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Exercises on Quantum Chromodynamics

Particle Physics II

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Assignment 1

Introduction

Exercise 1

The Gell-Mann/Okubo formula $2(m_N + m_\Xi) = 3m_\Lambda + m_\Sigma$ relates the masses of the baryon octet ignoring the small differences between e.g. $p - n$, $\Sigma^0 - \Sigma^\pm$, $\Xi^0 - \Xi^-$. Using the known masses from the particle data group (PDG - (http://pdg.lbl.gov/)), estimate the mass of Λ and figure out how much off from the real value you are.

Exercise 2

1. How many different mesons can you make out of 1, 2, 3, 4, 5, and 6 different quark flavours and what is the general formula? (Do not consider anti-particles)
2. Using the first four lightest quark flavours i.e. u, d, s, c, write down all the different mesons, ordered by charm content and associate them to a known particle.

Exercise 3

Which of the two decays is most likely to happen: $\Xi^- \to \Lambda + \pi^-$ or $\Xi^- \to n + \pi^-$? Please explain why and confirm it by looking at the particle data group page (http://pdg.lbl.gov/).

Exercise 4

Which reactions are possible and which are not, and why?

1. $p + \bar{p} \to \pi^+ + \pi^0$
2. $\eta \to \gamma + \gamma$
3. $\Sigma^0 \to \Lambda + \pi^0$
4. $\Sigma^- \to n + \pi^-$
5. $e^+ + e^- \to \mu^+ \mu^-$

1.1 Homework

Exercise 1[25]

The Gell-Mann/Okubo formula for the decouple is $m_\Delta - m_{\Sigma^*} = m_{\Sigma^*} - m_{\Xi^*} = m_{\Xi^*} - m_{\Omega^-}$. Use this formula to estimate the mass of the Ω baryon and figure out how much off from the real value you are. X^* states represent the average mass of the corresponding baryons.

Exercise 2[25]

1. How many different baryons can you make out of 1, 2, 3, 4, 5, and 6 different quark flavours and what is the general formula? (Do not consider anti-particles)
2. Using the first four lightest quark flavours i.e. u, d, s, c, write down all the different baryons, ordered by charm content.

Exercise 3[25]

Which of the following decays is most likely to happen: $D^0 \to K^- + \pi^+$, $D^0 \to K^+ + \pi^-$ or $D^0 \to \pi^- + \pi^+$? Please explain why and confirm it by looking at the particle data group page (http://pdg.lbl.gov/).

Exercise 4[25]

Which reactions are possible and which are not, and why?

1. $\mu^- \to e^- + \overline{\nu}_e$
2. $\Delta^+ \to p + \pi^0$
3. $\overline{\nu}_e + p \to n + e^+$
4. $p \to e^+ + \gamma$
5. $\Xi^- \to \Lambda + \pi^-$

Assignment 2

Symmetries and elements of group theory

Exercise 1

Show that invariance for translations in space leads to the conservation of momentum.

Exercise 2

Show that

$$\{1, i, -1, -i\} \cong \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$$

Exercise 3

1. Show that $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
2. Show that a matrix transform preserves the algebra of a Lie group. Representations that are related by similarity transformations are therefore called equivalent.
3. Show that a matrix transform preserves the product, determinant and trace, that is,

$$(AB)' = A'B', \quad \det(A') = \det(A) \quad \text{and} \quad \mathrm{Tr}(A') = \mathrm{Tr}(A).$$

Exercise 4

1. Instead of $|\mathrm{p}\rangle$ and $|\mathrm{n}\rangle$ we will write $|\mathrm{u}\rangle$ and $|\mathrm{d}\rangle$ to reflect isospin symmetry on the quark level. Verify that

$$I_3\,|\mathrm{u}\rangle = \tfrac{1}{2}|\mathrm{u}\rangle, \quad I_3\,|\mathrm{d}\rangle = -\tfrac{1}{2}|\mathrm{d}\rangle$$

   and that the Casimir operator $I^2 = I_1^2 + I_2^2 + I_3^2$ is a multiple of the unit operator, with

$$I^2\,|\mathrm{u}\rangle = \tfrac{3}{4}\,|\mathrm{u}\rangle, \qquad I^2\,|\mathrm{d}\rangle = \tfrac{3}{4}\,|\mathrm{d}\rangle$$

2. Define the step operators $I_\pm = I_1 \pm i I_2$ and verify that

$$I_+|\mathrm{u}\rangle = 0, \qquad I_+|\mathrm{d}\rangle = |\mathrm{u}\rangle, \qquad I_-|\mathrm{u}\rangle = |\mathrm{d}\rangle, \qquad I_-|\mathrm{d}\rangle = 0$$

Exercise 5

Two particles, each of spin 2 and third component 0, form a composite system whose orbital angular momentum is 0.

1. What is the probability for each of the states of the composite system?
2. Which state is the most probable?
3. Show that the probabilities add up to unity.

2.1 Homework

Exercise 1[20]

1. Show that invariance for translations in time leads to the conservation of energy.
2. Show that rotational invariance leads to the conservation of angular momentum.

Exercise 2[30]

1. Show that
$$\det[\exp(A)] = \exp[\mathrm{Tr}(A)]$$
for all matrices A that can be brought into diagonal form.
2. Show that $\tau_i\tau_j = \delta_{ij} + i\varepsilon_{ijk}\tau_k$. Together with the fact that the τ are Hermitian, we thus have $\tau_i^\dagger = \tau_i = \tau_i^{-1}$.
3. Show that $(a\cdot\tau)(b\cdot\tau) = a\cdot b + i\tau\cdot(a\times b)$ and, from this, that $(\theta\cdot\tau)^2 = |\theta|^2$.
4. Use the above, and the Taylor expansions of exp(), sin() and cos(), to show that $\exp(i\theta\cdot\tau) = \cos|\theta| + i(\hat{\theta}\cdot\tau)\sin|\theta|$. Here $\hat{\theta}$ is the unit vector along θ.

Exercise 3[20]

1. Show that the commutation relations of the set $\{I_\pm, I_3\}$ are
$$[I_3, I_+] = I_+ \qquad [I_3, I_-] = -I_- \qquad [I_+, I_-] = 2I_3$$
2. Show that $I_u = I_+I_-$ and $I_d = I_-I_+$ are counting operators in the sense that
$$I_u|u\rangle = |u\rangle \qquad I_u|d\rangle = 0 \qquad I_d|u\rangle = 0 \qquad I_d|d\rangle = |d\rangle$$

Exercise 4[30]

Consider a particle of spin 3/2 and another one of spin 2 that form a system whose orbital angular momentum is 0 and total spin is 5/2. If the z-component of the composite system is -1/2, what values would we get for the measurement of S_z and what is the probability for each? Show that they add up to unity?

Assignment 3

QCD Lagrangian

$$\boxed{\textbf{Exercise 1}}$$

The transformation property of A we find from the requirement $D'_\mu U = U D_\mu$:

$$(\partial_\mu + i g_\text{w} \tau \cdot A'_\mu)\, U \psi \underset{\text{Iwant}}{=} U(\partial_\mu + i g_\text{w} \tau \cdot A_\mu)\, \psi$$

Show that this gives the transformation rule:

$$\tau \cdot A'_\mu = U(\tau \cdot A_\mu) U^{-1} + \frac{i}{g_\text{w}}\, (\partial_\mu U) U^{-1}$$

3.1 Homework

$$\boxed{\textbf{Exercise 1}[40]}$$

The transformation rule for the gauge fields is

$$\tau \cdot A'_\mu = U(\tau \cdot A_\mu) U^{-1} + \frac{i}{g_\text{w}}\, (\partial_\mu U) U^{-1}.$$

Expand to first order $U \approx 1 - i g_\text{w}\, \tau \cdot \alpha$ and show that the transformation rule can be approximated by

$$\tau \cdot A'_\mu \approx \tau \cdot A_\mu + i g_\text{w}\, [\tau \cdot A_\mu\, , \tau \cdot \alpha] + \tau \cdot \partial_\mu \alpha$$

$$\boxed{\textbf{Exercise 2}[60]}$$

1. Use the expression for $(a \cdot \tau)(b \cdot \tau)$ on page 4 to evaluate the commutator

$$[\tau \cdot A_\mu\, , \tau \cdot \alpha] = -2i\tau\, (\alpha \times A_\mu)$$

2. Now substitute the commutator and multiply with τ^{-1} to get

$$A'_\mu \approx A_\mu + \partial_\mu \alpha + 2 g_\text{w}\, (\alpha \times A_\mu)$$

Assignment 4

Colour factors

Exercise 1

Calculate the colour factors for the octet $q\overline{q}$ configuration:

1. $B\overline{G}$
2. $(R\overline{R} - B\overline{B})/\sqrt{2}$

4.1 Homework

Exercise 1[40]

1. The λ matrices are normalised such that $\mathrm{Tr}(\lambda_a \lambda_b) = 2\delta_{ab}$. Check this for 4 matrices λ_a and λ_b.
2. Show that $\mathrm{Tr}(\lambda_c[\lambda_a, \lambda_b]) = 4if^c_{ab}$. By changing the order of the λ, and using $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, show that the structure constants f^c_{ab} are antisymmetric in the exchange of two indices.

Exercise 2[30]

Calculate the colour factors for the octet $q\overline{q}$ configuration $(R\overline{R} + B\overline{B} - 2G\overline{G})/\sqrt{6}$

Exercise 3[30]

Calculate the colour factors for the sextet qq configuration $(RB + BR)/\sqrt{2}$.

Assignment 5

Form factors

Exercise 1

For the elastic $e-p$ scattering in the lab frame (i.e. stationary proton) show that:

$$\langle |M_{if}|^2 \rangle = \frac{g_e^4}{4EE' \sin^4(\theta/2)} \Big(2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2) \Big)$$

where θ is the electron scattering angle, and E and $E^{'}$ are the incoming and outgoing energies of the electron.

Exercise 2

For the same process of exercise 1 (i.e. not the first homework exercise) show that:

$$E^{'} = \frac{ME}{M + E\Big(1 - \cos(\theta)\Big)} \Leftrightarrow E^{'} = \frac{E}{1 + \frac{2E}{M} \sin^2(\theta/2)}$$

5.1 Homework

Exercise 1[50]

Explain why R is defined as the ratio between $\sigma(e^- + e^+ \to \text{hadrons})$ and $\sigma(e^- + e^+ \to \mu^- + \mu^+)$ and not with respect to $\sigma(e^- + e^+ \to e^- + e^+)$ (write down all diagrams, if need be).

Exercise 2[50]

Starting from $\langle |M_{if}|^2 \rangle = \frac{g_e^4}{4EE' \sin^4(\theta/2)} \Big(2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2) \Big)$, show that

$$\frac{d\sigma}{d\Omega} = \Big[\frac{\alpha}{4ME \sin^2(\theta/2)} \Big]^2 \frac{E^{'}}{E} \Big(2K_1 \sin^2(\theta/2) + K_2 \cos^2(\theta/2) \Big)$$

Assignment 6

Deep inelastic scattering

Exercise 1

Calculate the centre-of-mass energies at SLAC (20 GeV electrons on stationary protons) and at HERA (27 GeV electrons on 800 GeV protons). You can neglect the electron mass and, at HERA, also the proton mass.

Exercise 2

All DIS kinematic variables can be determined from a measurement of the scattered electron energy E' and angle θ with respect to the incident beam. In particular, show that for fixed-target experiments (proton at rest and the electron coming in from the z direction) we have the relations

$$\begin{aligned} Q^2 &= 4EE' \sin^2(\theta/2) \\ \nu &= E - E' \\ x &= Q^2/(2M\nu) \\ y &= \nu/E \\ W^2 &= M^2 - Q^2 + 2M\nu \\ s &= M(M+2E) \approx 2ME \end{aligned}$$

6.1 Homework

Exercise 1[50]

Show that $Q^2 \approx xys$ for large $s \gg M^2$ (so that we can neglect the proton mass). What is, in this approximation, the largest Q^2 that can be reached at the SLAC experiments ($\sqrt{s} = 6.4$ GeV) and at HERA ($\sqrt{s} = 294$ GeV).

Exercise 2[50]

Show that:

1. $s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$
2. if we neglect the electron and proton mass,

$$
\begin{aligned}
Q^2 &= -t \\
x &= -t/(s+u) \\
y &= (s+u)/s \\
W^2 &= s+t+u
\end{aligned}
$$

Assignment 7

Asymptotic freedom

Exercise 1

Show that the term $q_\mu q_\nu J(q^2)$ does not contribute to the matrix element M_{if}, using the Dirac equations $(\not{p}-m)u=0$, $\bar{u}(\not{p}-m)=0$, $(\not{p}+m)v=0$ and $\bar{v}(\not{p}+m)=0$.

7.1 Homework

Exercise 1[70]

Show that

1. $q^2=(p_1-p_3)^2=(p_4-p_2)^2<0$,
2. for small and large Q^2

$$f\left(\frac{Q^2}{m_\mathrm{e}^2}\right)=\begin{cases}\frac{1}{5}\frac{Q^2}{m_\mathrm{e}^2} & \text{for} \quad Q^2\ll m_\mathrm{e}^2\\ \ln\left(\frac{Q^2}{m_\mathrm{e}^2}\right) & \text{for} \quad Q^2\gg m_\mathrm{e}^2\end{cases}$$

For this, you might need the integrals $\int_0^1 \mathrm{d}z\, z(1-z)=\frac{1}{6}$ and $\int_0^1 \mathrm{d}z\, z^2(1-z)^2=\frac{1}{30}$.

Exercise 2[30]

Calculate $\alpha(Q^2)$ for $Q^2=1000\ \mathrm{GeV}^2$.