Convex Analysis for Optimization

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Lecture 2

Course plan

- Week 1: Introduction to convexity
- Week 2: More on convex sets
- Week 3: Dual view of convex sets + more on convex functions
- Week 4: Dual view of convex functions
- Week 5: Duality and optimization
- Week 6: Introduction to algorithms, descend methods
- Week 7: Proximal methods, projected gradients
- Weeks 8 - 9: Fix point approach, averaged operators

Recap concepts of interior

Let $S \subseteq \mathbb{R}^n$

- Interior:
  $\mathrm{int}(S) := \{x \in S : \exists \text{ open ball } A \text{ such that } x \in A \subseteq S\}$
- Algebraic interior:
  $\mathrm{core}(S) := \{x \in S : \forall z \in \mathbb{R}^n \; \exists \delta > 0 \text{ such that } [x, x + \delta z] \subseteq S\}$

  $$\mathrm{core}(S) = \mathrm{int}(S) \text{ for convex sets of a finite dimension.}$$

- Relative interior:
  $\mathrm{ri}(S) := \{x \in S : \exists \text{ open ball } A \text{ such that } x \in A \cap \mathrm{aff}(S) \subseteq S\}$.

  $$\mathrm{core}(S) = \mathrm{int}(S) = \mathrm{ri}(S) \text{ for convex full-dim. sets of a finite dim.}$$

Recap line segment principle

Let $S \subseteq \mathbb{R}^n$ be a convex set. If $x \in \text{int}(S)$ (resp. $\text{ri}(S)$) and $y \in \text{cl}(S)$, then $[x, y) \subset \text{int}(S)$ (resp. $\text{ri}(S)$). In particular, $\text{int}(S)$ (resp. $\text{ri}(S)$) is a convex set. This is called “Line segment principle”.

Non-emptiness of relative interior

Let S be a nonempty convex set. Then
(a) $\mathrm{ri}(S)$ is a nonempty convex set
(b) $\mathrm{ri}(S)$ has the same affine hull as S.

Prolongation lemma

Let $S \subseteq \mathbb{R}^n$ be a non-empty convex set. Then

$$\mathrm{ri}(S) = \{x \in S : \forall y \in S \ \exists \varepsilon > 0 \text{ such that } x + \varepsilon(x - y) \in S\}$$

Interpretation: any line segment in S having $x \in \mathrm{ri}(S)$ as one of its endpoints can be prolonged a bit beyond x without leaving S.

Prolongation lemma and algebraic interior

Prolongation lemma (PL) extends the idea of $\mathrm{core}(S) = \mathrm{ri}(S)$ to not full-dimensional sets. The proof of $\mathrm{core}(S) = \mathrm{int}(S)$ is essentially the same as of PL, just take $y \in \mathbb{R}^n$ instead of $y \in S$.

A little bit of optimization

The set of *minimizers of a concave function on a convex set* S either belongs to the relative boundary of S ($\mathrm{rb}(S) := S \setminus \mathrm{ri}(S)$) or consists of the whole set S. Proof: prolongation lemma + concavity.

Operations on relative interior and closure

Let $S, \bar{S}$ be different non-empty convex sets

- $\mathrm{cl}(S) = \mathrm{cl}(\mathrm{ri}(S))$
- $\mathrm{ri}(S) = \mathrm{ri}(\mathrm{cl}(S))$
- $\mathrm{ri}(S) = \mathrm{ri}(\bar{S})$ if and only if $\mathrm{cl}(S) = \mathrm{cl}(\bar{S})$
- $\mathrm{ri}(S \times \bar{S}) = \mathrm{ri}(S) \times \mathrm{ri}(\bar{S})$
- $\mathrm{cl}(S \times \bar{S}) = \mathrm{cl}(S) \times \mathrm{cl}(\bar{S})$
- ...and many others, see the Textbook Section 1.3.1

Prove of $\mathbf{cl}(S) = \mathbf{cl}(\mathbf{ri}(S))$

Special cones

Convex cones based on a given set:

- Conic hull
- Recession cone
- Polar cone
- Dual cone
- Normal cone

Recession cone

Def: $d \in \mathbb{R}^n$ is a direction of recession of set $S \subseteq \mathbb{R}^n$ if
$x + \alpha d \in S \ \forall x \in S, \ \alpha \geq 0$

Def: Recession cone of set S consists of all its directions of recession:
$R_S = \text{rec}(S) = \{d \in \mathbb{R}^n : x + \alpha d \in S \ \forall x \in S, \ \alpha \geq 0\}.$

Useful facts about recession cone

Let $S \subseteq \mathbb{R}^n$ be non-empty, convex and closed

- The recession cone R_S is a convex and closed cone

- $R_S = \{d \in \mathbb{R}^n \colon \exists x \in S \text{ such that } x + \alpha d \in S \ \forall \alpha \geq 0\}$ (one x is enough)
- $R_S = \{0\} \iff S$ is bounded
- $R_S = R_{\mathrm{ri}(S)}$

Faces and extreme points

Def: a face of a convex set S is any convex $F \subseteq S$ whose elements cannot be in the relative interior of some line segment that lies outside of F but in S.

More formally: $F \subseteq S$ is face of S when $\alpha x + (1-\alpha)y \in F$ for some $x, y \in S$ and $\alpha \in (0,1)$ if an only if $x, y \in F$.

Def: extreme point of a convex set S is a face that consists of one point (0-dimensional face). Set of all extreme points of S is $\text{ext}(S)$.

Extreme directions and rays

Def: extreme direction of a convex set S is the direction of a face consisting of a half-line. The set of all extreme directions is $\mathrm{ext}_r(S)$.

Def: extreme rays of a closed convex cone K are its faces consisting of half-lines. These half-lines emanate from 0 in K's extreme directions.

By definition, $\mathrm{ext}_r(S) \subseteq \mathrm{ext}_r(R_S)$

Lineality space of set $S \subseteq \mathbb{R}^n$

Def: lineality space is the set of all directions in which S contains a whole line: $L_S := R_S \cap -R_S = \{d \in \mathbb{R}^n : x + \alpha d \in S \ \forall x \in S, \ \alpha \in \mathbb{R}\}$.

Def: S is pointed if $L_S = \{0\}$; usually defined for closed cones, then this is equivalent to having no straight lines contained in the cone.

Minkovsky theorem on convex sets

Minkovsky Thm: Let $S \subset \mathbb{R}^n$ be a closed pointed convex set. Then

$$S = \mathrm{conv}(\mathrm{ext}(S)) + R_S = \mathrm{conv}(\mathrm{ext}(S) \cup \mathrm{ext}_r(S)).$$

Corollary 1 (Krein-Milman Thm): Let $S \subset \mathbb{R}^n$ be compact and convex. Then $S = \mathrm{conv}(\mathrm{ext}(S))$.

Corollary 2: Let $S \subseteq \mathbb{R}^n$ be a closed, convex, pointed cone such that $S \neq \{0\}$. Then $S = \mathrm{cone}(\mathrm{ext}_r(S)) = R_S$.

Polar and dual cones of set $S \subseteq \mathbb{R}^n$

Polar cone: $S^\circ := \{x \in \mathbb{R}^n : x^\top y \leq 0 \ \forall y \in S\}$

Dual cone: $S^* := -S^\circ = \{x \in \mathbb{R}^n : x^\top y \geq 0 \ \forall y \in S\}$

Polar and dual cone properties

For a non-empty set S (so, no convexity or closedness assumed):

$$S^\circ = \mathrm{cl}(S)^\circ = \mathrm{conv}(S)^\circ = \mathrm{cone}(S)^\circ$$

Polar Cone Thm: $(S^\circ)^\circ = \mathrm{cl}(\mathrm{conv}(S))$ for a non-empty cone S

All above also holds for the dual cone (i.e., if we replace $\circ$ with $*$).

Farkas' lemma

Let $a_1, \ldots, a_m \in \mathbb{R}^n$. Then $\{x \in \mathbb{R}^n : a_j^\top x \geq 0 \ \forall j = 1, \ldots, m\}$ and $\text{cone}(a_1, \ldots, a_m)$ are closed convex cones dual to each other.

Note: Textbook uses $a_j^\top x \leq 0$, and so the cones become polar.

Interpretations of Farkas' lemma

Let $c, a_1, \ldots, a_m \in \mathbb{R}^n$. Then $c^\top x \geq 0$ for all $x \in S$, where

$$S := \{x \in \mathbb{R}^n : a_j^\top x \geq 0 \ \forall j = 1, \ldots, k, \ a_i^\top x = 0 \ \forall i = k+1, \ldots, m\}$$

if and only if

$$c = \sum_{j=1}^{k} a_j y_j + \sum_{i=k+1}^{m} a_i y_i \text{ for some } y \in \mathbb{R}^m, \ y_1, \ldots, y_k \geq 0.$$

Generalized Farkas' lemma

Let $K \subseteq \mathbb{R}^m$ be a closed convex cone, $c \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n \times m}$. Let the cone $\{Ay : y \in K^*\}$ be closed. Then $x^\top c \geq 0$ for all $x \in S$, where

$$S := \{x \in \mathbb{R}^n : A^\top x \in K\}$$

if and only if

$$c = Ay \text{ for some } y \in K^*.$$

Normal cone

Def: the normal cone of $S \subseteq \mathbb{R}^n$ in $x \in S$ is

$$N_S(x) := \{y \in \mathbb{R}^n : y^\top (z - x) \leq 0 \ \forall z \in S\}.$$

That is, the normal cone of S in x is $(S - x)^\circ$.

$N_S(x) = \{0\}$ if $x \in \text{int}(S)$; $N_S(x)$ contains at least one half-line otherwise.

Occurs in duality, optimality conditions.