Strategy-Proofness implies Minimal Participation if Voting is Costly*

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Abstract

We study a model in which agents with single-peaked preferences can participate in a costly voting procedure to determine the value of a one-dimensional variable. We show that, for all positive participation cost and all profiles of individual preferences, there exists a generically unique equilibrium with one single participant whenever the voting mechanism is strategy-proof, anonymous, and responsive in the sense that the outcome reacts to a unanimous move of the votes of all agents in the same direction. Since this single participant is never the median voter, we obtain as a corollary that no deterministic mechanism can implement the median if all participants vote according to their unique dominant strategy; even worse, the peak of the single participant in equilibrium is maximally far away from the median voter. By contrast, there are simple

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probabilistic strategy-proof mechanisms that yield the median as outcome. Inspired by this observation, we investigate if the median can be 'approximately implemented' by a natural deterministic counterpart, and come to a negative conclusion.

1 Introduction

One of the central possibility results in social choice theory, the median voter theorem, demonstrates that pairwise majority voting implements the median if voters' preferences are single-peaked. More generally, by a path-breaking result of Moulin (1980), there is an entire class of 'strategy-proof' social choice functions that are implementable in dominant strategies if all voters' preferences are single-peaked, the class of 'generalized' medians. Within this class the standard median has the important property that it minimizes the sum of the distances to the individual top alternatives. If individual utilities can be measured by objective distance, the median thus represents the utilitarian welfare optimum for the group of individuals.

However, this positive result assumes that voting is costless and that all voters therefore indeed participate in the collective decision making process. In this paper, we prove a general result on dominant strategy implementation under single-peakedness if voting is costly. We consider the class of anonymous voting rules that are strategy-proof for any given set of participants and satisfy a weak condition of responsiveness. We show that such voting rules can only have one of two possible outcomes: either the choice of the lowest, or the highest top alternative among the potential voters; moreover, in the (generically) unique dominant strategy equilibrium only one voter actually participates and casts a vote. This result holds independently of how participation costs are specified, as long as every voter prefers not to participate if the outcome does not change by her (unilateral) abstention.

Our main result thus uncovers a tension between two fundamental kinds of incentive properties of voting mechanisms: strategy-proofness, i.e. the incentive to reveal preferences truthfully, and participation, i.e. the incentive to invest the cost of casting one's vote. Strategy-proofness is desirable because without it the social outcome cannot be assessed vis-à-vis the true preferences of voters.[1] But for the same reason, participation is crucial because one cannot assess the merit of an alternative as

[1] For a closely related recent argument for strategy-proofness, see Dasgupta and Maskin (2019).

a *social* outcome if only few of the agents in fact provide information about their respective preferences even if these signals are truthful. Our analysis is based on the *pivotal voter model* (Downs, 1957; Palfrey and Rosenthal, 1983) which posits that rational agents will engage in a voting procedure if and only if the expected benefit of doing so exceeds its costs. Since the expected benefit can be positive only if an agent is indeed able to influence the outcome, the possibility of being pivotal is essential for a positive participation decision whenever participation comes even at a small but positive cost. On the other hand, strategy-proofness limits the extent to which an agent can influence the outcome; specifically, in our context strategy-proofness implies the property of *uncompromisingness*, i.e. voters cannot change the outcome by taking more extreme positions on the same side of the outcome, see e.g. Sprumont (1995).

Our analysis is embedded in a simple voting model with complete information. Our main departure from the bulk of the literature on the pivotal voter model is that we assume a rich one-dimensional space of alternatives in which individuals generically have different top alternatives. As we shall see, this radically changes the properties of equilibria. Of course, our assumption requires that there are more alternatives than voters, therefore our model applies to decisions in small committees and not to 'large' elections in which there are many more voters than alternatives. Agents take two decisions: whether or not to participate in the voting procedure, and if so, which vote to cast. We consider two versions of this general set-up: a sequential model and a simultaneous model. The sequential model has two stages: in the first stage agents simultaneously decide whether or not to participate in a committee, and in the second stage a simultaneous vote is cast by the committee members on the level of a one-dimensional variable. In the simultaneous model, the participation and voting decisions are made simultaneously by all agents. In either model, we assume that agents incur positive cost if and only if they in fact cast a vote. We shall see that while the two models may lead to different predictions in general, for our main result the timing of decisions is in fact not relevant. In order to ensure the existence of strategy-proof voting mechanisms, we will assume that voters' preferences are single-peaked. In this case, the class of all anonymous and strategy-proof mechanisms for a fixed number of participants has been characterized by Moulin (1980) as the *generalized median mechanisms*, or the median rule with 'phantom voters.'

As a simple example, think of a faculty meeting on a Friday afternoon at which the yearly expenditure shares, say, for research and teaching have to be determined (given a fixed budget). Each faculty member deliberates about whether or not to participate in the voting procedure. In the sequential model, one can think of the participation decision as being taken before the actual meeting. In the simultaneous model, there could be an announcement during the meeting that a vote would be taken after extensive discussion, and committee members may decide to leave early thereby abstaining from the collective decision. The assumption of complete information is strong but does not seem to be unrealistic in such a scenario; in fact, since all strategy-proof mechanisms only depend on the top alternative of each voter, it is sufficient to know each colleagues' top choice.

Our main result, Theorem 1, shows that in either version of the model there exists a generically unique equilibrium in which (at most) one single individual participates provided that the voting mechanism satisfies, in addition to strategy-proofness and anonymity, a condition of 'responsiveness' in the sense that the collective outcome reacts to a uniform strict increase (or strict decrease) of *all* votes. The identity of the single participant depends on the voting mechanism, but it is always either the agent with the lowest, or the agent with the highest top alternative.

The conclusion is in stark contrast to other voting mechanisms that are not strategy-proof. For instance, if the collective outcome is determined to be the *average* of the individual votes (Renault and Trannoy, 2005), every voter can shift the outcome by a positive amount and full participation is indeed an equilibrium if participation costs are sufficiently small. But while anonymous and responsive, taking the average does evidently not define a strategy-proof mechanism.

The intuition behind the single participation equilibrium under strategy-proofness is easily explained by looking at the case of two voters. Strategy-proofness and the responsiveness condition jointly imply that in the case of two voters the outcome must coincide with one of the two voters' top alternative; anonymity implies that it cannot depend on the identity of the voter, hence it must be either the lower or the higher top alternative. In the first case, if the agent with the lowest top alternative participates, no other agent has an incentive to participate (since costs are positive); in the latter case, the same holds if the agent with the highest top alternative participates. The proof that there are no other equilibria is more involved (see Section 3). Ultimately then, our main 'impossibility' result can be traced back to the fact that strategy-proofness necessitates the use of an asymmetric tie-breaking rule in the case of an even number of participants; and the counterfactual situation of an even number of participants is of course also relevant for the equilibrium even if the potential and actual number of voters is odd. This observation suggests that the incompatibility of strategy-proofness and (non-minimal) participation may vanish under a *probabilistic* and symmetric tie-breaking rule in the case of an even number of participants, and we show by means of a simple example that this is indeed true. A detailed analysis of the probabilistic case is however beyond the scope of the present paper and left to future work.

Relation to the Literature

The question of participation when voting is costly has been extensively discussed in the literature ever since the first formal formulation of the pivotal voter model by Palfrey and Rosenthal (1983), see among many others, Börgers (2004) for a seminal theoretical contribution and Levin and Palfrey (2007) for empirical results based on experimental data. The vast majority of the contributions in the literature studies the case of majority voting among two alternatives. Under complete information, a key issue is to analyze the equilibrium consequences of the fact that a large fraction of voters shares the same preferences. The resulting coordination problem is typically addressed by an analysis of mixed equilibria, see Nöldeke and Peña (2016); Mavridis and Serena (2018) for recent contributions. In the present paper, we set this problem aside by assuming that voters have distinct preferences (and preference tops) which is the generic case in our framework with a rich set of alternatives.

The paper closest to ours is Osborne *et al.* (2000). These authors also study costly voting in committees in a complete information environment with individuals that have single-peaked preferences. In their model, the authors assume that agents vote truthfully and show that 'extreme' voters are more likely to participate than 'moderate' voters. However, in the relevant results of Osborne *et al.* (2000) the outcome with an even number of participants is determined by the *symmetric* median, i.e. by the average of the two middle votes. But this specification does not give rise to a strategy-proof voting rule, and in fact sincere voting is no longer optimal in general. By contrast, in our model participants vote sincerely not by assumption but because truth-telling is a dominant strategy. While our main result does not strictly contradict the intuition of Osborne *et al.* (2000), it qualifies it in an important way. Under the responsiveness condition, the single participant is indeed always an 'extreme' voter: either the agent with the lowest or the agent with the highest top alternative; but as explained above, the rationale is not that the moderate voters cancel each other out. Moreover, without the responsiveness property, equilibria can occur in which only 'moderate' agents participate, see Section 3.1 below. In Section 4, we analyze optimal (but non-sincere) voting behavior under the symmetric median rule for an even number of participants. Theorem 2 provides a complete characterization of the subgame perfect equilibria of the sequential game if participation costs are sufficiently small but positive; taking the unit interval as outcome space, there are only two types of equilibria: the single participation of the voter whose peak is closest to the midpoint

0.5, or (almost) full participation with outcome 0.5.[2]

2 The model

We denote the set of agents by $N = \{1, 2, ..., n\}$. Each agent i is characterized by a single-peaked (ordinal) preference relation $\succcurlyeq_i$ over a compact interval in the reals which we assume to be normalized to unity, i.e. $[0, 1] \subseteq \mathbb{R}$. One possible interpretation is that each $0 \leq x \leq 1$ represents the expenditure share for a public project; but there are other, purely ordinal interpretations as well (e.g. potential positions on a political spectrum). In fact, none of our results depends on the assumption of a *continuous* space of alternatives; what is important is that there are sufficiently many more alternatives than agents.

Single-peakedness means that each agent i has a unique top alternative $0 \leq p_i \leq 1$ (the 'peak') such that, for all $x, y \in [0, 1]$, we have $x \succ_i y$ whenever $y < x \leq p_i$ or $p_i \leq x < y$.[3]

Agents decide whether or not to participate in a committee that decides on the level of the one-dimensional variable $x \in [0, 1]$ by a voting procedure. Each agent i faces a positive participation cost $c_i > 0$. This cost may vary from agent to agent, it may depend on the finally chosen outcome, and even on the set of the other participating agents; in fact, all what matters for our purpose is that these costs are strictly positive for all agents. In particular, we could allow the cost to be private information. For each possible non-empty set $K \subseteq N$ of participants, there is a social choice function $F^K(\succcurlyeq_1, ..., \succcurlyeq_{\#K}) \in [0, 1]$ that maps every profile of preferences of the agents in K to an outcome in $[0, 1]$. The collection $F = \{F^K\}_{\emptyset \neq K \subseteq N}$ of these social choice functions is referred to as a *voting mechanism*. The employed voting mechanism is

[2]In the latter case, if there is a voter with top alternative at 0.5 that voter will be the only one to abstain.

[3]Note that we do not need to make any assumptions about the comparisons of alternatives on different sides of the peak; in fact, the preference relation may even be incomplete and refrain from making such comparisons.

common knowledge among the agents.

In our model with endogenous participation, we need to specify agents' preferences $\widehat{\succcurlyeq}_i$ over pairs (x, K) of outcomes and sets of participants K who actually cast a vote. We denote by x_0 the (exogenously determined) outcome if nobody participates in the voting process, and will make the following assumptions. For all $i \in N$,

(i) the outcome x_0 is strictly worse than the most preferred outcome with single own participation, i.e.

$$(p_i, \{i\}) \mathrel{\widehat{\succ}_i} (x_0, \emptyset),$$

(ii) for every fixed set $K \neq \emptyset$ the preference over outcomes is given by $\succcurlyeq_i$, i.e.

$$(x, K) \mathrel{\widehat{\succcurlyeq}_i} (y, K) \iff x \succcurlyeq_i y,$$

(iii) for every fixed $x \in [0, 1]$, non-participation is strictly preferred to participation (and indifference with respect to the composition of the set of participants otherwise), i.e. for all $K, K' \neq \emptyset$,

$$\begin{aligned} \{i \notin K \quad \text{or } i \in K'\} &\implies (x, K) \mathrel{\widehat{\succcurlyeq}_i} (x, K'), \\ \{i \notin K \text{ and } i \in K'\} &\implies (x, K) \mathrel{\widehat{\succ}_i} (x, K'). \end{aligned}$$

Observe that, except for the 'non-triviality' condition (i), no assumptions are made about how agents compare an outcome without participation to a *strictly better* outcome with own participation; indeed, such trade-offs would determine the particular magnitude of participation cost which plays no role in our present analysis.

We will consider two variants of the model, a simultaneous and a sequential version. In the sequential version, agents first simultaneously decide whether or not to participate and vote simultaneously in a second stage after having observed who the other participants are. By contrast, in the simultaneous version, both the voting and participation decisions are made simultaneously. While the equilibria in general differ in the two models (see the discussion section below), the main conclusions of the present paper are robust with respect to the timing of decision.

In our main result, Theorem 1 in Section 3 below, we will require the voting mechanism to be anonymous and strategy-proof. The *anonymity* condition has two components: first, for each given set of participants K, the outcome under F^K is invariant with respect to permutations of the agents in K; secondly, the employed social choice function F^K should depend only on the number of agents in K. Using the latter condition, we can write F^k for all social choice functions F^K with $\#K = k$, and describe the voting mechanism $F = \{F^k\}_{1 \leq k \leq n}$ in terms of n social choice functions, one for each number of participants.

Strategy-proofness requires that truth-telling be a (weakly) dominant strategy for all participating agents: for all K, $i \in K$, $\succcurlyeq_i$, $\succcurlyeq_i'$, $\succcurlyeq_{K-i}$,

$$F^k(\succcurlyeq_i, \succcurlyeq_{K-i}) \;\; \succcurlyeq_i \;\; F^k(\succcurlyeq_i', \succcurlyeq_{K-i}),$$

where $k = \#K$ and $\succcurlyeq_{K-i}$ denotes any profile of preferences for the agents in K other than i.

By a famous result of Moulin (1980), the conditions of anonymity and strategy-proofness jointly imply that all social choice functions F^k are 'generalized medians' with $k+1$ so-called 'phantom voters.' Specifically, for each $k \in \{2, ..., n\}$, F^k only depends on the individual peaks, i.e. for some function $f^k : [0,1]^k \to [0,1]$

$$F^k(\succcurlyeq_1, ..., \succcurlyeq_k) \;\; = \;\; f^k(p_1, ..., p_k),$$

and there exist fixed values $\alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k \in [0,1]$ such that

$$f^k(p_1, ..., p_k) \;\; = \;\; med\{p_1, ..., p_k, \alpha_1^k, \alpha_2^k, ..., \alpha_{k+1}^k\}, \tag{2.1}$$

where *med* denotes the usual median operator and the p_i are the peaks of $\succcurlyeq_i$ for each participating agent i; note that there are $2k+1$, i.e. an odd number of values in (2.1). An important example is the standard median rule with an odd number of participants; in this case, half of the phantom voters are placed at 0 and half are placed at 1.

We will say that F^k, respectively f^k, satisfies *responsiveness* if, for all $p_1, ..., p_k$, $p'_1, ..., p'_k$,

$$p'_i > p_i \text{ for all } i \in K \quad \Rightarrow \quad f^k(p'_1, ..., p'_k) \neq f^k(p_1, ..., p_k).$$

Responsiveness can be viewed as a condition of 'local non-imposition:' if *every* agent desires a strictly higher (lower) outcome, the chosen alternative should move at least minimally.[4]

While arguably a weak and plausible condition, responsiveness does restrict the set of admissible voting mechanisms, as follows.

Observation. *The generalized median functions f^k in (2.1) satisfy responsiveness if and only if all 'phantom voters' are either at 0 or at 1, i.e. for all $j = 1, ..., k+1$, $\alpha^k_j \in \{0, 1\}$, and neither are all phantom voters located at 0 nor at 1. In particular, in this case the generalized median always coincides with one of the peaks of the agents and the corresponding voting mechanism is efficient.*

Proof. To verify this, suppose that, for some k and $j_0 \leq k+1$, one has $0 < \alpha^k_{j_0} < 1$. Clearly, for any given set of the other phantom voters α^k_j, $j \neq j_0$, one can choose peaks $p_i \in (0,1)$ all distinct from $\alpha^k_{j_0}$ such that $med\{p_1, ..., p_k, \alpha^k_1, ..., \alpha^k_{k+1}\} = \alpha^k_{j_0}$. But then the generalized median does not react to a sufficiently small uniform move of all peaks. The same is evidently true if all $k+1$ phantom voters are located either at 0 or at 1. Conversely, if all $k+1$ phantom voters are either at 0 or at 1, but not all of them at the same location, the generalized median must be one of the k peaks

[4]Intuitively, it should clearly move in the *same* direction; in the present context, this slightly stronger requirement is redundant because it follows from responsiveness plus strategy-proofness.

of the real agents; hence the underlying mechanism is efficient and must react to a uniform move of all peaks. □

It is well-known (Moulin, 1980) that under efficiency, the generalized median functions f^k in (2.1) can be assumed to have $k-1$ instead of $k+1$ phantom voters. Generalized medians for which all $k-1$ phantom voters are at one of the two extremes are also known as the *order statistics*. Specifically, the choice of the i-th lowest value of the $\{p_1, ..., p_k\}$ is referred to as the i-th order statistic, and corresponds to the generalized median in which $k-i$ phantom voters are at 0 and $i-1$ phantom voters are at 1, see Caragiannis *et al.* (2016) for further discussion.

In the present work, we are not focusing on the coordination problem that arises if several agents have the same top alternative. We therefore assume in all what follows that agents' peaks are in generic position, i.e. that no two peaks coincide: $p_i \neq p_j$ for all pairs $i, j \in N$ of distinct agents. If all social choice functions F^k are strategy-proof, voting truthfully is the unique (weakly) dominant strategy for every participant in the simultaneous game, and in every second-stage voting subgame of the sequential game. We will therefore assume that all participants who actually cast a vote submit their true peak. This assumption could be further justified by an appeal to an equilibrium refinement concept such as perfectness (Selten, 1975), or strong equilibrium (Aumann, 1959)[5].

3 Main result

The following is our main result.

Theorem 1. *Suppose that the voting mechanism is anonymous, strategy-proof and*

[5]In the implementation literature, there has been some discussion on the fact that that the median rule (as well as generalized medians) may have other Nash equilibria, in which agents do not follow their unique weakly dominant strategy; see, e.g., Yamamura and Kawasaki (2013); Núñez *et al.* (2020). For instance, if $k \geq 3$ and all agents cast exactly the same (non-truthful) vote nobody is pivotal, and hence such vote profile constitutes a Nash equilibrium. However, such equilibria are evidently neither robust against trembles, nor against deviations by coalitions of agents.

responsive, and that all individuals' voting costs are positive.

a) The simultaneous voting game has a unique perfect equilibrium in which exactly one agent participates.

b) The sequential voting game has a unique subgame perfect equilibrium in which all agents choose their unique (weakly) dominant strategy in the second stage; in this equilibrium, again exactly one agent participates.

In either model, the participating agent is either the individual with the highest peak, or the individual with the lowest peak.

Proof. The assumptions on the voting mechanism imply that, for all non-empty sets $K \subseteq N$ of participating agents, the outcome is determined by a generalized median with $\#K - 1$ phantom voters. Moreover, by anonymity, the set of phantom voters only depends on $k = \#K$.

We first show that this implies the existence of an equilibrium with a single participant. As before, denote by $p_1, ..., p_n$ the preference peaks of the agents, and assume without loss of generality that $p_1 < p_2 < ... < p_n$. For $\#K = 2$, there is one phantom voter α_1^2, and by the Observation in the previous section, we have either $\alpha_1^2 = 0$, or $\alpha_1^2 = 1$. Suppose the former, then the single participation of agent 1 (who reports truthfully) constitutes an equilibrium. Indeed, the outcome then is p_1 which by assumption is preferred by agent 1 to x_0 (the outcome if nobody participates). Every other agent has a higher peak and can thus not unilaterally change the outcome because $\alpha_1^2 = 0$; hence, if costs are positive each other agent prefers not to participate. The argument is completely symmetric if $\alpha_1^2 = 1$, in which case single participation of individual n is an equilibrium. Note that the argument applies to the dynamic model in the same way as to the simultaneous game.[6]

It remains to show that there are no other equilibria. By contradiction, suppose we have an equilibrium with the set $K \subseteq N$ of participants where $\#K > 1$. If this

[6]Of course, the complete strategy in the dynamic game also specifies, for each non-participant, truth-telling in all counterfactual participation situations.

situation constitutes an equilibrium, it is optimal for all $i \in K$ to participate; we will show that this is not possible. By re-numbering agents, we may assume that $K = \{1, ..., k\}$ and $p_1 < p_2 < ... < p_k$. By the Observation above, there exists $j \in K$ with $f^k(p_1, ..., p_k) = p_j$. First assume that $j = 1$, i.e. that the voter with the lowest peak gets her most preferred alternative. Then, voter k (the one with the highest peak among the participants) has an incentive to abstain; indeed, by the efficiency of f^{k-1} the outcome without voter k cannot be smaller than p_1. By a similar argument, one can show that $j \neq k$.

Thus, we must have that $1 < j < k$ for the individual j who receives her peak p_j. In this case, the assumed optimality of participation by agent 1 implies that

$$f^{k-1}(p_2, ..., p_k) \;=\; med\{p_2, ..., p_k, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} > p_j, \tag{3.2}$$

since otherwise agent 1 would prefer not to participate thereby saving the associated cost. Similarly, the assumed participation of agent k implies that

$$f^{k-1}(p_1, ..., p_{k-1}) \;=\; med\{p_1, ..., p_{k-1}, \alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\} < p_j. \tag{3.3}$$

Without agent 1 there are $j-1$ peaks that are below or equal to p_j. By (3.2), the generalized median $f^{k-1}(p_2, ..., p_k)$ with $k-1$ participants (i.e. agents 1 to $k-1$) is strictly above p_j; this implies that at most $(k-1-j)$ of the $k-2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at 0. Similarly, without agent k there are $k-j$ peaks above or equal to p_j. By (3.3), the generalized median $f^{k-1}(p_1, ..., p_{k-1})$ with $k-1$ participants (i.e. agents 2 to k) is strictly below p_j; this implies that at most $j-2$ of the $k-2$ phantom voters $\{\alpha_1^{k-1}, ..., \alpha_{k-2}^{k-1}\}$ can be located at 1. By the responsiveness, all of the $k-2$ phantom voters have to be located either at 0 or at 1. But we have just shown that under conditions (3.2) and (3.2) this is not possible since

$$(k-1-j) \;+\; (j-2) = k-3 \;<\; k-2.$$

Thus, there can be no equilibrium in which more that one agent participates. This concludes the proof of Theorem 1.

□

3.1 Discussion

In order to assess the scope and the robustness of Theorem 1, we now consider each of its assumptions. We explain why they are necessary for the conclusion and we discuss what happens if they were dropped.

Anonymity

Our anonymity condition has two components. First, it requires the voting mechanism not to depend on the 'names' of voters *for any given set of participants*; secondly, it requires that the same voting mechanism is employed for all subsets with the *same number* of participants. Arguably, both conditions are natural in the present context. The first part is a standard assumption in voting theory, and in fact Moulin's characterization of all strategy-proof mechanisms for single-peaked preferences in terms of phantom voters needs this assumption. In our present variable electorate context the second part also appears to be highly plausible. Importantly, it also guarantees the existence of an equilibrium in pure strategies (and its uniqueness). We show this by means of two simple examples, as follows.

Assume that all conditions of Theorem 1 are satisfied except the second part of the anonymity condition, and consider the following examples with $N = \{1, 2, 3\}$. Suppose that if the set of participants consists of agents 1 and 2, the outcome function $f^{\{1,2\}}$ chooses the higher peak, i.e. we have $\alpha_1^{\{1,2\}} = 1$ for the corresponding phantom voter; if the set of participants consists of agents 1 and 3, the outcome function $f^{\{1,3\}}$ chooses the lower peak, i.e. $\alpha_1^{\{1,3\}} = 0$ for the corresponding phantom voter; and finally, if the set of participants consists of agents 2 and 3, the outcome function $f^{\{2,3\}}$ chooses again the higher peak, i.e. $\alpha_1^{\{2,3\}} = 1$ for the corresponding phantom voter.

Evidently, this specification violates the (second part of the) anonymity condition. Suppose that agents are ordered so that $p_1 < p_2 < p_3$. For no agent single participation is an equilibrium: if agent 1 is the single voter, agent 2 has an incentive to join; if agent 2 is the single voter, agent 3 has an incentive to join; and if 3 is the single voter, agent 1 has an incentive to join. A situation with two participants cannot be an equilibrium either because, by the responsiveness condition, one of the two gets her peak in which case the other has an incentive to abstain and save the voting costs. Finally, full participation cannot be an equilibrium either. Indeed, suppose that all agents participate; again by the responsiveness condition, one of the agents must receive her peak. If agent 1 receives her peak, agent 2 has an incentive to abstain, because this would not change the outcome and agent 2 would save the participation cost; similarly, if agent 2 receives her peak, agent 3 has an incentive to abstain, and if agent 3 receives her peak, agent 1 has an incentive to abstain. Hence, in this example there is no equilibrium in pure strategies.

Here is an example in which there are several equilibria, including one with *full participation*. If all agents participate, the social choice function $f^{\{1,2,3\}}$ chooses the standard median, in other words, the corresponding phantom voters are given by $\alpha_1^{\{1,2,3\}} = 0$ and $\alpha_2^{\{1,2,3\}} = 1$; if agents 1 and 2 participate, the social choice function $f^{\{1,2\}}$ chooses the lower peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{1,2\}} = 0$; if agents 2 and 3 participate, the social choice function $f^{\{2,3\}}$ chooses the higher peak, i.e. the corresponding phantom voter is given by $\alpha_1^{\{2,3\}} = 1$. No matter how we specify the outcome in the case that the set of participants consists of agents 1 and 3, this already implies that for sufficiently small participation costs full participation is an equilibrium. Indeed, if all agents participate the agent with the median peak gets her peak and hence has no incentive to abstain if her participation costs are sufficiently small; for either of the other two agents, unilateral non-participation would move the outcome further away from their respective peak, so neither of them has an incentive to abstain as well. There also exists an additional single participation equilibrium.

Indeed, for the set of participants $\{1,3\}$ we either have $\alpha_1^{\{1,3\}} = 0$ or $\alpha_1^{\{1,3\}} = 1$. In the first case, single participation of the agent with the lowest peak is an equilibrium (since none of the other two agents can unilaterally change the outcome); in the second case, single participation of the agent with the highest peak is an equilibrium.

Responsiveness

Above, we have justified the responsiveness condition by an appeal to a 'local non-imposition' property: if all agents uniformly move in one direction, the outcome should not remain unchanged. We have also shown that this condition is equivalent to the property that all phantom voters should be at the two extreme points 0 or 1. There may exist an even deeper justification for the responsiveness condition in purely ordinal contexts. Indeed, if the set of alternatives is linearly ordered but in a purely ordinal way, any specific location of a phantom voter in the interior of the interval $[0,1]$ seems arbitrary. On the other hand, if cardinal information is available, such as in the example of the dividing a fixed budget, phantoms may be placed in the midpoint (at 0.5), or distributed uniformly in $[0,1]$ (the latter specification is also known as the 'uniform' or 'linear' median, see Jennings *et al.* (2020)). In any case, Theorem 1 fails without the responsiveness condition. As a simple example, consider the case of an even number of agents $N = \{1,2,...,2m\}$, and suppose that the phantom voters are located as follows: $m-1$ phantom voters are at 0, $m-1$ phantom voters are at 1, and one phantom voter is located in the interior, say at $x \in (0,1)$. Also, suppose that for any set of $2m-1$ participants the standard median is chosen as the outcome. Consider any distribution such that m agents have their peak below x and m agents have their peak above x. Then, if all individuals' costs are sufficiently small, full participation is an equilibrium. Indeed, the outcome under full participation is x, and for any agent unilateral abstention moves the outcome further away from her peak.

But even without the responsiveness condition, there often also exist profiles of peaks such that only one single agent participates in equilibrium. Specifically, let $p_1 < p_2 < ... < p_N$ be such that (i) $\{\alpha_j^k\}_{j=1,...,k+1}^{k=1,...,N} \cap [p_1, p_N] = \emptyset$, i.e. all phantom voters are either below the smallest peak or above the highest peak, and (ii) for all $k = 1, ..., N$, $\min_j\{\alpha_j^k\} < p_1$ and $p_N < \max_j\{\alpha_j^k\}$. (Note that condition (ii) is implied by efficiency of the voting mechanism.) Then, we obtain single participation as the unique equilibrium by the same logic as in the proof of Theorem 1.

Strategy-proofness

The assumption of strategy-proofness is essential for the conclusion of Theorem 1. Strategy-proofness together with a suitable refinement concept such as robustness against small trembles ('perfectness') implies that all participants will cast their vote truthfully. Evidently, the proof of Theorem 1 hinges on that property. In particular, also the fact that the equilibrium does not depend on individual costs (as long as they are all positive) crucially depends on the strategy-proofness (together with the responsiveness condition).

To illustrate this, suppose we can use cardinal information and employ the following *symmetric* version of the median rule. Let the peaks of the participants be ordered such that $p_1 < p_2 < ... < p_k$; for an odd number $k = 2m - 1$ the outcome is the standard median p_m, and for an even number $k = 2m$ the outcome is the midpoint between the two middle peaks $(p_m + p_{m+1})/2$. This rule is not strategy-proof, and therefore we cannot assume that the reported peaks coincide with the true peaks.[7] Consider the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$. Let $\tilde{p}_i$ be the reported peak by agent i, and assume that the participation costs of the two extreme agents 1 and 3 are small but positive. Then, the equilibrium depends, among other things, on the magnitude of the participation cost (i.e. the precise shape of the preferences) of the median agent. If the participation cost of agent 2 is sufficiently small,

[7] It is well-known that there exist no strategy-proof, anonymous and symmetric ('neutral') social choice functions on the domain of single-peaked preferences for an even number of individuals, see Moulin (1980, 1988).

full participation and truth-telling is an equilibrium in the simultaneous game; on the other hand, if agent 2 prefers the outcome 0.5 without own participation to the outcome 0.45 while participating, $\tilde{p}_1 = 0$ and $\tilde{p}_3 = 1$ is a (non-truthful) equilibrium.

Moreover, it is because of truthful voting in equilibrium that the conclusion of Theorem 1 is robust with respect to the timing of decisions. Indeed, in the sequential model participants' voting strategy may depend on the set of other participants which becomes common knowledge after all agents have made their participation decision. To illustrate this point, consider again the peak distribution $p_1 = 0.1$, $p_2 = 0.45$ and $p_3 = 0.9$ and the symmetric median rule, but now assume that agents move sequentially. In this case, full participation is no longer an equilibrium; indeed, if agent 3 does not participate, agents 1's and 2's optimal votes are $\tilde{p}_1 = 0$ and $\tilde{p}_2 = 0.9$, respectively, with the outcome 0.45. Since this is the same outcome as under full participation, agent 3 prefers to abstain whenever she has positive participation costs. However, participation of agents 1 and 2 with outcome 0.45 can also not be an equilibrium since then agent 1 has an incentive to abstain. In this example, if all agents have sufficiently small (but strictly positive) participation cost, the unique subgame perfect equilibrium of the sequential voting game is given by participation of agents 2 and 3 with votes $\tilde{p}_2 = 0$ and $\tilde{p}_3 = 1$, resulting in the outcome 0.5. A complete characterization of the equilibria under the symmetric median rule in the sequential model is provided in the next section under the assumption that in the second stage a strong Nash equilibrium is played.

4 On the possibility of implementing the median if voting is costly

One may interpret Theorem 1 as saying that, if voting is costly, no anonymous and deterministic voting rule can implement the median if all participants vote according to their unique dominant strategy. Specifically, we have the following corollary of

Theorem 1.

Corollary 1. *Suppose that the number of agents is odd. There does not exist an anonymous and strategy-proof voting mechanism that yields the median peak for all distributions of individual peaks if voting is costly and all actual participants vote according to their unique dominant strategy.*

To verify this simply note that by the remarks in Section 3.1, even without the responsiveness condition there always exist (generic) peak distributions for which the unique dominant strategy equilibrium involves the single participation either of the agent with the highest peak, or the agent with the lowest peak.

An important implicit assumption of our analysis is that we require voting mechanisms to be *deterministic*. One may argue that this is a strong assumption; and in fact, together with anonymity and strategy-proofness it forces an asymmetric tie-breaking rule in the case of an even number of participants. Consider the following probabilistic voting mechanism: for an odd number of participants choose the median peak, and for an even number choose a fair lottery that yields the two middle peaks with equal probability of 1/2. Moreover, suppose that all voters have expected utility preferences with a single-peaked von-Neumann-Morgenstern utility function. Then, for any fixed number of participants, it is the unique (weakly) dominant strategy to submit one's true preference peak as in the deterministic case. But unlike in the deterministic case, participation and truthful reporting now always changes the outcome to one's benefit at generic profiles.[8] In particular, at every generic profile with different peaks full participation with truthful reporting is an equilibrium if voting costs are sufficiently small.

With a fixed set of participants and without voting costs, strategy-proof probabilistic social choice rules with single-peaked preferences have been investigated by Ehlers *et al.* (2002) and Peters *et al.* (2014). A detailed analysis of the probabilistic

[8]We are grateful to Klaus Nehring and Martin Osborne who independently pointed this out to us.

case if voting is costly is beyond the scope of the present paper. But the question arises if one can implement the median in an 'approximate' sense using the symmetric median rule (and thus giving up strategy-proofness only for a fixed even number of participants). The following result provides a complete characterization of all subgame-perfect equilibria of the sequential game if costs are sufficiently small and answers this question to the negative. Note that in the case of an even number of participants truth-telling no longer constitutes a dominant strategy. To avoid the discussion of an artificial multiplicity of equilibria, we assume that in the second stage a strong Nash equilibrium is played (i.e. an equilibrium such that no subgroup of agents can profitably deviate); for every fixed number k of participants, a strong equilibrium exists (see Appendix). Say that a participant with peak $p_i \neq 0.5$ exhibits *extreme reporting* if she reports 0 if $p_i < 0.5$ and 1 if $p_i > 1$.

Theorem 2. *Consider the symmetric median rule and assume that individuals have sufficiently small but positive costs of participation. Then all subgame-perfect equilibria such that in the second stage a strong Nash equilibrium is played are of the following four types:*

> **Case 1.** *There exists an individual $j \in N$ with peak $p_j = 0.5$. Then single participation of individual j (who reports truthfully) is an equilibrium.*
>
> **Case 2.** *The peaks of all individuals are on the same side of 0.5, i.e. either $p_i \geq 0.5$ or $p_i \leq 0.5$ for all $i \in N$. Then single participation of the individual whose peak is closest to* 0.5 *(who reports truthfully) is an equilibrium.*
>
> **Case 3.** *The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them a peak strictly above* 0.5*. Then full participation with extreme reporting is an equilibrium and the outcome is* 0.5*.*
>
> **Case 4.** *The number of potential voters n is odd, there exists an individual j with peak $p_j = 0.5$ and the peaks of the other $n-1$ individuals are evenly split*

to both sides of 0.5*. Then it is an equilibrium that all individuals in* $N \setminus \{j\}$ *participate with extreme reporting; the resulting outcome is* 0.5*.*

In all other cases, there does not exist a pure strategy equilibrium such that in the second stage a strong Nash equilibrium is played.

(Proof in Appendix)

Note that there are cases in which the equilibrium outcome of the symmetric median rule is very far from the median peak. For instance, in Case 2 the median peak could be close to 0 while the equilibrium outcome is the highest peak which could be even at 0.5. The intuition behind this equilibrium is as follows. Suppose that only the voter with the highest peak participates and votes truthfully; clearly, this gives the best outcome for that voter (given her participation). If any of the remaining voters decides to participate, the voter with the highest peak can adjust her voting behavior in the sequential model and receive her peak again by optimally responding to the vote of the other participant. Thus, the outcome does not change hence none of the remaining voters has an incentive to participate. In Case 3, the outcome is always 0.5 while the symmetric median (i.e. the midpoint between the two middle peaks) could be arbitrarily close to 0.25 (resp. 0.75).

Finally, let us compare the findings of Theorem 2 with the intuition put forward by Osborne *et al.* (2000) that more extreme voters are more likely to participate. Cases 1, 3 and 4 do not confirm this intuition: in Case 1 the single participant is not determined be her being 'moderate' or 'extreme,' but simply by the fact that her peak is at 0.5; in Cases 3 and 4 we have (almost) full participation (albeit with extreme reporting). On the other hand, Case 2 comes closer to the intuition since the single participant is either the voter with the left-most or the right-most peak; however, within the spectrum of *conceivable* positions, this is also the voter with the most 'moderate' view among all potential voters since, by assumption, all of them are on the same side of 0.5.

5 Conclusion

Our main result reveals a strong tension between two kinds of incentive properties if voting is costly: participation and truthful reporting. If a voting mechanism is anonymous, strategy-proof and responsive the only equilibrium in which all participants follow their unique dominant strategy consists of the single participation of either the agent with the highest, or the agent with the lowest peak. In particular, there is no anonymous and strategy-proof deterministic voting mechanism that yields the median peak if participants vote according to their unique dominant strategy. While in this result the need for a tie-breaking rule in the case of an even number of actual participants plays an important role, it is remarkable that the result holds for any number of potential voters.

A possible way out of the problem is to consider probabilistic mechanisms, and indeed we have shown by means of example that a simple and natural strategy-proof probabilistic mechanism implements the median with full participation if voting costs are sufficiently small. A full fledged analysis of probabilistic mechanisms in the context of costly voting in committees appears to be a worthwhile subject for future work.

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Appendix: Proof of Theorem 2

We split the proof of Theorem 2 in two parts. Proposition 1 deals with Cases 1 and 2, while Proposition 2 proves the claim for Cases 3 and 4.

The following lemma ensures that in all subgames on the second stage of the sequential voting game there exists a strong Nash equilibrium and additionally, that if several strong Nash equilibria exist, they necessarily result in the same outcome.

Lemma 1. *Consider the symmetric median rule with a fixed number of participants k. There always exists a strong Nash equilibrium. Moreover,*

- *if k is odd, the outcome is the median of the peaks in all strong Nash equilibria.*
- *if k is even, the outcome is the median of $\{p_{\frac{k}{2}}, 0.5, p_{\frac{k}{2}+1}\}$ in all strong Nash equilibria, given that the peaks $p_1, ..., p_k$ are ordered.*

Proof. Let k be odd. Thus the symmetric median rule coincides with the median rule that is well-known to be strategy-proof. Hence there exists a strong Nash equilibrium, and in all strong equilibria the outcome is the median of the peaks of the participants.

Let k be even.

Case 1: $p_{\frac{k}{2}} < 0.5 < p_{\frac{k}{2}+1}$:

In this case, it is easily verified that there is a unique strong Nash equilibrium: the $k/2$ agents with peak lower than 0.5 vote for 0, while the $k/2$ agents with peak larger than 0.5 vote for 1, resulting in the outcome 0.5.

Case 2: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} \leq 50$:

In this case strong Nash equilibrium is not unique (unless $p_{\frac{k}{2}+1} = 50$) but all strong equilibria in fact result in the same outcome. In all strong equilibria the $k/2$ agents with the lowest peaks vote for 0, agent $\frac{k}{2}+1$ votes for $2 \cdot p_{\frac{k}{2}+1}$ (≤ 1), and all other agents submit a vote between $2 \cdot p_{\frac{k}{2}+1}$ and 1, resulting in the outcome $p_{\frac{k}{2}+1}$.

Case 3: $50 \leq p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

Analogously to Case 2, in all strong equilibria the $k/2$ individuals with the highest peaks vote for 1, individual $k/2$ votes for $2 \cdot p_{\frac{k}{2}} - 1$ (≥ 0), and all other individuals submit a vote between 0 and $2 \cdot p_{\frac{k}{2}} - 1$ resulting in the outcome $p_{\frac{k}{2}}$. □

As the symmetric median rule differentiates between odd and even number of participants, so do we in our equilibrium analysis, starting with odd number of participants.

Lemma 2. *Consider the symmetric median rule. There are no subgame-perfect equilibria in which a strong Nash equilibrium is played in the second stage with an odd number of participants greater than* 1.

Proof. By contradiction, let $k > 1$ be the number of participants and let k be odd. Then, in every (strong) Nash equilibrium, the median participant $i = \frac{k+1}{2}$ determines the outcome by truthfully revealing her peak $p_{\frac{k+1}{2}}$.

Case 1: $p_{\frac{k+1}{2}} = 0.5$:

Then, there are $\frac{k-1}{2}$ participants with a peak below 0.5 and $\frac{k-1}{2}$ participants with a peak above 0.5. Hence if agent $\frac{k+1}{2}$ abstains, the outcome will be the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+3}{2}}$ by Lemma 1, that is the outcome will be 0.5. Thus, since the outcome would not change, agent $\frac{k+1}{2}$ has an incentive to abstain.

Case 2: $p_{\frac{k+1}{2}} < 0.5$:

If an agent $i > \frac{k+1}{2}$ (i.e. an agent with peak above the outcome) abstains, the outcome becomes the median of $p_{\frac{k-1}{2}}$, 0.5 and $p_{\frac{k+1}{2}}$ by Lemma 1. But since $p_{\frac{k-1}{2}} < p_{\frac{k+1}{2}} < 0.5$, this means that the outcome does not change; hence, agent i will rather abstain.

Case 3: $p_{\frac{k+1}{2}} > 0.5$:

This case is symmetric to Case 2. □

With the help of Lemma 2, we can characterize all equilibria with an odd number of participants.

Proposition 1. *Consider the symmetric median rule. There exists a single participation equilibrium (with a strong Nash equilibrium played on the second stage) for all positive participation costs if and only if one of the following two conditions holds true:*

1. *There exists one individual $j \in N$ with a peak of $p_j = 0.5$.*

2. *The peaks of all individuals are on the same side of 0.5, i.e. either $p_i \geq 0.5$ for all $i \in N$ or $p_i \leq 0.5$ for all $i \in N$.*

In Case 1 the single participant is individual j, i.e. the individual with peak $p_j = 0.5$. In Case 2 the single participant is the individual whose peak is the closest to 0.5.

Proof. Case 1: It is easy to see that the existence of an individual j with peak $p_j = 0.5$ leads to a single participation equilibrium with this individual being the single participant. If j is the single participant, the outcome is $p_j = 0.5$. Whenever one individual joins, individual j will adapt her vote such that the outcome remains 0.5. Hence no individual (apart from j) has an incentive to participate. Moreover there is no other single participant equilibrium for all positive participation costs as individual j could move the outcome to 0.5 by joining which is profitable for her for sufficiently small participation costs.

Case 2: Assume that all peaks are on the same side of 0.5, w.l.o.g. assume $p_i \geq 0.5$ for all $i \in N$. Then by a similar argument one can show that there is only one single participation equilibrium in which individual 1 is the only participant. As before, if another individual decides to join her, she can adapt her vote such that the outcome doesn't change. Thus no individual has an incentive to join her. Moreover individual 1 has an incentive to join given that there is a different single participant as she can move the outcome to her peak which again is profitable for sufficiently small costs of participation. The case of all peaks below 0.5 can be dealt with by replacing individual 1 by individual n, who in this case is the individual with her peak closest to 0.5.

It remains to show that in all other cases there exists no single participation equilibrium for all positive participation costs. As neither Case 1 nor Case 2 apply, there is no individual with a peak of 0.5 and given ordered peaks we have $p_1 < 0.5 < p_n$, i.e. at least one individual with a peak strictly below 0.5 and at least one individual with a peak strictly above 0.5. Assume that there is a single participation equilibrium for all positive participation costs with an individual with a peak below 0.5 participating. Then individual n has an incentive to join, as she could move the outcome to 0.5 by participating, which is profitable for sufficiently small participation costs. Analogously, if there were a single participation equilibrium with an individual with peak above 0.5 participating, then individual 1 has an incentive to join for sufficiently small costs of participation. Hence there exists no single participation equilibrium for all positive participation costs if Cases 1 or 2 do not apply. □

One can easily see that if there exists a single-participation equilibrium for some participation cost, then this equilibrium remains an equilibrium if participation costs increase. Even with the lower costs of participation all non-participants preferred to abstain and the single-participant will never want to abstain. Thus we obtain that there are no other single-participation equilibria for small but strictly positive costs of participation, than the ones identified in Proposition 1.

We now turn to the analysis of an even number of participants, starting with a lemma stating that in a strong Nash equilibrium with a fixed even number of participants the outcome is 0.5. Moreover the peaks of the participants need to differ from 0.5 and need to be evenly distributed to both sides of 0.5.

Lemma 3. *Consider an equilibrium of the sequential participation game under the* *symmetric median rule with an even number of participants. Given that in the second stage participants play a strong Nash equilibrium, the outcome is* 0.5*. Moreover in all such equilibria half of the participants have a peak above and half of the participants have a peak below 0.5.*

Proof. Assume, by way of contradiction, that there exists an equilibrium of the required sort with an even number of participants k and an outcome that is different from 0.5. We will show that this is not possible. By Lemma 1, the outcome is the median of $p_{\frac{k}{2}}$, 0.5 and $p_{\frac{k}{2}+1}$. As the outcome is assumed to be different from 0.5, we must have either $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$ or $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$.

Case 1: $p_{\frac{k}{2}} < p_{\frac{k}{2}+1} < 0.5$:

In this case, Lemma 1 implies that the outcome is $p_{\frac{k}{2}+1}$. If an agent $i < \frac{k}{2} + 1$ (i.e. with a peak below $p_{\frac{k}{2}+1}$) abstains, then there are $\frac{k}{2} - 1$ participants with a peak below $p_{\frac{k}{2}+1}$ and $\frac{k}{2} - 1$ participants with a peak above $p_{\frac{k}{2}+1}$. Hence $j = \frac{k}{2} + 1$ is the median participant and the outcome is $p_{\frac{k}{2}+1}$. Since the outcome is unchanged, agent i has an incentive to abstain whenever her participation costs are positive.

Case 2: $0.5 < p_{\frac{k}{2}} < p_{\frac{k}{2}+1}$:

By a completely symmetric argument, one shows that in this case every agent $i > \frac{k}{2}$ (i.e. with a peak above $p_{\frac{n}{2}}$) has an incentive to abstain since this would again not change the outcome.

Hence we have shown that the outcome in equilibrium is 0.5. This implies directly that the number of individuals whose peak is below 0.5 needs to be same as the number of those individuals with a peak above 0.5. Furthermore there cannot be an individual with peak 0.5. Otherwise some individual will find it profitable to abstain as the outcome will not change as 0.5 will be the median vote after the abstention. □

With the result of Lemma 3, we obtain an equilibrium characterization result for an even number of participants. We find full participation equilibria when the number of individuals is even, and equilibria in which all but one individual participate, when the number of individuals is odd. We say that a participant with peak $p_i \neq 0.5$ exhibits *extreme reporting* if she reports 0 if $p_i < 0.5$ and 1 if $p_i > 1$. In both types of equilibria we find that all participants exhibit extrem reporting.

Proposition 2. *Consider the symmetric median rule. The only subgame-perfect equilibria (with a strong Nash equilibrium played on the second stage) with an even number of participants for small but positive participation costs are of one of the following types:*

- *The number of potential voters n is even with half of them having a peak strictly below 0.5 and half of them a peak strictly above* 0.5*. Then full participation with extreme reporting is an equilibrium and the outcome is* 0.5*.*

- *The number of potential voters n is odd, there exists an individual j with peak $p_j = 0.5$ and the peaks of the other $n-1$ individuals are evenly split to both sides of* 0.5*. Then it is an equilibrium that all individuals in $N \setminus \{j\}$ participate with extreme reporting; the resulting outcome is* 0.5*.*

In all other cases there does not exist an equilibrium of the specified form (with an even number of participants).

Proof. Let the peaks $p_1, \ldots, p_n$ be ordered. As peaks are generic, we get $\#\{i \in N : p_i < 0.5\} = \#\{i \in N : p_i > 0.5\} = \left\lfloor \frac{n}{2} \right\rfloor$, as there is at most one individual with a peak of 0.5. If n is odd, then $p_{(n+1)/2} = 0.5$.

Case 1: n is even.

By Lemma 3 we obtain that the outcome if all individuals participate is 0.5. If an individual i with peak below 0.5 abstains, the outcome shifts to $p_{(n/2)+1}$ which is worse for individual i for sufficiently small participation costs. Similarly, if an individual i' with peak above 0.5 abstains, the outcome changes to $p_{n/2}$ which again is worse for individual i' for sufficiently small participation costs. Hence full participation is an equilibrium given this distribution of peaks.

Case 2: n is odd.

By Lemma 3 we obtain that the outcome if all individuals except for individual $j = (n+1)/2$ participate is 0.5. By a similar argument one can show that no participant has an incentive to abstain for small but positive participation costs. As the outcome corresponds already to her peak, there is no incentive for individual j to participate, hence this constitutes an equilibrium.

It remains to show that the existence of these equilibrium implies the even distribution of the peaks required in the proposition.

Assume that there exists an equilibrium with full participation if n is even and with only one abstention if n is odd. As the number of participants is even in both cases, we know by Lemma 3 that there need to be as many individuals with a peak strictly below 0.5 as there are with peak strictly above 0.5. If n is odd, there exists one individual who did not participate and even for very small costs has no incentive to participate. This implies that the peak of this individual has to correspond to the outcome, which is 0.5.

It remains to show that there are no other equilibria with an even number of participants. Assume that for small costs of participation there exists an equilibrium with an even number of participants k, such that $k < n - 1$. By Lemma 3 we know that the outcome is 0.5. As $k < n - 1$ there are at least 2 individuals who do not participate. As peaks are generic at least one of those individuals has a peak that is different from 0.5. This individual can shift the outcome closer to her peak by participating. For (sufficiently) small participation costs this is profitable for her. Hence this cannot constitute an equilibrium. $\square$

The proof of Theorem 2 follows at once from combining Propositions 1 and 2. We conclude with two examples. The first demonstrates that there could be several equilibria of the sort required in Theorem 2 in the sequential model, the second shows that there could exist no equilibria in pure strategies.

Example 1. *Let $p_1 = 0.1$, $p_2 = 0.5$ and $p_3 = 0.7$. Then there exists an equilibrium of the sort required in Theorem 2 with one participant and an equilibrium with an even number of participants and outcome 0.5.*

If agent 2 is the only participant, the outcome is 0.5. Neither agent 1 nor 3 has an incentive to participate since the outcome would not change. If, on the other hand, agents 1 and 3 participate the outcome is again 0.5, hence agent 2 has no incentive to participate. If cost are sufficiently small, agents 1 and 3 indeed prefer to participate, since otherwise the outcome changes to 0.1 or 0.7 respectively.

Example 2. *Let $p_1 = 0.1$, $p_2 = 0.8$ and $p_3 = 0.9$. Then there is no equilibrium of the sort required in Theorem 2 (provided that costs of participation are small).*

With full participation the outcome would be the median of the votes cast, that is: 0.8. In that case agent 1 has an incentive to abstain, since without her the outcome remains unchanged: indeed, agent 2 would vote $\tilde{p}_2 = 0.6$ and agent 3 would vote of $\tilde{p}_3 = 1$ in the equilibrium of the subgame. But this situation cannot constitute an equilibrium either, since agent 3 would rather abstain. If agents 1 and 2, or agents 1 and 3 are the participants, the outcome is 0.5. But for small participation costs, the respective abstainer would prefers to join and change the result to 0.8. Finally, all single participation cases do not constitute an equilibrium since there always exists someone who abstains and could profitably change the outcome to 0.5 (provided that participation costs are small).