DOFs per node)

Kinematic: $$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} u_x' \\ u_y' - \theta \\ \theta' \end{Bmatrix} = \mathbf{B}\mathbf{a}^\mathrm{e} = \begin{bmatrix} N_1' & 0 & 0 & N_2' & 0 & 0 \\ 0 & N_1' & -N_1 & 0 & N_2' & -N_2 \\ 0 & 0 & N_1' & 0 & 0 & N_2' \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{Bmatrix} \qquad (\phi = 0)$$



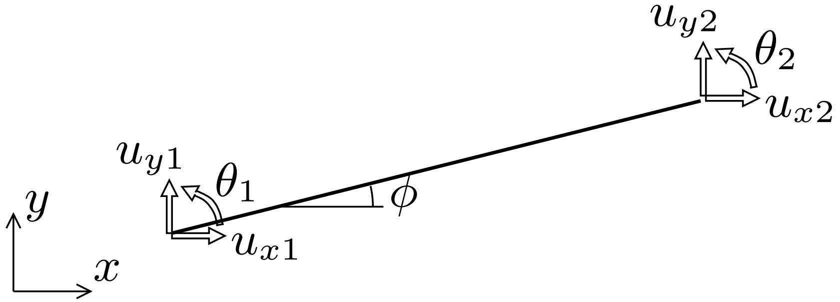


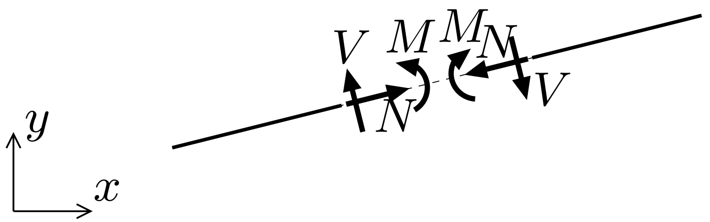
2-Node beam element

Extensible Timoshenko beam element (3 DOFs per node)

Kinematic: $$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}} = \begin{bmatrix} \cos\phi N_1' & \sin\phi N_1' & 0 & \cos\phi N_2' & \sin\phi N_2' & 0 \\ -\sin\phi N_1' & \cos\phi N_1' & -N_1 & -\sin\phi N_2' & \cos\phi N_2' & -N_2 \\ 0 & 0 & N_1' & 0 & 0 & N_2' \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

($\phi \neq 0 \quad \rightarrow \quad \varepsilon = u_x' \cos\phi + u_y' \sin\phi$ etc)





2-Node beam element

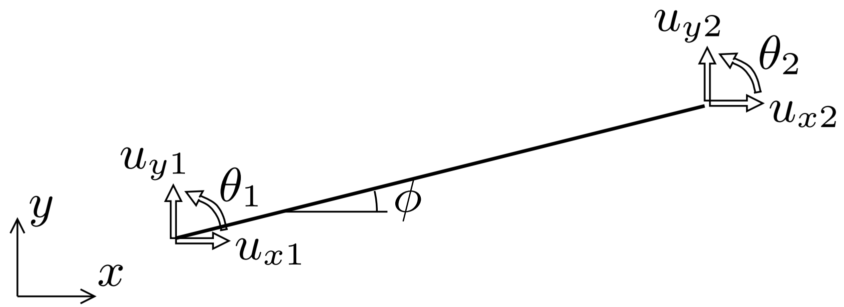
Extensible Timoshenko beam element (3 DOFs per node)

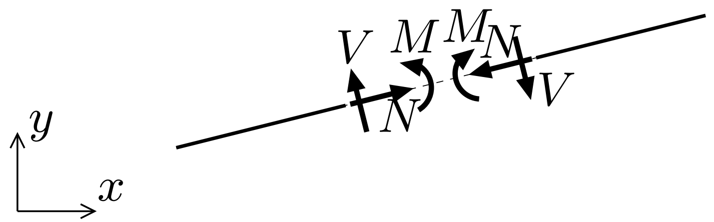
Kinematic: $\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}} = \begin{bmatrix} \cos\phi N_1' & \sin\phi N_1' & 0 & \cos\phi N_2' & \sin\phi N_2' & 0 \\ -\sin\phi N_1' & \cos\phi N_1' & -N_1 & -\sin\phi N_2' & \cos\phi N_2' & -N_2 \\ 0 & 0 & N_1' & 0 & 0 & N_2' \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{Bmatrix}$

$(\phi \neq 0 \quad \rightarrow \quad \varepsilon = u_x' \cos\phi + u_y' \sin\phi$ etc)

NB: Derivatives are defined along the element!

We have a 1D shape functions in 2D space





2-Node beam element

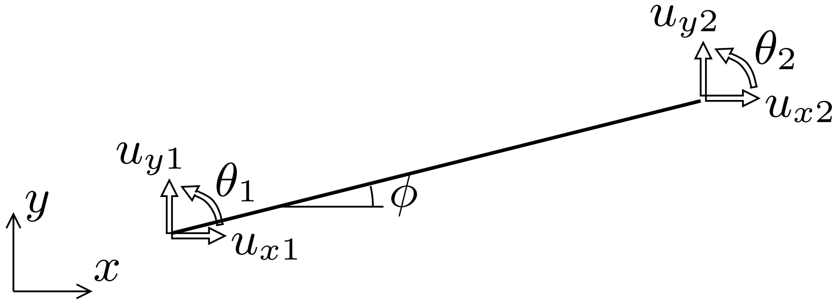
Extensible Timoshenko beam element (3 DOFs per node)

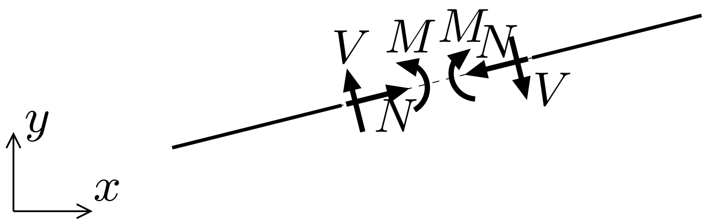
Kinematic: $$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}} = \frac{1}{L^{\mathrm{e}}} \begin{bmatrix} -\cos\phi & -\sin\phi & 0 & \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & -L^{\mathrm{e}}N_1 & -\sin\phi & \cos\phi & -L^{\mathrm{e}}N_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

(substituting $N_1' = -1/L^{\mathrm{e}}$ and $N_2' = 1/L^{\mathrm{e}}$)

NB: Derivatives are defined along the element!

We have a 1D shape functions in 2D space



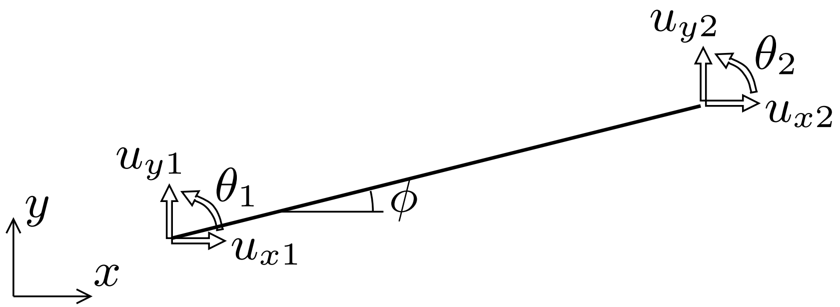


2-Node beam element

Extensible Timoshenko beam element (3 DOFs per node)

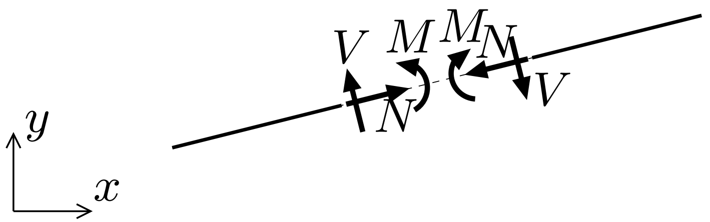
Kinematic: $$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}} = \frac{1}{L^{\mathrm{e}}} \begin{bmatrix} -\cos\phi & -\sin\phi & 0 & \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & -L^{\mathrm{e}}N_1 & -\sin\phi & \cos\phi & -L^{\mathrm{e}}N_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_{x1} \\ u_{y1} \\ \theta_1 \\ u_{x2} \\ u_{y2} \\ \theta_2 \end{Bmatrix}$$

Constitutive: $$\boldsymbol{\sigma} \equiv \begin{Bmatrix} N \\ V \\ M \end{Bmatrix} = \mathbf{D}\boldsymbol{\varepsilon} = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix}$$

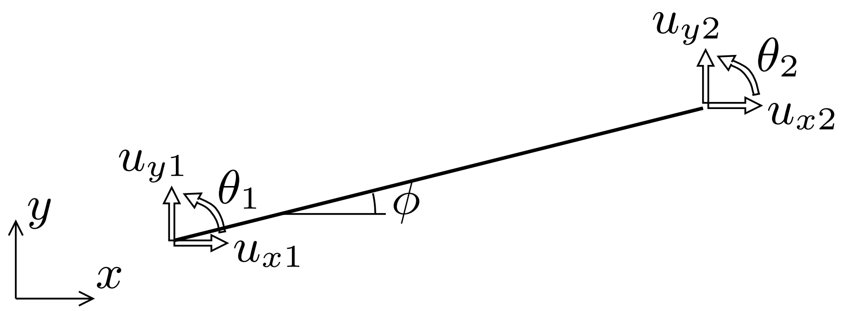


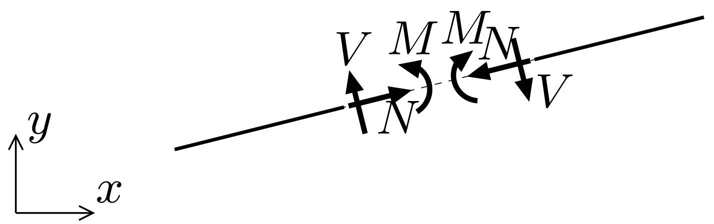
Equilibrium: $$\mathbf{f}^{\mathrm{e}}_{\mathrm{int}} = \int_{L^{\mathrm{e}}} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}x$$

Stiffness matrix (linear): $$\mathbf{K}^{\mathrm{e}} = \frac{\partial \mathbf{f}^{\mathrm{e}}_{\mathrm{int}}}{\partial \mathbf{a}^{\mathrm{e}}} = \int_{L^{\mathrm{e}}} \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}x$$



Relationship between strain and displacement





Linear, aligned:

$$\mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}^{\mathrm{e}}} = \begin{bmatrix} N_1' & 0 & 0 & N_2' & 0 & 0 \\ 0 & N_1' & -N_1 & 0 & N_2' & -N_2 \\ 0 & 0 & N_1' & 0 & 0 & N_2' \end{bmatrix}$$

Linear, arbitrary orientation:

$$\mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}^{\mathrm{e}}} = \frac{1}{L^{\mathrm{e}}} \begin{bmatrix} -\cos\phi & -\sin\phi & 0 & \cos\phi & \sin\phi & 0 \\ \sin\phi & -\cos\phi & -L^{\mathrm{e}}N_1 & -\sin\phi & \cos\phi & -L^{\mathrm{e}}N_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Geometrically nonlinear will need something like this, with updated orientation $\omega(\mathbf{a}^{\mathrm{e}})$:

$$\mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}^{\mathrm{e}}} = \frac{1}{L_0} \begin{bmatrix} -\cos\omega & -\sin\omega & 0 & \cos\omega & \sin\omega & 0 \\ \sin\omega & -\cos\omega & -L_0N_1 & -\sin\omega & \cos\omega & -L_0N_2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

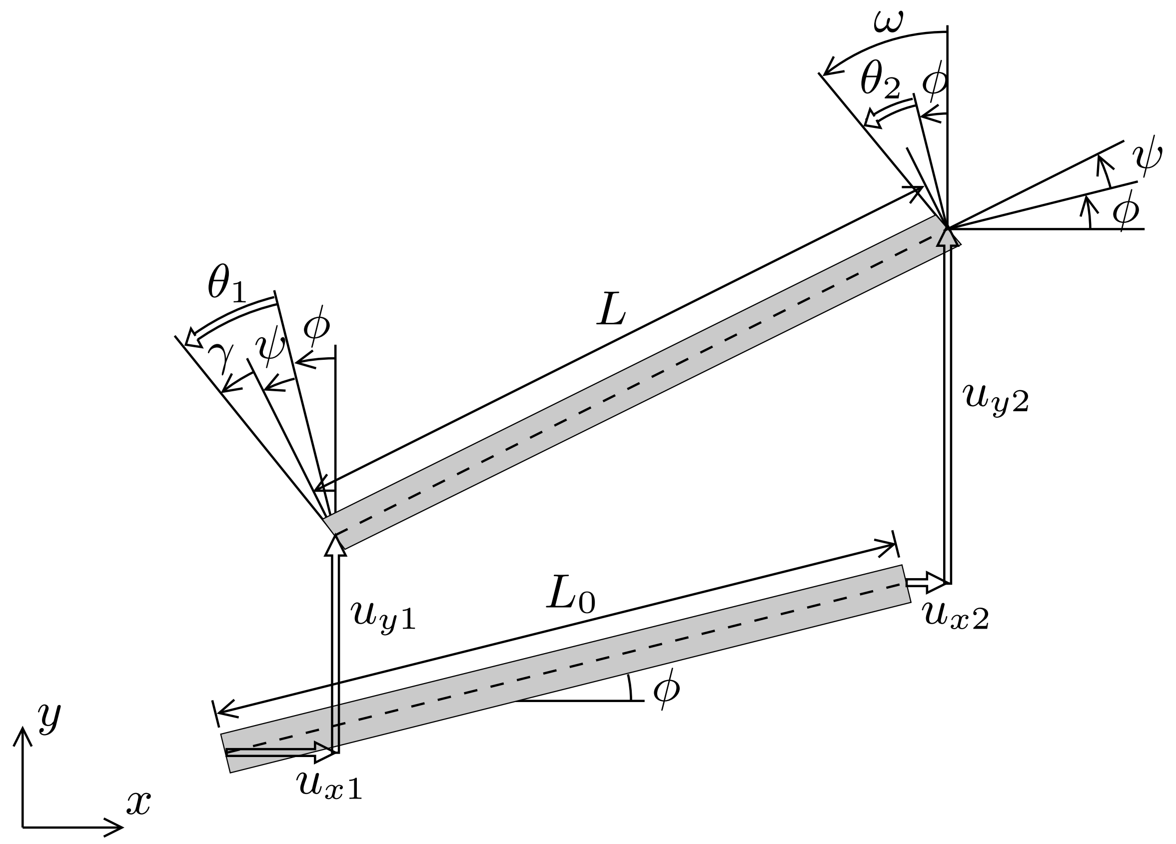
Geometrically nonlinear strain definition

Strain definition:

$$\varepsilon = \frac{L\cos(\theta-\psi)}{L_0} - 1$$

$$\gamma = \frac{L\sin(\theta-\psi)}{L_0}$$

$$\kappa = \theta'$$



Linearization:

$$\mathbf{B} = \frac{\partial \boldsymbol{\varepsilon}}{\partial \mathbf{a}^{\mathrm{e}}} = \frac{1}{L_0}\begin{bmatrix} -\cos\omega & -\sin\omega & L_0 N_1 \gamma & \cos\omega & \sin\omega & L_0 N_2 \gamma \\ \sin\omega & -\cos\omega & -L_0 N_1(1+\varepsilon) & -\sin\omega & \cos\omega & -L_0 N_2(1+\varepsilon) \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix}$$

Geometrically nonlinear finite element behavior

Strain becomes a nonlinear function of $\mathbf{a}^\mathrm{e}$

Constitutive: $$\boldsymbol{\sigma} \equiv \begin{Bmatrix} N \\ V \\ M \end{Bmatrix} = \mathbf{D}\boldsymbol{\varepsilon} = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix}$$

Equilibrium: $$\mathbf{f}_\mathrm{int}^\mathrm{e} = \int_{L^\mathrm{e}} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}x$$

Stiffness matrix: $$\mathbf{K}^\mathrm{e} = \frac{\partial \mathbf{f}_\mathrm{int}^\mathrm{e}}{\partial \mathbf{a}^\mathrm{e}} = \mathbf{K}_M^\mathrm{e} + \mathbf{K}_G^\mathrm{e}$$

$$\mathbf{K}_M^\mathrm{e} = \int_{L_0} \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}x$$

$$\mathbf{K}_G^\mathrm{e} = \int_{L_0} \boldsymbol{\sigma}^T \frac{\partial \mathbf{B}}{\partial \mathbf{a}^\mathrm{e}} \, \mathrm{d}x$$

Linear buckling analysis

For the nonlinear formulation, we have

$$\delta \mathbf{f}_{\text{int}} = \left[\int_L \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}x + \int_L \boldsymbol{\sigma}^T \frac{\partial \mathbf{B}}{\partial \mathbf{a}} \, \mathrm{d}x \right] \delta \mathbf{a}$$

Instability occurs when there is a $\delta \mathbf{a} \neq \mathbf{0}$ for which $\delta \mathbf{f}_{\text{int}} = \mathbf{0}$

- Assuming displacements are small up to buckling:
  $\mathbf{B}$ and $\partial \mathbf{B} / \partial \mathbf{a}$ can be evaluated at $\mathbf{a} = \mathbf{0}$
- Assuming all loads are proportional:
  Stress from unit load analysis: $\boldsymbol{\sigma} = \lambda \hat{\boldsymbol{\sigma}}$

Then, we can rephrase

$$\mathbf{K} = \int_L \mathbf{B}_0^T \mathbf{D} \mathbf{B}_0 \, \mathrm{d}x + \lambda \int_L \hat{\boldsymbol{\sigma}}^T \left(\frac{\partial \mathbf{B}}{\partial \mathbf{a}} \right)_0 \mathrm{d}x$$

or

$$\mathbf{K} = \mathbf{K}_M + \lambda \mathbf{K}_G$$

Linear buckling analysis

Assuming

- Displacements are small up to buckling
- All loads are proportional

Instability occurs when:

$$[\mathbf{K}_M + \lambda \mathbf{K}_G]\, \delta\mathbf{a} = 0 \quad \text{for} \quad \delta\mathbf{a} \neq \mathbf{0}$$

With $\mathbf{K}_M$ and $\mathbf{K}_G$ evaluated at $\mathbf{a} = 0$
and $\boldsymbol{\sigma}$ (in $\mathbf{K}_G$) from linear analysis with unit load

This is an eigenvalue problem, which can be solved for

- Critical load scale factor(s) λ
- Corresponding buckling mode(s) $\delta\mathbf{a}$

Nonlinear frame analysis with PyJive

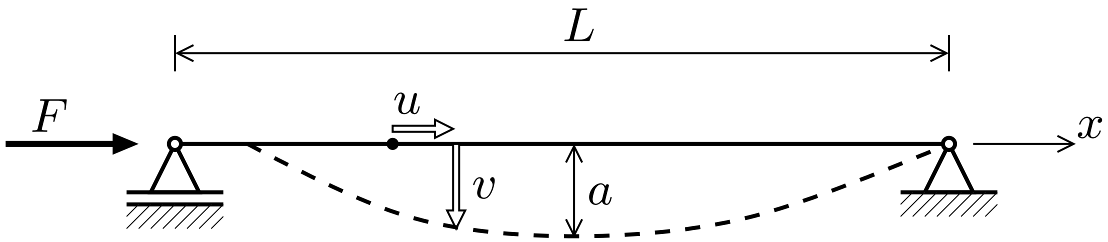
Two input files:

- `*.geom` – define geometry, discretization
- `*.pro` – define material parameters, boundary conditions, analysis options

Two main modules to be specified in input file

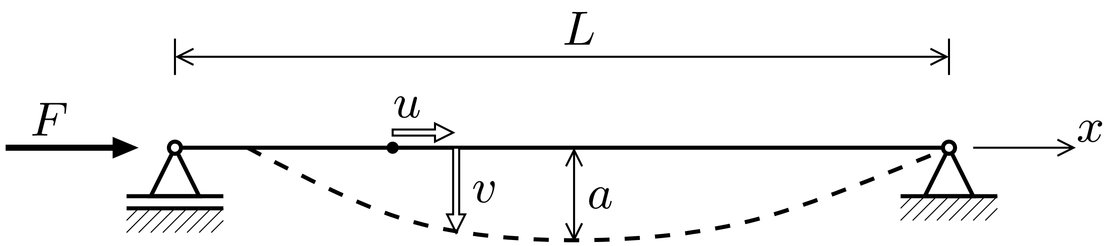
- `Nonlin` – incremental/iterative solver for nonlinear equilibrium problem
- `LinBuck` – linear buckling analysis

Linear buckling analysis: euler beam

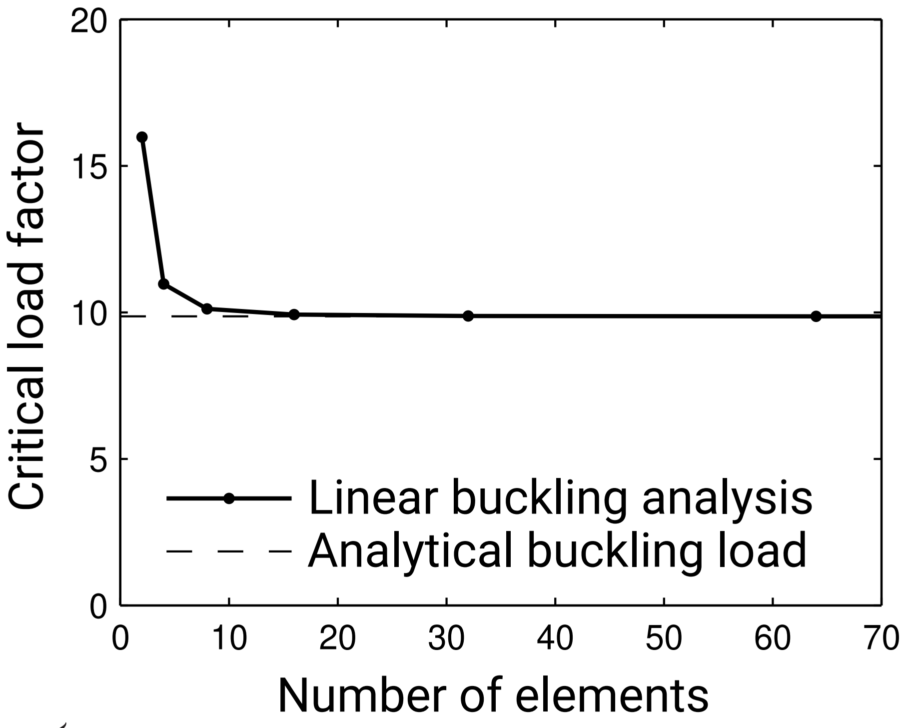


Mesh-refinement study with uniform mesh

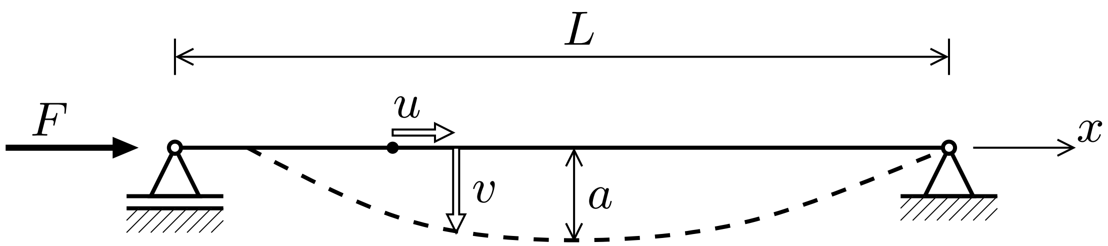
Linear buckling analysis: euler beam



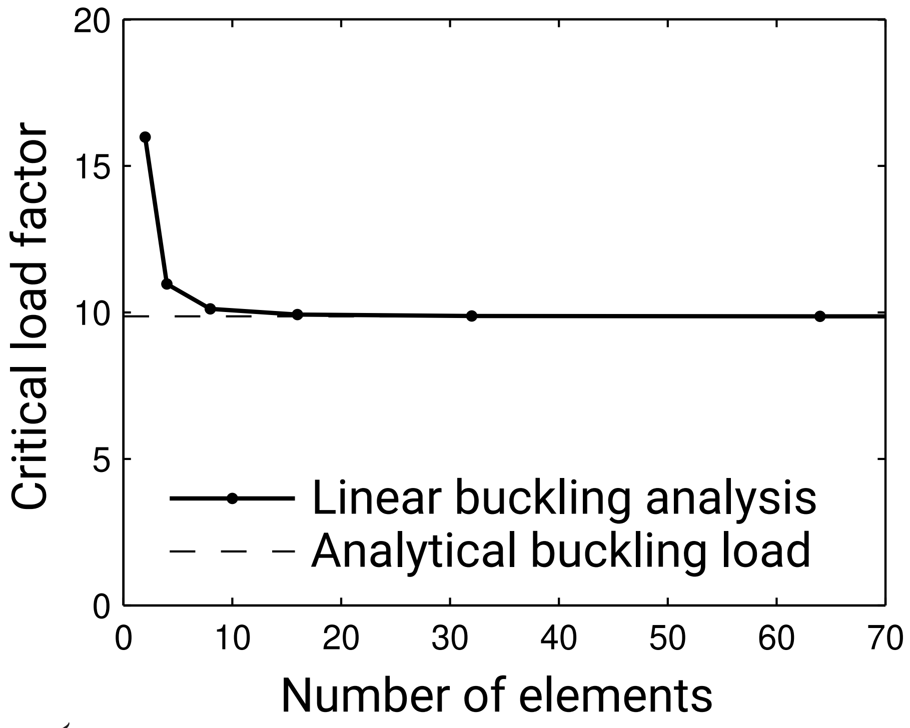
Mesh-refinement study with uniform mesh

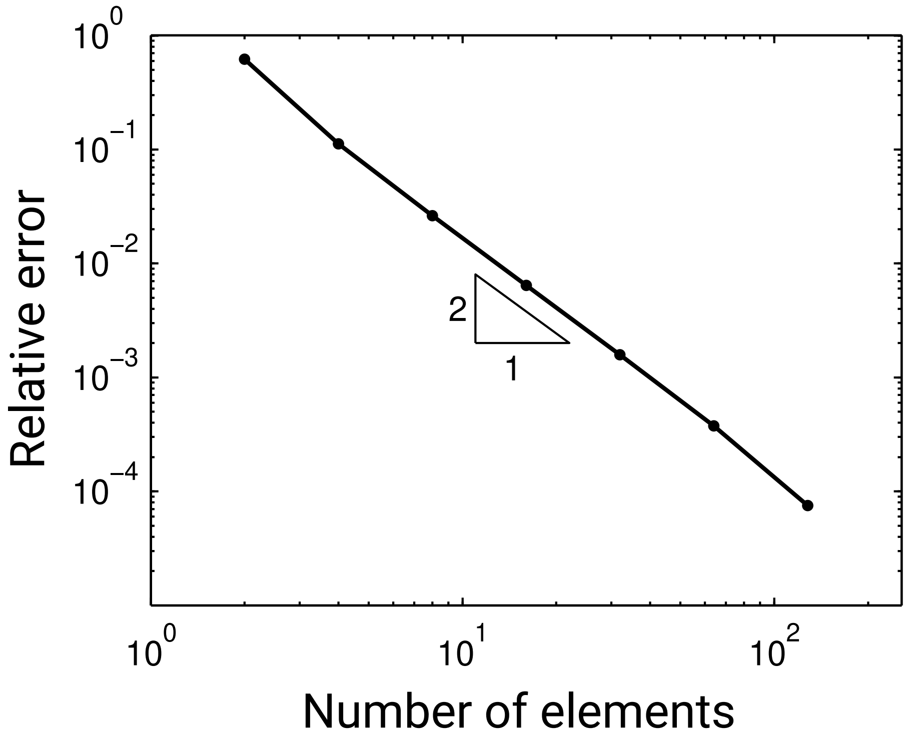


Linear buckling analysis: euler beam



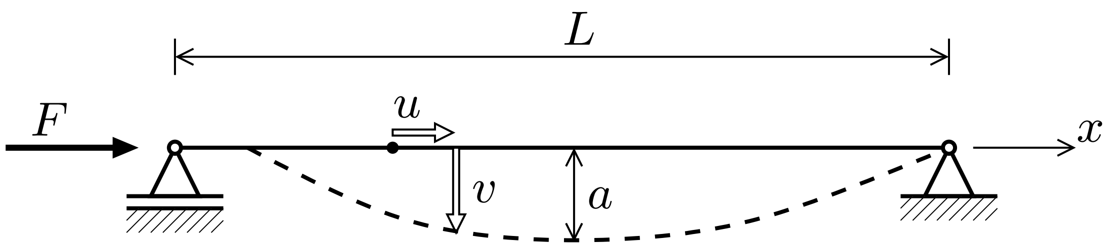
Mesh-refinement study with uniform mesh





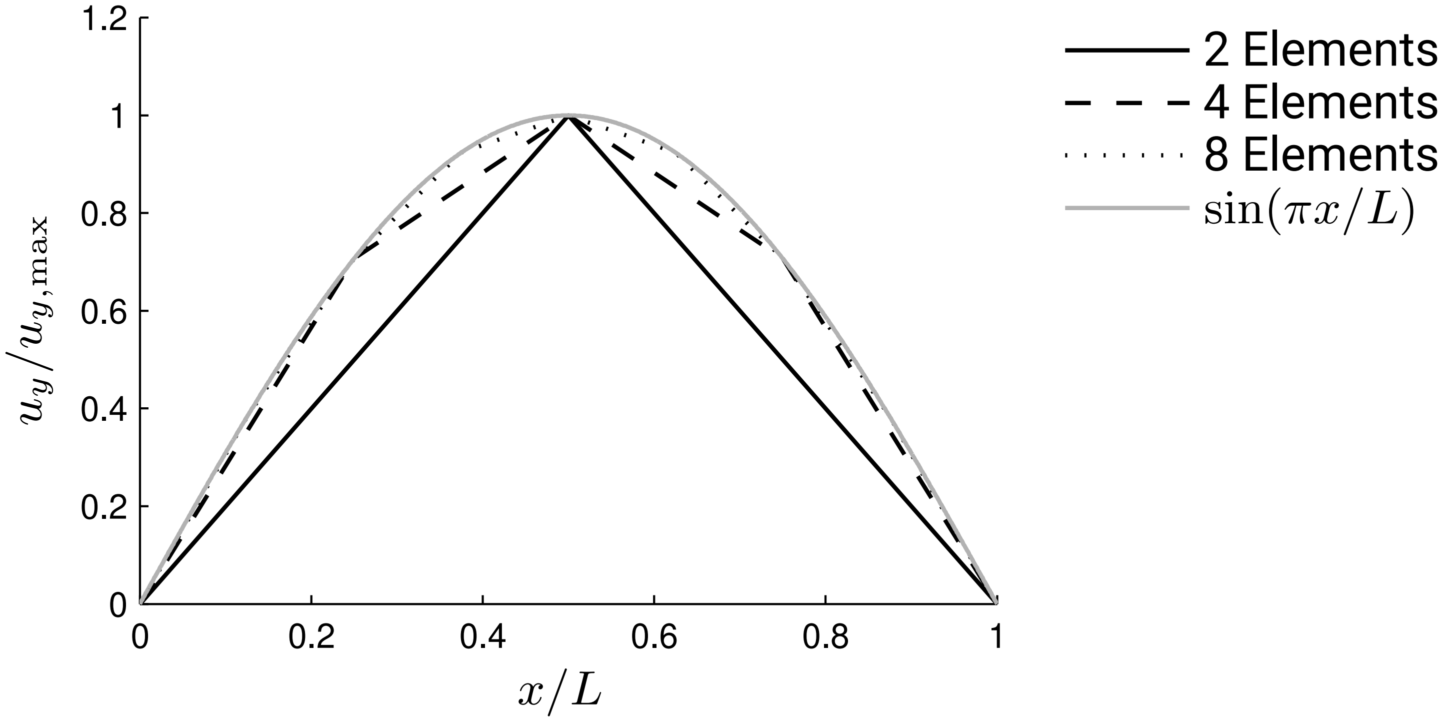
TU Delft

Linear buckling analysis: euler beam

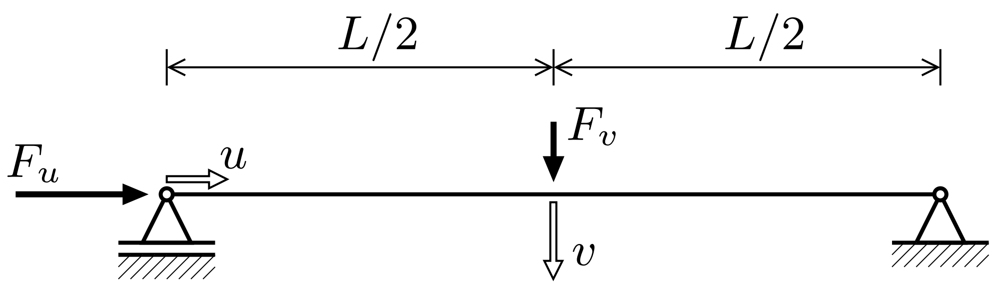


Mesh-refinement study with uniform mesh

First buckling mode

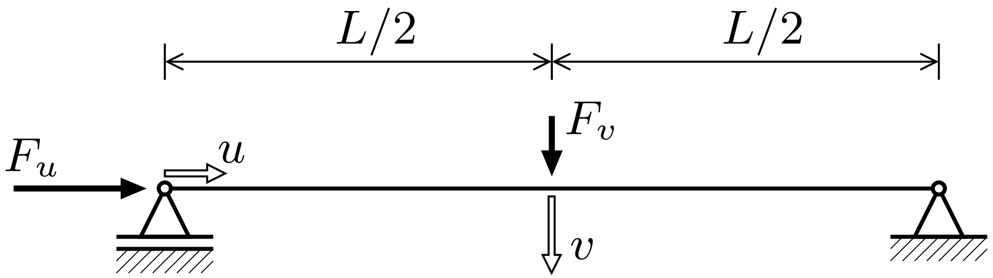


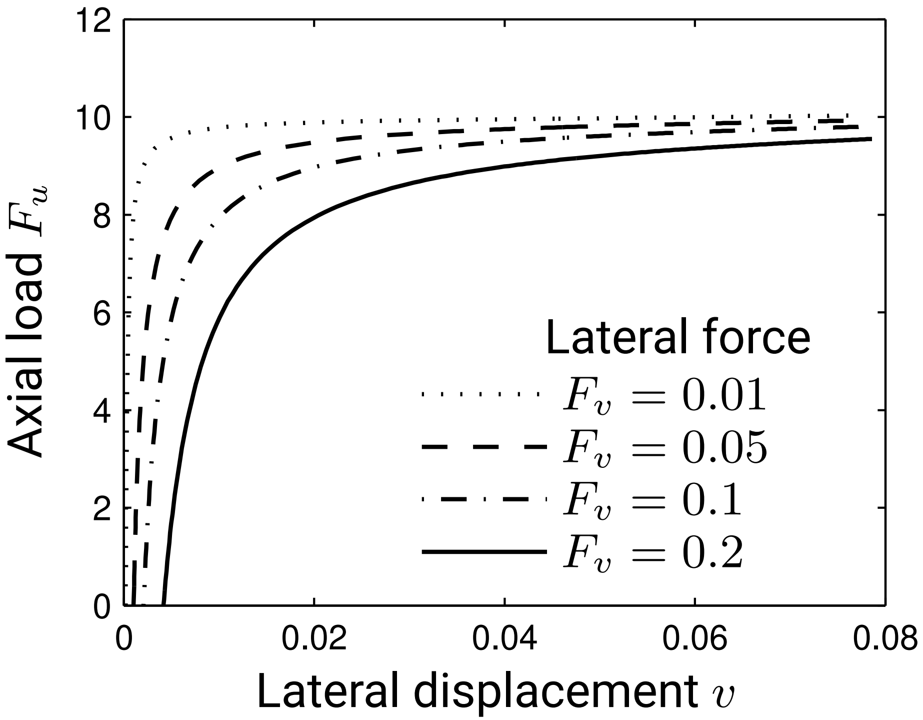
Nonlinear elastic analysis: euler beam



Perturbation with lateral load F_v

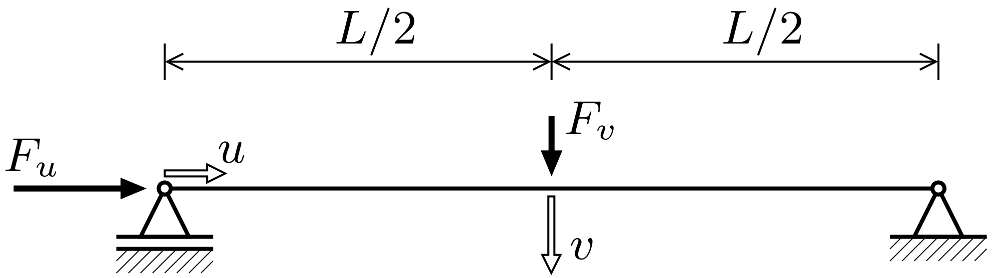
Nonlinear elastic analysis: euler beam



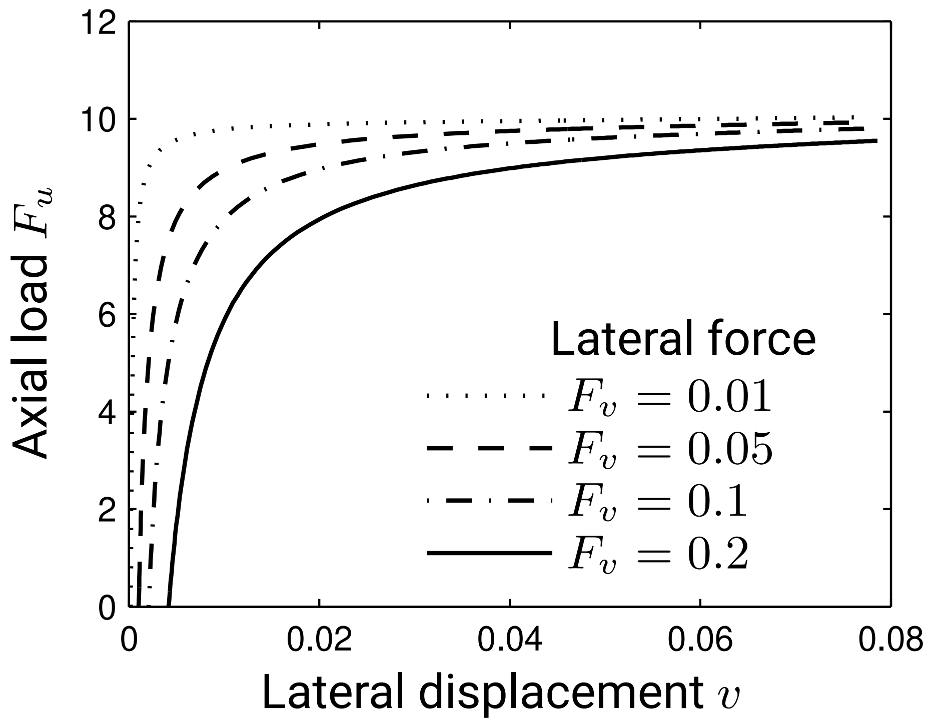


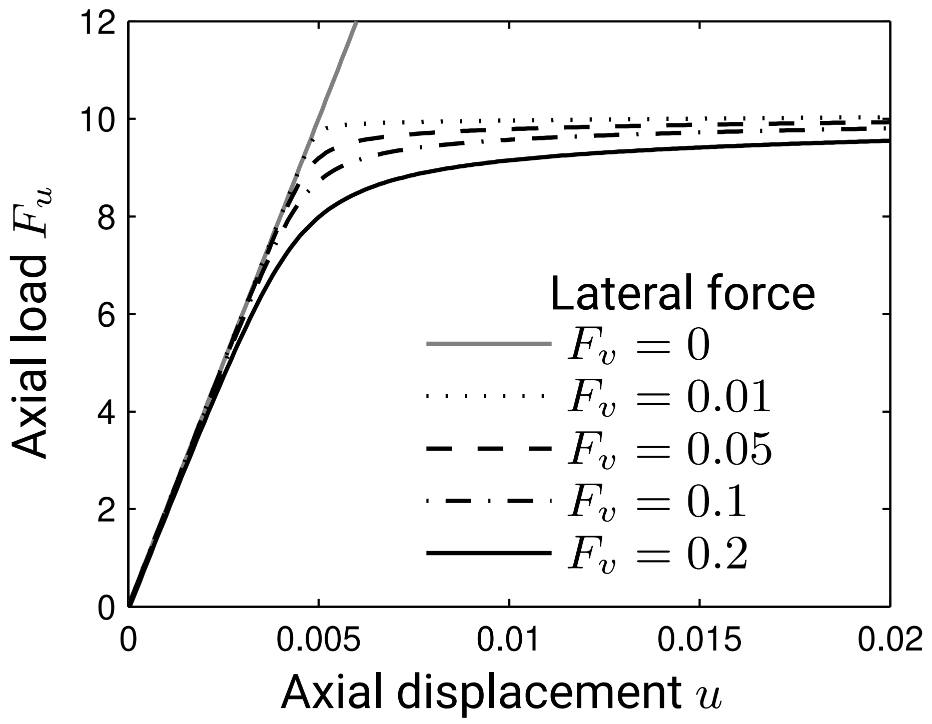
Perturbation with lateral load F_v

Nonlinear elastic analysis: euler beam



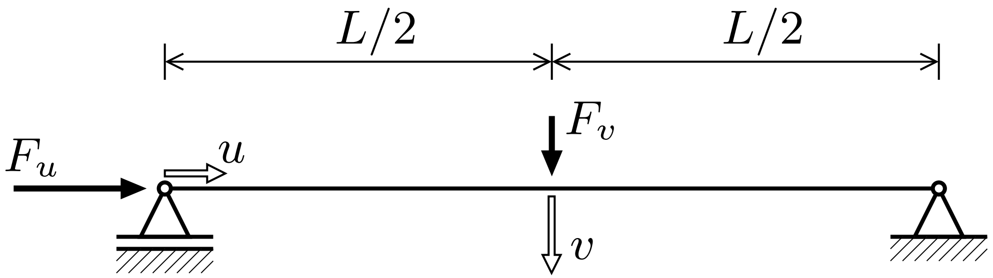
Perturbation with lateral load F_v

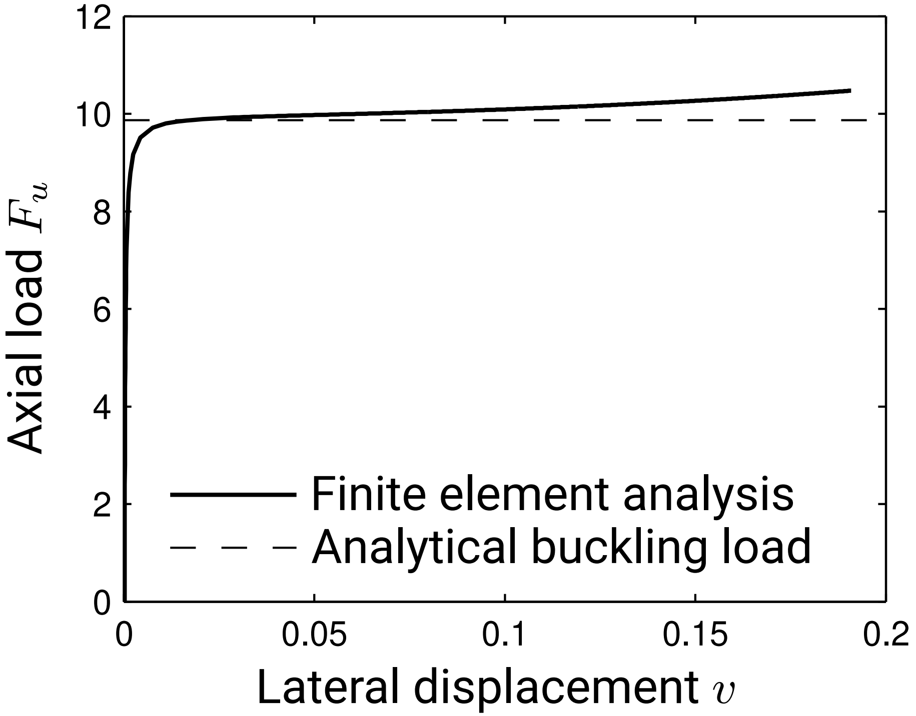




TU Delft

Nonlinear elastic analysis: euler beam



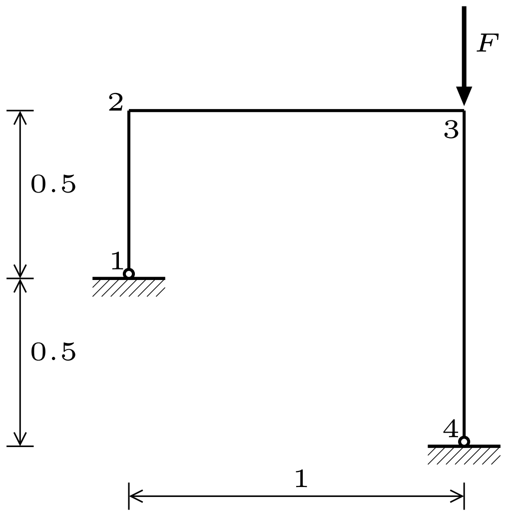


Post-buckling behavior

Going to larger lateral displacement

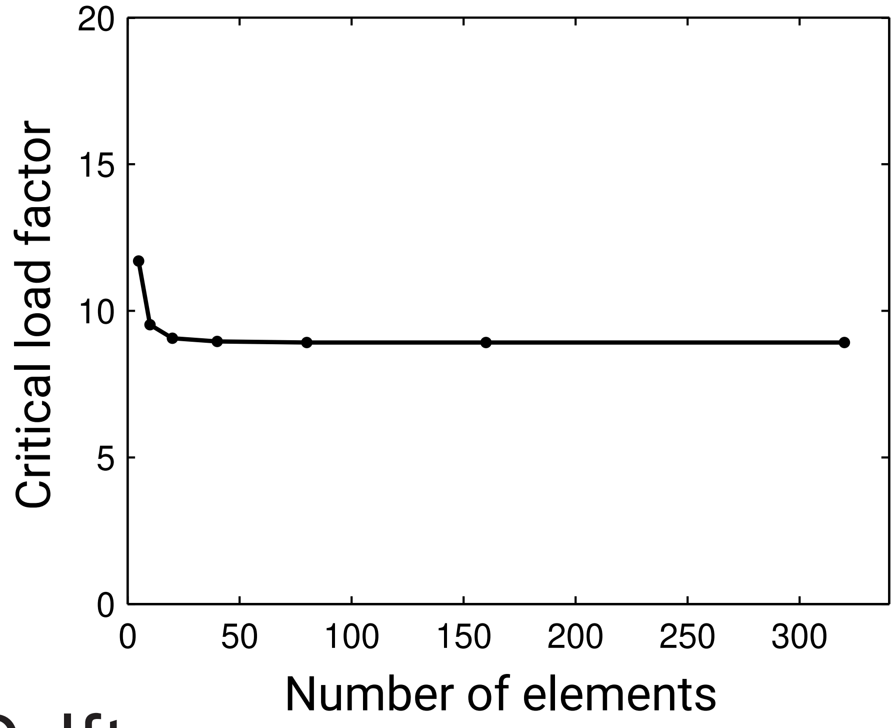
$F_v = 0.01$

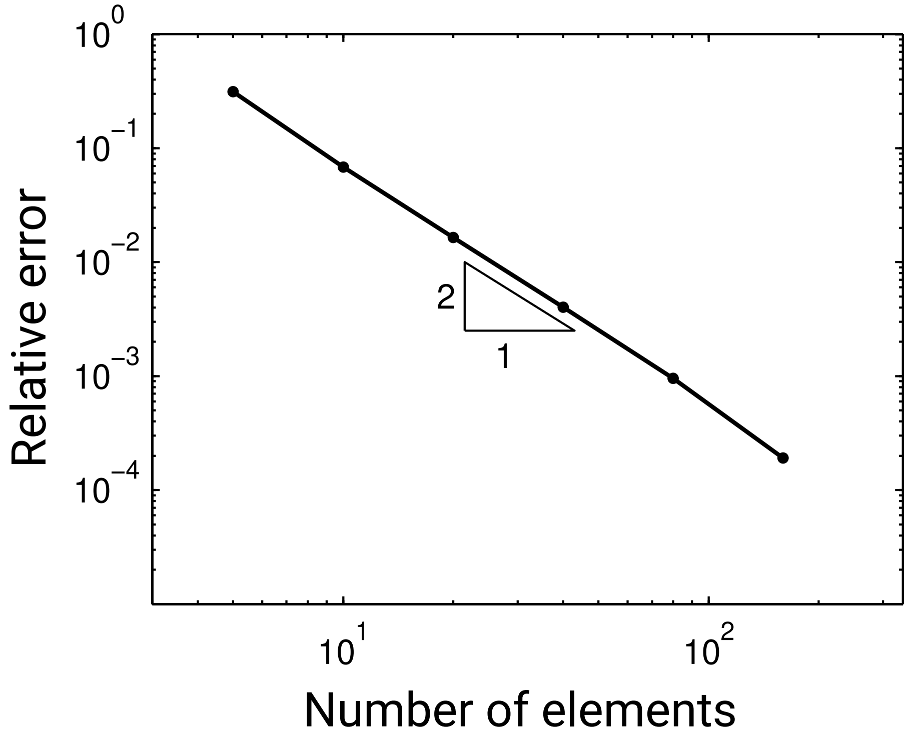
Linear buckling analysis: simple frame



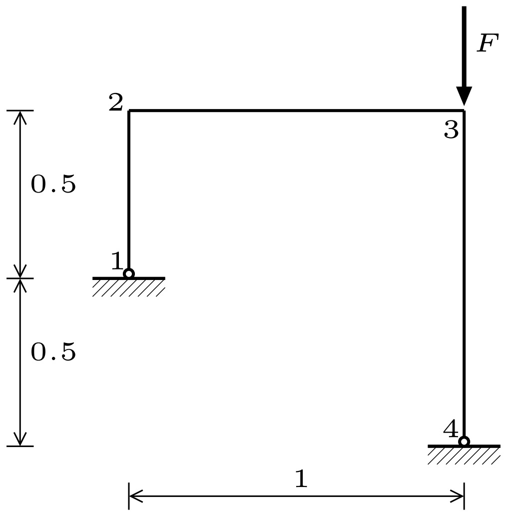
Frame with single point load

Mesh-refinement study





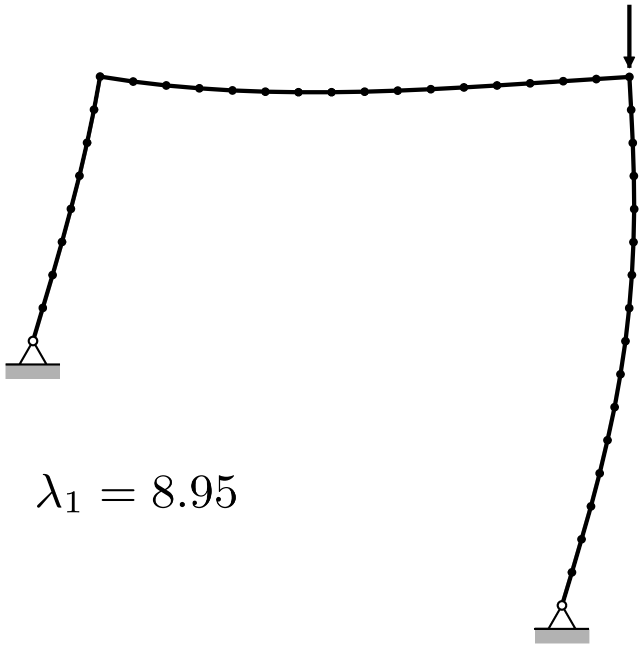
Linear buckling analysis: simple frame

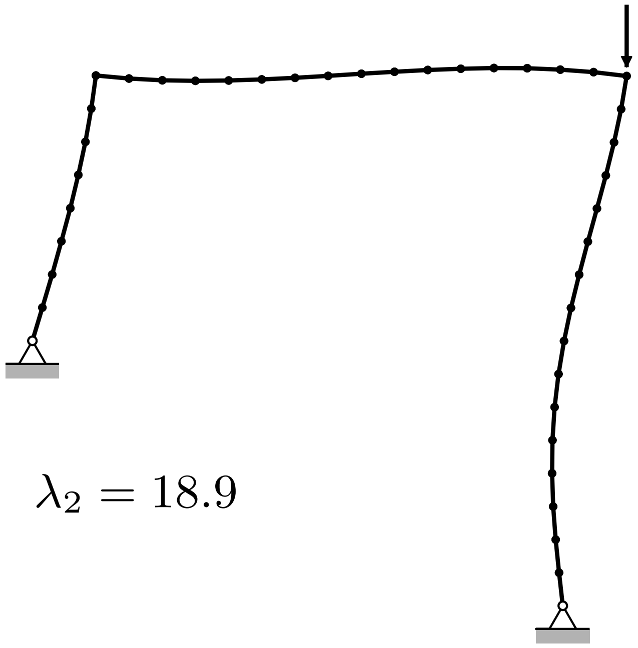


Frame with single point load

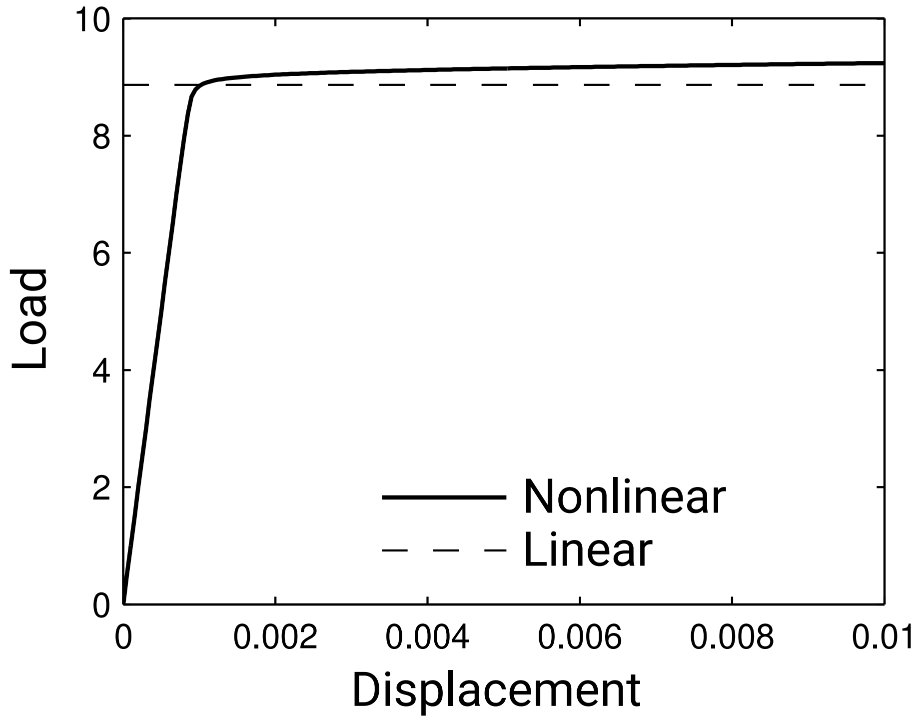
Mesh with 40 elements

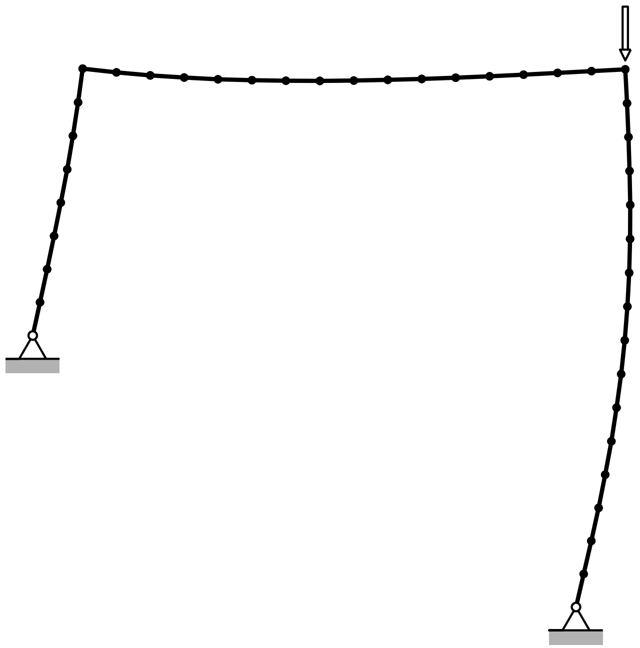
First and second buckling mode





Nonlinear elastic analysis: simple frame





Recap

Remarks on finite element method

- From beam to frame element
- Geometrically nonlinear analysis
- Linear buckling analysis

PyJive workshop

- Geometrically nonlinear analysis
- Linear buckling analysis
- Numerical results for basic cases