Post-doc challenge

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Challenge 1 (Strange unitary).

1. Find a $7 \times 7$ integer matrix $M$ such that the matrix $U$ with entries

$$U_{xy} = \frac{1}{\sqrt{7}} \exp \left( \frac{2\pi i}{6} M_{xy} \right)$$

is unitary. Yes, $U$ is a $7 \times 7$ unitary whose entries are 6th roots of unity!

2. For an unknown value of $x \in \{1, \ldots, 7\}$, let $O_x$ be a quantum oracle that provides access to the row $x$ of $M$: for any entry $y \in \{1, \ldots, 7\}$ and an arbitrary $a \in \{0, \ldots, 5\}$,

$$O_x |y\rangle |a\rangle = |y\rangle |(a + M_{xy}) \mod 6\rangle.$$

How many queries to $O_x$ are needed to determine which row $x$ it hides?

Challenge 2 (Broken Bernstein–Vazirani). Let $n \geq 3$ and $s \in \{0, 1\}^n$ be an unknown string. The standard phase oracle for the Bernstein–Vazirani problem is

$$O_s = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s} |x\rangle \langle x|.$$

Consider a broken oracle $O_s(\phi) = E_s(\phi)O_s$ where $\phi \in [0,2\pi)$ is some angle and $E_s(\phi)$ is an $s$-dependent diagonal error unitary that, for any $x \in \{0,1\}^n$, acts as

$$E_s(\phi)|x\rangle = \begin{cases} e^{i\phi} |x\rangle & \text{if } (x \neq s) \Leftrightarrow (x \cdot s = 0), \\ |x\rangle & \text{otherwise}, \end{cases}$$

where “$\Leftrightarrow$” denotes the logical XOR. Find an angle $\phi(n) \neq 0$ such that one query to the broken oracle $O_s(\phi(n))$ still lets you determine the string $s$ with certainty, promised that $s \neq 0$. 
Challenge 3 (Quantum averaging). Let \( \rho(\vec{r}) = \frac{1}{2}(I + xX + yY + zZ) \) where \( \vec{r} = (x, y, z) \in \mathbb{R}^3 \) and

\[
I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

are the Pauli matrices. Given \( \vec{r}_1, \ldots, \vec{r}_n \in \mathbb{R}^3 \) with norm at most one, the state

\[
\frac{1}{n} \sum_{i=1}^{n} \rho(\vec{r}_i)
\]

is called the naive average of states \( \rho(\vec{r}_i) \).

1. Show that, for two states, no quantum algorithm can beat the naive average. Namely, given unknown input states \( \rho(\vec{r}_1) \) and \( \rho(\vec{r}_2) \), no quantum algorithm can produce the state \( \rho(c \vec{r}_1 + \vec{r}_2) \) with \( c > 1 \). Curiously, one can actually achieve \( c < 0 \). What is the smallest \( c < 0 \) that you can achieve?

2. Show that, for three states, one can do better than the naive average. More specifically, let \( U \in U(2) \) be an unknown unitary and consider the states

\[
\rho_k(\theta) = U \rho(\theta, \frac{2\pi}{3}) U^\dagger, \quad k \in \{1, 2, 3\},
\]

where \( \rho(\theta, \varphi) = \rho(\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) \). Their naive average is

\[
\frac{1}{3} \sum_{k=1}^{3} \rho_k(\theta) = U \rho(0, 0, \cos \theta) U^\dagger.
\]

Find a quantum algorithm that (without knowing \( U \)) can implement the transformation

\[
\otimes_{k=1}^{3} \rho_k(\theta) \mapsto U \rho(0, 0, f(\theta)) U^\dagger,
\]

for some \( f(\theta) > \cos \theta \) for all \( \theta \in (0, \pi/2) \). What is the optimal \( f(\theta) \)?

3. More generally, let \( U \in U(2) \) again be unknown and let

\[
\rho_k = U \rho(\theta_k, \varphi_k) U^\dagger, \quad k \in \{1, 2, 3\},
\]

where the angles \( \theta_k \in \mathbb{R} \) are chosen independently from the normal distribution of mean 0 and variance \( \sigma^2 \) while \( \varphi_k \in [0, 2\pi) \) are chosen uniformly at random. When averaged over the choice of these angles, the naive average of these three states is \( U \rho(\vec{r}) U^\dagger \) where \( \vec{r} = (0, 0, e^{-\sigma^2/2}) \). What is the \( \vec{r} \) of their optimal quantum average?

Send your solutions to marozols@gmail.com with subject “Challenge”.