Post-doc challenge

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Challenge 1 (Strange unitary).

1. *Find a 7×7 integer matrix M such that the matrix U with entries*

$$U_{xy} = \frac{1}{\sqrt{7}} \exp\left(\frac{2\pi i}{6} M_{xy}\right)$$

is unitary. Yes, U is a 7×7 unitary whose entries are 6th roots of unity!

2. *For an unknown value of $x \in \{1, \dots, 7\}$, let O_x be a quantum oracle that provides access to the row x of M: for any entry $y \in \{1, \dots, 7\}$ and an arbitrary $a \in \{0, \dots, 5\}$,*

$$O_x |y\rangle |a\rangle = |y\rangle |(a + M_{xy}) \bmod 6\rangle.$$

How many queries to O_x are needed to determine which row x it hides?

Challenge 2 (Broken Bernstein–Vazirani). *Let $n \geq 3$ and $s \in \{0,1\}^n$ be an unknown string. The standard phase oracle for the Bernstein–Vazirani problem is*

$$O_s = \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s} |x\rangle\langle x|.$$

Consider a broken oracle $O_s(\varphi) = E_s(\varphi) O_s$ where $\varphi \in [0, 2\pi)$ is some angle and $E_s(\varphi)$ is an s-dependent diagonal error unitary that, for any $x \in \{0,1\}^n$, acts as

$$E_s(\varphi)|x\rangle = \begin{cases} e^{i\varphi}|x\rangle & \text{if } (x \neq s) \veebar (x \cdot s = 0), \\ |x\rangle & \text{otherwise,} \end{cases}$$

where "$\veebar$" denotes the logical XOR. Find an angle $\varphi(n) \neq 0$ such that one query to the broken oracle $O_s(\varphi(n))$ still lets you determine the string s with certainty, promised that $s \neq 0$.

Challenge 3 (Quantum averaging). *Let* $\rho(\vec{r}) = \frac{1}{2}(I + xX + yY + zZ)$ *where* $\vec{r} = (x, y, z) \in \mathbb{R}^3$ *and*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are the Pauli matrices. Given $\vec{r}_1, \ldots, \vec{r}_n \in \mathbb{R}^3$ *with norm at most one, the state*

$$\frac{1}{n}\sum_{i=1}^{n} \rho(\vec{r}_i) = \rho\left(\frac{1}{n}\sum_{i=1}^{n} \vec{r}_i\right)$$

is called the naive average of states $\rho(\vec{r}_i)$.

1. *Show that, for two states, no quantum algorithm can beat the naive average. Namely, given unknown input states* $\rho(\vec{r}_1)$ *and* $\rho(\vec{r}_2)$*, no quantum algorithm can produce the state* $\rho\big(c\,\frac{\vec{r}_1+\vec{r}_2}{2}\big)$ *with* $c > 1$*. Curiously, one can actually achieve* $c < 0$*. What is the smallest* $c < 0$ *that you can achieve?*

2. *Show that, for three states, one can do better than the naive average. More specifically, let* $U \in \mathrm{U}(2)$ *be an unknown unitary an consider the states*

$$\rho_k(\theta) = U\rho(\theta, \tfrac{2\pi k}{3})U^\dagger, \qquad k \in \{1, 2, 3\},$$

where $\rho(\theta, \varphi) = \rho(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$*. Their naive average is*

$$\frac{1}{3}\sum_{k=1}^{3} \rho_k(\theta) = U\rho(0, 0, \cos\theta)U^\dagger.$$

Find a quantum algorithm that (without knowing U) can implement the transformation

$$\bigotimes_{k=1}^{3} \rho_k(\theta) \mapsto U\rho(0, 0, f(\theta))U^\dagger,$$

for some $f(\theta) > \cos\theta$ *for all* $\theta \in (0, \pi/2)$*. What is the optimal* $f(\theta)$*?*

3. *More generally, let* $U \in \mathrm{U}(2)$ *again be unknown and let*

$$\rho_k = U\rho(\theta_k, \varphi_k)U^\dagger, \qquad k \in \{1, 2, 3\},$$

where the angles $\theta_k \in \mathbb{R}$ *are chosen independently from the normal distribution of mean 0 and variance* σ^2 *while* $\varphi_k \in [0, 2\pi)$ *are chosen uniformly at random. When averaged over the choice of these angles, the naive average of these three states is* $U\rho(\vec{r})U^\dagger$ *where* $\vec{r} = (0, 0, e^{-\sigma^2/2})$*. What is the* $\vec{r}$ *of their optimal quantum average?*

Send your solutions to `marozols@gmail.com` with subject "Challenge".