Tensor Graph Decomposition for Temporal Networks

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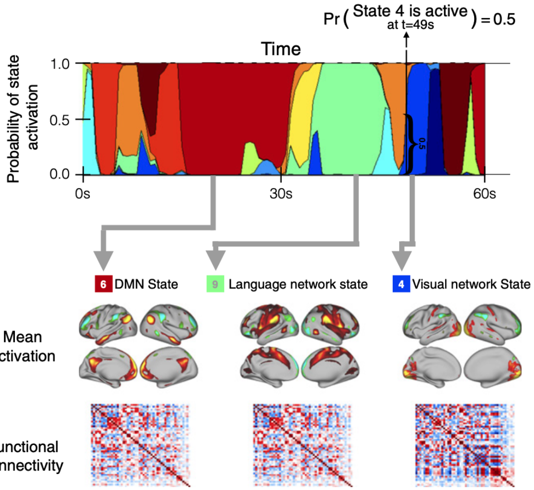
Motivation

- Dynamic Networks represent a sequence of changing topology Holme and Saramäki (2012); Casteigts et al. (2012)
- Can be caused by a myriad of factors
- Studied under network models Zhang et al. (2017) and inferred from spatio-temporal data Natali et al. (2022); Kalofolias et al. (2017)
- Sometimes unobserved due to privacy or constraints

Where do they occur?

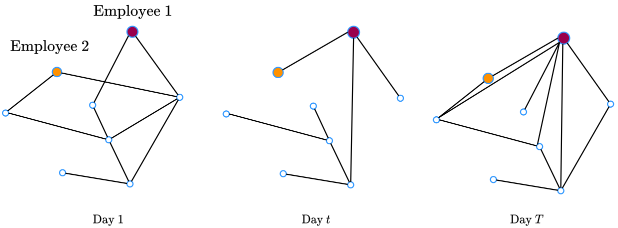
Time varying topology switching over a set of connectivity patterns [1]



[1] Figure taken from Vidaurre et al. (2017)

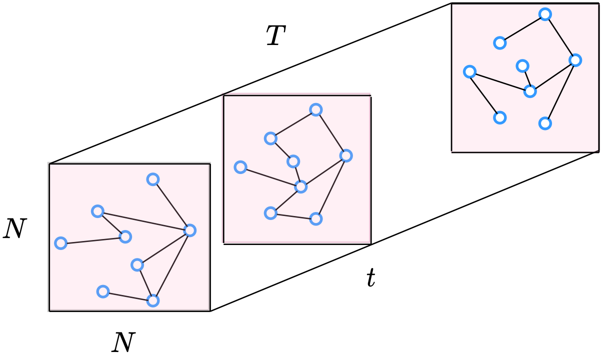
Where do they occur?

Communication pattern of a set of agents over a time period.



Representation through Tensors

- We can use three-dimensonal Tensors to represent the dynamic network



- Tensors allows the use of tools like Tensor Decompositions Kolda and Bader (2009); De Lathauwer et al. (2000)

Remarks

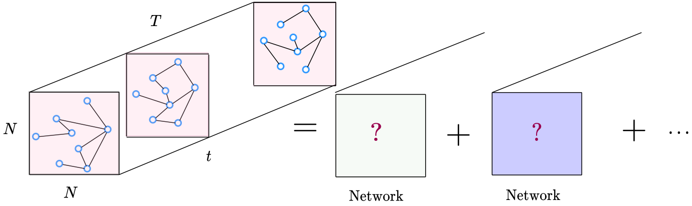
- Useful for downstream tasks dependent on embeddings Gujral et al. (2020)
- These decompositions are Low-Rank
- Low rank not always interpretable as a graph structure
- Data often not available

What we propose

- A two-way decomposition of this tensor
- Break down tensor as a sum of tensors
- Each component tensor corresponds to a mode or a graph scaled over the temporal dimension

Express the evolution of the structure as a sum of networks which can collectively capture this evolution

Illustration



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- Adjacency matrix at time t $\underline{\mathbf{X}}_{:,:,t} = \underline{\hat{\mathbf{X}}}_{:,:,t} + \underline{\mathbf{E}}_{:,:,t} = \sum_{r=1}^{R} c_{rt} \mathbf{A}_r + \underline{\mathbf{E}}_{:,:,t}$.

General Problem

$$\underset{\mathbf{A},\mathbf{C}}{\text{minimize}} \quad l(\underline{\mathbf{X}}, [[\mathbf{A}, \mathbf{C}]]) + \sum_{r=1}^{R} g(\mathbf{A}_r) + h(\mathbf{A}) + i(\mathbf{C})$$
$$\text{subject to } \mathbf{A}_r \in \mathcal{S},\ \mathbf{A} = [\mathbf{A}_1, \ldots, \mathbf{A}_R],\ \mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R] \tag{1}$$

Problem Statement

Given $\underline{\mathbf{X}}$, estimate $\{\mathbf{A}_r\}$ and $\{\mathbf{c}_r\}$ for a known R

Specifics

$$\begin{aligned}\underset{\mathbf{A},\mathbf{C}}{\text{minimize}} \quad & l(\underline{\mathbf{X}}, [[\mathbf{A}, \mathbf{C}]]) + \sum_{r=1}^{R} g(\mathbf{A}_r) + h(\mathbf{A}) + i(\mathbf{C}) \\ \text{subject to } & \mathbf{A}_r \in \mathcal{S},\ \mathbf{A} = [\mathbf{A}_1, \ldots, \mathbf{A}_R],\ \mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]\end{aligned}$$

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- $i(\mathbf{C}) = \sum_{r=1}^{R} ||\mathbf{D}\mathbf{c}_r||_2^2$

Tensor Graph Decomposition (TGD)

$$\min_{\mathbf{A},\mathbf{C}} \frac{1}{2}\left\|\underline{\mathbf{X}} - [[\mathbf{A}, \mathbf{C}]]\right\|_F^2 + \gamma \sum_{r=1}^{R} ||\mathbf{A}_r||_1 + \frac{\beta}{2} h(\mathbf{A}) + ||\mathbf{D}\mathbf{C}||_F^2$$

$$\text{subject to } \mathbf{A}_r \in \mathcal{S},\ \mathbf{A} = [\mathbf{A}_1, \ldots, \mathbf{A}_R],\ \mathbf{C} = [\mathbf{c}_1, \ldots, \mathbf{c}_R]$$

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Solving the TGD

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- Alternate minimization between **A** and **C**

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- Alternate minimization between **A** and **C**
- For **A**, alternate between each $\mathbf{A}_r$
- Solve for each $\mathbf{A}_r$ in the dual domain.

Solving the TGD

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- Alternate minimization between $\mathbf{A}$ and $\mathbf{C}$
- For $\mathbf{A}$, alternate between each $\mathbf{A}_r$
- Solve for each $\mathbf{A}_r$ in the dual domain.
- Closed form solution for each $\mathbf{C}$ update

Errors

Dataset	**TGD** $\gamma = 0.01$ $\beta = 0.01$	**TGD** $\gamma = 9$ $\beta = 16$	**Uncon**	**BTD**	**SBTD**
SBM	0.15	0.24	0.15	0.37	0.44
Enron	0.74	0.82	0.71	0.76	0.98

- NMSE score for all methods.

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Enron	0.74	0.82	0.71	0.76	0.98
SBM	0.92	0.93	0.58	0.48	0.48
Enron	0.18	0.40	0.07	0.03	0.03

- F1 score evaluated on 20 percent removed observations

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SBM	0.92	0.93	0.58	0.48	0.48
Enron	0.18	0.40	0.07	0.03	0.03
SBM	0.90	0.90	0.52	0.43	0.43
Enron	0.17	0.19	0.06	0.03	0.0

- F1 score over a fixed set of unobserved nodes.
- Higher NMSE for better F1
- Constraints help recover structure

Properties of modes

Graph entropy (no. of connected components) for the recovered components.

	$\beta = 0.01$	$\beta = 6$	$\beta = 16$
$\gamma = 0.01$	4.35(1)	4.29(1)	4.18(1.2)
$\gamma = 5$	4.2(1)	4.25(1.2)	4.12(1.26)
$\gamma = 9$	4.2(1.3)	4.19(1.4)	3.94(1.66)

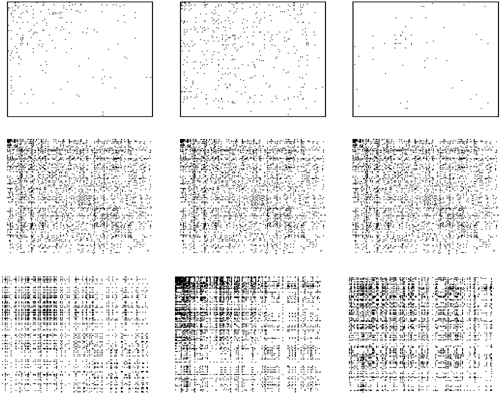
- Imposing graph structure tends to less diversity in degree
- Also leads to more connected components

Visualizing the modes

TGD

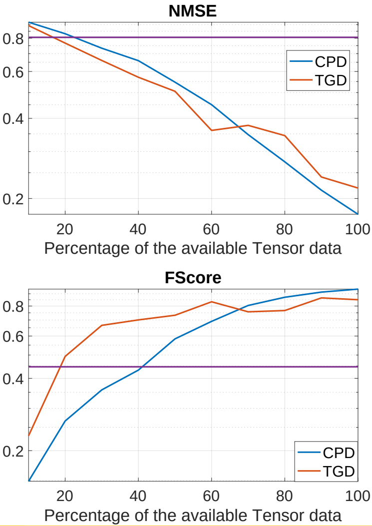
Uncon

SBTD

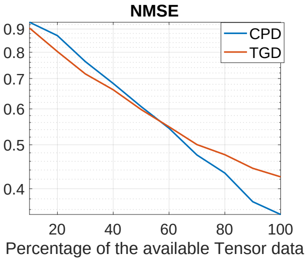


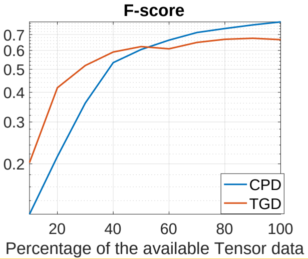
- TGD recovers different structures
- Unconstrained replicating same structure
- Symm BTD recovers typical low rank structure

SBM Graph with rewiring



Molene Temperature Sensor Graph





Summary

Conclusion

- A two-way decomposition on dynamic networks to recover structure.
- Helps recover structure, at the cost of reconstruction error.

Future Work

- Consider alternative priors on relationship between recovered modes.
- Conditions of Uniqueness, Identifiability and Recovery.

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