CIEM5110-2: FEM, lecture 2.1

Derivation of the finite element method for Timoshenko beam elements

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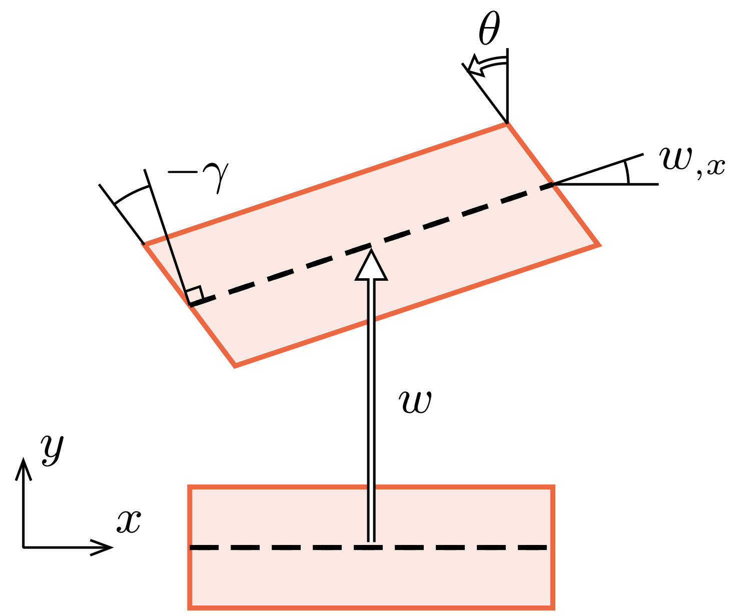
Objectives

This lecture focuses on the formulation of the finite elmenent method for Timoshenko beams

- Another example of how to go from PDEs to the FE formulation
- Specifically for a case with multiple fields and multiple equations
- Illustration of shear locking
- Stepping stone towards frame analysis for upcoming workshops

The basic ingredients

Equilibrium relations	$M_{,x} - V = 0$ $V_{,x} + q = 0$
Constitutive relations	$M = -EI\kappa$ $V = GA_{\mathrm{s}}\gamma$
Kinematic relations	$\kappa = \theta_{,x}$ $\gamma = w_{,x} - \theta$
Strong form equations	$-EI\theta_{,xx} - GA_{\mathrm{s}}\left(w_{,x} - \theta\right) = 0$ $GA_{\mathrm{s}}\left(w_{,xx} - \theta_{,x}\right) + q = 0$



Some remarks:

- Notations follow from Track Base except:
  - θ instead of φ for rotations
  - y (and w) are pointing upward
- The strong form has two coupled ODEs
- There are two unknown fields: w and θ

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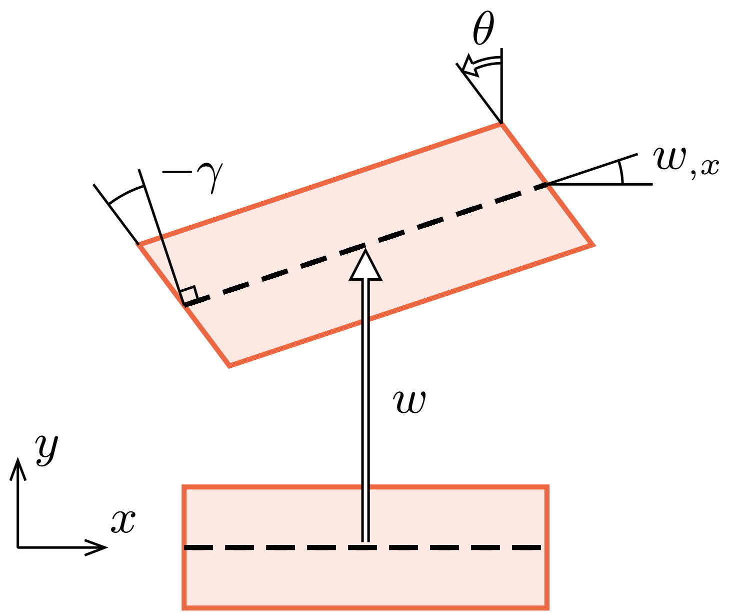

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The alternative would be an Euler beam:

- Single ODE but of 4th order
- Not treated in this unit
- FE derivation is given in the book

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Discretized form

The coupled structure remains visisble in the system of equations

$$\begin{bmatrix} \mathbf{K}_{\theta\theta} & \mathbf{K}_{\theta w} \\ \mathbf{K}_{w\theta} & \mathbf{K}_{ww} \end{bmatrix} \begin{bmatrix} \mathbf{a}_{\theta} \\ \mathbf{a}_{w} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{\theta} \\ \mathbf{f}_{w} \end{bmatrix}$$

with

$$\mathbf{K}_{\theta\theta} = \int_{\Omega} \mathbf{B}_{\theta}^{T} EI \mathbf{B}_{\theta} + \mathbf{N}_{\theta}^{T} GA_s \mathbf{N}_{\theta} \,\mathrm{d}\Omega$$

$$\mathbf{K}_{\theta w} = -\int_{\Omega} \mathbf{N}_{\theta}^{T} GA_s \mathbf{B}_{w} \,\mathrm{d}\Omega$$

$$\mathbf{K}_{w\theta} = -\int_{\Omega} \mathbf{B}_{w}^{T} GA_s \mathbf{N}_{\theta} \,\mathrm{d}\Omega$$

$$\mathbf{K}_{ww} = \int_{\Omega} \mathbf{B}_{w}^{T} GA_s \mathbf{B}_{w} \,\mathrm{d}\Omega$$

$$\mathbf{f}_{\theta} = \int_{\Gamma_M} \mathbf{N}_{\theta}^{T} T \,\mathrm{d}\Gamma$$

$$\mathbf{f}_{w} = \int_{\Gamma_V} \mathbf{N}_{w}^{T} F \,\mathrm{d}\Gamma + \int_{\Omega} \mathbf{N}_{w}^{T} q \,\mathrm{d}\Omega$$

Extensible beam element

Combining the Timoshenko element with a bar element:

$$\begin{bmatrix} \mathbf{K}^{\mathrm{e}}_{uu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{\mathrm{e}}_{ww} & \mathbf{K}^{\mathrm{e}}_{w\theta} \\ \mathbf{0} & \mathbf{K}^{\mathrm{e}}_{\theta w} & \mathbf{K}^{\mathrm{e}}_{\theta\theta} \end{bmatrix} \begin{bmatrix} \mathbf{a}_u \\ \mathbf{a}_w \\ \mathbf{a}_\theta \end{bmatrix} = \begin{bmatrix} \mathbf{f}_u \\ \mathbf{f}_w \\ \mathbf{f}_\theta \end{bmatrix}$$

with

$$\mathbf{K}_{uu} = \int_\Omega \mathbf{B}_u^T EA \mathbf{B}_u \,\mathrm{d}\Omega$$

$$\mathbf{f}_u = \int_{\Gamma_N} \mathbf{N}_u^T F_x \,\mathrm{d}\Gamma + \int_\Omega \mathbf{N}_u^T q_x \,\mathrm{d}\Omega$$

$$\mathbf{f}_w = \int_{\Gamma_V} \mathbf{N}_w^T F_y \,\mathrm{d}\Gamma + \int_\Omega \mathbf{N}_w^T q_y \,\mathrm{d}\Omega$$

$$\mathbf{f}_\theta = \int_{\Gamma_M} \mathbf{N}_\theta^T T \,\mathrm{d}\Gamma$$

Now there are three coupled PDEs

$$EAu_{,xx} + q_x = 0$$

$$-EI\theta_{,xx} - GA_{\mathrm{s}}\left(w_{,x} - \theta\right) = 0$$

$$GA_{\mathrm{s}}\left(w_{,xx} - \theta_{,x}\right) + q_y = 0$$

Alternative formulation

Discretized form from the derivation

$$\mathbf{K}^{\mathrm{e}} = \begin{bmatrix} \mathbf{K}^{\mathrm{e}}_{uu} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{K}^{\mathrm{e}}_{ww} & \mathbf{K}^{\mathrm{e}}_{w\theta} \\ \mathbf{0} & \mathbf{K}^{\mathrm{e}}_{\theta w} & \mathbf{K}^{\mathrm{e}}_{\theta\theta} \end{bmatrix}, \quad \mathbf{a}^{\mathrm{e}} = \begin{bmatrix} \mathbf{a}^{\mathrm{e}}_{u} \\ \mathbf{a}^{\mathrm{e}}_{w} \\ \mathbf{a}^{\mathrm{e}}_{\theta} \end{bmatrix}$$

with

$$\mathbf{K}^{\mathrm{e}}_{uu} = \int_\Omega \mathbf{B}_u^T EA \mathbf{B}_u \,\mathrm{d}\Omega$$

$$\mathbf{K}^{\mathrm{e}}_{ww} = \int_\Omega \mathbf{B}_w^T GA_s \mathbf{B}_w \,\mathrm{d}\Omega$$

$$\mathbf{K}^{\mathrm{e}}_{w\theta} = -\int_\Omega \mathbf{B}_w^T GA_s \mathbf{N}_\theta \,\mathrm{d}\Omega$$

$$\mathbf{K}^{\mathrm{e}}_{\theta w} = -\int_\Omega \mathbf{N}_\theta^T GA_s \mathbf{B}_w \,\mathrm{d}\Omega$$

$$\mathbf{K}^{\mathrm{e}}_{\theta\theta} = \int_\Omega \mathbf{B}_\theta^T EI \mathbf{B}_\theta + \mathbf{N}_\theta^T GA_s \mathbf{N}_\theta \,\mathrm{d}\Omega$$

Alternative: collect all deformations in a single vector

$$\boldsymbol{\varepsilon} \equiv \begin{Bmatrix} \varepsilon \\ \gamma \\ \kappa \end{Bmatrix} = \begin{Bmatrix} u_{,x} \\ w_{,x} - \theta \\ \theta_{,x} \end{Bmatrix} = \mathbf{B}\mathbf{a}^{\mathrm{e}}$$

with

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_u & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_w & -\mathbf{N}_\theta \\ \mathbf{0} & \mathbf{0} & \mathbf{B}_\theta \end{bmatrix}$$

The stiffness matrix takes a familiar form

$$\mathbf{K}^{\mathrm{e}} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \,\mathrm{d}\Omega$$

with

$$\mathbf{D} = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & EI \end{bmatrix}$$

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Shear locking (2-node Timoshenko beam element)

Objective: pure bending deformation ($\kappa \neq 0, \varepsilon = 0, \gamma = 0$):

Node 1: $u_1 = w_1 = 0, \qquad \theta_1 = -\dfrac{L^{\mathrm{e}}}{2}\kappa, \qquad N_1 = 1 - \dfrac{x}{L^{\mathrm{e}}}$

Node 2: $u_2 = w_2 = 0, \qquad \theta_2 = \dfrac{L^{\mathrm{e}}}{2}\kappa, \qquad N_2 = \dfrac{x}{L^{\mathrm{e}}}$

Interpolations: $u(x) = 0, \qquad w(x) = 0, \qquad \theta(x) = N_i\theta_i = \kappa x - \dfrac{L^{\mathrm{e}}}{2}\kappa$

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Strains: $\kappa = \theta_{,x} = \kappa$ ✓, $\qquad \varepsilon = u_{,x} = 0$ ✓, $\qquad \gamma = w_{,x} - \theta = -\kappa x + \frac{L^{\mathrm{e}}}{2}\kappa$ ✗

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Solution: reduced integration

- Only evaluate shear strain at the centere (where $\gamma = 0$ above)
- In Timoshenko beam: for terms that are not related to γ, 1 point is enough for exact integration
- Using only 1 point removes shear locking without side effects

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Alternative solution: mixed interpolation (quadratic for w, linear for θ)

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