Operads in unstable global homotopy theory

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Abstract

The goal of this thesis is to study the homotopy theory of algebras over operads in global unstable homotopy theory, the homotopy theory of spaces which have simultaneous compatible actions of all compact Lie groups. We first study orthogonal spaces which have an additional action by a fixed group G. We define two classes of morphisms, the G-global equivalences and the G-flat cofibrations, and prove some properties about them. The main result gives a model structure on the category of algebras over any operad in orthogonal spaces, without the usual cofibrancy condition. We also give a simple characterization of when a map of operads induces a Quillen equivalence between those model structures, and a relevant example.

Contents

Introduction

The goal of this thesis is to study the homotopy theory of algebras over operads in unstable global homotopy theory. Equivariant homotopy theory is the study of spaces with G-actions for a group G, and G-equivariant continuous maps between them. Interest in it has recently seen an increase due to its connections to other areas of mathematics.

Global homotopy theory is the homotopy theory of spaces which have simultaneous compatible actions by all compact lie groups. As a model for unstable global homotopy theory we use the category of orthogonal spaces, which are continuous functors from the category of linear isometries to $\underline{\mathrm{Top}}$, with the positive global model structure of [Sch18]. Each orthogonal space has an underlying G-space for each compact lie group G, and a morphism of orthogonal spaces induces a G-equivariant map between their underlying G-spaces. A global equivalence is a morphism which induces G-weak homotopy equivalences between the underlying G-spaces.

An operad is a tool used to describe algebraic structures in various contexts. We can define operads in any symmetric monoidal category. An operad $\mathcal{O}$ consists of a series of objects $\mathcal{O}_n$ which each represents a collection of of n-ary operations, and a composition law. An algebra over a given operad is an object together with an interpretation of the operad as operations on the object.

If $\mathscr{C}$ is a symmetric monoidal cofibrantly generated model category, and given an operad $\mathcal{O}$ in $\mathscr{C}$, one is often interested in lifting the model structure of $\mathscr{C}$ to a model structure on the algebras over $\mathcal{O}$. In full generality this is possible if the operad is cofibrant and $\mathscr{C}$ satisfies the monoid axiom (see [Spi01]). The first main theorem of this thesis states that the situation is much simpler in the case of orthogonal spaces.

Theorem I. *Let $\mathcal{O}$ be any operad in $\mathcal{Spc}$ the category of orthogonal spaces, with the positive global model structure. Then there is a cofibrantly generated model category structure on $\mathcal{Alg}(\mathcal{O})$ the category of algebras over $\mathcal{O}$, where the forgetful functor $U_{\mathcal{Alg}(\mathcal{O})}$ creates the weak equivalences and fibrations, and sends cofibrations in $\mathcal{Alg}(\mathcal{O})$ to h-cofibrations in $\mathcal{Spc}$.*

An operad is composed of objects $\mathcal{O}_n$ with an action by the symmetric group Σ_n. Therefore to prove the results of this thesis we need to study the homotopy theory of orthogonal spaces which have an additional action by a fixed compact Lie group G, or G-orthogonal spaces. To this end we will define two classes of morphisms between G-orthogonal spaces, the G-global equivalences and the G-flat cofibrations.

We will not check that the G-global equivalences and the G-flat cofibrations are part of a model structure on the category of G-orthogonal spaces, because we will not need such a model structure. However the results that we prove are enough to obtain the structure of a cofibration category, or category of cofibrant objects, on the category of G-orthogonal spaces.

The category of orthogonal spaces has a symmetric monoidal product, the box product, and this box product is fully homotopical, that is the box product of two global equivalences is a global equivalence. The box product of G-orthogonal spaces is similarly well behaved, since the box product of a G-global equivalence and a K-global equivalence is a $(G \times K)$-global equivalence. This is the main result that allows us to omit the cofibrancy assumption from Theorem I. Another fact that makes Theorem I possible is that n-fold box products of generating orthogonal spaces are Σ_n-free.

We also study morphisms of operads, and the functors they induce between the categories of algebras. Any morphism of operads induces a Quillen adjunction, and in the second main theorem of this thesis we give a simple necessary and sufficient condition for this adjunction to be a Quillen equivalence.

Theorem II. *Let $g\colon \mathcal{O} \to \mathcal{P}$ be a morphism of operads in $\mathcal{Spc}$. Then we have that the pair $(g_!, g^*)$ is a Quillen equivalence between their respective categories of algebras if and only if for each $n \geq 0$ the morphism $g_n\colon \mathcal{O}_n \to \mathcal{P}_n$ is a Σ_n-global equivalence.*

This is related to the work of Blumberg and Hill in [BH15] on operads in G-spaces. The preferred notion of weak equivalence between operads in G-spaces is that of a graph equivalence, where for each continuous homomorphism $\phi\colon G \to \Sigma_n$, the map g_n induces a weak homotopy equivalence on points fixed by ϕ. A Σ_n-global equivalence is roughly a morphism that induces graph equivalences between the underlying $G \times \Sigma_n$-spaces for each G. If we abstractly think of an operad in orthogonal spaces as a collection of G-operads for each G, then a morphism of operads that satisfies the conditions of Theorem II would give graph equivalences between these G-operads.

Theorem II also shows that the naive notion of what the analog of an E_∞-operad in orthogonal spaces is (that each $\mathcal{O}_n$ is globally contractible) is not the correct one. Instead we need to ask that each $\mathcal{O}_n$ is Σ_n-globally contractible, and we call such operads global E_∞-operads.

By Theorem II the category of algebras over a global E_∞-operad $\mathcal{O}$ is Quillen equivalent to the category of strictly associative and strictly commutative monoids in orthogonal spaces. This also implies that any algebra X over $\mathcal{O}$ is connected to a strictly associative and strictly commutative monoid via a zigzag of global equivalences of $\mathcal{O}$-algebras, which implies certain things about the homotopy of X, like the existance of transfers on homotopy sets.

Overview

The first chapter of this thesis deals with some preliminary theory required for the rest of the document. We first develop the theory of operads and their algebras in a general symmetric monoidal category, following [Fre09]. We then state a result on lifting a model structure to the category of algebras over a monad. After that we prove a refinement, Theorem 1.2.2, for the case where the monad comes from an operad, and we have a bigger class of cofibrations to work with. We will later use this refinement to prove Theorem I, using the h-cofibrations, or morphisms with the homotopy extension property.

The main condition of Theorem 1.2.2 asks that cobase changes, in the category of algebras over the operad, of generating cofibrations are h-cofibrations, and that cobase changes of generating acyclic cofibrations are global equivalences. Pushouts in the category of algebras over an operad are not at all straightforward (the forgetful functor preserves filtered colimits and reflexive coequalizers, but not all colimits) and this is what causes most of the work needed to apply Theorem 1.2.2.

After this we expose the basics of the theory of orthogonal spaces. The specific structure of the generating (acyclic) cofibrations of the positive global model structure is important to prove the results that we need, so we make it explicit.

The second chapter is devoted to G-orthogonal spaces, G-flat cofibrations, and G-global equivalences. We define them and prove the results that we will need later. One of the things that we need to do, is to determine some conditions under which taking Σ_n-orbits of a Σ_n-global equivalence yields a global equivalence. In [GG16] morphisms which yield weak equivalences when taking Σ_n-orbits are considered in a general setting, but we have to construct the Σ_n-global equivalences to use that the box product is fully homotopical.

We denote by G-h-cofibrations the morphisms with the homotopy extension property in the category of G-orthogonal spaces. We check that the relevant properties of preservation of G-global equivalences along G-flat cofibrations also hold for G-h-cofibrations. This is helpful because we can then work just with G-h-cofibrations, and the box product with any G-orthogonal space preserves G-h-cofibrations.

The third chapter deals first with checking that the main condition of Theorem 1.2.2 holds for any operad in orthogonal spaces. This then gives the proof of Theorem I. We then study morphisms of operads of orthogonal spaces and prove Theorem II. We also define global E_∞-operads and prove some facts about them.

In the last chapter we talk about an example of an operad $\mathcal{O}$ on orthogonal spaces derived from the little-disks operad. We conjecture that this operad is a global E_∞-operad, that is, that the category of algebras over this operad is Quillen equivalent to the category of strictly associative and strictly commutative monoids in orthogonal spaces.

Notation and conventions

Throughout this document, whenever we talk about a space we will be referring to a compactly generated weak Hausdorff topological space. We use $\underline{\mathrm{Top}}$ to refer to the category of such spaces. In the rare cases where we refer to a general topological space, we do so explicitly.

Let G be a topological group. A G-space is a space X with an associative and unital continuous action $G \times X \to X$. We denote the category of G-spaces by $\underline{\mathrm{GTop}}$. For any set F of closed subgroups of G, we say that a morphism $f\colon X \to Y$ of G-spaces is an F-equivalence (F-fibration) if for any $H \in F$ the restriction of f to the H-fixed points $f^H\colon X^H \to Y^H$ is a weak homotopy equivalence (respectively a Serre fibration).

For each F set of subgroups of G there is a cofibrantly generated model structure on $\underline{\mathrm{GTop}}$ with the F-equivalences as weak equivalences and the F-fibrations as fibrations (see [Sch18, Proposition B.7]). We refer to the cofibrations of this model structure as F-cofibrations. If the set F is the set of all closed subgroups of G we instead use G-weak homotopy equivalences, G-fibrations and G-cofibrations to refer to these classes of morphisms.

By default, an inner product space refers to a real inner product space, and for a compact Lie group G, when we mention a G-representation we always refer to an orthogonal G-representation in an inner product space, finite dimensional unless stated otherwise. We write Σ_n for the symmetric group on n elements.

To talk about small objects in a category, we will follow the convention of [Hov07, Section 2.1.1], which is not universal. For an ordinal λ, a λ-sequence in a cocomplete category $\mathscr{C}$ is a colimit preserving functor $Y\colon \lambda \to \mathscr{C}$. We call the morphism $Y(0) \to \mathrm{colim}_{\beta\in\lambda} Y(\beta)$ the transfinite composition of Y. We say that a subcategory $\mathscr{D} \subset \mathscr{C}$ is closed under transfinite composition if, for every λ-sequence Y such that for each $\beta \in \lambda$ the morphism $Y(\beta) \to Y(\beta+1)$ is in $\mathscr{D}$, the transfinite composition of Y is in $\mathscr{D}$.

For any cardinal κ, we say that an ordinal λ is κ-filtered if its cofinality is strictly bigger than κ.

Given a cardinal κ (and a subcategory $\mathscr{D} \subset \mathscr{C}$ closed under transfinite composition), we say that an object $X \in \mathscr{C}$ is κ-small (relative to $\mathscr{D}$) if $\mathscr{C}(X,-)$ preserves all colimits of λ-sequences (with image in $\mathscr{D}$), with λ a κ-filtered ordinal. We say that X is small (relative to $\mathscr{D}$) if it is κ-small (relative to $\mathscr{D}$) for some cardinal κ. We say that X is finite (relative to $\mathscr{D}$) if it is κ-small (relative to $\mathscr{D}$) for some finite cardinal κ.

By default, when we say that an object has an action by a group G, we mean a left G-action. The main exception is for operads and symmetric objects, where each $\mathcal{O}_n$ has a right Σ_n-action by convention. We will often turn left actions into right actions and vice versa by precomposing with the antihomomorphism $(-)^{-1}$.

We will often use i_l to refer to the boundary map $\partial D^l \to D^l$ in $\underline{\mathrm{Top}}$, for each $l \geq 0$. Similarly we will use j_l for the inclusion $[0,1]^l \to [0,1]^l \times [0,1]$ for $l \geq 0$.

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Chapter 1

Preliminaries

1.1 Basics of operads

We now develop the theory of symmetric objects, operads and their algebras in a general setting, following [Fre09].

Let $\mathscr{C}$ denote a cocomplete symmetric monoidal category, where the tensor product preserves all colimits in both variables.

Definition 1.1.1 (Symmetric objects)**.** The category of symmetric objects in $\mathscr{C}$, denoted by Σ_*-$\mathscr{C}$, has as objects sequences in $\mathscr{C}$ $\{M(n)\}_{n\in\mathbb{N}}$ where each $M(n)$ has a right Σ_n-action. Morphisms in this category are sequences of morphisms in $\mathscr{C}$ that commute with the Σ_n-actions. Equivalently, Σ_*-$\mathscr{C}$ is also the functor category $\underline{\mathrm{Fun}}(\coprod_{n\in\mathbb{N}}\underline{\Sigma_n},\mathscr{C})$, where $\underline{\Sigma_n}$ is the groupoid associated to Σ_n.

We can define two different monoidal structures on Σ_*-$\mathscr{C}$, the first is the tensor product of symmetric objects, which gives a symmetric monoidal structure. On level n it is

$$(M\otimes N)(n)=\coprod_{p+q=n}\Sigma_n\otimes_{\Sigma_p\times\Sigma_q}(M(p)\otimes N(q))$$

where the tensoring of $\mathscr{C}$ over $\underline{\mathrm{Set}}$ is given by $S\otimes C=C^{\otimes|S|}$, and we quotient by the diagonal $(\Sigma_p\times\Sigma_q)$-action which is the inverse of the one on $M(p)\otimes N(q)$ obtained from being symmetric objects, and the one on Σ_n obtained from the canonical embedding $\Sigma_p\times\Sigma_q\to\Sigma_n$ where a permutation from Σ_p acts on the first p elements and one from Σ_q on the last q. On morphisms it is also defined through this formula.

Proposition 1.1.2 (Tensor product symmetric monoidal structure on Σ_*-$\mathscr{C}$)**.** *The category of symmetric objects Σ_*-$\mathscr{C}$ is equipped with the structure of a symmetric monoidal category tensored over $\mathscr{C}$. The bifunctor is the tensor product previously defined, and the unit is $\mathbf{1}_{\Sigma_*\text{-}\mathscr{C}}=\mathbf{1}$, where $\mathbf{1}(0)$ is $\mathbf{1}_{\mathscr{C}}$ and $\mathbf{1}(n)$ is the initial object $\emptyset$ for each $n\geq 1$.*

The details can be found in [Fre09, 2.1.5 and 2.1.7].

We also have generating symmetric objects F_n for each $n\in\mathbb{N}$. $F_n(n)$ is $\Sigma_n\otimes\mathbf{1}_{\mathscr{C}}$, and F_n is $\emptyset$ on all other levels. For these generating symmetric objects, we have that $M(n)\cong\mathrm{Hom}_{\Sigma_*\text{-}\mathscr{C}}(F_n,M)$

[Fre09, Proposition 2.1.13]. We also denote F_1 by I, and we have that $F_n = I^{\otimes n}$.

The second monoidal structure on Σ_*-$\mathscr{C}$ is the composition product, which is not symmetric. Its unit is I. It is defined from the tensor product in Σ_*-$\mathscr{C}$ as

$$M \circ N = \coprod_{n \in \mathbb{N}} M(n) \otimes_{\Sigma_n} N^{\otimes n}$$

where the tensoring of Σ_*-$\mathscr{C}$ over $\mathscr{C}$ is just the tensor product on each level, and the Σ_n-action is the given one on $M(n)$ and permuting the terms on $N^{\otimes n}$.

Proposition 1.1.3 (Composition monoidal structure on Σ_*-$\mathscr{C}$)**.** *The category of symmetric objects Σ_*-$\mathscr{C}$ is equipped with the structure of a monoidal category, where the bifunctor is the composition product previously defined, and the unit is I.*

The details can be found in [Fre09, 2.2.1 and 2.2.2].

Let $\mathscr{D}$ be a symmetric monoidal category tensored over $\mathscr{C}$, where the tensor product over $\mathscr{C}$ preserves all colimits in $\mathscr{C}$ In our case $\mathscr{D}$ will usually be $\mathscr{C}$ itself. Then we can assign to each symmetric object M on $\mathscr{C}$ a functor $\mathcal{F}(M)\colon \mathscr{D} \to \mathscr{D}$, by:

$$\mathcal{F}(M)(X) = \coprod_{n \in \mathbb{N}} M(n) \otimes_{\Sigma_n} X^{\otimes n} \tag{1}$$

Proposition 1.1.4. *The previous construction yields a functor $\mathcal{F}\colon \Sigma_*\text{-}\mathscr{C} \to \underline{\mathrm{Fun}}(\mathscr{D}, \mathscr{D})$. Furthermore, $\mathcal{F}(I) = Id_{\mathscr{D}}$ and $\mathcal{F}(M \circ N) = \mathcal{F}(M) \circ \mathcal{F}(N)$. This means that $\mathcal{F}$ is a strong monoidal functor into $\underline{\mathrm{Fun}}(\mathscr{D}, \mathscr{D})$ with the monoidal structure given by composition of functors. Additionally $\mathcal{F}$ preserves all colimits.*

Proof. Functoriality comes from the functoriality of the tensor product and of colimits. For $\mathcal{F}(M \circ N) = \mathcal{F}(M) \circ \mathcal{F}(N)$ see [Fre09, Proposition 2.2.1]. Colimits in both Σ_*-$\mathscr{C}$ and $\underline{\mathrm{Fun}}(\mathscr{D}, \mathscr{D})$ are computed in $\mathscr{C}$ and $\mathscr{D}$ respectively. Then commutativity of colimits and the fact that the tensor product preserves all colimits in $\mathscr{C}$ imply that $\mathcal{F}$ preserves colimits. □

Note that when $\mathscr{D} = \Sigma_*$-$\mathscr{C}$, $\mathcal{F}(M)(N) = M \circ N$. We can consider the full subcategory $(\Sigma_*\text{-}\mathscr{C})_0 \subset \Sigma_*\text{-}\mathscr{C}$ of objects $M \in \Sigma_*\text{-}\mathscr{C}$ with $M(n) = \emptyset$ for $n > 0$. This is canonically isomorphic to $\mathscr{C}$ by sending and object $X \in \mathscr{C}$ to the symmetric object $\iota(X)$ which is X on level 0 and $\emptyset$ otherwise. Then $\iota(\mathcal{F}(M)(X)) = M \circ \iota(X) = \mathcal{F}(M)(\iota(X))$ (note that here the first $\mathcal{F}$ refers to the functor associated to M on $\mathscr{C}$ and the second on Σ_*-$\mathscr{C}$).

Proposition 1.1.5. *For each $M \in \Sigma_*\text{-}\mathscr{C}$ the functor $\mathcal{F}(M)\colon \mathscr{D} \to \mathscr{D}$ preserves filtered colimits and reflexive coequalizers.*

Proof. [Fre09, Proposition 2.4.1]. □

That is, if we take $\mathcal{F}$ as a bifunctor $\Sigma_*\text{-}\mathscr{C} \times \mathscr{D} \to \mathscr{D}$, it preserves all colimits on the first variable, but only filtered colimits and reflexive coequalizers on the second.

Definition 1.1.6 (Operad)**.** An operad on $\mathscr{C}$ is a monoid object in the monoidal category $(\Sigma_*\text{-}\mathscr{C}, \circ, I)$. That is, a symmetric object in $\mathscr{C}$ $\mathcal{O} \in \Sigma_*\text{-}\mathscr{C}$ together with morphisms of symmetric objects $\mu\colon \mathcal{O} \circ \mathcal{O} \to \mathcal{O}$ multiplication and $\eta\colon I \to \mathcal{O}$ unit such that the following associativity and unit diagrams commute:

$$\begin{tikzcd} I \circ \mathcal{O} \arrow[r, "\eta\circ Id"] \arrow[dr, "Id"'] & \mathcal{O} \circ \mathcal{O} \arrow[d, "\mu"] & \mathcal{O} \circ I \arrow[l, "Id\circ\eta"'] \arrow[dl, "Id"] \\ & \mathcal{O} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{O} \circ \mathcal{O} \circ \mathcal{O} \arrow[r, "\mu\circ Id"] \arrow[d, "Id\circ\mu"'] & \mathcal{O} \circ \mathcal{O} \arrow[d, "\mu"] \\ \mathcal{O} \circ \mathcal{O} \arrow[r, "\mu"] & \mathcal{O} \end{tikzcd}$$

Unraveling this definition to the level of $\mathscr{C}$, we obtain that the multiplication morphism consists of suitably equivariant maps:

$$(\mathcal{O} \circ \mathcal{O})(n) = \coprod_{k\in\mathbb{N}, n_1+\cdots+n_k=n} \mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \cdots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n)$$

The unit is an element $\mathbf{1}_{\mathscr{C}} \to \mathcal{O}(1)$, and these maps have to make the equivalent associativity and unit diagrams commute.

Given a category $\mathscr{C}$, a monad is a monoid object in $\underline{\text{Fun}}(\mathscr{D}, \mathscr{D})$. Explicitly this is a functor $F\colon \mathscr{C} \to \mathscr{C}$, and two natural transformations $\eta\colon Id \Rightarrow F$ and $\mu\colon F \circ F \Rightarrow F$, such that the associativity and unit diagrams similar to those for an operad commute.

Since $\mathcal{F}$ is strong monoidal, we have that if $\mathcal{O}$ is an operad in $\mathscr{C}$, then $\mathcal{F}(\mathcal{O})\colon \mathscr{D} \to \mathscr{D}$ is a monad in $\mathscr{D}$.

Definition 1.1.7 (Left modules, right modules and algebras over an operad)**.** We define left and right modules on $\Sigma_*\text{-}\mathscr{C}$ over an operad, using that an operad is a monoid object on $(\Sigma_*\text{-}\mathscr{C}, \circ, I)$. An algebra on $\mathscr{D}$ over the operad $\mathcal{O}$ is an algebra over the monad $\mathcal{F}(\mathcal{O})$. Explicitly this is an object $A \in \mathscr{D}$ and a morphism $\zeta_A\colon \mathcal{F}(\mathcal{O})(A) \to A$ in $\mathscr{D}$ compatible with the multiplication and unit of the operad, in the sense that $\zeta_A \circ \mathcal{F}(\eta)(A) = Id_A$ and $\zeta_A \circ \mathcal{F}(\mathcal{O}, \zeta_A) = \zeta_A \circ \mathcal{F}(\mu, A)$.

If $\mathscr{D} = \mathscr{C}$, then we can alternatively define an algebra as an object $X \in \mathscr{C}$ together with a left module structure on $\iota(X)$ over the operad $\mathcal{O}$. These two definitions are equivalent since a map $M \circ \iota(X) = \iota(\mathcal{F}(M)(X)) \to \iota(X)$ is the same as a map $\mathcal{F}(M)(X) \to X$, and the associativity and unit diagrams required to commute are similarly equivalent.

We denote the category of algebras on $\mathscr{C}$ over an operad $\mathcal{O}$ by $\mathcal{Alg}(\mathcal{O})$, and the categories of left and right modules by $\mathcal{Lmod}(\mathcal{O})$ and $\mathcal{Rmod}(\mathcal{O})$ respectively.

There are forgetful functors for each of these categories, respectively $U_{\mathcal{Alg}(\mathcal{O})}\colon \mathcal{Alg}(\mathcal{O}) \to \mathscr{C}$, $U_{\mathcal{Lmod}(\mathcal{O})}\colon \mathcal{Lmod}(\mathcal{O}) \to \Sigma_*\text{-}\mathscr{C}$ and $U_{\mathcal{Rmod}(\mathcal{O})}\colon \mathcal{Rmod}(\mathcal{O}) \to \Sigma_*\text{-}\mathscr{C}$, and there are free functors left adjoint to these $F_{\mathcal{Alg}(\mathcal{O})}\colon \mathscr{C} \to \mathcal{Alg}(\mathcal{O})$, $F_{\mathcal{Lmod}(\mathcal{O})}\colon \Sigma_*\text{-}\mathscr{C} \to \mathcal{Lmod}(\mathcal{O})$ and $F_{\mathcal{Rmod}(\mathcal{O})}\colon \Sigma_*\text{-}\mathscr{C} \to \mathcal{Rmod}(\mathcal{O})$. $F_{\mathcal{Alg}(\mathcal{O})}$ is obtained from the functor associated to $\mathcal{O}$, $\mathcal{F}(\mathcal{O})$, by $F_{\mathcal{Alg}(\mathcal{O})}(X) = \mathcal{F}(\mathcal{O})(X)$. This is the same as the free algebra over the monad $\mathcal{F}(\mathcal{O})$.

1.2 Model categories in categories of algebras

From now on, let $\mathscr{C}$ be a cofibrantly generated model category which is also a symmetric monoidal category, and such that the monoidal product preserves all colimits in each variable.

We now deal with the conditions needed to lift the model structure on $\mathscr{C}$ through the forgetful functor $U_{\mathcal{Alg}(\mathcal{O})}\colon \mathcal{Alg}(\mathcal{O}) \to \mathscr{C}$ to the category of algebras over a given operad. For categories of algebras over monads on $\mathscr{C}$ we have the following result from [SS00, Lemma 2.3]:

Theorem 1.2.1. *Let $\mathscr{C}$ be a cofibrantly generated model category with I and J sets of generating cofibrations and acyclic cofibrations respectively, and $T\colon \mathscr{C} \to \mathscr{C}$ a monad in $\mathscr{C}$. If T preserves filtered colimits, then $\mathcal{Alg}(T)$ has all colimits. Then let $F_{\mathcal{Alg}(T)}$ denote the free T-algebra functor and let $I_T = F_{\mathcal{Alg}(T)}(I)$, $J_T = F_{\mathcal{Alg}(T)}(J)$. Let I_T-reg and J_T-reg denote the regular I_T and J_T cofibrations in $\mathcal{Alg}(T)$ respectively. Those are the transfinite compositions of cobase changes in $\mathcal{Alg}(T)$ of morphisms in I_T and J_T respectively.*

Then if the domains of morphisms in I_T and J_T are small with respect to I_T-reg and J_T-reg respectively, and every morphism in J_T-reg is a weak equivalence in $\mathscr{C}$, we obtain that $\mathcal{Alg}(T)$ is a cofibrantly generated model category where $U_{\mathcal{Alg}(T)}$ creates the weak equivalences and fibrations and I_T and J_T are generating sets of cofibrations and trivial cofibrations.

Note that limits in $\mathcal{Alg}(T)$ are computed in $\mathscr{C}$ since $U_{\mathcal{Alg}(T)}$ is a right adjoint. For colimits, by [Bor94, Proposition 4.3.6] it is enough for T to commute with filtered colimits to have all colimits in $\mathcal{Alg}(T)$, and since filtered colimits are preserved by T, they are also preserved by $U_{\mathcal{Alg}(T)}$ by [Bor94, Proposition 4.3.2]. This is the reason why the previous theorem requires that T preserves filtered colimits. In the case where T is obtained from an operad on $\mathscr{C}$ we have that by Proposition 1.1.5 T commutes with filtered colimits.

For $\mathcal{O}$ an operad, we will shorten the associated monad $\mathcal{F}(\mathcal{O})$ to also $\mathcal{O}$ when referring to things like $I_{\mathcal{F}(\mathcal{O})}$ and so on. The idea of the previous theorem is to define weak equivalences (fibrations) in $\mathcal{Alg}(\mathcal{O})$ to be those morphisms f such that $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a weak equivalence (respectively fibration) in $\mathscr{C}$, and then define cofibrations to be the morphisms with the left lifting property with respect to trivial fibrations. On the other hand we could also define the (trivial) cofibrations in $\mathcal{Alg}(\mathcal{O})$ to be the morphisms generated by I_T (respectively J_T). The last condition ensures that the classes of weak equivalences, fibrations, and cofibrations defined by these two different methods are actually the same, and then the proof of the theorem uses one of these two approaches to check each of the model category axioms.

Note that usually, in a category with a model structure and a monoidal structure, two compatibility conditions between these two structures are required. These are the pushout product axiom, and that the unit is cofibrant, and if they are satisfied we call the category a (symmetric) monoidal (cofibrantly generated) model category. We don't ask that they hold for $\mathscr{C}$, in fact in the positive global model structure on orthogonal spaces on which we focus the unit is not cofibrant.

We have the following refinement of the previous theorem, which applies to operads in $\mathscr{C}$ if the domains of the generating (acyclic) cofibrations I and J are small with respect to not just the morphisms in $I_{\mathcal{O}}$-reg and $J_{\mathcal{O}}$-reg respectively, but also with respect to a more general class of cofibrations, $Hcof$, which when we apply the result will be defined by having the homotopy extension property. This refinement is inspired by and similar to [Fre09, Proposition 11.1.14].

Theorem 1.2.2. *Let $\mathscr{C}$ be a symmetric monoidal category which is also a cofibrantly generated model category with sets of generating cofibrations and acyclic cofibrations I and J, and such that the monoidal product preserves all colimits in each variable. Also let $Hcof$ be a class of* *morphisms in $\mathscr{C}$ which satisfies the following:*

1. *$Hcof$ is closed under retracts and transfinite compositions.*
2. *The domains of the generating (acyclic) cofibrations I (J) of $\mathscr{C}$ are small with respect to $Hcof$*
3. *Transfinite compositions of morphisms that are both in $Hcof$ and weak equivalences are weak equivalences*

Then fix any operad $\mathcal{O}$ in $\mathscr{C}$. Assume that for each pushout in $\mathcal{Alg}(\mathcal{O})$ of the form

$$\begin{tikzcd} F_{\mathcal{Alg}(\mathcal{O})}(X) \arrow[r, "F_{\mathcal{Alg}(\mathcal{O})}(i)"] \arrow[d] & F_{\mathcal{Alg}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd} \tag{2}$$

the following holds:

- *If $i \in I$ then $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is in $Hcof$.*
- *If $i \in J$ then $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a weak equivalence.*

Then the conditions of Theorem 1.2.1 are satisfied, so that $\mathcal{Alg}(\mathcal{O})$ is a cofibrantly generated model category where $U_{\mathcal{Alg}(T)}$ creates the weak equivalences and fibrations and $I_{\mathcal{O}}$ and $J_{\mathcal{O}}$ are generating sets of cofibrations and trivial cofibrations. Furthermore $U_{\mathcal{Alg}(\mathcal{O})}$ sends cofibrations to morphisms in $Hcof$.

Proof. The monad associated to an operad preserves filtered colimits, so we have left to check that the domains of morphisms in $I_{\mathcal{O}}$ and $J_{\mathcal{O}}$ are small with respect to $I_{\mathcal{O}}$-reg and $J_{\mathcal{O}}$-reg respectively, and every morphisms in $J_{\mathcal{O}}$-reg is a weak equivalence in $\mathscr{C}$.

Let X be the domain of a morphisms in I or J, λ an ordinal, and $V : \lambda \to \mathcal{Alg}(\mathcal{O})$ a λ-sequence which lands in $I_{\mathcal{O}}$-reg or $J_{\mathcal{O}}$-reg respectively. Remember that $U_{\mathcal{Alg}(\mathcal{O})}$ preserves filtered colimits. Then we have that

$$\operatorname*{colim}_{\lambda} \operatorname{Hom}_{\mathcal{Alg}(\mathcal{O})}(F_{\mathcal{Alg}(\mathcal{O})}(X), V) \cong \operatorname*{colim}_{\lambda} \operatorname{Hom}_{\mathscr{C}}(X, U_{\mathcal{Alg}(\mathcal{O})} \circ V) \to$$
$$\operatorname{Hom}_{\mathscr{C}}(X, \operatorname*{colim}_{\lambda} U_{\mathcal{Alg}(\mathcal{O})} \circ V) \cong \operatorname{Hom}_{\mathscr{C}}(X, U_{\mathcal{Alg}(\mathcal{O})}(\operatorname*{colim}_{\lambda} V)) \cong \operatorname{Hom}_{\mathcal{Alg}(\mathcal{O})}(F_{\mathcal{Alg}(\mathcal{O})}(X), \operatorname*{colim}_{\lambda} V)$$

assuming that $U_{\mathcal{Alg}(\mathcal{O})}$ sends morphisms in $I_{\mathcal{O}}$-reg and $J_{\mathcal{O}}$-reg to $Hcof$. So we will now prove this.

Morphisms in $I_{\mathcal{O}}$-reg are transfinite compositions of cobase changes of morphisms with the form $F_{\mathcal{Alg}(\mathcal{O})}(i)$ like in Diagram (2). $U_{\mathcal{Alg}(\mathcal{O})}$ preserves transfinite compositions, so for g a morphisms in $I_{\mathcal{O}}$-reg, $U_{\mathcal{Alg}(\mathcal{O})}(g)$ is a transfinite composition of morphisms that by our assumptions are in $Hcof$, and so their transfinite composition is also in $Hcof$.

Cofibrations in $\mathcal{Alg}(\mathcal{O})$ are defined as those morphisms with the left lifting property with respect to morphisms which in $\mathscr{C}$ are both weak equivalences and fibrations. By adjointness, the morphisms which in $\mathscr{C}$ are both weak equivalences and fibrations are precisely those that have the right lifting property with respect to $I_{\mathcal{O}}$, since I are the generating cofibrations of $\mathscr{C}$.

Since $U_{\mathcal{Alg}(\mathcal{O})}$ sends morphisms in $I_{\mathcal{O}}$-reg to $Hcof$, the domains of $I_{\mathcal{O}}$ are small with respect to $I_{\mathcal{O}}$-reg, so we can apply the small object argument to them. Let f be a cofibration in $\mathcal{Alg}(\mathcal{O})$. The small object argument gives a factorization $g \circ h$ such that $h \in I_{\mathcal{O}}$-reg and g has the right lifting property against $I_{\mathcal{O}}$. That means that f has the left lifting property against g, which gives that f is a retract of h. Since $Hcof$ is closed under retracts and $U_{\mathcal{Alg}(\mathcal{O})}(h) \in Hcof$, $U_{\mathcal{Alg}(\mathcal{O})}(f) \in Hcof$.

So $U_{\mathcal{Alg}(\mathcal{O})}$ sends cofibrations in $\mathcal{Alg}(\mathcal{O})$ to $Hcof$. $J_{\mathcal{O}}$ are cofibrations in $\mathcal{Alg}(\mathcal{O})$ by adjointness again, and since the class of cofibrations is saturated, $J_{\mathcal{O}}$-reg are also cofibrations, and they are sent to $Hcof$ by $U_{\mathcal{Alg}(\mathcal{O})}$.

Lastly, every morphism in $J_{\mathcal{O}}$-reg is a transfinite composition of cobase changes of morphisms with the form $F_{\mathcal{Alg}(\mathcal{O})}(j)$ like in Diagram (2) where $j \in J$ is a generating acyclic cofibration. These cobase changes are by our hypothesis weak equivalences, and by the previous discussion $F_{\mathcal{Alg}(\mathcal{O})}(j)$ are cofibrations in $\mathcal{Alg}(\mathcal{O})$, therefore also f, and $U_{\mathcal{Alg}(\mathcal{O})}(f)$ are in $Hcof$. So since $U_{\mathcal{Alg}(\mathcal{O})}(f)$ preserves transfinite compositions, $U_{\mathcal{Alg}(\mathcal{O})}(f)(J_{\mathcal{O}}$-reg) are transfinite compositions of morphisms that are both in $Hcof$ and weak equivalences, and so they are weak equivalences. □

Note that indeed we didn't require the pushout product axiom for the proof of this theorem. However, the pushout product axiom or something similar will in general be required in order to actually check the conditions of this theorem.

Remark 1.2.3. In [Spi01, Theorem 4] it is proven that in a general symmetric monoidal cofibrantly generated model category which satisfies the monoid axiom, and for any cofibrant operad $\mathcal{O}$, there is a cofibrantly generated model structure on $\mathcal{Alg}(\mathcal{O})$ where the forgetful functor creates weak equivalences and fibrations.

In contrast, we will use Theorem 1.2.2 to obtain a model structure on $\mathcal{Alg}(\mathcal{O})$ for an operad on orthogonal spaces, and due to the properties of orthogonal spaces as a model for unstable global homotopy theory, we will obtain this model structure for any operad.

1.3 Unstable global homotopy theory

We now turn to the main goal of this thesis, which is to study the homotopy theory of the algebras over any operad in unstable global homotopy theory. We will give a model structure for the category of algebras over any such operad.

By unstable global homotopy theory we mean the homotopy theory of spaces which have simultaneous and compatible actions of all compact Lie groups. A model for this is the category of orthogonal spaces. These are $\underline{\mathrm{Top}}$-enriched functors $\underline{\mathrm{L}} \to \underline{\mathrm{Top}}$, where $\underline{\mathrm{L}}$ denotes the $\underline{\mathrm{Top}}$-enriched category where the objects are inner product spaces, and the morphisms are the linear isometric embeddings. We use $\mathcal{Spc}$ to denote the $\underline{\mathrm{Top}}$-enriched category of orthogonal spaces (See [Sch18, Definition 1.1.1]). Note the similarity of this definition to the definition of orthogonal spectra as enriched functors.

For an inner product space V, evaluating an orthogonal space X on V yields an $O(V)$-space $X(V)$ with the action of $\psi \in O(V)$ given by $X(\psi)$. So if we have a compact Lie group G, and V is a G-representation, then $X(V)$ similarly has a G-action. In this sense, orthogonal spaces have actions by all compact Lie groups.

For a compact Lie group G, let $\mathcal{U}_G$ be a complete G-universe (a countably infinite dimensional orthogonal representation with non-zero fixed points, and such that for each finite dimensional G-representation V, a countably infinite direct sum of copies of V embeds G-equivariantly into $\mathcal{U}_G$). Let $s(\mathcal{U}_G)$ denote the poset of finite dimensional subrepresentations of $\mathcal{U}_G$. Then we define the underlying G-space of X as $X(\mathcal{U}_G) = \operatorname{colim}_{V \in s(\mathcal{U}_G)} X(V)$. Therefore fixing a complete G-universe yields a functor $\mathcal{S}pc \to \underline{\mathrm{GTop}}$.

A global equivalence of orthogonal spaces is roughly a morphism which induces G-weak homotopy equivalences, on the homotopy colimits over all G-representations, for each compact Lie group G. If the orthogonal spaces are closed, this is equivalent to asking that the induced map on the underlying G-spaces is a G-weak homotopy equivalence, see [Sch18, Definition 1.1.16] and [Sch18, Proposition 1.1.17], or Lemma 2.2.3 for the analogous result for G-global equivalences.

There is however a more explicit definition of global equivalence, which we proceed to motivate. Given a map of spaces $f\colon X \to Y$, [May99, 9.6 Lemma] says that f is a weak homotopy equivalence if and only if for each $l \geq 0$ and pair of maps $\alpha\colon \partial D^l \to X, \beta\colon D^l \to Y$ such that $\beta \circ i_l = f \circ \alpha$, there is a map $\lambda\colon D^l \to X$ such that $\lambda \circ i_l = \alpha$ and such that $f \circ \lambda$ is homotopic relative ∂D^l to β.

In general for a similar setup of maps α, β, we will refer to the following commutative diagram as a lifting problem

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, "i_l"] & X \arrow[d, "f"] \\ D^l \arrow[r, "\beta"] & Y \end{tikzcd}$$

and we will say that any map $\lambda\colon D^l \to X$ such that $\lambda \circ i_l = \alpha$ and such that $f \circ \lambda$ is homotopic to β relative ∂D^l solves the lifting problem, and if there exists any such λ we will say that the lifting problem has a solution.

The more explicit definition of global equivalence given in [Sch18, Definition 1.1.2] is the following:

Definition 1.3.1 (Global equivalence of orthogonal spaces)**.** A morphism f of $\mathcal{S}pc$ is a global equivalence if for each K compact lie group and every K-representation V and $l \geq 0$, the following holds: For any lifting problem

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & X(V)^K \arrow[d, "f(V)^K"] \\ D^l \arrow[r, "\beta"] & Y(V)^K \end{tikzcd}$$

there is a K-equivariant linear isometric embedding $\psi\colon V \to W$ into W a K-representation such that there is a morphism $\lambda\colon D^l \to X(W)^K$ which solves the lifting problem $(X(\psi)^K \circ \alpha, Y(\psi)^K \circ \beta)$. This explicitly means that in the diagram

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & X(V)^K \arrow[r, "X(\psi)^K"] & X(W)^K \arrow[d, "f(W)^K"] \\ D^l \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^K \arrow[r, "Y(\psi)^K"] & Y(W)^K \end{tikzcd}$$

the upper left triangle commutes, and the lower right triangle commutes up to homotopy relative to ∂D^l.

On $\mathcal{S}pc$ there are two similar cofibrantly generated model structures whose weak equivalences are precisely the global equivalences, the global model structure [Sch18, Theorem 1.2.21] and the positive global model structure [Sch18, Theorem 1.2.23]. We would like to lift these model structures to $\mathcal{A}lg(\mathcal{O})$ using Theorem 1.2.2. It will turn out that the positive global model structure is the better choice.

Later on we will need to prove some things about the generating cofibrations and acyclic cofibrations of the positive global model structure on $\mathcal{S}pc$, so we write them out explicitly now and fix some notation.

Remark 1.3.2 (Semifree orthogonal spaces). For a compact Lie group G and a G-representation V, the evaluation functor $\mathcal{S}pc \to \underline{\mathrm{GTop}}$ has a left adjoint $L_{G,V}$ which for a G-space A when evaluated at W is $\underline{\mathrm{L}}(V,W) \times_G A$. We denote the orthogonal space $L_{G,V}*$ by $L_{G,V}$.

Remark 1.3.3 (Generating (acyclic) cofibrations). The set of generating cofibrations of the positive global model structure I is a set of representatives of the isomorphism classes of morphisms $L_{G,V} \times i_l$ for G a compact Lie group, $V \neq 0$ a faithful G-representation, and $l \geq 0$, where the map i_l is $\partial D^l \to D^l$.

The set of generating acyclic cofibrations of the positive global model structure is $J \cup K$, where J is a set of representatives of the isomorphism classes of morphisms $L_{G,V} \times j_l$ for G a compact Lie group, $V \neq 0$ a faithful G-representation, and $l \geq 0$, where the map j_l is $[0,1]^l \to [0,1]^{l+1}$.

To describe the set K, we consider a compact Lie group G, and G-representations V and W, then we have the morphism of representable orthogonal spaces $\rho_{V,W} \colon \underline{\mathrm{L}}(V \oplus W, -) \to \underline{\mathrm{L}}(V,-)$ given by restricting the embeddings to V. Denote the morphism

$$\rho_{V,W}/G \colon \underline{\mathrm{L}}(V \oplus W, -)/G = L_{G,V\oplus W} \to \underline{\mathrm{L}}(V,-)/G = L_{G,V}$$

by $\rho_{G,V,W}$, and by $\iota_{\rho_{G,V,W}} : L_{G,V\oplus W} \to M_{\rho_{G,V,W}}$ the inclusion into the mapping cylinder.

Let κ be a set of representatives of isomorphism classes of triples (G,V,W) consisting of a compact Lie group G, a faithful G-representation $V \neq 0$, and a G-representation W. Then the set K is

$$K = \bigcup_{(G,V,W)\in\kappa} \{\, \iota_{\rho_{G,V,W}} \Box i_l : l \geq 0 \,\}$$

where $\iota_{\rho_{G,V,W}} \Box i_l$ denotes the pushout-product of the map of orthogonal spaces $\iota_{\rho_{G,V,W}}$ and the map of spaces i_l.

The notation I, J and K for the sets of generating (acyclic) cofibrations of the positive global model structure on $\mathcal{S}pc$ will be used throughout this document.

If we remove everywhere in the last remark the requirement that $V \neq 0$ we obtain the generating (acyclic) cofibrations for the global model structure.

We will later need to know that the sources of all generating (acyclic) cofibrations of the (positive) global model structure are finite with respect to a class of maps bigger than the cofibrations.

The h-cofibrations of $\mathcal{S}pc$ are the maps that have the homotopy extension property.

Lemma 1.3.4 (Small sources). *In the positive global model structure on orthogonal spaces the sources of all generating (acyclic) cofibrations are finite with respect to the class of maps that are levelwise closed embeddings, and since h-cofibrations are levelwise closed embeddings, also with respect to the class of h-cofibrations.*

Proof. We first check that for any G compact Lie group, V a faithful G-representation and A a compact space, the orthogonal space $L_{G,V} \times A$ is finite with respect to morphisms which are levelwise closed embeddings.

Colimits with the shape of a filtered poset and built out of closed embeddings of compactly generated weak Hausdorff spaces can be computed in the category of all topological spaces (see [Sch18, Proposition A.14 (ii)]). Weak Hausdorff spaces are T_1, so by [Hov07, Proposition 2.4.2] we have that maps from compact spaces into the colimit of a λ-sequence of closed embeddings (for λ a limit ordinal) factor through some stage $\beta \in \lambda$. Therefore compact spaces are finite in $\underline{\mathrm{Top}}$ relative closed embeddings.

Taking G-fixed points commutes with filtered colimits along G-equivariant maps which are closed embeddings (see [Sch18, Proposition B.1 ii)]). Therefore by the semifreeness property of $L_{G,V} \times A$, and since colimits in $\mathcal{Spc}$ are computed levelwise, we know that for each limit ordinal λ and each λ-sequence $\{X_\beta\}_{\beta\in\lambda}$ of levelwise closed embeddings, we have that

$$\begin{aligned}\mathcal{Spc}(L_{G,V} \times A, \operatorname*{colim}_{\beta\in\lambda} X_\beta) \cong \underline{\mathrm{Top}}(A, (\operatorname*{colim}_{\beta\in\lambda} X_\beta)(V)^G) \cong \underline{\mathrm{Top}}(A, \operatorname*{colim}_{\beta\in\lambda}(X_\beta(V)^G)) \cong \\ \operatorname*{colim}_{\beta\in\lambda} \underline{\mathrm{Top}}(A, (X_\beta(V)^G)) \cong \operatorname*{colim}_{\beta\in\lambda} \mathcal{Spc}(L_{G,V} \times A, X_\beta)\end{aligned}$$

So for a generating cofibration $i \in I$, its source is of the form $L_{G,V} \times \partial D^l$, so it is finite relative levelwise closed embeddings. Similarly the source of a generating acyclic cofibration $j \in J$ is $L_{G,V} \times [0,1]^l$, so it is also finite relative levelwise closed embeddings.

For a generating acyclic cofibration $k \in K$, fix a homeomorphism $\varepsilon\colon D^l \cong (\partial D^l \times [0,1]) \cup_{\partial D^l\times\{1\}} (D^l \times \{1\})$. Then we have the following isomorphisms of orthogonal spaces, where the first one is the source of k:

$$L_{G,V\oplus W} \times D^l \cup_{L_{G,V\oplus W}\times\partial D^l} M_{\rho_{G,V,W}} \times \partial D^l \cong \tag{3}$$

$$L_{G,V\oplus W} \times ((\partial D^l \times [0,1]) \cup_{\partial D^l\times\{1\}} (D^l \times \{1\})) \cup_{L_{G,V\oplus W}\times\partial D^l\times\{0\}} L_{G,V} \times \partial D^l \quad \cong \tag{4}$$

$$L_{G,V\oplus W} \times D^l \cup_{L_{G,V\oplus W}\times\partial D^l} L_{G,V} \times \partial D^l \tag{5}$$

We first use the homeomorphism ε, and then distribute the leftmost $L_{G,V\oplus W}$ into the left pushout and commute the two pushouts.

The orthogonal space (5) is a finite colimit of objects that are finite relative the levelwise closed embeddings. Therefore, using that in $\underline{\mathrm{Set}}$ finite limits commute with limit-ordinal shaped colimits, we can see that the source of k is also finite.

h-cofibrations of orthogonal spaces are levelwise h-cofibrations of spaces, which are closed embeddings on the category of compactly generated weak Hausdorff spaces. Therefore h-cofibrations are levelwise closed embeddings. □

Chapter 2

G-orthogonal spaces

To study the homotopy theory of operads in $\mathcal{S}pc$ we have to consider objects $\mathcal{O}_n$ in $\mathcal{S}pc$ which have a right action by the symmetric group Σ_n. Throughout the rest of this thesis let G be a compact Lie group, and let $\underline{\mathrm{G}}$-$\mathcal{S}pc$ denote the category of $\underline{\mathrm{Top}}$-enriched G-objects in $\mathcal{S}pc$, which we call G-orthogonal spaces.

To check that the conditions of Theorem 1.2.2 hold on $\mathcal{S}pc$ for any operad, we will need to construct three classes of morphisms in $\underline{\mathrm{G}}$-$\mathcal{S}pc$, which we will refer to by G-global equivalences, G-flat cofibrations, and G-h-cofibrations. We will need some facts about these classes, although note that we will not prove that these form the cofibrations and weak equivalences of any model structure.

2.1 G-flat cofibrations

Let G be a compact Lie group. Consider the isomorphism of $\underline{\mathrm{Top}}$-enriched categories:

$$\mathrm{Fun}(\underline{\mathrm{L}} \times \underline{\mathrm{G}}, \underline{\mathrm{Top}}) \cong \mathrm{Fun}(\underline{\mathrm{G}}, \mathrm{Fun}(\underline{\mathrm{L}}, \underline{\mathrm{Top}})) = \underline{\mathrm{G}}\text{-}\mathcal{S}pc$$

We will construct a level model structure on this $\underline{\mathrm{Top}}$-enriched functor category using the results of the appendix C of [Sch18]. In this section we will first give the mentioned model structure, and then state the properties of G-flat cofibrations that will be relevant later, and prove them if necessary.

We have that $\mathscr{D} = \underline{\mathrm{L}} \times \underline{\mathrm{G}}$ is a skeletally small symmetric monoidal $\underline{\mathrm{Top}}$-enriched category. On $\mathscr{D}$ we have a dimension function $|-|$ on the objects given by the dimension of the inner product space of $\underline{\mathrm{L}}$. This function satisfies that if $|V| < |W|$ then $\mathscr{D}(V, W) = \emptyset$ and if $|V| = |W|$ then V and W are isomorphic on $\mathscr{D}$. We fix an object of each dimension, $\mathbb{R}^m$ for each $m \geq 0$. Then by [Sch18, Construction C.13] we obtain a skeleton filtration for each object of $\mathrm{Fun}(\underline{\mathrm{L}} \times \underline{\mathrm{G}}, \underline{\mathrm{Top}})$, similar to the one for $\mathcal{S}pc$.

Let $\mathscr{D}_{\leq m} \subset \mathscr{D}$ be the full subcategory of inner product spaces of dimension less than or equal to m. Let l_m denote the left adjoint to the restriction functor for $\mathcal{S}pc$. Similarly let $l_m^{\mathscr{D}} \colon \mathrm{Fun}(\mathscr{D}_{\leq m}, \underline{\mathrm{Top}}) \to \mathrm{Fun}(\mathscr{D}, \underline{\mathrm{Top}})$ denote the left adjoint to the restriction in the case of G-orthogonal spaces.

Lemma 2.1.1. *For each $Z\colon \mathscr{D}_{\leq m} \to \underline{\mathrm{Top}}$, $l_m^{\mathscr{D}}(Z)$ is naturally isomorphic as a G-orthogonal space to $l_m(Z)$ with the inherited G-action.*

Proof. Colimits in $\mathrm{Fun}(\mathscr{D}, \underline{\mathrm{Top}})$ can be computed in $\underline{\mathrm{Top}}$. For a functor $Z\colon \mathscr{D}_{\leq m} \to \underline{\mathrm{Top}}$ and an inner product space V, the G-orthogonal space $l_m^{\mathscr{D}}(Z)(V)$ is given by the coequalizer

$$\coprod_{0\leq j\leq k\leq m} \underline{\mathrm{L}}(\mathbb{R}^k, V) \times G \times \underline{\mathrm{L}}(\mathbb{R}^j, \mathbb{R}^k) \times G \times Z(\mathbb{R}^j) \rightrightarrows \coprod_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V) \times G \times Z(\mathbb{R}^i)$$

where the first arrow is given by $(\phi, g, \psi, g', z) \mapsto (\phi\circ\psi, gg', z)$ and the second one is given by $(\phi, g, \psi, g', z) \mapsto (\phi, g, Z(\psi, g')(z))$.

We have the following diagram, where each row and column is a coequalizer diagram. The vertical columns are just the split coequalizer diagrams for $X \times_G G \cong X$ for X a G-space. The bottom row is precisely the coequalizer diagram that describes $l_m(Z)(V)$.

$$\begin{tikzcd}
\coprod\limits_{0\leq j\leq k\leq m} \underline{\mathrm{L}}(\mathbb{R}^k, V) \times \underline{\mathrm{L}}(\mathbb{R}^j, \mathbb{R}^k) \times G \times G \times Z(\mathbb{R}^j) \arrow[r, shift left, "\partial_0^L"] \arrow[r, shift right, "\partial_1^L"'] \arrow[d, shift right, "\partial_1^G"'] \arrow[d, shift left, "\partial_0^G"] \arrow[dr, dotted, shift left, "\partial_0"] \arrow[dr, dotted, shift right, "\partial_1"'] & \coprod\limits_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V) \times G \times G \times Z(\mathbb{R}^i) \arrow[r, "b"] \arrow[d, shift right, "\partial_1^G"'] \arrow[d, shift left, "\partial_0^G"] & l_m(Z)(V) \times G \times G \arrow[d, shift right, "\partial_1^G"'] \arrow[d, shift left, "\partial_0^G"] \\
\coprod\limits_{0\leq j\leq k\leq m} \underline{\mathrm{L}}(\mathbb{R}^k, V) \times \underline{\mathrm{L}}(\mathbb{R}^j, \mathbb{R}^k) \times G \times Z(\mathbb{R}^j) \arrow[u, bend left, "e"] \arrow[r, shift left, "\partial_0^L"] \arrow[r, shift right, "\partial_1^L"'] \arrow[d, "a"] & \coprod\limits_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V) \times G \times Z(\mathbb{R}^i) \arrow[r, "b"] \arrow[d, "a"] \arrow[dr, dotted] & l_m(Z)(V) \times G \arrow[d, "a"] \\
\coprod\limits_{0\leq j\leq k\leq m} \underline{\mathrm{L}}(\mathbb{R}^k, V) \times \underline{\mathrm{L}}(\mathbb{R}^j, \mathbb{R}^k) \times Z(\mathbb{R}^j) \arrow[r, shift left, "\partial_0^L"] \arrow[r, shift right, "\partial_1^L"'] & \coprod\limits_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V) \times Z(\mathbb{R}^i) \arrow[r, "b"] & l_m(Z)(V)
\end{tikzcd}$$

The ∂_0^L arrows are composition in $\underline{\mathrm{L}}$, the ∂_0^G arrows are multiplication in G, and the ∂_1^L and ∂_1^G arrows are the action of either $\underline{\mathrm{L}}$ or G respectively. The diagram commutes in the sense that if we remove all the arrows labeled with ∂_1 it commutes, the same holds for ∂_0. The arrow e inserts the unit of G in the second copy of G, so that $\partial_0^G \circ e = id$ and $\partial_1^G \circ e = id$.

We are interested in checking that the diagonal diagram is also a coequalizer diagram. Given a map

$$f\colon \coprod_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V) \times G \times Z(\mathbb{R}^i) \to X$$

such that $f\circ\partial_0 = f\circ\partial_1$, we have that

$$f\circ\partial_0^L = f\circ\partial_0^L\circ\partial_0^G\circ e = f\circ\partial_1^L\circ\partial_1^G\circ e = f\circ\partial_1^L$$

Thus

$$f\circ\partial_0^G\circ\partial_0^L = f\circ\partial_1^G\circ\partial_1^L = f\circ\partial_1^L\circ\partial_1^G = f\circ\partial_0^L\circ\partial_0^G = f\circ\partial_1^G\circ\partial_0^L$$

and since ∂_0^L is surjective, there is an unique $f'\colon \coprod_{0\leq i\leq m} \underline{\mathrm{L}}(\mathbb{R}^i, V)\times Z(\mathbb{R}^i) \to X$ such that $f'\circ a = f$.

Then

$$f'\circ\partial_0^L\circ a = f'\circ a\circ\partial_0^L = f'\circ a\circ\partial_1^L = f'\circ\partial_1^L\circ a$$

so $f'\circ\partial_0^L = f'\circ\partial_1^L$ and there is an unique $f''\colon l_m(Z)(V)\to X$ such that $f''\circ b = f'$. Therefore the diagonal diagram is also a coequalizer diagram. $\square$

The m-skeleton of X a G-orthogonal space is then $\mathrm{sk}^m_{\mathscr{D}} X = l^{\mathscr{D}}_m(X^{\leq m})$. The mth latching object of X is $\mathcal{L}^{\mathscr{D}}_m(X) = \mathrm{sk}^{\mathscr{D}}_{m-1} X(\mathbb{R}^m)$. By the previous result, these are exactly the skeletons and latching objects of X as a plain orthogonal space, with the induced G-action.

Definition 2.1.2 (G-flat cofibrations)**.** A morphism $f\colon X \to Y$ of $\underline{\mathrm{G}}$-$\mathcal{S}pc$ is a G-flat cofibration if the latching morphisms $\nu_m f\colon X(\mathbb{R}^m) \cup_{\mathcal{L}_m(X)} \mathcal{L}_m(Y) \to Y(\mathbb{R}^m)$ are $(O(m) \times G)$-cofibrations for each $m \geq 0$.

We say that $X \in \underline{\mathrm{G}}$-$\mathcal{S}pc$ is G-flat if $* \to X$ is a G-flat cofibration.

Next, as input to obtain the G-level model structure we fix the usual model structures for each $m \geq 0$ on the categories of spaces with an action by $\mathscr{D}(\mathbb{R}^m, \mathbb{R}^m) = O(m) \times G$. We need to check the consistency condition of [Sch18, Definition C.22]. Let $m, n \geq 0$ and let i be an $O(m) \times G$-acyclic cofibration. Then we need to check that $(\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G) \times_{O(m)\times G} i$ is a weak equivalence in the model structure of $(O(m+n) \times G)$-spaces.

Lemma 2.1.3 (Consistency condition)**.** *For each $m, n \geq 0$ and each acyclic cofibration i in $\underline{(\mathrm{O(m)} \times \mathrm{G})\mathrm{Top}}$, the morphism $(\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G) \times_{O(m)\times G} i$ is an $O(m+n) \times G$-acyclic cofibration.*

Proof. The functor $(\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G) \times_{O(m)\times G} -$ is a left adjoint to the functor given by $\mathrm{Map}(\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G, -)^{O(m+n)\times G}$. Therefore we only need to check that it sends the generating acyclic cofibrations to acyclic cofibrations.

The generating acyclic cofibrations of $\underline{(\mathrm{O(m)} \times \mathrm{G})\mathrm{Top}}$ are of the form $((O(m) \times G)/H) \times j_l$, for a closed subgroup $H \leq O(m) \times G$ and $l \geq 0$. Then the functor takes this generating acyclic cofibration to $((\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/H) \times j_l$.

Then we consider $\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ as an $(O(m+n) \times G \times O(m) \times G)$-space, where the component $O(m+n) \times G$ acts on the left, and $O(m) \times G$ originally acts on the right so we precompose with $(-)^{-1}$ to obtain a left action. $\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ is homeomorphic to a Stiefel manifold, and the action is smooth. Illman's theorem [Ill83, p. 7.2] provides a $(O(m+n) \times G \times O(m) \times G)$-CW-structure, so $\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G$ is cofibrant.

Then by [Sch18, B.14 (i)] and [Sch18, B.14 (iii)], $(\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/H$ is $(O(m+n) \times G)$-cofibrant, and so $((\underline{\mathrm{L}}(\mathbb{R}^m, \mathbb{R}^{m+n}) \times G)/H) \times j_l$ is an acyclic $(O(m+n) \times G)$-cofibration. □

Since the consistency condition is satisfied, we obtain a level model structure on $\underline{\mathrm{G}}$-$\mathcal{S}pc$.

Theorem 2.1.4 (G-level model structure)**.** *There is a topological cofibrantly generated model structure on the category $\underline{\mathrm{G}}$-$\mathcal{S}pc$ of orthogonal spaces with an action by the compact Lie group G, which we call the G-level model structure. The cofibrations are the G-flat cofibrations. The weak equivalences (respectively the fibrations) are those morphisms f such that for each $m \geq 0$ and each closed subgroup $H \leq O(m) \times G$, the map $f(\mathbb{R}^m)^H$ is a weak homotopy equivalence (respectively a Serre fibration).*

Proof. Such a model structure with the G-flat cofibrations as cofibrations exists by [Sch18, Proposition C.23 (i)]. It is cofibrantly generated by [Sch18, Proposition C.23 (iii)] because each of the model structures on $\underline{(\mathrm{O(m)} \times \mathrm{G})\mathrm{Top}}$ is cofibrantly generated.

The functor $\underline{\mathrm{G}}\text{-}\mathcal{S}pc \to \underline{(\mathrm{O(m)} \times \mathrm{G})\mathrm{Top}}$ given by evaluation at $\mathbb{R}^m$ has a left adjoint, which we denote by F_m, and it is given by $A \mapsto (\underline{\mathrm{L}}(\mathbb{R}^m, -) \times G) \times_{O(m)\times G} A$. The generating cofibrations obtained from [Sch18, Proposition C.23 (iii)] are those of the form $F_m(i)$ where i is a generating cofibration of $\underline{(\mathrm{O(m)} \times \mathrm{G})\mathrm{Top}}$, which are of the form $((O(m) \times G)/H) \times i_l$ for a closed subgroup $H \leq O(m) \times G$ and $l \geq 0$.

Therefore these generating cofibrations have the form $((\underline{\mathrm{L}}(\mathbb{R}^m, -) \times G)/H) \times i_l$ for a closed subgroup $H \leq O(m) \times G$ and $l \geq 0$. We denote by I_G this set of generating cofibrations. Similarly the set of generating acyclic cofibrations of the G-level model structure is $\{ ((\underline{\mathrm{L}}(\mathbb{R}^m, -) \times G)/H) \times j_l : m, l \geq 0, H \leq O(m) \times G \}$.

Each G-orthogonal space of the form $(\underline{\mathrm{L}}(\mathbb{R}^m, -) \times G)/H$ is G-flat because $F_m(((O(m) \times G)/H) \times i_0)$ is a generating cofibration. Then by [Sch18, Proposition B.5] with $\mathcal{G} = \{ (\underline{\mathrm{L}}(\mathbb{R}^m, -) \times G)/H : m \geq 0, H \leq O(m) \times G \}$ and $\mathcal{Z} = \emptyset$ we have that this model structure is topological. □

Corollary 2.1.4.1. *The class of G-flat cofibrations is closed under coproducts, transfinite composition, cobase change and retracts.*

Remark 2.1.5. In $\mathcal{S}pc$ the box product is constructed as a Day convolution product on $\mathrm{Fun}(\underline{\mathrm{L}}, \underline{\mathrm{Top}})$. On $\underline{\mathrm{L}} \times \underline{\mathrm{G}}$ however there is no clear analogous monoidal structure, so we cannot construct a Day convolution product on $\mathrm{Fun}(\underline{\mathrm{L}} \times \underline{\mathrm{G}}, \underline{\mathrm{Top}})$. However the symmetric monoidal structure on $\mathcal{S}pc$ gives a pointwise symmetric monoidal structure on $\mathrm{Fun}(\underline{\mathrm{G}}, \mathrm{Fun}(\underline{\mathrm{L}}, \underline{\mathrm{Top}}))$, the pointwise box product. This symmetric monoidal structure is additionally closed because the box product on orthogonal spaces is closed.

In this way we can also define the box product of a G-orthogonal space and a K-orthogonal space as the $(G \times K)$-orthogonal space given by the pointwise box product.

Lemma 2.1.6. *For a continuous homomorphism between compact Lie groups $\alpha\colon K \to G$ and a G-flat cofibration $f\colon X \to Y$, then $\alpha^*(f)$ the restriction along α of f is a K-flat cofibration*

Proof. By Lemma 2.1.1 $\alpha^*(\mathrm{sk}_m(X))$ and $\mathrm{sk}_m(\alpha^*(X))$ are naturally isomorphic as K-orthogonal spaces. Therefore for the latching morphism $\nu_m(f)$ we have that $\nu_m(\alpha^*(f)) = \alpha^*(\nu_m(f))$, and by [Sch18, Proposition B.14 (i)] $\nu_m(\alpha^*(f))$ is an $(O(m) \times K)$-cofibration. □

Remark 2.1.7. The G-orthogonal spaces $\mathscr{D}(V, -)/H = (\underline{\mathrm{L}}(V, -) \times G)/H$, which we will denote $L_{H,V;G}$, for an inner product space V and a closed subgroup $H \leq O(V) \times G$, are special. They have a certain "freeness" condition, namely they are the representing objects for the functors $(-)(V)^H$ given by evaluating at V and then taking H-fixed points of the $(O(V) \times G)$-action. We will sometimes refer to them as the semifree G-orthogonal spaces, since they have the same property as the semifree orthogonal spaces $L_{H,V}$.

Explicitly the natural isomorphism of the functors $\underline{\mathrm{G}}\text{-}\mathcal{S}pc(L_{H,V;G}, -)$, $(-)(V)^H : \underline{\mathrm{G}}\text{-}\mathcal{S}pc \to \underline{\mathrm{Top}}$ is given by, in one direction $f \mapsto f(V)([id_V, e])$. For the other direction, given a point $y_0 \in Y(V)^H$, there is a morphism of G-orthogonal spaces f given by:

$$\begin{aligned} (\underline{\mathrm{L}}(V, W) \times G)/H \to & Y(W) \\ [\psi, g] \mapsto & Y(\psi)(gy_0) \end{aligned}$$

Analogously to the case of the semifree orthogonal spaces, the box product of a semifree G-orthogonal space and a semifree K-orthogonal space has a nice structure, as a $(G\times K)$-orthogonal space it is isomorphic to a semifree $(G\times K)$-orthogonal space. Note however that the box product of two G-orthogonal spaces with the G-action given by restriction along the diagonal will not be a semifree G-orthogonal space in general.

Proposition 2.1.8. *For compact Lie groups G and K, inner product spaces V, V' and closed subgroups $H \leq O(V) \times G$ and $H' \leq O(V') \times K$ we have that $L_{H,V;G} \boxtimes L_{H',V';K}$ is isomorphic as a $(G \times K)$-orthogonal space to $L_{H\times H',V\oplus V';G\times K}$.*

Proof. We will define inverse $(G \times K)$-equivariant morphisms using on one hand the universal property of the box product, and on the other hand the freeness of $L_{H\times H',V\oplus V';G\times K}$, just like the case for G, K trivial of [Sch18, Example 1.3.3].

The universal bimorphism evaluated at V, V' $i_{V,V'}\colon (\underline{\mathrm{L}}(V,V) \times G)/H \times (\underline{\mathrm{L}}(V',V') \times K)/H' \to (L_{H,V;G} \boxtimes L_{H',V';K})(V \oplus V')$ is $(O(V) \times G \times O(V') \times K)$-equivariant when considering the left action. It is $(O(V) \times O(V'))$-equivariant because i is a bimorphism of orthogonal spaces, and $(G \times K)$-equivariant because we set the box product of G-orthogonal spaces as the pointwise box product. Then since the point $[id_V, e_G] \times [id_{V'}, e_K]$ is fixed by $H \times H'$ it gives a point on $(L_{H,V;G}\boxtimes L_{H',V';K})(V\oplus V')^{H\times H'}$, and by freeness we obtain a morphism $\alpha\colon L_{H\times H',V\oplus V';G\times K} \to L_{H,V;G} \boxtimes L_{H',V';K}$.

The inverse is obtained from the bimorphism given by, for each W, W' inner product spaces:

$$\begin{aligned}(\underline{\mathrm{L}}(V,W) \times G)/H \times (\underline{\mathrm{L}}(V',W') \times K)/H' \to& (\underline{\mathrm{L}}(V \oplus V', W \oplus W') \times G \times K)/(H \times H')\\ [\psi, g] \times [\psi', k] \mapsto& [\psi \oplus \psi', g, k]\end{aligned}$$

These $(O(W) \times G \times O(W') \times K)$-equivariant maps induce a morphism of $(G \times K)$-orthogonal spaces $\beta\colon L_{H,V;G} \boxtimes L_{H',V';K} \to L_{H\times H',V\oplus V';G\times K}$.

For each W and W', if we precompose $\alpha \circ \beta(W \oplus W')$ with $i_{W,W'}$, we obtain a map that sends $[\psi, g] \times [\psi', k]$ to $(L_{H,V;G} \boxtimes L_{H',V';K})(\psi \oplus \psi')((g, k)i_{W,W'}([id_V, e_G] \times [id_{V'}, e_K]))$, which is just $i_{W,W'}([\psi, g] \times [\psi', k])$ since $i_{W,W'}$ is $(G \times K)$-equivariant and i is a bimorphism. Therefore $\alpha \circ \beta(W \oplus W') \circ i_{W,W'} = i_{W,W'}$ and so $\alpha \circ \beta$ is the identity.

$\beta \circ \alpha$ is a morphism out of a semifree $(G \times K)$-orthogonal space, associated to the point $\beta \circ \alpha(V \oplus V')([id_{V\oplus V'}, e_{G\times K}])$ in $L_{H\times H',V\oplus V';G\times K}(V \oplus V')^{H\times H'}$. Then we have:

$$\begin{aligned}\beta \circ \alpha(V \oplus V')([id_{V\oplus V'}, e_{G\times K}]) =&\\ \beta(V \oplus V')((L_{H,V;G} \boxtimes L_{H',V';K})(id_{V\oplus V'})(e_{G\times K} i_{V,V'}([id_V, e_G] \times [id_{V'}, e_K]))) =&\\ \beta(V \oplus V')(i_{V,V'}([id_V, e_G] \times [id_{V'}, e_K])) = [id_V \oplus id_{V'}, e_G \times e_K] = [id_{V\oplus V'}, e_{G\times K}]&\end{aligned}$$

The penultimate equality is due to $\beta \circ i_{V,V'}$ being precisely the bimorphism used to construct β. Then $\beta \circ \alpha(W)$ sends $[\psi, (g, k)] \in (\underline{\mathrm{L}}(V \oplus V', W) \times G \times K)/(H \times H')$ to $L_{H\times H',V\oplus V';G\times G}(\psi)((g, k)[id_{V\oplus V'}, e_{G\times K}]) = [\psi, (g, k)]$ and therefore $\beta \circ \alpha$ is the identity. □

Proposition 2.1.9. *The pushout product of a G-flat cofibration and a K-flat cofibration is a $(G \times K)$-flat cofibration.*

Proof. Given a generating G-flat cofibration $f = L_{H,\mathbb{R}^m;G} \times i_l$ and a generating K-flat cofibration $g = L_{H',\mathbb{R}^n;K} \times i_k$, then their pushout product is by Proposition 2.1.8 $(G \times K)$-isomorphic to $L_{H\times H',\mathbb{R}^{m+n};G\times K} \times (i_l \square i_k)$. Additionally $i_l \square i_k$ is homeomorphic to i_{l+k}, and so $f \square g$ is a generating $(G \times K)$-flat cofibration.

Therefore since the box product of orthogonal spaces is closed, [Hov07, Lemma 4.2.4] implies that the pushout product of a G-flat cofibration and a K-flat cofibration is a $G\times K$-flat cofibration. □

Corollary 2.1.9.1. *The pushout product of two G-flat cofibrations is a G-flat cofibration.*

Proof. By the previous proposition, it is a $(G \times G)$-flat cofibration, and by Lemma 2.1.6 it is a G-flat cofibration. □

Corollary 2.1.9.2. *The pushout product with respect to $\times$ of a G-flat cofibration and a G-cofibration of G-spaces is a G-flat cofibration.*

Proof. Let $f = G/H \times i_{l'}$ be a generating G-cofibration, then as a morphism of constant orthogonal spaces it is a generating G-flat cofibration with $V = 0$. Therefore a G-cofibration of G-spaces is a G-flat cofibration, and by the previous corollary we have this corollary. □

Since the category $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ is tensored with $\underline{\mathrm{Top}}$, we can define a homotopy of morphisms of G-orthogonal spaces f and g to be a morphism $H\colon X \times [0,1] \to Y$ in $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ such that $H(-,0) = f$ and $H(-,1) = g$. Thus we can also define what a G-homotopy equivalence of orthogonal spaces is.

We can also consider the class of h-cofibrations, the morphisms which have the homotopy extension property. The map $f\colon X \to Y$ has the homotopy extension property if and only if there is a retraction in $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ for the induced morphism $X \times [0,1] \cup_X Y \to Y \times [0,1]$. In $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ all objects are fibrant so by [Sch18, Corollary A.30 (iii)] each G-flat cofibration is an h-cofibration of $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$. We will refer to the h-cofibrations in $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ as G-h-cofibrations.

Remark 2.1.10. On $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ the G-h-cofibrations can be equivalently defined as those morphisms that have the left lifting property with respect to $\mathrm{ev}_0\colon X^{[0,1]} \to X$ for all $X \in \underline{\mathrm{G}}\text{-}\mathcal{S}pc$. This implies that the class of G-h-cofibrations is closed under coproducts, transfinite composition, cobase change and retracts.

Lemma 2.1.11. *Let G be a compact Lie group and $H \leq G$ a closed normal subgroup. For a G-flat cofibration (respectively a G-h-cofibration) of orthogonal spaces $f\colon X \to Y$, the morphism $f/H\colon X/H \to Y/H$ is a (G/H)-flat cofibration (respectively a (G/H)-h-cofibration) of orthogonal spaces.*

Proof. $\mathrm{sk}^m(X)$ is a left Kan extension along $\underline{\mathrm{L}}_{\leq m} \subset \underline{\mathrm{L}}$ of $X^{\leq n}$, so it preserves colimits and since $X^{\leq m}/H = (X/H)^{\leq m}$, we have that $\mathrm{sk}_m X/H \cong \mathrm{sk}_m(X/H)$ are naturally and G/H-equivariantly isomorphic.

Then under this isomorphism the latching map of f/H corresponds to taking H orbits of the latching map of f, and since the latter is an $(O(m) \times G)$-cofibration, the latching map of f/H is an $(O(m) \times G/H)$-cofibration by [Sch18, Proposition B.14 (iii)], and so f/H is a (G/H)-flat cofibration.

Suppose that we have a retraction in $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ $r\colon Y \times [0,1] \to X \times [0,1] \cup_X Y$. Then r/H is a retraction $Y/H \times [0,1] \to X/H \times [0,1] \cup_{X/H} Y/H$, and so f/H is an G/H-h-cofibration. □

2.2 G-global equivalences

Given two compact Lie groups G and K, a space X with a $(K \times G)$-action, a closed subgroup $L \leq K$, and a continuous homomorphism $\phi\colon L \to G$, we denote by X^ϕ the space of points of X fixed by the graph subgroup $\phi = \{ (k, \phi(k)) : k \in L \} \leq K \times G$. We will also refer to the L action on X obtained by restricting the $(K \times G)$-action through $(L \leq K, \phi)$ as the twisted L-action.

We denote the set of graph subgroups of $K \times G$, for continuous homomorphisms $\phi\colon L \to G$ with $L \leq K$ a closed subgroup, by $\mathscr{F}(K, G)$. These graph subgroups are precisely the closed subgroups $\Gamma \leq K \times G$ such that $\Gamma \cap \{e_K\} \times G = \{e_{K \times G}\}$.

Definition 2.2.1 (G-global equivalence)**.** For a compact Lie group G, a morphism f of $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ is a G-global equivalence if for each compact lie group$K$, continuous homomorphism $\phi\colon K \to G$, orthogonal K-representation V and $l \geq 0$, the following holds: For any lifting problem

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & X(V)^\phi \arrow[d, "f(V)^\phi"] \\ D^l \arrow[r, "\beta"] & Y(V)^\phi \end{tikzcd}$$

there is a K-equivariant linear isometric embedding $\psi\colon V \to W$ into a K-representation W such that there is a morphism $\lambda\colon D^l \to X(W)^\phi$ which solves the lifting problem $(X(\psi)^\phi \circ \alpha, Y(\psi)^\phi \circ \beta)$. This explicitly means that in the diagram

$$\begin{tikzcd} \partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & X(V)^\phi \arrow[r, "X(\psi)^\phi"] & X(W)^\phi \arrow[d, "f(W)^\phi"] \\ D^l \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V)^\phi \arrow[r, "Y(\psi)^\phi"] & Y(W)^\phi \end{tikzcd}$$

the upper left triangle commutes, and the lower right triangle commutes up to homotopy relative to ∂D^l.

The previous definition is the most concrete one (Note the similarity to the definition of global equivalence Definition 1.3.1). However, morally the definition of a G-global equivalence, just as in the case of global equivalences, is meant to capture that for each compact lie group K, the map induced between the homotopy colimits over all K-representations is an $\mathscr{F}(K, G)$-weak homotopy equivalence. The following lemma analogous to [Sch18, Proposition 1.1.7] makes this explicit.

For a compact Lie group K, we say that a nested sequence of K-representations

$$V_0 \subset V_1 \subset \cdots \subset V_n \subset \ldots$$

is exhaustive if each K-representation isometrically embeds into some V_n.

Lemma 2.2.2. *A morphism $f\colon X \to Y$ in G-$\mathcal{S}pc$ is a G-global equivalence if and only if for each compact Lie group K and each exhaustive sequence of K-representations $\{V_i\}_{i\in\mathbb{N}}$, the map induced on the mapping telescopes of the sequences of $(K\times G)$-spaces and $(K\times G)$-equivariant maps $X(V_i)$ and $Y(V_i)$*

$$\mathrm{tel}_i\, f(V_i)\colon\ \mathrm{tel}_i\, X(V_i) \to \mathrm{tel}_i\, Y(V_i)$$

is an $\mathscr{F}(K,G)$-weak homotopy equivalence of $K\times G$-spaces.

Proof. First we assume that for each compact Lie group K and each exhaustive sequence of orthogonal K-representations, the map induced on the mapping telescopes is an $\mathscr{F}(K,G)$-weak homotopy equivalence of $(K\times G)$-spaces. Any compact Lie group K has an exhaustive sequence of representations $\{V_i\}_{i\in\mathbb{N}}$, so for any K-representation V, continuous homomorphism $\phi\colon K\to G$ and lifting problem (α,β), since $\{V_i\}_{i\in\mathbb{N}}$ is exhaustive, we can embed V into some V_n, and so we assume that $V = V_n$.

Denote by $c_{X,n}$ the $(K\times G)$-equivariant canonical map $X(V_n)\to \mathrm{tel}_i\, X(V_i)$. Also let $\mathrm{tel}_{[0,n]}\, X(V_i)$ denote the truncated mapping telescope, $\pi_{X,n}$ the $(K\times G)$-equivariant canonical projection $\mathrm{tel}_{[0,n]}\, X(V_i)\to X(V_n)$, and by abuse of notation let $c_{X,n}$ also denote the canonical map $X(V_n)\to \mathrm{tel}_{[0,n]}\, X(V_i)$. For $n\le m$, let $c_{X,n,m}$ denote the inclusion of truncated mapping telescopes $\mathrm{tel}_{[0,n]}\, X(V_i)\to \mathrm{tel}_{[0,m]}\, X(V_i)$, and $c_{X,n,\infty}$ the inclusion $\mathrm{tel}_{[0,n]}\, X(V_i)\to \mathrm{tel}_i\, X(V_i)$.

Taking fixed points commutes with the construction of the mapping telescopes, so $(\mathrm{tel}_i\, X(V_i))^\phi \cong \mathrm{tel}_i\, X(V_i)^\phi$. Since $\mathrm{tel}_i\, f(V_i)^\phi$ is a weak homotopy equivalence, by [May99, 9.6 Lemma] there is a solution λ to the lifting problem $(c^\phi_{X,n}\circ\alpha, c^\phi_{Y,n}\circ\beta)$.

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, "i_l"'] & X(V_n)^\phi \arrow[r, "c^\phi_{X,n}"] & \mathrm{tel}_i\, X(V_i)^\phi \arrow[d, "\mathrm{tel}_i\, f(V_i)^\phi"] \\
D^l \arrow[r, "\beta"'] \arrow[urr, dashed, "\lambda"] & Y(V_n)^\phi \arrow[r, "c^\phi_{Y,n}"'] & \mathrm{tel}_i\, Y(V_i)^\phi
\end{tikzcd}$$

Both λ and the relative homotopy H that witnesses that $c^\phi_{Y,n}\circ\beta$ and $\mathrm{tel}_i\, f(V_i)^\phi\circ\lambda$ are homotopic have compact domains, and since the mapping telescopes are colimits along the closed inclusions $c^\phi_{X,n,m}$, both λ and H factor through some stage $m\ge n$ with $\psi\colon V_n\to V_m$, $\lambda'\colon D^l\to \mathrm{tel}_{[0,m]}\, X(V_i)^\phi$ and $H'\colon D^l\times[0,1]\to \mathrm{tel}_{[0,m]}\, Y(V_i)^\phi$. Then $\pi^\phi_{X,m}\circ\lambda'$ and $\pi^\phi_{X,m}\circ H'$ give a solution to the lifting problem $(X(\psi)^\phi\circ\alpha, X(\psi)^\phi\circ\beta)$, so that f is a G-global equivalence.

Now assume that f is a G-global equivalence. Fix a compact Lie group K, a closed subgroup $L\le K$, a continuous homomorphism $\phi\colon L\to G$, and a exhaustive sequence of K-representations $\{V_i\}_{i\in\mathbb{N}}$. We have to check that $\mathrm{tel}_i\, f(V_i)^\phi$ is a weak homotopy equivalence.

For a lifting problem (α,β) for $\mathrm{tel}_i\, f(V_i)^\phi$, since ∂D^l and D^l are compact, α and β factor through some stage n, as $\alpha'\colon \partial D^l\to \mathrm{tel}_{[0,n]}\, X(V_i)^\phi$ and $\beta'\colon D^l\to \mathrm{tel}_{[0,n]}\, Y(V_i)^\phi$.

For each n, there is a homotopy from the identity on $\mathrm{tel}_{[0,n]}\, X(V_i)$ to $c_{X,n}\circ\pi_{X,n}$, which is $(K\times G)$-equivariant and natural on X. By [Sch18, Lemma 1.1.5] this means that there is a solution of the lifting problem (α', β') if there is a solution to the lifting problem $(c^{\phi}_{X,n} \circ \pi^{\phi}_{X,n} \circ \alpha', c^{\phi}_{Y,n} \circ \pi^{\phi}_{Y,n} \circ \beta')$.

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha'"] \arrow[d, "i_l"] & \mathrm{tel}_{[0,n]} X(V_i)^{\phi} \arrow[r, "\pi^{\phi}_{X,n}"] \arrow[d, "\mathrm{tel}_{[0,n]} f(V_i)^{\phi}"] & X(V_n)^{\phi} \arrow[r, "c^{\phi}_{X,m}"] \arrow[d, "f(V_m)^{\phi}"] & \mathrm{tel}_{[0,n]} X(V_i)^{\phi} \arrow[d, "\mathrm{tel}_{[0,n]} f(V_i)^{\phi}"] \\
D^l \arrow[r, "\beta'"'] & \mathrm{tel}_{[0,n]} Y(V_i)^{\phi} \arrow[r, "\pi^{\phi}_{Y,n}"'] & Y(V_n)^{\phi} \arrow[r, "c^{\phi}_{Y,m}"'] & \mathrm{tel}_{[0,n]} Y(V_i)^{\phi}
\end{tikzcd}$$

This lifting problem, after evaluating at some larger $m \geq n$ with embedding $\psi\colon V_n \to V_m$, has as solution $c^{\phi}_{X,n} \circ \lambda$, where λ is a solution to the lifting problem $(X(\psi)^{\phi} \circ \pi^{\phi}_{X,n} \circ \alpha', Y(\psi)^{\phi} \circ \pi^{\phi}_{Y,n} \circ \beta')$. This λ exists because f is a G-global equivalence and the sequence of underlying L-representations $\{V_i\}_{i\in\mathbb{N}}$ is an exhaustive sequence of L-representations by [BD85, III Theorem 4.5].

Explicitly, $c^{\phi}_{X,n} \circ \lambda$ is a solution of the lifting problem $(c^{\phi}_{X,n} \circ X(\psi)^{\phi} \circ \pi^{\phi}_{X,n} \circ \alpha', c^{\phi}_{Y,n} \circ Y(\psi)^{\phi} \circ \pi^{\phi}_{Y,n} \circ \beta')$, and since $\pi^{\phi}_{X,m} \circ c^{\phi}_{X,n,m} = X(\psi)^{\phi} \circ \pi^{\phi}_{X,n}$, it is also a solution of $(c^{\phi}_{X,n} \circ \pi^{\phi}_{X,m} \circ c^{\phi}_{X,n,m} \circ \alpha', c^{\phi}_{Y,n} \circ \pi^{\phi}_{Y,m} \circ c^{\phi}_{Y,n,m} \circ \beta')$, and so by the previously mentioned homotopy, $(c^{\phi}_{X,n,m} \circ \alpha', c^{\phi}_{Y,n,m} \circ \beta')$ has a solution λ'.

Note that we didn't obtain a solution to (α', β'), but since $c^{\phi}_{X,m,\infty} \circ c^{\phi}_{X,n,m} = c^{\phi}_{X,n,\infty}$, the map $c^{\phi}_{X,m,\infty} \circ \lambda'$ is a solution of the original lifting problem $(\alpha, \beta) = (c^{\phi}_{X,m,\infty} \circ c^{\phi}_{X,n,m} \circ \alpha', c^{\phi}_{Y,m,\infty} \circ c^{\phi}_{Y,n,m} \circ \beta')$. $\square$

An orthogonal space X is said to be closed if for each linear isometric embedding ψ we have that $X(\psi)$ is a closed embedding. For closed orthogonal spaces, there is a simpler characterization of G-global equivalences. For each compact Lie group K we fix a complete K universe $\mathcal{U}_K$, and denote by $s(\mathcal{U}_K)$ the poset of finite dimensional subrepresentations of $\mathcal{U}_K$.

For each G-orthogonal space X we can consider the $(K\times G)$-space $X(\mathcal{U}_K) = \mathrm{colim}_{V\in s(\mathcal{U}_K)} X(V)$. We will usually refer to this as the underlying $(K\times G)$-space of the G-orthogonal space X. Then we have the analogue of [Sch18, Proposition 1.1.17] for G-orthogonal spaces:

Lemma 2.2.3. *A morphism $f\colon X \to Y$ in G-$\mathcal{S}pc$ between closed orthogonal spaces is a G-global equivalence if and only if for each K compact Lie group the map induced on underlying $(K\times G)$-spaces*

$$f(\mathcal{U}_K)\colon X(\mathcal{U}_K) \to Y(\mathcal{U}_K)$$

is an $\mathscr{F}(K,G)$-weak homotopy equivalence of $(K\times G)$-spaces.

Proof. The colimit $\mathrm{colim}_{V\in s(\mathcal{U}_K)} X(V)$ can be written as a sequential colimit $\mathrm{colim}_{i\in\mathbb{N}} X(V_i)$ for a nested sequence of finite dimensional subrepresentations $\{V_i\}_{i\in\mathbb{N}}$ of $\mathcal{U}_K$ which cover all of $\mathcal{U}_K$. This is a colimit of $(K\times G)$-spaces along closed embeddings because X and Y are closed.

Then for each $\phi \in \mathscr{F}(K,G)$ taking ϕ-fixed points commutes with this colimit along closed embeddings.

Since additionally ∂D^l and D^l are compact, a lifting problem for $(\mathrm{colim}_{i\in\mathbb{N}} f(V_i))^{\phi}$ factors through some stage n of the sequential colimit. Then it can be seen that if f is a G-global equivalence the map $(\mathrm{colim}_{i\in\mathbb{N}} f(V_i))^{\phi}$ is a weak homotopy equivalence. We are using that if $L \leq K$ is a closed subgroup, the underlying L-representation of $\mathcal{U}_K$ is a complete L-universe.

If $f(\mathcal{U}_K)^\phi$ is a weak homotopy equivalence for each such K and ϕ, then any K representation V embeds into some V_i by $\psi\colon V \to V_i$ and therefore for any lifting problem for $f(V_i)^\phi$ there is some $j \geq i$ and a solution for the associated lifting problem on $f(V_j)^\phi$, and so f is a G-global equivalence. □

Now we proceed with a technical lemma which we will use to prove two propositions that deal with what happens to G-global equivalences when taking orbits or inducing from a subgroup. The nice thing is that for finite groups no cofibrancy is required anywhere, although freeness of the G-action on both source and target is required in the orbits case.

Lemma 2.2.4. *Let H be a finite group and K and G compact Lie groups. Assume that we have equivariant maps of $(K \times G \times H)$-spaces $f\colon X \to Y$ and $g\colon Y \to Z$ such that Z is Hausdorff and H-free. Then we have that $f/H\colon X/H \to Y/H$ is an $\mathscr{F}(K, G)$-weak homotopy equivalence if and only if f is an $\mathscr{F}(K, G \times H)$-weak homotopy equivalence.*

Proof. First note that since Z is H-free, so are X and Y. For any closed subgroup $L \leq K$ and continuous homomorphism $\phi\colon L \to G$, [Sch18, Proposition B.17] gives a natural homeomorphism for X, Y and Z:

$$\coprod_{[\psi]} X^\psi/C(\psi) \to (X/H)^\phi$$

The disjoint union on the left is indexed by the conjugacy classes of continuous homomorphisms $\psi\colon \Gamma(\phi) \to H$. $C(\psi)$ denotes the centralizer of the image of ψ on H. Here we write $\Gamma(\phi)$ for the graph subgroup of ϕ for clarity.

Fix an homomorphism $\phi\colon L \to G$. An homomorphism $\psi\colon \Gamma(\phi) \to H$, as a subgroup of $K \times G \times H$, has elements $(k, \phi(k), \psi(k, \phi(k))$ for $k \in L$, so $\psi \in \mathscr{F}(K, G \times H)$. In the other direction, for a $\psi \in \mathscr{F}(K, G \times H)$, let ϕ be the homomorphism $\pi_G \circ \psi\colon L \to G$ where $\pi_G\colon G \times H \to G$ is the projection. Then $\Gamma(\psi)$ is a graph subgroup of $\Gamma(\phi) \times H$, so that ψ can be seen as a homomorphism $\Gamma(\phi) \to H$.

We know that a disjoint union of maps is a weak homotopy equivalence if and only each of the maps is a weak homotopy equivalence. Therefore we have that f/H is an $\mathscr{F}(K, G)$-weak homotopy equivalence if and only if for each $\psi \in \mathscr{F}(K, G \times H)$ the map $f^\psi/C(\psi)$ is a weak homotopy equivalence.

For each $\psi \in \mathscr{F}(K, G \times H)$, the centralizer of the image of ψ, $C(\psi) \leq H$, is finite and Z^ψ is $C(\psi)$-free and a closed subspace of Z so Hausdorff. Therefore the $C(\psi)$-action on Z^ψ is properly discontinuous, and since f^ψ and g^ψ are $C(\psi)$-equivariant, the $C(\psi)$-actions on X^ψ and Y^ψ are also properly discontinuous.

This means that $X^\psi \to X^\psi/C(\psi)$ and $Y^\psi \to Y^\psi/C(\psi)$ are covering maps, and since f^ψ is $C(\psi)$-equivariant, it induces a map of coverings. Then we consider the long exact sequence of homotopy groups for these covering maps. We have that $f^\psi/C(\psi)$ is a weak homotopy equivalence if and only if f^ψ is a weak homotopy equivalence, which can be seen by using the five lemma and checking explicitly on π_0 and π_1.

Thus we finally obtain that f/H is an $\mathscr{F}(K, G)$-weak homotopy equivalence if and only if f is an $\mathscr{F}(K, G \times H)$-weak homotopy equivalence. □

This next proposition is similar to [SS12, Lemma 8.1].

Proposition 2.2.5. *Let H be a finite group and G a compact Lie group. Consider two $(G \times H)$-equivariant morphisms of orthogonal spaces $f\colon X \to Y$ and $g\colon Y \to Z$, where for Z we know that for each inner product space V the space $Z(V)$ is Hausdorff and H-free. Then $f/H\colon X/H \to Y/H$ is a G-global equivalence if and only if f is a $(G \times H)$-global equivalence.*

Proof. By Lemma 2.2.2 $f/H\colon X/H \to Y/H$ is a G-global equivalence if and only if for each compact Lie group K and exhaustive sequence of K-representations $\{V_i\}_{i\in\mathbb{N}}$ the map

$$\mathrm{tel}_i\, f/H(V_i)\colon\ \mathrm{tel}_i\, X/H(V_i) \to \mathrm{tel}_i\, Y/H(V_i)$$

is an $\mathscr{F}(K, G)$-weak homotopy equivalence.

Taking H-orbits commutes with colimits and $- \times [0, 1]$, so it commutes with taking mapping telescopes, therefore $\mathrm{tel}_i\, f/H(V_i) \cong \mathrm{tel}_i\, f(V_i)/H$. Now f and g induce $(K \times G \times H)$-equivariant maps on mapping telescopes:

$$\mathrm{tel}_i\, X(V_i) \xrightarrow{\mathrm{tel}_i\, f(V_i)} \mathrm{tel}_i\, Y(V_i) \xrightarrow{\mathrm{tel}_i\, g(V_i)} \mathrm{tel}_i\, Z(V_i)$$

Since each $Z(V)$ is Hausdorff and H-free, so is $\mathrm{tel}_i\, Z(V_i)$. Additionally by Lemma 2.2.2 f is a $(G\times H)$-global equivalence if and only if $\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K, G\times H)$-weak homotopy equivalence for each K and $\{V_i\}_{i\in\mathbb{N}}$. By Lemma 2.2.4 $\mathrm{tel}_i\, f(V_i)/H$ is an $\mathscr{F}(K, G)$-weak homotopy equivalence if and only if $\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K, G\times H)$-weak homotopy equivalence, which yields the result. □

Proposition 2.2.6. *Given a compact Lie group G, a finite subgroup $H \le G$, and an H-global equivalence $f\colon X \to Y$, we have that the morphism $G \times_H f$ is a G-global equivalence.*

Proof. We first need to check that $G \times f$ is a $(G \times H)$-global equivalence, for the action where G acts on the left on the G factor, and H acts both on the right on the G factor and on the left on the f factor.

Consider a compact Lie group K and a exhaustive sequence of K-representations $\{V_i\}_{i\in\mathbb{N}}$. $G\times -$ commutes with colimits and $-\times[0, 1]$, so it commutes with taking mapping telescopes, therefore it suffices to check that $G \times \mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K, G \times H)$-weak homotopy equivalence.

For any continuous homomorphism $\phi\colon K \to G \times H$, the image of the graph subgroup under the projection $\pi_{K\times H}\colon K \times G \times H \to K \times H$ is the graph subgroup of $\pi_H \circ \phi$, for π_H the projection $G \times H \to H$. Therefore $(\mathrm{tel}_i\, f(V_i))^{\phi} = (\mathrm{tel}_i\, f(V_i))^{\pi_H\circ\phi}$, and the latter is a weak homotopy equivalence since $\mathrm{tel}_i\, f(V_i)$ is an $\mathscr{F}(K, H)$-weak homotopy equivalence. Then $(G\times \mathrm{tel}_i\, f(V_i))^{\phi} = G^{\phi} \times \mathrm{tel}_i\, f(V_i)^{\phi}$ is also a weak homotopy equivalence.

Lastly, the projection $G\times Y \to G$ is a $(G\times H)$-equivariant map, where again G acts on G on the left and H acts on the right. With this action G is H-free and Hausdorff, so by Proposition 2.2.5, $G \times_H f$ is a G-global equivalence. □

We now check some general properties about G-global equivalences.

Lemma 2.2.7. *For compact Lie groups G, H, and a continuous homomorphism $\psi\colon H \to G$, we have the following properties:*

i) (2-out-of-6) For three composable morphisms of G-orthogonal spaces $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to W$, such that $g \circ f$ and $h \circ g$ are G-global equivalences, we have that f, g, h and $h \circ g \circ f$ are G-global equivalences.

ii) If $f\colon X \to Y$ is a G-global equivalence and g is homotopic to f through G-equivariant morphisms of orthogonal spaces, then g is a G-global equivalence.

iii) For a G-orthogonal space X, and an H-global equivalence $f\colon Y \to Z$, the morphism $X \times f$ is a $(G \times H)$-global equivalence.

iv) For a G-global equivalence $f\colon X \to Y$ and an H-global equivalence $f'\colon X' \to Y'$, the morphism $f \times f'$ is a $(G \times K)$-global equivalence.

v) For a G-global equivalence $f\colon X \to Y$ the restriction $\psi^* f$ is a K-global equivalence.

Proof. i) For each compact Lie group K and exhaustive sequence of K-representations $\{V_i\}_{i\in\mathbb{N}}$ we have that by Lemma 2.2.2 $\mathrm{tel}_i(g \circ f)(V_i) = \mathrm{tel}_i\, g(V_i) \circ \mathrm{tel}_i\, f(V_i)$ and $\mathrm{tel}_i(h \circ g)(V_i) = \mathrm{tel}_i\, h(V_i) \circ \mathrm{tel}_i\, g(V_i)$ are $\mathscr{F}(K,G)$-weak homotopy equivalences, and since the class of $\mathscr{F}(K,G)$-weak homotopy equivalences satisfies the 2-out-of-6 property, $\mathrm{tel}_i\, f(V_i)$, $\mathrm{tel}_i\, g(V_i)$, $\mathrm{tel}_i\, h(V_i)$ and $\mathrm{tel}_i(h \circ g \circ f)(V_i)$ are also $\mathscr{F}(K,G)$-weak homotopy equivalences and so by Lemma 2.2.2 again f, g, h and $h \circ g \circ f$ are G-global equivalences.

ii) If $H\colon X \times [0,1] \to Y$ is a homotopy through G-equivariant morphisms of orthogonal spaces it induces a homotopy through $(K \times G)$-equivariant maps on mapping telescopes for each compact Lie group K and each exhaustive sequence of representations. Then by Lemma 2.2.2 and because a map $(K \times G)$-homotopic to an $\mathscr{F}(K,G)$-weak homotopy equivalence is an $\mathscr{F}(K,G)$-weak homotopy equivalence, we see that g is also a G-global equivalence.

iii) Consider any compact Lie group K and continuous homomorphism $\phi\colon K \to G \times H$, an orthogonal K-representation V, and a lifting problem $\alpha\colon \partial D^l \to ((X \times Y)(V))^\phi$ and $\beta\colon D^l \to ((X \times Z)(V))^\phi$. Since $(X \times Y)(V) = X(V) \times Y(V)$, if we consider the trivial H-action on $X(V)$ and the trivial G-action on $Y(V)$ and $Z(V)$, we have that $(X \times Y)(V)^\phi = X(V)^\phi \times Y(V)^\phi$, and similarly $(X \times Z)(V)^\phi = X(V)^\phi \times Z(V)^\phi$.

We have that $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$, and since f is an H-global equivalence there is a K-equivariant linear isometric embedding $\eta\colon V \to W$ and λ a solution to the lifting problem α_2, β_2. That is $Y(\eta)^\phi \circ \alpha_2 = \lambda \circ i_l$ and $f(W)^\phi \circ \lambda$ is homotopic relative ∂D^l to $Z(W)^\phi \circ \beta_2$. Then on the original lifting problem, after postcomposing by η, we have a solution $(X(\eta)^\phi \circ \beta_1, \lambda)$, and so $X \times f$ is a $(G \times H)$-global equivalence.

iv) $f \times f' = (Y \times f') \circ (f \times X')$ and each of these is a $(G \times H)$-global equivalence by the previous point.

v) Given any compact Lie group K and continuous homomorphism $\phi\colon K \to H$, an orthogonal K-representation V, and a lifting problem $\alpha\colon \partial D^l \to (\psi^* X(V))^\phi$ and $\beta\colon D^l \to (\psi^* Y(V))^\phi$, we have that $(\psi^* X(V))^\phi = X(V)^{\psi \circ \phi}$, and the same thing is true for Y. Then since f is a G-global equivalence the lifting problem has a solution after possibly embedding V in some bigger K-representation W. □

We now turn to the box product of orthogonal spaces, to check that it works well with G-global equivalences. Specifically that it is fully homotopical with no cofibrancy assumptions required, just like with respect to the global equivalences.

Lemma 2.2.8. *Given $F\colon \underline{\mathrm{L}} \to \underline{\mathrm{L}}$ a continuous endofunctor, a natural transformation $\eta\colon Id \Rightarrow F$, and a G-orthogonal space X, then the morphism $X \circ \eta\colon X \to X \circ F$ is a G-global equivalence.*

Proof. We will use the fact that for each compact Lie group K and each K-representation V the two embeddings $F(\eta_V), \eta_{F(V)}\colon F(V) \to F(F(V))$ are homotopic relative to $\eta_V\colon V \to F(V)$ through K-equivariant linear isometric embeddings. This is proven in the proof of the equivalent result where X is just an orthogonal space on [Sch18, Theorem 1.1.10].

Then given a compact Lie group K, an K-representation V, a continuous homomorphism $\phi\colon K \to G$ and a lifting problem (α, β) like in the following diagram, we can see that the linear isometric embedding η_v and the map β provide a solution, since first the upper left triangle commutes by construction.

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, "i_l"] & X(V)^K \arrow[r, "X(\eta_V)^K"] \arrow[d, "X(\eta_V)^K"'] & X(F(V))^K \arrow[d, "X(\eta_{F(V)})^K"] \arrow[dl, equal] \\
D^l \arrow[r, "\beta"] & X(F(V))^K \arrow[r, "X(F(\eta_V))^K"'] & X(F(F(V)))^K
\end{tikzcd}$$

For the lower right triangle, $F(\eta_V)$ and $\eta_{F(V)}$ are homotopic through K-equivariant linear isometric embeddings, therefore $X(F(\eta_V))$ and $X(\eta_{F(V)})$ are homotopic through K-equivariant maps when considered with the twisted K-action, and $X(F(\eta_V))^K$ and $X(\eta_{F(V)})^K$ are homotopic. Since the original homotopy was relative to η_V, and $\beta \circ i_l = X(\eta_V)^K \circ \alpha$, the obtained homotopy between $X(F(\eta_V))^K \circ \beta$ and $X(\eta_{F(V)})^K \circ \beta$ is relative i_l. Thus $X(\eta)$ is a G-global equivalence. □

Given a G-orthogonal space X and a K-orthogonal space Y, we can construct a bimorphism $(X, Y) \to X \times Y$ via:

$$X(V) \times Y(W) \xrightarrow{X(\iota_1)\times Y(\iota_2)} X(V \oplus W) \times Y(V \oplus W) = (X \times Y)(V \oplus W)$$

This bimorphism yields a morphism of orthogonal spaces:

$$\rho_{X,Y}\colon X \boxtimes Y \to X \times Y$$

[Sch18, Theorem 1.3.2 (i)] states that this is a global equivalence of underlying orthogonal spaces. We will now rewrite that proof to check that it is additionally a $(G \times K)$-global equivalence.

Proposition 2.2.9. *Given a G-orthogonal space X and a K-orthogonal space Y, the morphism of orthogonal spaces $\rho_{X,Y}$ is a $(G \times K)$-global equivalence.*

Proof. Consider the endofunctor $\mathrm{sh}\colon \underline{\mathrm{L}} \to \underline{\mathrm{L}}$ that sends V to $V \oplus V$. We have two natural transformations $\iota_1, \iota_2\colon Id \Rightarrow \mathrm{sh}$ given by the embeddings into the first and second factor respectively. We also denote by sh the functor of orthogonal spaces given by precomposing with sh, $\mathrm{sh}(X) = X \circ \mathrm{sh}$.

The universal bimorphism i that exhibits $X \boxtimes Y$ as the box product of X and Y gives a morphism of orthogonal spaces $\lambda\colon X \times Y \to \mathrm{sh}(X \boxtimes Y)$ through the maps $i_{V,V}\colon X(V) \times Y(V) \to (X \boxtimes Y)(V \oplus V) = (\mathrm{sh}(X \boxtimes Y))(V)$. In the following diagram we will check that $\lambda \circ \rho_{X,Y}$ and $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ are $(G \times K)$-global equivalences and then use the 2-out-of-6 property 2.2.7 i) to obtain that $\rho_{X,Y}$ is a $(G \times K)$-global equivalence.

$$X \boxtimes Y \xrightarrow{\rho_{X,Y}} X \times Y \xrightarrow{\lambda} \mathrm{sh}(X \boxtimes Y) \xrightarrow{\mathrm{sh}(\rho_{X,Y})} \mathrm{sh}(X \times Y)$$

We have that $\mathrm{sh}(\rho_{X,Y}) \circ \lambda$ evaluated at V is the same as the bimorphism associated to $\rho_{X,Y}$ on level V, V, by the constructions of λ and $\rho_{X,Y}$. That is $X(\iota_1)(V) \times Y(\iota_2)(V)$, where each is respectively a G-global equivalence or a K-global equivalence by Lemma 2.2.8, and then their product is a $(G \times K)$-global equivalence by Lemma 2.2.7 iv).

Next we use that $\lambda \circ \rho_{X,Y}$ is homotopic through $(G \times K)$-equivariant morphisms to $(X \boxtimes Y)(\iota_1)$. The homotopy given in the proof of [Sch18, Theorem 1.3.2 (i)] is through $(G \times K)$-equivariant morphisms, and $(X \boxtimes Y)(\iota_1)$ is a $(G \times K)$-global equivalence by Lemma 2.2.8, so by Lemma 2.2.7 ii) $\lambda \circ \rho_{X,Y}$ is a $(G \times K)$-global equivalence. □

Corollary 2.2.9.1. *For a G-global equivalence $f\colon X \to Y$ and a K-global equivalence $f'\colon X' \to Y'$, the morphism $f \boxtimes f'$ is a $(G \times K)$-global equivalence. If $K = G$ then $f \boxtimes f'$ is a G-global equivalence. Therefore for any $X \in \underline{\mathrm{G}}\text{-}\mathcal{S}pc$, the functor $X \boxtimes -$ preserves G-global equivalences.*

Proof. $\rho_{Y,Y'} \circ (f \boxtimes f') = (f \times f') \circ \rho_{X,X'}$, and $\rho_{Y,Y'}$ and $\rho_{X,X'}$ are $(G \times K)$-global equivalences by Proposition 2.2.9, and $f \times f'$ is one by Lemma 2.2.7 iv).

If $K = G$ by restricting along the diagonal homomorphism $\Delta\colon G \to G \times G$ and using Lemma 2.2.7 v) we have that $f \boxtimes f'$ is a G-global equivalence and $X \boxtimes -$ preserves G-global equivalences. □

2.3 Interplay of G-global equivalences with G-h-cofibrations

We will refer to the following result as the gluing lemma for G-global equivalences.

Lemma 2.3.1 (Gluing lemma)**.** *For a compact Lie group G, and a commutative diagram of G-orthogonal spaces*

$$\begin{tikzcd} Y \arrow[d, "\beta"] & X \arrow[l, "f"'] \arrow[r, "g"] \arrow[d, "\alpha"] & Z \arrow[d, "\gamma"] \\ Y' & X' \arrow[l, "f'"'] \arrow[r, "g'"] & Z' \end{tikzcd}$$

where α, β, γ are G-global equivalences, and f and f' are G-h-cofibrations, then the morphism induced on the pushouts $Y \cup_X Z \to Y' \cup_{X'} Z'$ is a G-global equivalence.

Proof. Consider a compact Lie group K and an exhaustive sequence of K-representations $\{V_i\}_{i \in \mathbb{N}}$. We have the following diagram of equivariant morphisms of $K \times G$-spaces:

$$\begin{tikzcd} \mathrm{tel}_i\, Y(V_i) \arrow[d, "\mathrm{tel}_i\, \beta(V_i)"] & \mathrm{tel}_i\, X(V_i) \arrow[l, "\mathrm{tel}_i\, f(V_i)"'] \arrow[r, "\mathrm{tel}_i\, g(V_i)"] \arrow[d, "\mathrm{tel}_i\, \alpha(V_i)"] & \mathrm{tel}_i\, Z(V_i) \arrow[d, "\mathrm{tel}_i\, \gamma(V_i)"] \\ \mathrm{tel}_i\, Y'(V_i) & \mathrm{tel}_i\, X'(V_i) \arrow[l, "\mathrm{tel}_i\, f'(V_i)"'] \arrow[r, "\mathrm{tel}_i\, g'(V_i)"] & \mathrm{tel}_i\, Z'(V_i) \end{tikzcd}$$

Here by Lemma 2.2.2 $\mathrm{tel}_i\,\alpha(V_i)$, $\mathrm{tel}_i\,\beta(V_i)$ and $\mathrm{tel}_i\,\gamma(V_i)$ are $\mathscr{F}(K,G)$-weak homotopy equivalences, and the formation of mapping telescopes commutes with pushouts, retracts and $-\times[0,1]$, so $\mathrm{tel}_i\,f(V_i)$ and $\mathrm{tel}_i\,f'(V_i)$ are h-cofibrations of $(K\times G)$-spaces. Therefore by the Gluing lemma for $\mathscr{F}(K,G)$-weak homotopy equivalences (see for example [Sch18, Proposition B.6]) the induced map on the pushouts of the mapping telescopes is also an $\mathscr{F}(K,G)$-weak homotopy equivalence. Since again taking mapping telescopes commutes with pushouts, this means that $Y\cup_X Z\to Y'\cup_{X'} Z'$ is a G-global equivalence. □

Corollary 2.3.1.1. *For a compact Lie group G, and a pushout diagram of G-orthogonal spaces*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d] \\ X' \arrow[r, "f'"] \arrow[ur, phantom, "\ulcorner"] & Y' \end{tikzcd}$$

where f is a G-global equivalence and either f or g is a G-h-cofibration, then f' is a G-global equivalence.

Proof. Apply the previous proposition to the diagram:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"] & X \arrow[r, "g"] \arrow[d, equal] & X' \arrow[d, equal] \\ Y & X \arrow[l, "f"'] \arrow[r, "g"] & X' \end{tikzcd}$$

□

With this last corollary we can check that with the G-h-cofibrations and the G-global equivalences, $\underline{\mathrm{G}}$-$\mathcal{S}pc$ forms a cofibration category.

Proposition 2.3.2. *$\underline{\mathrm{G}}$-$\mathcal{S}pc$, together with the G-h-cofibrations and the G-global equivalences, forms a cofibration category, also called a category of cofibrant objects.*

Proof. We check the axioms as listed on [Sch13]. By the characterization of G-h-cofibrations of Remark 2.1.10, they include the isomorphisms, are closed under composition and cobase change, and any morphism from the initial object is one. Then the rest of (C1) is straightforward. For (C2), we checked that G-global equivalences satisfy the 2-out-of-3 property on Lemma 2.2.7 i). For (C3), $\underline{\mathrm{G}}$-$\mathcal{S}pc$ is cocomplete and we just checked on Corollary 2.3.1.1 that G-global equivalences are preserved by cobase changes of G-h-cofibrations.

Lastly, we can use the G-level model structure on $\underline{\mathrm{G}}$-$\mathcal{S}pc$ of Theorem 2.1.4 to factor any morphism into a G-flat cofibration (which is therefore a G-h-cofibration) and a morphism f such that for each $m\geq 0$ and closed subgroup $H\leq O(m)\times G$ the map $f(\mathbb{R}^m)^H$ is a weak homotopy equivalence.

For any compact Lie group K and any K-representation V there is a linear isometry $\psi\colon V\to\mathbb{R}^m$. Then for any graph subgroup $\phi\leq K\times G$, conjugation by ψ induces a continuous homomorphism $\alpha\colon K\to O(m)$, and we also have an induced natural (on the orthogonal space) homeomorphism $X(V)^\phi\cong X(\mathbb{R}^m)^{(\alpha\times G)(\phi)}$. Then we obtain that any lifting problem for f has a solution after applying ϕ, and so f is a G-global equivalence. □

Note that the G-flat cofibrations and the G-global equivalences do not make $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$ a cofibration category, since not every G-orthogonal space is G-flat. If we restrict to the full subcategory of G-flat orthogonal spaces however we do obtain a cofibration category by the same argument as before.

Corollary 2.3.2.1. *For morphisms of G-orthogonal spaces $f\colon X_1 \to Y_1$ and $g\colon X_2 \to Y_2$ such that f is a G-global equivalence and a G-h-cofibration, their pushout product $f\square g$ is a G-global equivalence.*

Similarly, if $f\colon X_1 \to Y_1$ is a morphism of G-orthogonal spaces and $g\colon X_2 \to Y_2$ is a map of G-spaces, and either f is a G-global equivalence and a G-h-cofibration, or g is a G-weak homotopy equivalence and a G-h-cofibration, their pushout product $f\square g$ is a G-global equivalence.

Proof. By Lemma 2.2.7 iii) we have that $f\boxtimes X_2$ and $f\boxtimes Y_2$ are G-global equivalences. Since $f\boxtimes X_2$ is also a G-h-cofibration, by Corollary 2.3.1.1 the morphism α is also a G-global equivalence, so by the 2-out-of-3 property so is $f\square g$.

$$\begin{tikzcd}
X_1 \boxtimes X_2 \arrow[r, "X_1\boxtimes g"] \arrow[d, "f\boxtimes X_2"'] & X_1 \boxtimes Y_2 \arrow[d, "\alpha"] \arrow[ddr, bend left, "f\boxtimes Y_2"] & \\
Y_1 \boxtimes X_2 \arrow[r] \arrow[drr, bend right, "Y_1\boxtimes g"'] & P \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f\square g"] & \\
& & Y_1 \boxtimes Y_2
\end{tikzcd}$$

The same is true if g is a map of G-spaces, since the product of an orthogonal space with a space is the same as the box product with the associated constant orthogonal space, and a G-weak homotopy equivalence between constant orthogonal spaces is a G-global equivalence, and similarly a G-h-cofibration of spaces is a G-h-cofibration between constant orthogonal spaces. $\square$

Proposition 2.3.3. *For a compact Lie group G, and λ a limit ordinal, consider two λ-sequences in $\underline{\mathrm{G}}\text{-}\mathcal{S}pc$, which are colimit preserving functors $X\colon \lambda \to \underline{\mathrm{G}}\text{-}\mathcal{S}pc$ and $Y\colon \lambda \to \underline{\mathrm{G}}\text{-}\mathcal{S}pc$, and a natural transformation f between them. Then if for each $\beta \in \lambda$ the morphisms $g_\beta\colon X_\beta \to X_{\beta+1}$ and $h_\beta\colon Y_\beta \to Y_{\beta+1}$ are G-h-cofibrations and the morphism $f_\beta\colon X_\beta \to Y_\beta$ is a G-global equivalence, the morphism induced on the colimits $\mathrm{colim}_{\beta\in\lambda} f_\beta\colon \mathrm{colim}_{\beta\in\lambda} X_\beta \to \mathrm{colim}_{\beta\in\lambda} Y_\beta$ is a G-global equivalence.*

Proof. By Lemma 2.2.2 it is enough to check that for each compact Lie group K and exhaustive sequence of K-representations $\{V_i\}_{i\in I}$ the map $\mathrm{tel}_i(\mathrm{colim}_{\beta\in\lambda} f_\beta)(V_i)$ is an $\mathscr{F}(K,G)$-weak homotopy equivalence. The construction of the mapping telescopes commutes with taking colimits, so this map is isomorphic to $\mathrm{colim}_{\beta\in\lambda}(\mathrm{tel}_i f_\beta(V_i))$.

For each $\beta \in \lambda$ the map $\mathrm{tel}_i f_\beta(V_i)$ is an $\mathscr{F}(K,G)$-weak equivalence, and the maps $\mathrm{tel}_i g_\beta(V_i)$ and $\mathrm{tel}_i h_\beta(V_i)$ are h-cofibrations of $K\times G$-spaces, and so in particular h-cofibrations of underlying compactly generated weak Hausdorff spaces, and therefore closed embeddings.

For each $\phi \in \mathscr{F}(K,G)$ taking ϕ-fixed points commutes with filtered colimits along closed embeddings. Colimits with the shape of a filtered poset and built out of closed embeddings of compactly generated weak Hausdorff spaces can be computed in the category of all topological spaces (see [Sch18, Proposition A.14 (ii)]). Weak Hausdorff spaces are T_1, so by [Hov07, Proposition 2.4.2] we have that maps from compact spaces (∂D^l and D^l in this case) into the colimit of a λ-sequence of closed embeddings (for λ a limit ordinal) factor through some stage $\beta \in \lambda$. Therefore compact spaces are finite in $\underline{\mathrm{Top}}$ relative closed embeddings.

This implies that, for the λ-sequences given by $(\mathrm{tel}_i\, g_\beta(V_i))^\phi$ and $(\mathrm{tel}_i\, h_\beta(V_i))^\phi$, which consist of closed embeddings, and the natural transformation between them given by the maps $(\mathrm{tel}_i\, f_\beta(V_i))^\phi$ which are weak homotopy equivalences, the map induced on the colimits $\mathrm{colim}_{\beta\in\lambda}(\mathrm{tel}_i\, f_\beta(V_i))^\phi \cong (\mathrm{colim}_{\beta\in\lambda}(\mathrm{tel}_i\, f_\beta(V_i)))^\phi$ is a weak homotopy equivalence. Therefore $\mathrm{tel}_i(\mathrm{colim}_{\beta\in\lambda} f_\beta)(V_i)$ is an $\mathscr{F}(K,G)$-weak homotopy equivalence. □

Corollary 2.3.3.1. *A transfinite composition of morphisms in $\underline{\mathrm{G}}$-$\mathcal{S}pc$ that are G-h-cofibrations and G-global equivalences is a G-global equivalence.*

Proof. We check this via transfinite induction on the ordinal λ. Let $Y\colon \lambda \to \underline{\mathrm{G}}$-$\mathcal{S}pc$ be a λ-sequence such that for each $\beta \in \lambda$ the morphism $h_\beta\colon Y_\beta \to Y_{\beta+1}$ is a G-h-cofibration and a G-global equivalence. The base case and the case where λ is a successor ordinal hold because composition of two G-global equivalences is a G-global equivalence.

If λ is a limit ordinal, set $X\colon \lambda \to \underline{\mathrm{G}}$-$\mathcal{S}pc$ as the constant functor $X_\beta = Y_0$. Define a natural transformation $f\colon X \Rightarrow Y$ by letting f_β be the morphism $Y_0 \to Y_\beta$. This is the transfinite composition of Y restricted to $\beta+1$. Then by the induction hypothesis f_β is a G-global equivalence for each $\beta \in \lambda$. Then we use Proposition 2.3.3 to obtain that $\mathrm{colim}_{\beta\in\lambda} f_\beta$ is a G-global equivalence, but this morphism is precisely the transfinite composition of Y. □

Chapter 3

Homotopy of algebras over a global operad

3.1 Model structure on the category of algebras

Our goal now is to finally check the conditions of Theorem 1.2.2 on cobase changes in $\mathcal{Alg}(\mathcal{O})$ of what will be the generating (acyclic) cofibrations of $\mathcal{Alg}(\mathcal{O})$, for operads in $\mathcal{Spc}$. We focus now first on the generating cofibrations. Throughout this section let $\mathcal{O}$ denote an operad on $\mathcal{Spc}$, with no further conditions assumed.

We first check an auxiliary lemma.

Lemma 3.1.1. *For $i\colon X \to Y$ a cofibration (injective morphism) of simplicial sets, and a finite group G such that X and Y are equipped with a G-action and the morphism i is G-equivariant, the map $|i|\colon |X| \to |Y|$ is a G-cofibration between G-cofibrant objects.*

Proof. $|X|$ and $|Y|$ are CW complexes with the typical CW structure associated to the geometric realization functor for simplicial sets, and $|i|$ is a relative CW complex. To check that each $|X|$ and $|Y|$ are G-CW complexes and $|i|$ is a relative G-CW complex, we need to check that for each $g \in G$ and open cell γ of $|Y|$, either $g\gamma \cap \gamma = \emptyset$, or the g action restricted to γ is the identity.

Cells in $|Y|$ come from non-degenerate simplices of Y. If $g \in G$ sends a non-degenerate $y \in Y_k$ to a different simplex, which also has to be non-degenerate because the G action commutes with the degeneracy maps, then the two associated cells of the same dimension are different. If $gy = y$, the G action on $Y_{k,ndg} \times \Delta^k$ is the identity when restricted to $\{y\} \times \Delta^k$, so the action of g restricted to the cell associated to y is the identity. $\square$

Corollary 3.1.1.1. *For $i\colon X \to Y$ a cofibration (injective morphism) of simplicial sets, the n-fold product $|i|^{\times n}$ and the n-fold pushout product $|i|^{\square n}$ of $|i|$ are Σ_n-cofibrations between Σ_n-cofibrant spaces.*

Proof. The geometric realization functor preserves products and pushouts, and the Σ_n action on $|Y|^{\times n}$ is the same as the one obtained from the Σ_n-action on $Y^{\times n}$ by applying the geometric realization functor. Therefore $|i|^{\times n} = |i^{\times n}|$ and $|i|^{\square n} = |i^{\square n}|$ are geometric realizations of Σ_n

equivariant cofibrations of simplicial sets, and by Lemma 3.1.1 they are Σ_n-cofibrations between Σ_n-cofibrant spaces. $\square$

Then we check the condition of Theorem 1.2.2 for the generating (acyclic) cofibrations in I and J.

Proposition 3.1.2. *Given a generating cofibration of the positive global model structure of the form $i \in I$, and a pushout in $\mathcal{Alg}(\mathcal{O})$ of the form*

$$\begin{tikzcd} F_{\mathcal{Alg}(\mathcal{O})}(X) \arrow[r, "F_{\mathcal{Alg}(\mathcal{O})}(i)"] \arrow[d] & F_{\mathcal{Alg}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

the morphism $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is an h-cofibration of orthogonal spaces.

Proof. Consider the filtration of [SS12, Proposition A.16] with $k = 0$, where $U_0^{\mathcal{O}} = U_{\mathcal{Alg}(\mathcal{O})}$. $i = L_{G,V} \times i_l$, so for each $j \geq 1$, we have the following pushout in $\mathcal{Spc}$:

$$\begin{tikzcd} U_j^{\mathcal{O}}(A) \boxtimes_{\Sigma_j} Q_{j-1}^j(i) \arrow[r, "U_j^{\mathcal{O}}(A)\boxtimes_{\Sigma_j} i^{\square j}"] \arrow[d] & U_j^{\mathcal{O}}(A) \boxtimes_{\Sigma_j} (L_{G,V} \times D^l)^{\boxtimes j} \arrow[d] \\ P_{j-1}U_0^{\mathcal{O}}(B) \arrow[r, "f_j"] & P_j U_0^{\mathcal{O}}(B) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We have that $i^{\square j} = L_{G,V}^{\boxtimes j} \times i_l^{\square j}$. The map $i_l^{\square j}$ is a Σ_j-cofibration of spaces by Corollary 3.1.1.1 (so it is also a Σ_j-h-cofibration). Thus the same is true for $U_j^{\mathcal{O}}(A) \boxtimes L_{G,V}^{\boxtimes i} \times i_l^{\square j}$, and so $U_j^{\mathcal{O}}(A) \boxtimes_{\Sigma_j} i^{\square j}$ is an h-cofibration of orthogonal spaces by Lemma 2.1.11. Then so is f_j, and then $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is an infinite composition of h-cofibrations, and therefore an h-cofibration. $\square$

Proposition 3.1.3. *Given a generating acyclic cofibration of the positive global model structure of the form $j \in J$, and a pushout in $\mathcal{Alg}(\mathcal{O})$ of the form*

$$\begin{tikzcd} F_{\mathcal{Alg}(\mathcal{O})}(X) \arrow[r, "F_{\mathcal{Alg}(\mathcal{O})}(j)"] \arrow[d] & F_{\mathcal{Alg}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

the morphism $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a global equivalence.

Proof. Consider again the filtration of [SS12, Proposition A.16] with $k = 0$, where $U_0^{\mathcal{O}} = U_{\mathcal{Alg}(\mathcal{O})}$. $j = L_{G,V} \times j_l$, so for each $i \geq 1$, we have the following pushout in $\mathcal{Spc}$:

$$\begin{tikzcd} U_i^{\mathcal{O}}(A) \boxtimes_{\Sigma_i} Q_{i-1}^i(j) \arrow[r, "U_i^{\mathcal{O}}(A)\boxtimes_{\Sigma_i} j^{\square i}"] \arrow[d] & U_i^{\mathcal{O}}(A) \boxtimes_{\Sigma_i} (L_{G,V} \times [0,1]^{l+1})^{\boxtimes i} \arrow[d] \\ P_{i-1}U_0^{\mathcal{O}}(B) \arrow[r, "f_i"] & P_i U_0^{\mathcal{O}}(B) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

We have that $j^{\square i} = L_{G,V}^{\boxtimes i} \times j_l^{\square i}$. The map $j_l^{\square i}$ is a Σ_i-cofibration of spaces by Corollary 3.1.1.1 (so it is also a Σ_i-h-cofibration) and a Σ_i-homotopy equivalence. Thus the same is true for $U_i^{\mathcal{O}}(A) \boxtimes L_{G,V}^{\boxtimes i} \times j_l^{\square i}$, and so $U_i^{\mathcal{O}}(A) \boxtimes_{\Sigma_i} j^{\square i}$ is an h-cofibration and homotopy equivalence of orthogonal spaces by Lemma 2.1.11, and therefore also a global equivalence. Then so is f_i, and then $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is an infinite composition of morphisms which are h-cofibrations and global equivalences, so it is a global equivalence. □

We now consider the generating acyclic cofibrations in K. First we have a result that, for a morphism f, reduces the task of checking that $f^{\square n}$ is a Σ_n-global equivalence for all n to checking that $f^{\boxtimes n}$ is one for all n, assuming that the morphisms $f^{\square n}$ are Σ_n-h-cofibrations.

Proposition 3.1.4. *Let $f\colon X \to Y$ be a morphism of orthogonal spaces such that for each $n \geq 1$ the morphism $f^{\boxtimes n}$ is a Σ_n-global equivalence, and such that for each $n \geq 1$ the morphism $f^{\square n}$ is a Σ_n-h-cofibration. Then for each $n \geq 1$ the morphism $f^{\square n}$ is a Σ_n-global equivalence.*

Proof. We will proceed by strong induction. For the base case, $f^{\square 1} = f^{\boxtimes 1} = f$ which is a Σ_1-global equivalence, that is, a global equivalence.

Assuming the result holds for all $i < n$, we decompose $f^{\boxtimes n}$ using the filtration of [SS12, Lemma A.8] with $X_0 = X_2 = X$ and $X_1 = Y$.

$$X^{\boxtimes n} = Q_0^n(f) \longrightarrow Q_1^n(f) \longrightarrow \dots \longrightarrow Q_{n-1}^n(f) \xrightarrow{f^{\square n}} Q_n^n(f) = Y^{\boxtimes n}$$

The last step of the filtration is precisely $f^{\square n}$.

For each step $1 \leq i < n$ there is a Σ_n-equivariant pushout diagram of orthogonal spaces:

$$\begin{tikzcd}
\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes Q_{i-1}^i(f) \arrow[r, "\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes f^{\square i}"] \arrow[d] & \Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes Y^{\boxtimes i} \arrow[d] \\
Q_{i-1}^n(f) \arrow[r] & Q_i^n(f) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

By Corollary 2.2.9.1 and the induction hypothesis we have that $X^{\boxtimes n-i} \boxtimes f^{\square i}$ is a $(\Sigma_{n-i} \times \Sigma_i)$-global equivalence. Then by Proposition 2.2.6 $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes f^{\square i}$ is a Σ_n-global equivalence. Additionally the functor $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes -$ preserves colimits and $- \times [0,1]$, so applying it to the Σ_i-equivariant retraction that witnesses that $f^{\square i}$ is a Σ_i-h-cofibration yields that $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} X^{\boxtimes n-i} \boxtimes f^{\square i}$ is a Σ_n-h-cofibration.

By Corollary 2.3.1.1 this means that $Q_{i-1}^n(f) \to Q_i^n(f)$ is a Σ_n-global equivalence for each $1 \leq i < n$. Since so is $f^{\boxtimes n}$, by the 2-out-of-6 property for Σ_n-global equivalences $f^{\square n}$ is a Σ_n-global equivalence. □

Proposition 3.1.5. *For each compact Lie group G, faithful G-representation $V \neq 0$, each G-representation W, and each $n \geq 1$ let $\iota_{\rho_{G,V,W}}$ be the morphism given in Remark 1.3.3. Then we have that $\iota_{\rho_{G,V,W}}^{\square n}$ is a Σ_n-flat cofibration.*

Proof. We use Lemma A.1 to decompose $\iota_{\rho_{G,V,W}}^{\square n}$ into

$$Q_{n-1}^{n}(\iota_{\rho_{G,V,W}}) = K_{n,-1}(\rho_{G,V,W}) \to K_{n,0}(\rho_{G,V,W}) \to \cdots \to K_{n,n}(\rho_{G,V,W}) = (M_{\rho_{G,V,W}})^{\boxtimes n}$$

Then we have that for each $0 \leq j \leq n$ the morphism $K_{n,j-1}(\rho_{G,V,W}) \to K_{n,j}(\rho_{G,V,W})$ is a Σ_n-equivariant cobase change of $\Sigma_n \times_{\Sigma_{n-j}\times\Sigma_j} (L_{G,V}^{\boxtimes n-j} \boxtimes (L_{G,V\oplus W} \times i_1)^{\square j}$, where we use the convention that $(L_{G,V\oplus W} \times i_1)^{\square 0}$ is the unique morphism $\emptyset \to *$ from the initial object to the monoidal unit.

For each $0 \leq j \leq n$, $(L_{G,V\oplus W} \times i_1)^{\square j} = L_{G^j,V^j\oplus W^j} \times i_j$, where $L_{G^j,V^j\oplus W^j}$ is Σ_j-flat because it is isomorphic to $(L(V^j \oplus W^j, -) \times \Sigma_j)/H$, where $H \leq O(V^j \oplus W^j) \times \Sigma_j$ given by

$$H = \{\, (((g_1,\ldots,g_j)\circ\sigma) \oplus ((g_1,\ldots,g_j)\circ\sigma), \sigma^{-1}) : (g_1,\ldots,g_j) \in G^j, \sigma \in \Sigma_j \,\}$$

and since i_j is a Σ_j-cofibration of spaces, by Corollary 2.1.9.2 $(L_{G,V\oplus W} \times i_1)^{\square j}$ is a Σ_j-flat cofibration.

Similarly, $L_{G,V}^{\boxtimes n-j} \cong L_{G^{n-j},V^{n-j}}$ is also Σ_{n-j}-flat, so that $L_{G,V}^{\boxtimes n-j} \boxtimes (L_{G,V\oplus W} \times i_1)^{\square j}$ is a $(\Sigma_{n-j}\times\Sigma_j)$-flat cofibration. By Lemma 2.1.11, the morphism $\Sigma_n \times_{\Sigma_{n-j}\times\Sigma_j} (L_{G,V}^{\boxtimes n-j} \boxtimes (L_{G,V\oplus W} \times i_1)^{\square j}$ is a Σ_n-flat cofibration, and therefore for each $0 \leq j \leq n$ the morphism $K_{n,j-1}(\rho_{G,V,W}) \to K_{n,j}(\rho_{G,V,W})$ is also a Σ_n-flat cofibration.

Then we have that $\iota_{\rho_{G,V,W}}^{\square n}$ is a Σ_n-flat cofibration. □

Lemma 3.1.6. *For $f\colon X \to Y$ a homotopy equivalence between orthogonal spaces and $n \geq 1$, $f^{\boxtimes n}$ is a Σ_n-homotopy equivalence of orthogonal spaces, and therefore a Σ_n-global equivalence.*

Proof. Let $g\colon Y \to X$ be an homotopy inverse to f, H an homotopy between $f\circ g$ and Id_X and H' an homotopy between $g\circ f$ and Id_Y. Then for each $n \geq 1$, $H^{\boxtimes n}\colon X^{\boxtimes n} \times [0,1]^n \cong (X \times [0,1])^{\boxtimes n} \to Y^{\boxtimes n}$ is a Σ_n equivariant map of orthogonal spaces, and so $H^{\boxtimes n} \circ (X^{\boxtimes n} \times \Delta)\colon X^{\boxtimes n} \times [0,1] \to Y^{\boxtimes n}$ is a Σ_n equivariant homotopy between $(f\circ g)^{\boxtimes n}$ and $Id_X^{\boxtimes n}$, where $\Delta\colon [0,1] \to [0,1]^n$ is the diagonal. Same thing applies to H'. □

Proposition 3.1.7. *For a compact Lie group G, a faithful G-representation $V \neq 0$, a G-representation W, and for each $n > 0$, the Σ_n-equivariant morphism $\rho_{G,V,W}^{\boxtimes n}\colon L_{G,V\oplus W}^{\boxtimes n} \to L_{G,V}^{\boxtimes n}$ is a Σ_n-global equivalence.*

Proof. By [Sch18, Example 1.3.3] the orthogonal space $L_{G,V\oplus W}^{\boxtimes n}$ is closed and isomorphic to $L_{G^n,(V\oplus W)^n}$ and similarly $L_{G,V}^{\boxtimes n}$ is isomorphic to L_{G^n,V^n}.

V^n is a faithful $(\Sigma_n \wr G)$-representation, so for each compact Lie group K by [Sch18, Proposition 1.1.26 (ii)] the restriction map

$$\rho_{V^n,W^n}(\mathcal{U}_K)\colon \underline{\mathrm{L}}((V\oplus W)^n, \mathcal{U}_K) \to \underline{\mathrm{L}}(V^n, \mathcal{U}_K)$$

is a $(K \times (\Sigma_n \wr G))$-homotopy equivalence. Using that $\underline{\mathrm{L}}(V^n, \mathcal{U}_K) \cong \mathrm{colim}_{V' \in s(\mathcal{U}_K)} \underline{\mathrm{L}}(V^n, V')$ and since $-/G^n$ preserves colimits we have that $\rho_{G,V,W}^{\boxtimes n}(\mathcal{U}_K) \cong \rho_{V^n,W^n}(\mathcal{U}_K)/G^n$ is a $(K \times \Sigma_n)$-homotopy equivalence, therefore an $\mathscr{F}(K,\Sigma_n)$-weak homotopy equivalence, and so $\rho_{G,V,W}^{\boxtimes n}$ is a Σ_n-global equivalence. □

Proposition 3.1.8. *For each compact Lie group G, faithful G-representation $V \neq 0$, each G-representation W, and each $n \geq 1$, the morphism $\iota^{\square n}_{\rho_{G,V,W}}$ is a Σ_n-global equivalence.*

Proof. If we let $\pi_{\rho_{G,V,W}} : Mc_{\rho_{G,V,W}} \to L_{G,V}$ denote the projection of the mapping cylinder onto the target, then $\pi_{\rho_{G,V,W}} \circ \iota_{\rho_{G,V,W}} = \rho_{G,V,W}$. The morphism $\pi_{\rho_{G,V,W}}$ is an homotopy equivalence, so by Lemma 3.1.6 the morphism $\pi^{\boxtimes n}_{\rho_{G,V,W}}$ is a Σ_n-global equivalence. By Proposition 3.1.7 so is $\rho^{\boxtimes n}_{G,V,W}$, so that by the 2-out-of-3 property $\iota^{\boxtimes n}_{\rho_{G,V,W}}$ is a Σ_n-global equivalence.

Then by Proposition 3.1.5 we can use Proposition 3.1.4 to obtain that for each $n \geq 1$ the morphism $\iota^{\square n}_{\rho_{G,V,W}}$ is a Σ_n-global equivalence. □

We can now put together all the previous results to check that:

Proposition 3.1.9. *Given a generating acyclic cofibration of type $k \in K$, and a pushout in $\mathcal{Alg}(\mathcal{O})$ of the form*

$$\begin{tikzcd} F_{\mathcal{Alg}(\mathcal{O})}(X) \arrow[r, "F_{\mathcal{Alg}(\mathcal{O})}(k)"] \arrow[d] & F_{\mathcal{Alg}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

the morphism $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a global equivalence.

Proof. The generating acyclic cofibration $k \in K$ is of the form $\iota_{\rho_{G,V,W}} \square i_l$ for a compact lie group G, a faithful G-representation $V \neq 0$, a G-representation W, and $l \geq 0$. We also use $X = (L_{G,V\oplus W} \times D^l) \cup_{L_{G,V\oplus W}\times \partial D^l} (M_{\rho_{G,V,W}} \times \partial D^l)$ and $Y = M_{\rho_{G,V,W}} \times D^l$ for the source and target of k.

Consider again the filtration of [SS12, Proposition A.16], where $U^{\mathcal{O}}_0 = U_{\mathcal{Alg}(\mathcal{O})}$. For each $i \geq 1$, we have the following pushout in $\mathcal{Spc}$:

$$\begin{tikzcd} U^{\mathcal{O}}_i(A) \boxtimes_{\Sigma_i} Q^i_{i-1}(k) \arrow[r, "U^{\mathcal{O}}_i(A)\boxtimes_{\Sigma_i} k^{\square i}"] \arrow[d] & U^{\mathcal{O}}_i(A) \boxtimes_{\Sigma_i} (Y)^{\boxtimes i} \arrow[d] \\ P_{i-1}U^{\mathcal{O}}_0(B) \arrow[r, "f_i"] & P_i U^{\mathcal{O}}_0(B) \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

For each $i \geq 1$, by Proposition 3.1.8 the morphism $\iota^{\square i}_{\rho_{G,V,W}}$ is a Σ_i-global equivalence, and by Proposition 3.1.5 it is a Σ_i-flat cofibration. Then $k^{\square i} = \iota^{\square i}_{\rho_{G,V,W}} \square i^{\square i}_l$ is a Σ_i-flat cofibration by Corollary 2.1.9.1 and Corollary 3.1.1.1. It is also a Σ_i-global equivalence by Corollary 2.3.2.1

Then $U^{\mathcal{O}}_i(A) \boxtimes k^{\square i}$ is a Σ_i-h-cofibration and so $U^{\mathcal{O}}_i(A) \boxtimes_{\Sigma_i} k^{\square i}$ is an h-cofibration of orthogonal spaces by Lemma 2.1.11.

Since $k^{\square i}$ is a Σ_i-global equivalence by Corollary 2.2.9.1 the morphism $U^{\mathcal{O}}_i(A) \boxtimes k^{\square i}$ is also a Σ_i-global equivalence. Consider the Σ_i-orthogonal space $L^{\boxtimes i}_{G,V} \cong L_{G^i,V^i}$. For each inner product space U the group G^i acts freely (since V is faithful), smoothly and properly (since G^i is compact) on $\underline{\mathrm{L}}(V^i, U)$, as long as $|U| \geq |V|^i$. Therefore $L_{G^i,V^i}(U) = \underline{\mathrm{L}}(V^i, U)/G^i$ is Hausdorff, and since V^i is a faithful Σ_i-representation, $L_{G^i,V^i}(U)$ is also Σ_i-free.

If $|W| < |V|^i$, $\underline{\mathrm{L}}(V^i, W)$ is empty, so in particular $L_{G^i,V^i}(W)$ is still Hausdorff and Σ_i-free.

Now we denote by π the projection of the mapping cylinder $M_{\rho_{G,V,W}} \to L_{G,V}$, then $* \boxtimes \pi^{\boxtimes i} \times *\colon U_i^{\mathcal{O}}(A) \boxtimes (M_{\rho_{G,V,W}})^{\boxtimes i} \times (D^l)^{\times i} \to * \boxtimes L_{G,V}^{\boxtimes i} \times * = L_{G^i,V^i}$ is a Σ_i-equivariant map of orthogonal spaces, and so by Proposition 2.2.5 we have that $U_i^{\mathcal{O}}(A) \boxtimes_{\Sigma_i} k^{\Box i}$ is a global equivalence.

Therefore each f_i is also an h-cofibration and a global equivalence, and then $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is an infinite composition of morphisms which are h-cofibrations and global equivalences, so it is a global equivalence. □

All the previous results come together to give a model structure on $\mathcal{Alg}(\mathcal{O})$.

Theorem 3.1.10 (Theorem I)**.** *If $\mathcal{O}$ is any operad in $\mathcal{Spc}$ the category of orthogonal spaces, with the positive global model structure, then there is a cofibrantly generated model structure on $\mathcal{Alg}(\mathcal{O})$ the category of algebras over $\mathcal{O}$, where the forgetful functor $U_{\mathcal{Alg}(\mathcal{O})}$ creates the weak equivalences and fibrations, and sends cofibrations in $\mathcal{Alg}(\mathcal{O})$ to h-cofibrations in $\mathcal{Spc}$.*

Proof. We want to apply Theorem 1.2.2. Let $Hcof$ be the class of h-cofibrations of orthogonal spaces. Then by Remark 2.1.10 with $G = e$ the class $Hcof$ is closed under retracts and transfinite compositions, by Lemma 1.3.4 condition 2. is satisfied, and by Corollary 2.3.3.1 with $G = e$ condition 3. is satisfied.

Then consider a pushout in $\mathcal{Alg}(\mathcal{O})$ of the form:

$$\begin{tikzcd} F_{\mathcal{Alg}(\mathcal{O})}(X) \arrow[r, "F_{\mathcal{Alg}(\mathcal{O})}(i)"] \arrow[d] & F_{\mathcal{Alg}(\mathcal{O})}(Y) \arrow[d] \\ A \arrow[r, "f"] & B \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

If $i \in I$ is a generating cofibration of the positive global model structure we checked on Proposition 3.1.2 that $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is in $Hcof$. If $i \in J$ is a generating acyclic cofibration, then on Proposition 3.1.3 we checked that $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a global equivalence, and lastly if $i \in K$ is a generating acyclic cofibration then on Proposition 3.1.9 we checked that also $U_{\mathcal{Alg}(\mathcal{O})}(f)$ is a global equivalence.

Thus all the conditions of Theorem 1.2.2 are satisfied, which means that then the conditions of Theorem 1.2.1 are satisfied. Then $\mathcal{Alg}(\mathcal{O})$ is a cofibrantly generated model category, where $U_{\mathcal{Alg}(\mathcal{O})}$ creates the weak equivalences and fibrations, the generating cofibrations are the maps $F_{\mathcal{Alg}(\mathcal{O})}(i)$ for $i \in I$, the generating acyclic cofibrations are $F_{\mathcal{Alg}(\mathcal{O})}(j)$ where $j \in J \cup K$, and by Theorem 1.2.2 the forgetful functor $U_{\mathcal{Alg}(\mathcal{O})}$ sends cofibrations to h-cofibrations. □

Remark 3.1.11. Let $\mathcal{OP}$-$\mathscr{C}$ denote the category of operads on $\mathscr{C}$. Then there is an adjunction between symmetric objects and operads (F_{op}, U_{op}), where $F_{op}\colon \Sigma_*\text{-}\mathscr{C} \to \mathcal{OP}\text{-}\mathscr{C}$ and $U_{op}\colon \mathcal{OP}\text{-}\mathscr{C} \to \Sigma_*\text{-}\mathscr{C}$. The operad $F_{op}(M)$ is called the free operad associated with M.

A semi model structure (or J-semi model structue) is a slightly weaker notion than that of a model structure, see the definition in [Spi01, Definition 1]. Then for any symmetric monoidal cofibrantly generated model category $\mathscr{C}$, in [Spi01, Theorem 3] it is checked that $\mathcal{OP}$-$\mathscr{C}$ has the structure of a cofibrantly generated semi model structure. The fibrations and the weak equivalences are the morphisms of operads $g\colon \mathcal{O} \to \mathcal{P}$ such that for each $n \geq 0$ the morphism $g_n\colon \mathcal{O}_n \to \mathcal{P}_n$ is a fibration or a weak equivalence respectively.

In [Spi01, Theorem 4] it is proven that in a symmetric monoidal cofibrantly generated model category which satisfies the monoid axiom (in orthogonal spaces it is satisfied, see [Sch18, Proposition 1.4.13]), and for any cofibrant operad $\mathcal{O}$, there is a cofibrantly generated model structure on $\mathcal{Alg}(\mathcal{O})$ where the forgetful functor creates weak equivalences and fibrations.

In contrast, the previous theorem proves that in the case of orthogonal spaces no such cofibrancy condition is required.

Cofibrant operads in $\mathcal{OP}$-$\mathscr{C}$ are generally built out of cells of the form $F_{op}(\Sigma_n \otimes i)$, for i a generating cofibration of $\mathscr{C}$. This implies additionally that a cofibrant operad $\mathcal{O}$ in $\mathcal{Spc}$ would be such that each $\mathcal{O}_n$ is Σ_n-free. That $\mathcal{O}$ is cofibrant is quite restrictive, in fact in other contexts, usually if one wants to study most examples of operads and the homotopy of its algebras, one has to first take a cofibrant replacement in $\mathcal{OP}$-$\mathscr{C}$.

A morphism of cofibrant operads g which is a weak equivalence on $\mathcal{OP}$-$\mathscr{C}$ will induce a Quillen equivalence between their categories of algebras (see for example [Fre09, Theorem 12.5.A]). We will see in the following section that for arbitrary operads in orthogonal spaces there is a necessary and sufficient condition for g to induce a Quillen equivalence. This will also show why it is not enough to simply take a cofibrant replacement of an operad in orthogonal spaces $\mathcal{O}$ to obtain the correct homotopy theory of $\mathcal{Alg}(\mathcal{O})$.

3.2 Morphisms of operads which induce Quillen equivalences

We return now temporarily to the general setting of a symmetric monoidal category $(\mathscr{C}, \otimes, *)$, where the tensor product preserves all colimits on both variables. We want to consider a map of operads $g\colon \mathcal{O} \to \mathcal{P}$, that is a morphism of monoids on $(\Sigma_*\text{-}\mathscr{C}, \circ, I)$, or equivalently a morphisms of symmetric objects in $\mathscr{C}$ which respects the multiplication and unit.

Such a morphism of operads induces an adjoint pair of functors between their respective categories of algebras. The restriction functor $g^*\colon \mathcal{Alg}(\mathcal{P}) \to \mathcal{Alg}(\mathcal{O})$ is the right adjoint, and the extension functor $g_!\colon \mathcal{Alg}(\mathcal{O}) \to \mathcal{Alg}(\mathcal{P})$ its left adjoint. See [Fre09, Section 3.3.5] for the details.

We are interested in determining, for operads in orthogonal spaces, under which conditions on g this adjoint pair is a Quillen equivalence between the model structures on the categories of algebras constructed on Theorem 3.1.10.

The restriction functor is very straightforward. For an algebra X over $\mathcal{P}$, then X is an algebra over the monad $\mathcal{F}(\mathcal{P})$ with structure map $\zeta_X\colon \mathcal{F}(\mathcal{P})(X) \to X$. The morphism of operads g induces a natural transformation $\theta\colon \mathcal{F}(\mathcal{O}) \Rightarrow \mathcal{F}(\mathcal{P})$, and then X is a $\mathcal{P}$-algebra with structure map $\zeta_X \circ \theta(X)$. The explicit structure of the extension functor will not be relevant, only that it is left adjoint to the restriction functor.

In this section, we will need to consider in more detail the functors $U_k^{\mathcal{O}}$ from [SS12, Proposition 10.1], for $k \geq 0$, where $U_0^{\mathcal{O}} = U_{\mathcal{Alg}(\mathcal{O})}$. The functor $U_k^{\mathcal{O}}$ goes from $\mathcal{Alg}(\mathcal{O})$ to $\underline{\Sigma_k}$-$\mathcal{Spc}$.

Remark 3.2.1. Let $\mathcal{O}, \mathcal{P}$ be two operads, and let g be a morphism of operads $g\colon \mathcal{O} \to \mathcal{P}$. For a general $\mathcal{O}$-algebra X, a $\mathcal{P}$-algebra Y, and a map of $\mathcal{O}$-algebras $\gamma\colon X \to g^*(Y)$, we would like to construct a map $g_{k,\gamma}\colon U_k^{\mathcal{O}}(X) \to U_k^{\mathcal{P}}(Y)$ in $\underline{\Sigma_k}$-$\mathcal{Spc}$, in a way that is natural on γ, and in a way that preserves filtered colimits. It is important to note that the morphism $g_{k,\gamma}$ will not in general be $U_k^{\mathcal{O}}(\gamma)$. In fact $U_k^{\mathcal{O}} \circ g^*$ is generally not the same as $U_k^{\mathcal{P}}$, so they won't have the same target. Only for $k = 0$ will $g_{0,\gamma}$ and $U_0^{\mathcal{O}}(\gamma)$ be actually equal.

Consider the construction of the functors $\mathcal{O}(-,k)\colon \mathcal{S}pc \to \underline{\Sigma_k}\text{-}\mathcal{S}pc$ in [SS12, Section A.9], for an operad $\mathcal{O}$ and $k \geq 0$:

$$\mathcal{O}(X,k) = \coprod_{n\in\mathbb{N}} \mathcal{O}(n+k) \boxtimes_{\Sigma_n} X^{\otimes n}$$

Note that $\mathcal{O}(-,0) = \mathcal{F}(\mathcal{O})$. The morphism of operads g induces natural transformations $\theta_k\colon \mathcal{O}(-,k) \Rightarrow \mathcal{P}(-,k)$, with $\theta_0 = \theta$.

That $\gamma\colon X \to g^*(Y)$ is a map of $\mathcal{O}$-algebras precisely means that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(X,0) \arrow[r, "\zeta_X"] \arrow[d, "\mathcal{O}(\gamma,0)"] & X \arrow[drr, "\gamma"] & & \\
\mathcal{O}(Y,0) \arrow[r, "\theta(Y)"] & \mathcal{P}(Y,0) \arrow[rr, "\zeta_Y"] & & Y
\end{tikzcd} \tag{6}$$

Then also consider the construction of the functors $U_k^{\mathcal{O}}$ in [SS12, Definition A.10], as the following coequalizer:

$$\mathcal{O}(\mathcal{O}(X,0),k) \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} \mathcal{O}(X,k) \longrightarrow U_k^{\mathcal{O}}(X)$$

γ induces Σ_k-equivariant maps between the coequalizer diagrams that define $U_k^{\mathcal{O}}(X)$ and $U_k^{\mathcal{P}}(Y)$, these are $\mathcal{P}(\gamma,k) \circ \theta_k(X) = \theta_k(Y) \circ \mathcal{O}(\gamma,k)\colon \mathcal{O}(X,k) \to \mathcal{P}(Y,k)$, and the similar one from $\mathcal{O}(\mathcal{O}(X,0),k)$ to $\mathcal{P}(\mathcal{P}(Y,0),k)$. Then they commute with the morphism ∂_1 of the coequalizer, which is induced by the structure map ζ, because of the commutativity of Diagram (6) and the naturality of θ_k. They commute with ∂_0 because ∂_0 is natural and because g is a map of operads, so it preserves their multiplication and unit, which are used to construct ∂_0. Then the induced map on the coequalizers is $g_{k,\gamma}$.

This construction preserves filtered colimits because the functors of type $\mathcal{O}(-,k)$ preserve them, which is the case because tensor powers preserve filtered colimits.

We now go back to the case of $\mathcal{S}pc$, the category of orthogonal spaces with the positive global model structure.

Lemma 3.2.2. *For any morphism $g\colon \mathcal{O} \to \mathcal{P}$ of operads in $\mathcal{S}pc$, and considering the model structures on $\mathcal{A}lg(\mathcal{O})$ and $\mathcal{A}lg(\mathcal{P})$ obtained on Theorem 3.1.10, the restriction functor g^* preserves and reflects fibrations and weak equivalences. Thus the pair $(g_!, g^*)$ is a Quillen adjunction.*

Proof. The restriction functor is the identity on the underlying objects of the algebras, that is $U_{\mathcal{A}lg(\mathcal{O})} \circ g^* = U_{\mathcal{A}lg(\mathcal{P})}$. Since on $\mathcal{A}lg(\mathcal{O})$ the fibrations and the weak equivalences are precisely those morphisms f such that $U_{\mathcal{A}lg(\mathcal{O})}(f)$ is a fibration (respectively a weak equivalence), we have that g^* preserves and reflects fibrations, acyclic fibrations and weak equivalences.

That the right adjoint g^* preserves fibrations and acyclic fibrations is one of the possible characterizations of Quillen adjunctions. $\square$

Lemma 3.2.3. *Let $F\colon \mathscr{C} \rightleftarrows \mathscr{D} : G$ be a Quillen adjunction, such that the right adjoint G preserves and creates weak equivalences. Then the pair (F, G) is a Quillen equivalence if and only if for each for each cofibrant $A \in \mathscr{C}$ the adjunction unit η_A is a weak equivalence.*

Proof. The pair (F, G) is a Quillen equivalence if and only if for each cofibrant $A \in \mathscr{C}$ and fibrant $B \in \mathscr{D}$, a morphism of $\mathscr{C}$ of the form $\gamma\colon A \to G(B)$ is a weak equivalence if and only if its adjoint morphism $F(A) \to B$ is one. We can decompose γ as $A \to G(F(A)) \to G(B)$. G preserves and reflects weak equivalences, so if η_A is a weak equivalence then by the 2-out-of-3 property $\gamma\colon A \to G(B)$ is a weak equivalence if and only if its adjoint morphism $F(A) \to B$ is one.

In the other direction, Let $A \in \mathscr{C}$ be cofibrant, and let B be a fibrant replacement of $F(A)$ in $\mathscr{D}$, given by $\delta\colon F(A) \to B$. We have that δ is a weak equivalence, so if the pair (F, G) is a Quillen equivalence then $A \to G(B)$ the adjoint morphism of δ is also a weak equivalence. Since G preserves weak equivalences, the unit $\eta_A\colon A \to G(F(A))$ is a weak equivalence. $\square$

Note that since the composition of left adjoints is again a left adjoint, and as we saw in the proof of Lemma 3.2.2, $U_{\mathcal{Alg}(\mathcal{O})} \circ g^* = U_{\mathcal{Alg}(\mathcal{P})}$, we have that $g_! \circ F_{\mathcal{Alg}(\mathcal{O})}$ is naturally isomorphic to $F_{\mathcal{Alg}(\mathcal{P})}$.

Now we give the characterization of when a morphism of operads in $\mathcal{Spc}$ gives a Quillen equivalence between their respective categories of algebras.

Theorem 3.2.4 (Theorem II)**.** *Let $g\colon \mathcal{O} \to \mathcal{P}$ be a morphism of operads in $\mathcal{Spc}$. Then we have that the pair $(g_!, g^*)$ is a Quillen equivalence between their respective categories of algebras if and only if for each $n \geq 0$ the morphism $g_n\colon \mathcal{O}_n \to \mathcal{P}_n$ is a Σ_n-global equivalence.*

Proof. We first check that if the condition is satisfied then for each cofibrant $A \in \mathcal{Alg}(\mathcal{O})$ the unit $\eta_A\colon A \to g^*(g_!(A))$ is a weak equivalence in $\mathcal{Alg}(\mathcal{O})$, that is, a global equivalence of underlying orthogonal spaces.

First assume that A is the colimit of a λ-sequence of morphisms $\{f_\beta\}_{\beta\in\lambda}$, where each f_β is a cobase change of a morphism of the form $F_{\mathcal{Alg}(\mathcal{O})}(i_\beta)$ for $i_\beta \in I$, $i_\beta\colon X_\beta \to Y_\beta$ a generating flat cofibration of orthogonal spaces. We want to check that $U_{\mathcal{Alg}(\mathcal{O})}(\eta_A)$ is a global equivalence.

If we evaluate the unit of the adjunction η on the λ-sequence that gives rise to A we obtain the following diagram:

$$\begin{tikzcd}
A_0 = \mathcal{O}_0 \arrow[r, "f_0"] \arrow[d, "\eta_{A_0}"] & A_1 \arrow[r, "f_1"] \arrow[d, "\eta_{A_1}"] & \dots \arrow[r] & A_\beta \arrow[r, "f_\beta"] \arrow[d, "\eta_{A_\beta}"] & \dots \arrow[r] & {} \\
g^*(g_!(A_0)) \arrow[r, "g^*(g_!(f_0))"] & g^*(g_!(A_1)) \arrow[r, "g^*(g_!(f_1)))"] & \dots \arrow[r] & g^*(g_!(A_\beta)) \arrow[r, "g^*(g_!(f_\beta))"] & \dots \arrow[r] & {}
\end{tikzcd}$$

We apply $U_{\mathcal{Alg}(\mathcal{O})}$ to the whole diagram. By Proposition 3.1.2 for each $\beta \in \lambda$, the morphism $U_{\mathcal{Alg}(\mathcal{O})}(f_\beta)$ is an h-cofibration. $g_!$ preserves pushouts, so $g_!(f_\beta)$ is a cobase change of $g_!(F_{\mathcal{Alg}(\mathcal{O})}(i_\beta))$ which is isomorphic to $F_{\mathcal{Alg}(\mathcal{P})}(i_\beta)$. Since $U_{\mathcal{Alg}(\mathcal{O})} \circ g^* = U_{\mathcal{Alg}(\mathcal{P})}$, we know that $U_{\mathcal{Alg}(\mathcal{O})}(g^*(g_!(f_\beta)))$ is an h-cofibration by applying Proposition 3.1.2 now in $\mathcal{Alg}(\mathcal{P})$.

If each of the $U_{\mathcal{Alg}(\mathcal{O})}(\eta_{A_\beta})$ is a global equivalence, since $U_{\mathcal{Alg}(\mathcal{O})}$ preserves filtered colimits, we have that $U_{\mathcal{Alg}(\mathcal{O})}(\eta_A)$ is $\operatorname{colim}_{\beta\in\lambda} U_{\mathcal{Alg}(\mathcal{O})}(\eta_{A_\beta})$, then by Proposition 2.3.3 with $G = e$ the morphism $U_{\mathcal{Alg}(\mathcal{O})}(\eta_A)$ is a global equivalence, and we are done.

To check that each $U_{\mathcal{Alg}(\mathcal{O})}(\eta_{A_\beta})$is a global equivalence we follow the proof of the similar statement [SS12, Lemma 9.13]. We will check this by induction, but we in fact need to work with a stronger property. For each β and each $k \geq 0$, let $g_{k,\beta}$ be the morphism $g_{k,\eta_{A_\beta}}$ constructed on Remark 3.2.1. We will check by induction on β that for each $k \geq 0$ the morphism $g_{k,\beta} \colon U_k^{\mathcal{O}}(A_\beta) \to U_k^{\mathcal{P}}(g_!(A_\beta))$ is a Σ_k-global equivalence. For $k = 0$ this reduces to our desired result.

For the base case, remember that the initial object of $\mathcal{Alg}(\mathcal{O})$ is $\mathcal{O}_0$, since it is $F_{\mathcal{Alg}(\mathcal{O})}(\emptyset)$. Take $A_0 = \mathcal{O}_0 = F_{\mathcal{Alg}(\mathcal{O})}(\emptyset)$. Then by [SS12, Lemma A.13] the Σ_k-orthogonal space $U_k^{\mathcal{O}}(F_{\mathcal{Alg}(\mathcal{O})}(\emptyset))$ is isomorphic to $\mathcal{O}(\emptyset, k)$, and $\mathcal{O}(\emptyset, k)$ equals $\mathcal{O}_k$. Similarly $g_!(F_{\mathcal{Alg}(\mathcal{O})}(\emptyset)$ is isomorphic to $F_{\mathcal{Alg}(\mathcal{P})}(\emptyset)$, and then $U_k^{\mathcal{P}}(F_{\mathcal{Alg}(\mathcal{P})}(\emptyset)) \cong \mathcal{P}(\emptyset, k) = \mathcal{P}_k$, and under these identifications, the morphism $g_{k,0}$ corresponds to g_k, which is a Σ_k-global equivalence.

Remarkably, no conditions on the morphism of operads g are required anywhere else on the proof.

Then we check the induction step, for a successor ordinal $\beta+1$. For this we will use the filtration of [SS12, Proposition A.16], in the same way that it is used in the proof of [SS12, Lemma 9.13]. Assume that for each $k \geq 0$ the morphism $g_{k,\beta}$ is a Σ_k-global equivalence.

$$\begin{tikzcd}
U_k^{\mathcal{O}}(A_\beta) = F_0 U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[r] \arrow[d, "g_{k,\beta}"] & F_1 U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \operatorname*{colim}_{j\in\mathbb{N}} F_j U_k^{\mathcal{O}}(A_{\beta+1}) = U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[d, "g_{k,\beta+1}"] \\
U_k^{\mathcal{P}}(g_!(A_\beta)) = F_0 U_k^{\mathcal{P}}(g_!(A_{\beta+1})) \arrow[r] & F_1 U_k^{\mathcal{P}}(g_!(A_{\beta+1})) \arrow[r] & \dots \arrow[r] & \operatorname*{colim}_{j\in\mathbb{N}} F_j U_k^{\mathcal{P}}(g_!(A_{\beta+1})) = U_k^{\mathcal{P}}(g_!(A_{\beta+1}))
\end{tikzcd} \tag{7}$$

Each horizontal map is a cobase change of $U_{j+k}^{\mathcal{O}}(A_\beta) \boxtimes_{\Sigma_j} i_\beta^{\square j}$ or $U_{j+k}^{\mathcal{P}}(g_!(A_\beta)) \boxtimes_{\Sigma_j} i_\beta^{\square j}$, since $f_\beta \colon A_\beta \to A_{\beta+1}$ is a cobase change of $F_{\mathcal{Alg}(\mathcal{O})}(i_\beta)$. Therefore each horizontal map is then a Σ_k-h-cofibration since the j-fold pushout product of i_β is a Σ_k-h-cofibration, and these are closed under cobase changes.

Each vertical map is obtained from the previous by the following morphism of pushout diagrams:

$$\begin{tikzcd}
F_{j-1} U_k^{\mathcal{O}}(A_{\beta+1}) \arrow[d] & U_{j+k}^{\mathcal{O}}(A_\beta) \boxtimes_{\Sigma_j} Q_{j-1}^j(i_\beta) \arrow[l] \arrow[r] \arrow[d] & U_{j+k}^{\mathcal{O}}(A_\beta) \boxtimes_{\Sigma_j} (Y_\beta)^{\boxtimes j} \arrow[d] \\
F_{j-1} U_k^{\mathcal{P}}(g_!(A_{\beta+1})) & U_{j+k}^{\mathcal{P}}(g_!(A_\beta)) \boxtimes_{\Sigma_j} Q_{j-1}^j(i_\beta) \arrow[l] \arrow[r] & U_{j+k}^{\mathcal{P}}(g_!(A_\beta)) \boxtimes_{\Sigma_j} (Y_\beta)^{\boxtimes j}
\end{tikzcd}$$

The right horizontal maps are Σ_k-h-cofibrations. By the induction hypothesis $g_{j+k,\beta} \colon U_{j+k}^{\mathcal{O}}(A_\beta) \to U_{j+k}^{\mathcal{P}}(g_!(A_\beta))$ is a Σ_{j+k}-global equivalence. By Proposition 2.2.5 and Corollary 2.2.9.1, and using the same arguments as in the proof of Proposition 3.1.9, we can then check that the two rightmost vertical maps are Σ_k-global equivalences.

Then we can use induction on j and the Gluing Lemma 2.3.1 to check that each vertical map of (7) is also a Σ_k-global equivalence. Since each horizontal map of Diagram (7) is a Σ_k-h-cofibration, by Proposition 2.3.3 we have that $g_{k,\beta+1}$ is a Σ_k-global equivalence.

If β is a limit ordinal, we just need to use Proposition 2.3.3, and that $g_{k,\beta}$ is the map induced on the colimits $\operatorname{colim}_{\alpha\in\beta} g_{k,\alpha}$ since its construction preserves filtered colimits.

Then we have proven that $g_{k,\beta}$ is a Σ_k-global equivalence for each k and β. Setting $k = 0$ we have our original intended result, and therefore $U_{\mathcal{Alg}(\mathcal{O})}(\eta_A)$ is a global equivalence.

If $A \in \mathcal{Alg}(\mathcal{O})$ is cofibrant, then it is a retract of an algebra A' of the kind we were considering before, and the unit η_A is a retract of $\eta_{A'}$. Since retracts preserve weak equivalences, η_A is a weak equivalence in $\mathcal{Alg}(\mathcal{O})$.

We now prove that if $(g_!, g^*)$ is a Quillen equivalence then for each $n \geq 0$ the morphism g_n is a Σ_n-global equivalence. Consider the free orthogonal space $\underline{\mathrm{L}}(\mathbb{R}, -)$, which is positively flat. Then $F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -))$ is cofibrant in $\mathcal{Alg}(\mathcal{O})$. Since $(g_!, g^*)$ is a Quillen equivalence the unit $\eta_{F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R},-))} \colon F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -)) \to g^*(g_!(F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -))))$ is a weak equivalence, so its underlying morphism of orthogonal spaces is a global equivalence.

The $\mathcal{P}$-algebra $g_!(F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -)))$ is naturally isomorphic to $F_{\mathcal{Alg}(\mathcal{P})}(\underline{\mathrm{L}}(\mathbb{R}, -))$. After composing with g^* of this isomorphism, the unit is $F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -)) \to g^*(F_{\mathcal{Alg}(\mathcal{P})}(\underline{\mathrm{L}}(\mathbb{R}, -)))$, and its underlying morphism of orthogonal spaces is precisely $\theta(\underline{\mathrm{L}}(\mathbb{R}, -)) \colon F_{\mathcal{Alg}(\mathcal{O})}(\underline{\mathrm{L}}(\mathbb{R}, -)) \to F_{\mathcal{Alg}(\mathcal{P})}(\underline{\mathrm{L}}(\mathbb{R}, -))$.

So we know that $\coprod_{n \in \mathbb{N}} g_n \boxtimes_{\Sigma_n} \underline{\mathrm{L}}(\mathbb{R}, -)^{\boxtimes n}$ is a global equivalence. Therefore each $g_n \boxtimes_{\Sigma_n} \underline{\mathrm{L}}(\mathbb{R}, -)^{\boxtimes n}$ is a global equivalence. If $n = 0$ we obtain that g_0 is a global equivalence. For each $n \geq 1$, $\underline{\mathrm{L}}(\mathbb{R}, -)^{\boxtimes n} \cong \underline{\mathrm{L}}(\mathbb{R}^n, -)$, and the orthogonal space $\underline{\mathrm{L}}(\mathbb{R}^n, -)$ is Σ_n-free and Hausdorff on each inner product space V. Thus by Proposition 2.2.5 the morphism $g_n \boxtimes \underline{\mathrm{L}}(\mathbb{R}^n, -)$ is a Σ_n-global equivalence for each $n \geq 1$.

The morphisms $\rho_{\mathcal{O}_n, \underline{\mathrm{L}}(\mathbb{R}^n,-)}$ and $\rho_{\mathcal{P}_n, \underline{\mathrm{L}}(\mathbb{R}^n,-)}$ are Σ_n-global equivalences by Proposition 2.2.9 and Lemma 2.2.7 v). By the 2-out-of-3 property of Σ_n-global equivalences we obtain that $g_n \times \underline{\mathrm{L}}(\mathbb{R}^n, -)$ is a Σ_n-global equivalence.

$$\begin{tikzcd}
\mathcal{O}_n \boxtimes \underline{\mathrm{L}}(\mathbb{R}^n, -) \arrow[r, "g_n \boxtimes \underline{\mathrm{L}}(\mathbb{R}^n, -)"] \arrow[d, "\rho_{\mathcal{O}_n, \underline{\mathrm{L}}(\mathbb{R}^n,-)}"'] & \mathcal{P}_n \boxtimes \underline{\mathrm{L}}(\mathbb{R}^n, -) \arrow[d, "\rho_{\mathcal{P}_n, \underline{\mathrm{L}}(\mathbb{R}^n,-)}"] \\
\mathcal{O}_n \times \underline{\mathrm{L}}(\mathbb{R}^n, -) \arrow[r, "g_n \times \underline{\mathrm{L}}(\mathbb{R}^n, -)"] & \mathcal{P}_n \times \underline{\mathrm{L}}(\mathbb{R}^n, -)
\end{tikzcd}$$

We want to check that g_n is a Σ_n-global equivalence. Consider a compact Lie group K, a K-representation V, and a continuous homomorphism $\phi \colon K \to \Sigma_n$. Let $(\alpha \colon \partial D^l \to \mathcal{O}_n(V)^\phi, \beta \colon D^l \to \mathcal{P}_n(V)^\phi)$ be a lifting problem.

$\mathbb{R}^n$ is an orthogonal Σ_n-representation, where $\sigma \in \Sigma_n$ acts by permuting the canonical basis. The action of $\sigma \in \Sigma_n$ that permutes the factors on $\underline{\mathrm{L}}(\mathbb{R}, -)^{\boxtimes n} \cong \underline{\mathrm{L}}(\mathbb{R}^n, -)$ is precisely precomposition with σ^{-1}. We pull back the Σ_n-action on $\mathbb{R}^n$ through ϕ to turn $\mathbb{R}^n$ into a K-representation, and let ψ_V be the K-equivariant summand embedding $V \to V \oplus \mathbb{R}^n$.

Then we have that $\underline{\mathrm{L}}(\mathbb{R}^n, V \oplus \mathbb{R}^n)^\phi$ is non-empty, since the summand embedding $\psi_n \colon \mathbb{R}^n \to V \oplus \mathbb{R}^n$ is fixed by ϕ. Thus the map $\psi_n \colon * \to \underline{\mathrm{L}}(\mathbb{R}^n, V \oplus \mathbb{R}^n)^\phi$ gives the following lifting problem:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, "i_l"'] & \mathcal{O}_n(V)^\phi \arrow[r, "\mathcal{O}_n(\psi_V)^\phi"] \arrow[d, "g_n(V)^\phi"'] & \mathcal{O}_n(V \oplus \mathbb{R}^n)^\phi \arrow[r, "\mathcal{O}_n(V \oplus \mathbb{R}^n)^\phi \times \psi_n"] \arrow[d, "g_n(\psi_V)^\phi"'] & \mathcal{O}_n(V \oplus \mathbb{R}^n)^\phi \times \underline{\mathrm{L}}(\mathbb{R}^n, V \oplus \mathbb{R}^n)^\phi \arrow[d, "g_n(\psi_V)^\phi \times \underline{\mathrm{L}}(\mathbb{R}^n, V \oplus \mathbb{R}^n)^\phi"] \\
D^l \arrow[r, "\beta"] & \mathcal{P}_n(V)^\phi \arrow[r, "\mathcal{P}_n(\psi_V)^\phi"] & \mathcal{P}_n(V \oplus \mathbb{R}^n)^\phi \arrow[r, "\mathcal{P}_n(V \oplus \mathbb{R}^n)^\phi \times \psi_n"] & \mathcal{P}_n(V \oplus \mathbb{R}^n)^\phi \times \underline{\mathrm{L}}(\mathbb{R}^n, V \oplus \mathbb{R}^n)^\phi
\end{tikzcd}$$

Since $g_n \boxtimes \underline{\mathrm{L}}(\mathbb{R}^n, -)$ is a Σ_n-global equivalence, after embedding $V \oplus \mathbb{R}^n$ into some bigger K-representation W, this lifting problem has a solution $\lambda \colon D^l \to \mathcal{O}_n(W)^\phi \times \underline{\mathrm{L}}(\mathbb{R}^n, W)^\phi$. This means that postcomposing λ with the projection $\mathcal{O}_n(W)^\phi \times \underline{\mathrm{L}}(\mathbb{R}^n, W)^\phi \to \mathcal{O}_n(W)^\phi$ gives a solution of the lifting problem. Therefore g_n is a Σ_n-global equivalence. $\square$

Remark 3.2.5. This previous theorem generalizes, in the setting of orthogonal spaces, the general result that between cofibrant operads, a morphism of operads g induces a Quillen equivalence if the underlying morphism of each g_n is a weak equivalence (see [Fre09, 12.5.A]). For orthogonal spaces, and a morphism g between operads which are not necessarily cofibrant, we require the stronger condition that each g_n is not just a global equivalence, but also a Σ_n-global equivalence.

Given an operad in orthogonal spaces $\mathcal{O}$, we could take a cofibrant replacement of it in the semi model category $\mathcal{OP}\text{-}\mathcal{S}pc$. This would be a cofibrant operad $\mathcal{O}'$ and a morphism of operads $g\colon \mathcal{O}' \to \mathcal{O}$ such that each g_n is a global equivalence. But as we just saw, this g will not induce a Quillen equivalence between the categories of algebras of $\mathcal{O}$ and $\mathcal{O}'$, unless each g_n is additionally a Σ_n-global equivalence. This means that simply taking a cofibrant replacement$\mathcal{O}'$ in $\mathcal{OP}\text{-}\mathcal{S}pc$ of an operad $\mathcal{O}$ and considering the model structure given on $\mathcal{A}lg(\mathcal{O}')$ by general results doesn't give the correct homotopy theory of the algebras over $\mathcal{O}$.

Additionally, we can see that we cannot have a better cofibrant replacement functor $F^c\colon \mathcal{OP}\text{-}\mathcal{S}pc \to \mathcal{OP}\text{-}\mathcal{S}pc$, with a natural transformation $\eta\colon F^c \Rightarrow Id_{\mathcal{OP}\text{-}\mathcal{S}pc}$ such that each $\eta(\mathcal{O})_n$ is a Σ_n-global equivalence. Assume that this were the case, then consider a morphism of operads $g\colon \mathcal{O} \to \mathcal{O}'$ which satisfies that each g_n is a global equivalence, but doesn't satisfy that each g_n is a Σ_n-global equivalence. We have that each $F^c(g)_n$ is a global equivalence by the 2-out-of-3 property, so $F^c(g)$ induces a Quillen equivalence between $\mathcal{A}lg(F^c(\mathcal{O}))$ and $\mathcal{A}lg(F^c(\mathcal{O}'))$ because $F^c(\mathcal{O})$ and $F^c(\mathcal{O}')$ are cofibrant operads. The morphisms of operads $\eta(\mathcal{O})$ and $\eta(\mathcal{O}')$ would also induce Quillen equivalences by Theorem 3.2.4, But this would imply that g induces a Quillen equivalence between the categories of algebras, which contradicts the only if part of Theorem 3.2.4.

This means that in order to study the genuine homotopy theory of global operads, we cannot restrict ourselves to looking only at cofibrant operads.

3.3 Global E_∞-operads

Let $*$ be the terminal symmetric object on $\mathcal{S}pc$. Explicitly each $*_n$ is $*$ the constant one point orthogonal space. $*$ is also an operad in a trivial way, and it is in fact the terminal operad. Algebras over $*$ are precisely the commutative monoids on $\mathcal{S}pc$ with respect to the box product, which are called commutative orthogonal monoid spaces or ultracommutative monoids on [Sch18, Definition 1.4.14]. Note that if you expand the unit and multiplication maps, you obtain that a commutative monoid on $\mathcal{S}pc$ is precisely a lax symmetric monoidal functor $(\underline{\mathrm{L}}, \oplus) \to (\underline{\mathrm{Top}}, \times)$.

Definition 3.3.1. A global E_∞-operad is an operad $\mathcal{O}$ on $\mathcal{S}pc$ such that each $\mathcal{O}_n$ is Σ_n-globally equivalent to $*$ with the trivial Σ_n-action.

By Theorem 3.2.4 of the previous section, if $\mathcal{O}$ is a global E_∞-operad and g is the morphism of operads $\mathcal{O} \to *$, then the induced Quillen adjunction $(g_!, g^*)$ is a Quillen equivalence between $\mathcal{A}lg(\mathcal{O})$ and $\mathcal{A}lg(*)$, the category of ultracommutative monoids.

Lemma 3.3.2. *If $\mathcal{O}$ is a global E_∞-operad and A is an algebra over $\mathcal{O}$, then there is a zigzag natural in A of morphisms of $\mathcal{O}$-algebras from A to an ultracommutative monoid B, such that the underlying morphisms of orthogonal spaces are global equivalences.*

Proof. Let $g\colon \mathcal{O} \to *$, and let $\alpha\colon A' \to A$ be a cofibrant replacement of A on $\mathcal{Alg}(\mathcal{O})$. Then $U_{\mathcal{Alg}(\mathcal{O})}(\alpha)$ is a global equivalence, and the adjunction unit for A', $\eta_{A'}\colon A' \to g^*(g_!(A'))$, is a global equivalence on $\mathcal{Spc}$ by the proof of Theorem 3.2.4.

This is natural because factorizations in $\mathcal{Alg}(\mathcal{O})$ are functorial since they are obtained from the small object argument. □

This lemma lets us say some things about algebras over global E_∞-operads. For each orthogonal space X, one can define the G-equivariant homotopy set $\pi_0^G(X)$ for each compact Lie group G (see [Sch18, Definition 1.5.5]). These equivariant homotopy sets are contravariantly functorial on G, each continuous homomorphism $\alpha\colon K \to G$ induces a restriction map $\alpha^*\colon \pi_0^G \to \pi_0^K$.

We also have that the equivariant homotopy sets of an ultracommutative monoid R have certain additional operations, namely for each G a commutative binary operation $\pi_0^G(R) \times \pi_0^G(R) \to \pi_0^G(R)$ induced by the multiplication map, which turns $\pi_0^G(R)$ into a commutative monoid, and such that the restriction maps are monoid homomorphisms.

Additionally we also have power operations $[m]\colon \pi_0^G(R) \to \pi_0^{\Sigma_m \wr G}(R)$ (see [Sch18, Construction 2.2.3]) which turn $\pi_0^{(-)}(R)$ into a global power monoid ([Sch18, Definition 2.2.8]). These power operations then induce ([Sch18, Construction 2.2.29]) transfer maps $\mathrm{tr}_H^G\colon \pi_0^H(R) \to \pi_0^G(R)$ for each finite index closed subgroup $H \leq G$.

A global equivalence induces natural bijections on the equivariant homotopy sets. Thus by Lemma 3.3.2 we have the following proposition:

Proposition 3.3.3. *If $\mathcal{O}$ is a global E_∞-operad and A is an algebra over $\mathcal{O}$, then for each compact Lie group G, the homotopy set $\pi_0^G(A)$ is a commutative monoid. Additionally for each G and each $m \geq 1$ there are power operations $[m]\colon \pi_0^G(A) \to \pi_0^{\Sigma_m \wr G}(A)$ that turn $\pi_0^{(-)}(A)$ into a global power monoid, and which induce transfer maps $\mathrm{tr}_H^G\colon \pi_0^H(A) \to \pi_0^G(A)$ for each finite index closed subgroup $H \leq G$.*

Chapter 4

An example

4.1 A global operad

We now give an example of an operad $\mathcal{O}$ in $\mathcal{S}pc$, derived from the little disks operad. The little k-disks operad is the operad of rectilinear embeddings of k-dimensional disks into another k-dimensional disk. The limit of these operads is an E_∞-operad in spaces.

Construction 4.1.1. For the operad $\mathcal{O}$, each $\mathcal{O}_n$ is on an inner product space V the space of rectilinear embeddings of n copies of the open unit disk $D(V)$ into the unit disk itself. This is given by n points on $D(V)$ and n positive radiuses less than or equal to 1, that is, $\mathcal{O}_n(V)$ is the following, with the subspace topology:

$$\begin{gathered}\mathcal{O}_n(V) = \{\, (v_1, \dots, v_n, r_1, \dots, r_n) \in D(V)^n \times (0,1]^n : \\ \text{the map } x \in D(V)_i \mapsto v_i + r_i x \text{ is an embedding } \textstyle\coprod_{i=1}^n D(V)_i \to D(V) \,\}\end{gathered}$$

We have the following equivalent condition for the points and radiuses $(v_1, \dots, v_n, r_1, \dots, r_n)$ to give an embedding:

$$\forall v, v' \in D(V), 1 \le i \neq j \le n, \text{ it holds that } v_i + r_i v \neq v_j + r_j v' \text{ and } |v_i| + r_i \le 1 \tag{8}$$

Or also equivalently:

$$\forall\, 1 \le i \neq j \le n, \text{ it holds that } |v_i - v_j| \ge r_i + r_j \text{ and } |v_i| + r_i \le 1 \tag{9}$$

The structure map for $\psi\colon V \to W$ a linear isometric embedding is given by

$$(v_1, \dots, v_n, r_1, \dots, r_n) \mapsto (\psi(v_1), \dots, \psi(v_n), r_1, \dots, r_n)$$

which is a map $\mathcal{O}_n(V) \to \mathcal{O}_n(W)$, because ψ is a linear isometric embedding and because of the third condition (9). Functoriality is straightforward, so $\mathcal{O}_n$ is an orthogonal space. Furthermore, it is closed because each $\mathcal{O}_n(\psi)\colon \mathcal{O}_n(V) \to \mathcal{O}_n(W)$ is the restriction of a closed embedding. $\mathcal{O}_0$ is just $*$, the constant one point orthogonal space. Given an element $\sigma \in \Sigma_n$, the σ-action is given by the map $\mathcal{O}_n(V) \to \mathcal{O}_n(V)$ that permutes both the centers and the radiuses by σ, sending a tuple $(v_1, \dots, v_n, r_1, \dots, r_n)$ to $(v_{\sigma(1)}, \dots, v_{\sigma(n)}, r_{\sigma(1)}, \dots, r_{\sigma(n)})$. This is the same as precomposing by σ if we understand points in $\mathcal{O}_n(V)$ as maps $\{1, \dots, n\} \to D(V) \times (0,1]$. Therefore we have a right Σ_n-action on each $\mathcal{O}_n$, so $\mathcal{O}$ is a symmetric object in $\mathcal{S}pc$.

Now we construct the operadic structure. The unit is an element of $\mathcal{O}_1$ and it is given by a map from the monoidal unit $\eta\colon * \to \mathcal{O}_1$, which on level V is given by $* \mapsto (0,1) \in D(V) \times (0,1]$, representing the identity $id_{D(V)}$.

For each $k \geq 0$ and $n_1, \dots, n_k \geq 0$, let $m = n_1 + \dots + n_k$. Composition is given by maps $\circ\colon \mathcal{O}_k \boxtimes \mathcal{O}_{n_1} \boxtimes \dots \boxtimes \mathcal{O}_{n_k} \to \mathcal{O}_m$, obtained from multimorphisms given on levels $V_0, \dots, V_k$ by:

$$\begin{aligned} \mathcal{O}_k(V_0) \times \mathcal{O}_{n_1}(V_1) \times \dots \times \mathcal{O}_{n_k}(V_k) &\to \mathcal{O}_m(V_0 \oplus \dots \oplus V_k) \\ ((v_{0,1}, \dots, v_{0,k}, r_{0,1}, \dots, r_{0,k}), \dots, (v_{k,1}, \dots, v_{k,n_k}, r_{k,1}, \dots, r_{k,n_k})) &\mapsto (v_0, \dots, v_m, r_0, \dots, r_m) \end{aligned}$$

where for each $i = n_{j-1} + p$ we set $v_i \in V_0 \oplus \dots \oplus V_k$ as $v_i = (v_{0,j}, 0, \dots, 0, r_{0,j} v_{j,p}, 0, \dots, 0)$ and let $r_i = r_{0,j} r_{j,p}$.

This is indeed an element of $\mathcal{O}_{n_1+\dots+n_k}(V_0 \oplus \dots \oplus V_k)$, which can be checked using the second condition (8).

This is a multimorphism of orthogonal spaces since the actions of embeddings $\psi_j\colon V_j \to V_j'$ are on both sides given by applying each embedding to the respective $v_{j,p}$.

In fact, note that we have a little disks operad for each V inner product space, whose spaces are precisely the $\mathcal{O}_n(V)$, with composition $\circ_V$, and these are natural on V. To obtain the composition maps $\circ$ of $\mathcal{O}$ we are simply first applying the maps given by $\mathcal{O}_k(V_0 \to V_0 \oplus \dots \oplus V_k)$ and $\mathcal{O}_{n_j}(V_j \to V_0 \oplus \dots \oplus V_k)$, and then and then applying the composition $\circ_{V_0 \oplus \dots \oplus V_k}$.

More geometrically, we are considering the k disks in $V_0 \oplus \dots \oplus V_k$ given by the centers$(v_{0,1}, \dots, v_{0,k})$ in V_0, with radiuses $(r_{0,1}, \dots, r_{0,k})$, and then inside the jth disk, we are taking the subdisks like in the usual $V_0 \oplus \dots \oplus V_k$-disks operad given by $(v_{j,1}, \dots, v_{j,n_j}, r_{j,1}, \dots, r_{j,n_j})$, along the direction of V_j.

Therefore associativity follows from the naturality of $\circ_V$ on V, and the associativity of each composition $\circ_V$.

The obtained map is Σ_k-equivariant since on the left hand side Σ_k acts on the $v_{0,1}, \dots, v_{0,k}$, $r_{0,1}, \dots, r_{0,k}$, and permutes the blocks $(v_{j,1}, \dots, v_{j,n_j}, r_{j,1}, \dots, r_{j,n_j})$ by acting on j, which on the right hand side v_i, r_i gives exactly the action of the shuffle of the k blocks in $\Sigma_{n_1+\dots+n_k}$. Similarly it is $\Sigma_{n_1} \times \dots \times \Sigma_{n_k}$-equivariant because on both sides it acts as if permuting now the index p in the assignments $v_i = (v_{0,j}, 0, \dots, v_{j,p}, \dots, 0)$, and $r_i = r_{0,j} r_{j,p}$.

For the unit axiom, the composite $* \boxtimes \mathcal{O}_n \to \mathcal{O}_1 \boxtimes \mathcal{O}_n \to \mathcal{O}_n$, given by $\eta \boxtimes Id$ and then $\circ$, is on levels V_0, V_1 exactly $\mathcal{O}_n(V_1 \mapsto V_0 \otimes V_1)\colon\ * \times \mathcal{O}_n(V_1) \mapsto \mathcal{O}_n(V_0 \otimes V_1)$, which is the canonical isomorphism $* \boxtimes \mathcal{O}_n \to \mathcal{O}_n$, and the same holds for $\mathcal{O}_n \boxtimes *^{\boxtimes n} \to \mathcal{O}_n \boxtimes \mathcal{O}_1^{\boxtimes n} \to \mathcal{O}_n$.

The little disks operad is an E_∞-operad in spaces. With a similar reasoning we can prove that for the operad $\mathcal{O}$ that we just constructed, each $\mathcal{O}_n$ is globally contractible.

Proposition 4.1.2. *Each $\mathcal{O}_n$ is globally contractible.*

Proof. To check that $\mathcal{O}_n \to *$ is a global equivalence, we need to for each compact lie group G, orthogonal G-representation V and $l \geq 0$, find a solution for each lifting problem:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{O}_n(V)^G \arrow[d] \\
D^l \arrow[r] & *
\end{tikzcd}$$

Set ψ as the embedding $V \to V \oplus \mathbb{R}^{l+2}$, with trivial G-action on $\mathbb{R}^{l+2}$. Since V^G is a subspace of V, we have that $\mathcal{O}_n(V)^G \cong \mathcal{O}_n(V^G)$ because the fixed tuples $(v_1, \ldots, v_n, r_1, \ldots, r_n)$ are precisely those where every v_i lies on V^G, and similarly $\mathcal{O}_n(V \oplus \mathbb{R}^{l+2})^G \cong \mathcal{O}_n(V^G \oplus \mathbb{R}^{l+2})$. Furthermore, after fixing an isomorphism $V^G \oplus \mathbb{R}^{l+2} \cong \mathbb{R}^{|V^G|+l+2}$, $\mathcal{O}_n(V^G \oplus \mathbb{R}^{l+2})$ is weakly homotopically equivalent to the ordered configuration space $\mathrm{Conf}_n(D(\mathbb{R}^{|V^G|+l+2}))$ by the following Lemma 4.1.3. $\mathrm{Conf}_n(D(\mathbb{R}^{|V^G|+l+2}))$ is $(|V^G| + l + 2 - 2)$-connected by [FN62], so we have a solution to the lifting problem:

$$\begin{tikzcd}
\partial D^l \arrow[r, "\alpha"] \arrow[d, hook] & \mathcal{O}_n(V^G) \arrow[r, "\mathcal{O}_n(\phi^G)"] & \mathcal{O}_n(V^G \oplus \mathbb{R}^{l+2}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Conf}_n(D(\mathbb{R}^{|V^G|+l+2})) \\
D^l \arrow[rru, dashed] \arrow[rr] & & * &
\end{tikzcd}$$

Therefore each $\mathcal{O}_n$ is globally equivalent to $*$. $\square$

Lemma 4.1.3. *For each $n, m \geq 1$, the map $\pi\colon \mathcal{O}_n(\mathbb{R}^m) \to \mathrm{Conf}_n(D(\mathbb{R}^m))$ obtained from restricting the projection $p\colon D(\mathbb{R}^m)^n \times (0,1]^n \to D(\mathbb{R}^m)^n$ to $\mathcal{O}_n(\mathbb{R}^m)$ is a Serre fibration and a weak homotopy equivalence.*

Proof. First construct a map $c\colon \mathrm{Conf}_n(D(\mathbb{R}^m)) \to (0,2]^{n^2}$ by $(v_j) \mapsto (t_{j,j'})$ with $t_{j,j'} = |v_j - v_{j'}|$ if $j \neq j'$ and $t_{j,j} = 2 - 2|v_j|$. We denote the projection of $\mathcal{O}_n(\mathbb{R}^m) \to (0,1]^n$ onto the radiuses by r, and by X the subset:

$$X = \{\, (t_{j,j'}, r_j) \in (0,2]^{n^2} \times (0,1]^n : \forall 1 \leq j, j' \leq n \quad r_j + r_{j'} \leq t_{j,j'} \,\}$$

$\pi_1\colon X \subset (0,2]^{n^2} \times (0,1]^n \to (0,2]^{n^2}$ is the projection onto the first factor.

$\pi_2\colon X \subset (0,2]^{n^2} \times (0,1]^n \to (0,1]^n$ is the projection onto the second factor.

Then since $\pi_1 \circ ((c \circ p) \times r) = c \circ p$, the following diagram commutes.

$$\begin{tikzcd}
\mathcal{O}_n(\mathbb{R}^m) \arrow[r, "(c\circ p)\times r"] \arrow[d, "p"] & X \arrow[d, "\pi_1"] \\
\mathrm{Conf}_n(D(\mathbb{R}^m)) \arrow[r, "c"] & (0,2]^{n^2}
\end{tikzcd}$$

It is also a pullback, which can be seen on the point set level. We want to check that π_1 here is a Serre fibration and has contractible fibers. For the second part, we see that the fiber at $(t_{j,j'})$ is a convex subset of $(0,1]^n$. The map $s\colon (0,2]^{n^2} \to X$ given by $(t_{j,j'}) \mapsto (t_{j,j'}, s_j)$ where $s_j = \min_{1 \leq j' \leq n}\{\frac{t_{j,j'}}{2}\}$ is a section of π_1, so each fibre is contractible.

To check that π_1 is a Serre fibration, consider the following lifting diagram:

$$\begin{tikzcd} \{0\} \times [0,1]^l \arrow[r, "f"] \arrow[d] & X \arrow[d, "\pi_1"] \\ {[0,1] \times [0,1]^l} \arrow[r, "g"] \arrow[ur, dashed, "\lambda"] & (0,2]^{n^2} \end{tikzcd}$$

Let $\pi_2 \circ f = m$, and set $\mu \colon [0,1] \times [0,1]^l \to \mathbb{R}_n$ to be

$$\mu_j(t,x) = \min\{m_j(0,x), \min_{1 \le j' \le n}\{m_j(0,x) - 1/2|g(0,x)_{j,j'} - g(t,x)_{j,j'}|\}\}$$

Then for each $1 \le j, j' \le n$, $(t,x) \in [0,1] \times [0,1]^l$, we have that

$$\mu_j(t,x) + \mu_{j'}(t,x) \le m_j(0,x) + m_{j'}(0,x) - |g(0,x)_{j,j'} - g(t,x)_{j,j'}| \le$$

$$g(0,x)_{j,j'} - |g(0,x)_{j,j'} - g(t,x)_{j,j'}| \le g(t,x)_{j,j'}$$

so μ satisfies the bounds, and $\mu(0,x) = m(0,x)$.

The problem is that μ_j can be negative. But since $[0,1]^l$ is compact we can find an $\epsilon > 0$ such that $\pi_2(\mu([0,\epsilon] \times [0,1]^l)) \subset (\mathbb{R}_{>0})^n$, and then if we call $\delta \colon [0,1] \to [0,1]$ the function given by $\delta(t) = t/\epsilon$ on $[0,\epsilon]$ and 1 on $[\epsilon, 1]$, then $\lambda(t,x) = (1-\delta(t))[g(t,x) \times \mu(t,x)] + \delta(t)s(g(t,x))$ is a lift of the diagram. $\square$

4.2 A candidate for a global E_∞-operad.

We conjecture that the operad that we constructed in the previous section is actually a global E_∞-operad. To check this we would need to prove that for each n the morphism $\mathcal{O}_n \to *$ is a Σ_n-global equivalence, not just a global equivalence like we already proved. $\mathcal{O}$ has some useful properties, like how each $\mathcal{O}_n$ is Σ_n-free, and this result would let us rectify algebras over $\mathcal{O}$, which might be a more general class, into ultracommutative monoids in $\mathcal{S}pc$.

Given K a compact Lie group, V a K-representation, and $\phi \colon K \to \Sigma_n$ we want to consider $\mathcal{O}_n(V)^\phi$, and check that any map $\partial D^l \to \mathcal{O}_n(V)^\phi$ is nullhomotopic possibly after embedding V into some bigger representation $V \to W$. Tuples in $\mathcal{O}_n(V)^\phi$ have to consist of points in $V^{\ker\phi}$, and the K-action on V restricts to a K-action on $V^{\ker\phi}$ which factors through $\operatorname{Im}\phi \le \Sigma_n$.

Then we have $\mathcal{O}_n(V)^\phi = \mathcal{O}_n(V^{\ker\phi})^{\operatorname{Im}\phi}$, where $\operatorname{Im}\phi$ acts both by the Σ_n-action on $\mathcal{O}_n$ and the $(\operatorname{Im}\phi)$-action on $V^{\ker\phi}$. As in the previous section, this is weakly equivalent to $\operatorname{Conf}_n(D(V)^{\ker\phi})^{\operatorname{Im}\phi}$ where again $\operatorname{Im}\phi$ acts both by the Σ_n-action that permutes the points and the $(\operatorname{Im}\phi)$-action on $V^{\ker\phi}$.

We can further write $\operatorname{Conf}_n(D(V)^{\ker\phi})^{\operatorname{Im}\phi}$ as a product of orbit configuration spaces. Orbit configuration spaces are configuration spaces of m points on a space X with a free action by the group G where the points are not allowed to share an orbit. We use the notation $\operatorname{Conf}^G_m(X)$ for these orbit configuration spaces.

Proposition 4.2.1. *For each compact Lie group K, each K-representation V, each $n \ge 1$, and each continuous homomorphism $\phi \colon K \to \Sigma_n$, the space $\operatorname{Conf}_n(D(V)^{\ker\phi})^{\operatorname{Im}\phi}$ is naturally (on*

V) homeomorphic to $\prod_{\gamma\in\Gamma} \mathrm{Conf}_{m_\gamma}^{(N_{\mathrm{Im}\,\phi}\gamma)/\gamma}((D(V)^{\ker\phi})_\gamma)$ *where* Γ *is a set of representatives of conjugacy classes of subgroups of* $\mathrm{Im}\,\phi$*,* $(D(V)^{\ker\phi})_\gamma$ *is the subspace of points whose stabilizer is* γ*, and* $m_\gamma = |1 \leq i \leq n\colon (\mathrm{Im}\,\phi)_i \in [\gamma]|$ *where* $(\mathrm{Im}\,\phi)_i$ *is the stabilizer of* i *in the action of* $\mathrm{Im}\,\phi$ *on* $\{1,\ldots,n\}$*.*

The proof of this proposition is long and very technical, so it has been omitted.

To check that $\mathcal{O}$ is a global E_∞-operad we would need to check that each map from ∂D^l into $\mathrm{Conf}_n(V^{\ker\phi})^{\mathrm{Im}\,\phi}$ is nullhomotopic after possibly embedding V in some bigger K-representation.

The terminal operad $*$ itself is already a global E_∞-operad, but one of the benefits of having more examples would be to be able to check that orthogonal spaces which are not ultracommutative monoids are still algebras over a non-trivial global E_∞-operad.

Appendices

A Iterated pushout products of mapping cylinder inclusions

Let $(\mathscr{C}, \otimes, \mathbf{1}_{\mathscr{C}})$ be a closed cocomplete symmetric monoidal category which is tensored over Top. For a morphism $f\colon X \to Y$, we want to obtain a filtration, similar to the one of [SS12, Lemma A.8], that decomposes $\iota_f^{\square n}$, the n-fold pushout product of the inclusion $\iota_f\colon X \to M_f$ into the mapping cylinder. Let $\mathcal{Q}$, $\mathcal{K}$ and $\mathcal{P}$ be respectively the finite categories/posets:

$$0 \longrightarrow 4 \qquad \begin{tikzcd} 0 \arrow[r] & 1 \\ & 2 \arrow[u] \arrow[r] & 3 \end{tikzcd} \qquad \begin{tikzcd} 1 \\ 2 \arrow[u] \arrow[r] & 3 \end{tikzcd}$$

We can write M_f as the $\mathcal{K}$-shaped colimit of the functor $Z\colon \mathcal{K} \to \mathscr{C}$ given by the diagram:

$$\begin{tikzcd} X \arrow[r, "\iota_X"'] \arrow[rr, bend left, "\iota_f"] & X \amalg Y \arrow[r] & M_f \\ & X \amalg X \arrow[u, "id_X \amalg f"] \arrow[r, "X \times i_1"'] \arrow[ur, phantom, "\lrcorner"] & X \times [0,1] \arrow[u] \end{tikzcd} \tag{1}$$

As mentioned in [SS12, Lemma A.7], the n-fold pushout product of ι_f can be identified with $Q^n_{n-1}(\iota_f) \to M_f^{\otimes n}$, where $Q^n_{n-1}(\iota_f)$ is the colimit $\operatorname{colim}_{\alpha \in \mathcal{Q}^n_{n-1}} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n}$, and $\mathcal{Q}^n_{n-1}$ is the full subcategory of $\mathcal{Q}^{\times n}$ of objects $(\alpha_1, \dots, \alpha_n)$ which have at least one component equal to 0, and $Z_0 = X$ and $Z_4 = M_f$.

For each $0 \le j \le i \le n$, let $\mathcal{K}^n_{i,j} \subset \mathcal{K}^{\times n}$ denote the full subcategory of those objects $\alpha = (\alpha_1, \dots, \alpha_n)$ such that either (at least one component is 0 and at most i components are 3) or (at most j components are 3). For $j = -1$ and $0 \le i \le n$ let $\mathcal{K}^n_{i,j} \subset \mathcal{K}^{\times n}$ be those such that at least one component is 0 and at most i components are 3, and let $\mathcal{K}^n_{-1,-1} = \emptyset$ be the empty subcategory.

We write $K_{i,j}(Z)$ for the colimit $\operatorname{colim}_{\alpha \in \mathcal{K}^n_{i,j}} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n}$. Note that for $-1 \le j < i \le n-1$, $\mathcal{K}^n_{i+1,j} \cap \mathcal{K}^n_{i,j+1} = \mathcal{K}^n_{i,j}$ and $\mathcal{K}^n_{i+1,j} \cup \mathcal{K}^n_{i,j+1} = \mathcal{K}^n_{i+1,j+1}$ as subcategories, not just as subsets of objects, which implies that the following square diagram is a pushout diagram:

$$\begin{tikzcd}
K_{i,j}(Z) \arrow[r] \arrow[d] & K_{i,j+1}(Z) \arrow[d] \\
K_{i+1,j}(Z) \arrow[r] & K_{i+1,j+1}(Z)
\end{tikzcd}$$

Note also that for each $-1 \leq j \leq n-1$ $\mathcal{K}^n_{n-1,j} = \mathcal{K}^n_{n,j}$. This all fits together into the following commutative diagram:

$$\begin{tikzcd}
K_{-1,-1}(Z) = \emptyset \arrow[d] \arrow[dr] & & & & & \\
K_{0,-1}(Z) \arrow[r] \arrow[d] & K_{0,0}(Z) \arrow[d] \arrow[dr] & & & & \\
K_{1,-1}(Z) \arrow[r] \arrow[d] & K_{1,0}(Z) \arrow[r] \arrow[d] & K_{1,1}(Z) \arrow[d] \arrow[dr] & & & \\
\dots \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \dots \arrow[r] \arrow[d] & \dots \arrow[d] \arrow[dr] & & \\
K_{n-1,-1}(Z) \arrow[r] \arrow[d, equal] & K_{n-1,0}(Z) \arrow[r] \arrow[d, equal] & K_{n-1,1}(Z) \arrow[r] \arrow[d, equal] & \dots \arrow[r] \arrow[d, equal] & K_{n-1,n-1}(Z) \arrow[d, equal] \arrow[dr] & \\
K_{n,-1}(Z) \arrow[r] & K_{n,0}(Z) \arrow[r] & K_{n,1}(Z) \arrow[r] & \dots \arrow[r] & K_{n,n-1}(Z) \arrow[r] & K_{n,n}(Z)
\end{tikzcd} \tag{2}$$

Since the monoidal product preserves colimits on each variable, we can see that $K_{n,n}(Z) = \operatorname{colim}_{\alpha \in \mathcal{K}^n} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n} \cong (\operatorname{colim}_{\mathcal{K}} Z)^{\otimes n} = (M_f)^{\otimes n}$. We now check that $K_{n,-1}(Z)$ is precisely $Q^n_{n-1}(\iota_f)$ and the composition of all of the morphisms in the bottom row is precisely $\iota_f^{\square n}$.

$\mathcal{K}^n_{n,-1}$ is the full subcategory of those objects where at least one component is 0. For each $U \subsetneq \{1, \dots, n\}$ let $\mathcal{K}_U$ denote the full subcategory given by $\{\, \alpha \in \mathcal{K}^n \colon \alpha_i = 0 \ \forall i \notin U \,\}$. Then the $\mathcal{K}_U$ cover $\mathcal{K}^n_{n,-1}$ as subcategories, and by the universal property that defines the colimit there is a natural isomorphism $\operatorname{colim}_{\alpha \in \mathcal{K}^n_{n,-1}} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n} \cong \operatorname{colim}_{U \subsetneq \{1,\dots,n\}} (\operatorname{colim}_{\alpha \in \mathcal{K}_U} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n})$.

Since $\otimes$ preserves colimits, for each $U \subsetneq \{1, \dots, n\}$, $\operatorname{colim}_{\alpha \in \mathcal{K}_U} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n}$ is isomorphic to $Z_{\gamma^U_1} \otimes \cdots \otimes Z_{\gamma^U_n}$ where $\gamma^U_i = 0$ if $i \notin U$ and $\gamma^U_i = 4$ if $i \in U$, and $Z_0 = X$ and $Z_4 = M_f$. Therefore we obtain that $K_{n,-1}(Z) \cong \operatorname{colim}_{U \subsetneq \{1,\dots,n\}} (Z_{\gamma^U_1} \otimes \cdots \otimes Z_{\gamma^U_n}) = Q^n_{n-1}(\iota_f)$. Since the n-fold pushout product is the unique morphism obtained from the universal property of the colimits, it is also the composition of the bottom row of Diagram (2).

We can now put all of this together in the following lemma:

Lemma A.1. *Let $\mathscr{C}$ be a cocomplete symmetric monoidal category where the monoidal product preserves all colimits in both variables, which is tensored over* Top. *For a morphism $f \colon X \to Y$, let $\iota_f \colon X \to M_f$ denote the inclusion of X into the mapping cylinder of f.*

Then there is a filtration $Q^n_{n-1}(\iota_f) = K_{n,-1}(f) \to K_{n,0}(f) \to \cdots \to K_{n,n}(f) = (M_f)^{\otimes n}$ that decomposes $\iota_f^{\square n}$, the n-fold pushout product of ι_f, such that for each $0 \leq i \leq n$ the morphism

$K_{n,i-1}(f) \to K_{n,i}(f)$ is a Σ_n-equivariant cobase change of $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} Y^{\otimes n-i} \otimes (X \times i_1)^{\square i}$, where we use the convention that $(X \times i_1)^{\square 0}$ is the unique morphism $\emptyset \to *$ from the initial object to the monoidal unit.

Proof. The only thing left to check is that $K_{n,i-1}(f) \to K_{n,i}(f)$ is a cobase change of said map.

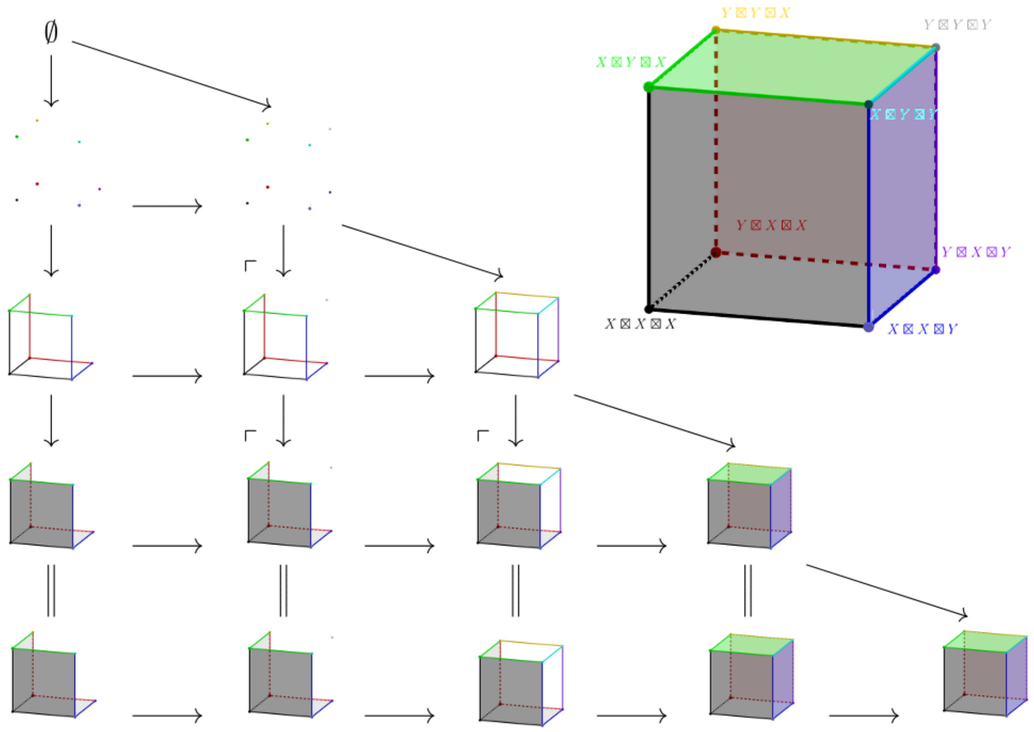
The sequence given by the diagonal of Diagram (2) is precisely the sequence $P_i^n(X \times i_1, id_X \amalg f)$ of [SS12, Lemma A.8] applied to the pushout square of Diagram (1). For each $i \geq 0$, $\mathcal{K}_{i,i}^n$ is the full subcategory of $\mathcal{K}^{\times n}$ of objects where at most i components are 3. We write $\mathcal{P}_i^n$ for the full subcategory of $\mathcal{P}^{\times n}$ of objects where at most i components are 3. If we regard $\mathcal{P}$ as a full subcategory of $\mathcal{K}$, then $\mathcal{P}_i^n \subset \mathcal{K}_{i,i}^n$ is a terminal subcategory in the sense of [Hir03, Definition 14.2.1]. By [Hir03, Theorem 14.2.5] this means that the canonical morphism $\operatorname{colim}_{\alpha\in\mathcal{P}_i^n} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n} \to \operatorname{colim}_{\alpha\in\mathcal{K}_{i,i}^n} Z_{\alpha_1} \otimes \cdots \otimes Z_{\alpha_n}$ is an isomorphism.

This means that by [SS12, Lemma A.8] for each $i \geq 1$ the morphism $K_{i-1,i-1}(Z) \to K_{i,i}(Z)$ is a Σ_n-equivariant cobase change of $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} (X \amalg Y)^{\otimes(n-i)} \otimes (X \times i_1)^{\square i}$.

Since $\otimes$ preserves colimits, $(X \amalg Y)^{\otimes i} \cong \coprod_{\beta\in\{0,1\}^i} B_{\beta_1} \otimes \cdots \otimes B_{\beta_i} \cong (\coprod_{\beta\in\{0,1\}^i, \beta\neq(1,\ldots,1)} B_{\beta_1} \otimes \cdots \otimes B_{\beta_i}) \amalg Y \otimes \cdots \otimes Y = Q_{i-1}^i(\iota_X) \amalg Y^{\otimes i}$, where $B_0 = X$ and $B_1 = Y$, and ι_X is the inclusion $X \to X \amalg Y$. Note that $\mathcal{K}_{0,0}^n$ has a terminal object, $(1,\ldots,1)$, so $K_{0,0}(Z)$ is precisely $(X \amalg Y)^{\otimes n}$, and with this identification $K_{0,-1}(Z)$ is $Q_{i-1}^i(\iota_X) = \coprod_{\beta\in\{0,1\}^i,\beta\neq(1,\ldots,1)} B_{\beta_1} \otimes \cdots \otimes B_{\beta_i}$.

We can decompose $(Q_{n-i-1}^{n-i}(\iota_X) \otimes (X \times i_1)^{\square i}) \amalg Y^{\otimes n-i} \otimes (X \times i_1)^{\square i}$ into $[(Q_{n-i-1}^{n-i}(\iota_X) \otimes (X \times [0,1])^{\otimes i}) \amalg Y^{\otimes n-i} \otimes (X \times i_1)^{\square i}] \circ [(Q_{n-i-1}^{n-i}(\iota_X) \otimes (X \times i_1)^{\square i}) \amalg Y^{\otimes n-i} \otimes Q_{i-1}^i(X \times i_1)]$, and we obtain that the morphism $K_{i,i-1}(Z) \to K_{i,i}(Z)$ is a cobase change of $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} Y^{\otimes n-i} \otimes (X \times i_1)^{\square i}$. By Diagram (2) $K_{n,i-1}(Z) \to K_{n,i}(Z)$ is also a cobase change of $\Sigma_n \times_{\Sigma_{n-i}\times\Sigma_i} Y^{\otimes n-i} \otimes (X \times i_1)^{\square i}$. If we adopt the convention that $g^{\square 0} = \emptyset \to *$ for any g then it also holds for $i = 0$. $\square$

Remark A.2. An illustration of the case $n = 3$. Each "point" of the cube is a copy of the product given by its color, so for example each point in the vertex, face and two edges colored green is a copy of $X \otimes Y \otimes X$.



B List of Notation

$\mathbf{1}_{\mathscr{C}}$ - Unit of the monoidal category $\mathscr{C}$.

$\mathcal{A}lg(\mathcal{O})$ - The category of algebras over the operad $\mathcal{O}$ 1.1.7.

$\boxtimes$ - The box product 2.1.5.

Conf_n - The configuration space of n points.

Conf_n^G - The orbit configuration space of n points in a free G-space.

$\mathcal{F}(-)$ - The functor associated to a symmetric object (1).

$F_{\mathcal{A}lg}(\mathcal{O})$ - The free algebra functor $\mathscr{C} \to \mathcal{A}lg(\mathcal{O})$.

$\mathscr{F}(K, G)$ - The set of graph subgroups given by homomorphisms $K \to G$ Section 2.2.

$\mathrm{Fun}(-,-)$ - The functor category between two ordinary categories, or the enriched functor category between two enriched categories.

$(g_!, g^*)$ - The Quillen adjunction associated to a morphism of operads ϕ 3.2.2.

$\underline{\mathrm{G}}$ - One-object-groupoid representing a group G.

I - Unit of Σ_*-$\mathscr{C}$ with the composition monoidal structure 1.1.3.

ι_f - The inclusion of the mapping cylinder of $f\colon X \to Y$, $\iota_f\colon X \to M_f$.

i_l - The canonical inclusion $\partial D^l \to D^l$

j_l - The canonical inclusion $[0,1]^l \to [0,1]^l \times [0,1]$

$\underline{\mathrm{L}}$ - Category of real inner product spaces and linear isometric embeddings.

$L_{G,V}$ - A semifree orthogonal space 1.3.2.

$L_{H,V;G}$ - A semifree G-orthogonal space 2.1.7.

M_f - Mapping cylinder of f.

$Q^n_{n-1}(f)$ - The source of the n-fold pushout product of f $f^{\square n}$ [SS12, Lemma A.7].

$\rho_{G,V,W}$ - The restriction morphism $L_{G,V\oplus W} \to L_{G,V}$ 1.3.3.

$\underline{\mathrm{Set}}$ - Category of sets.

Σ_*-$\mathscr{C}$ - The category of symmetric sequences in a symmetric monoidal category $\mathscr{C}$ 1.1.1.

tel_i - The mapping telescope of a sequence of maps.

$\underline{\mathrm{Top}}$ - Category of compactly generated weak Hausdorff topological spaces.

$U_{\mathcal{A}lg}(\mathcal{O})$ - The forgetful functor $\mathcal{A}lg(\mathcal{O}) \to \mathscr{C}$.

$U_k^{\mathcal{O}}$ - See 3.2.1.

$\Sigma_n \wr G$ - The wreath product of Σ_n and G. Note that we mean $\Sigma_n \wr_{\{1,\dots,n\}} G$ and not $\Sigma_n \wr_{\Sigma_n} G$.

$\otimes$, $\circ$ - In Σ_*-$\mathscr{C}$, the tensor product 1.1.2 and the composition product 1.1.3.

$\square$ - the pushout product of two morphisms.

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