Fermi's golden rule

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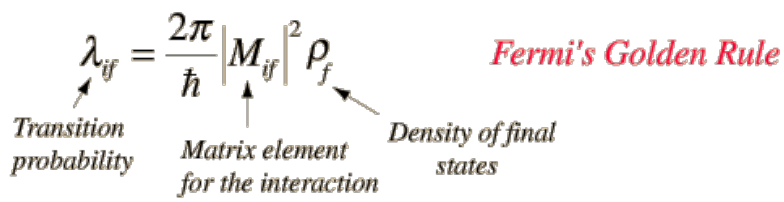

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Chapter 1

Fermi's golden rule

In this chapter we are going to describe the golden rule of Fermi for different processes. Before doing so, let's review a couple of important ingredients that we are going to encounter when dealing with particle interactions.

Let's imagine that we have N_0 particles that decay and after dt the relevant number becomes N. We define as decay rate Γ the probability that a particle will decay per unit time:

$$\frac{dN}{dt} = -\Gamma N$$

$$\frac{1}{N} dN = -\Gamma dt$$

$$\int_{N_0}^{N(t)} \frac{1}{N'} dN' = -\Gamma \int_{t_0}^{t} dt'$$

$$[lnN]_{N_0}^{N(t)} = -\Gamma (t - t_0)$$

$$lnN(t) - lnN_0 = -\Gamma \Delta t$$

$$ln\frac{N(t)}{N_0} = -\Gamma \Delta t$$

$$N(t) = N_0 e^{-\Gamma \Delta t}$$

We define as mean (or average) lifetime τ, the time it takes for a sample of particles to reach the value of $N(\tau) = N_0/e$. It turns out that:

$$\tau = \frac{1}{\Gamma}$$

If a particle has more than one decay mode, then

$$\Gamma_{tot} = \sum_{i=1}^{n} \Gamma_i$$

$$\tau = \frac{1}{\Gamma_{tot}}$$

In the case of a decay, an important property is the so-called branching ratio that defines the fraction of particles of a given type that decay by each mode. This is defined by

$$B.R. = \frac{\Gamma_i}{\Gamma_{tot}}$$

A different category of particle interactions is related to scattering. The relevant quantity that reflects the probability of such an interaction is the so-called cross-section. The cross-section depends on the nature of the interaction e.g. we can speak about elastic, inelastic, total or inclusive cross-section. This latter is calculated as the sum of every individual contribution, such that

$$\sigma_{tot} = \sum_{i=1}^{n} \sigma_i$$

In the following, we are going to study some basic examples.

Example A: Extract the differential cross-section for a particle that bounces elastically off a hard sphere of radius R

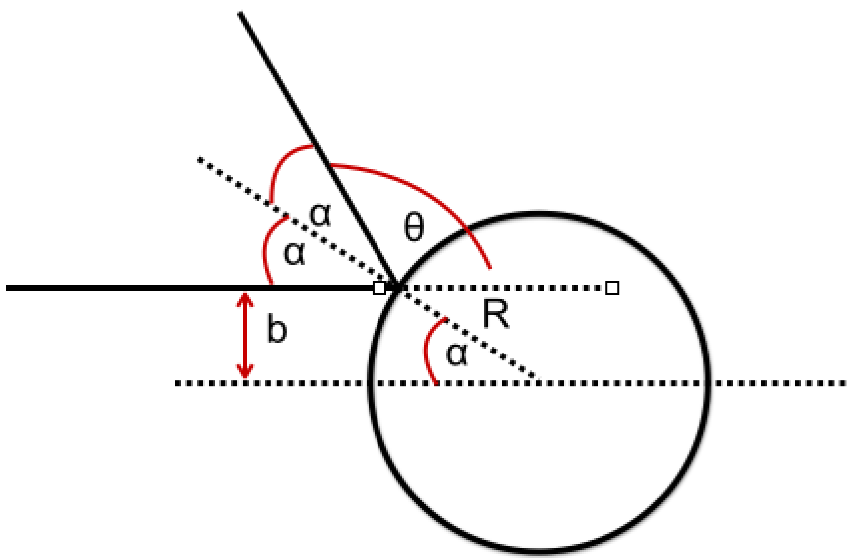

Fig. 1.1: Scattering of a particle on a hard sphere.

$$\frac{b}{R} = \sin(\alpha) \Leftrightarrow b = R\sin(\alpha)$$

But $2\alpha + \theta = \pi$, which means that

$$b = R\sin(\frac{\pi}{2} - \frac{\theta}{2}) \Leftrightarrow b = R\cos(\frac{\theta}{2}) \tag{1.0.1}$$

If one considers the azimuthal (φ) and polar (θ) angles, the differential cross-section is defined as $d\sigma/d\Omega$:

$$d\sigma = |b \cdot db \cdot d\varphi|$$

$$d\Omega = |\sin(\theta) \cdot d\theta \cdot d\varphi|$$

$$\frac{d\sigma}{d\Omega} = |\frac{b \cdot db}{\sin(\theta) \cdot d\theta}| \tag{1.0.2}$$

Equation 1.0.2 with the help of Eq. 1.0.1 can be written as:

$$\frac{d\sigma}{d\Omega} = |\frac{b}{\sin(\theta)} \cdot \frac{R}{2} \cdot \sin(\frac{\theta}{2})|$$

$$\Leftrightarrow \frac{d\sigma}{d\Omega} = |\frac{R^2}{2} \cdot \frac{\cos(\frac{\theta}{2}) \cdot \sin(\frac{\theta}{2})}{\sin(\theta)}|$$

$$\Leftrightarrow \frac{d\sigma}{d\Omega} = |\frac{R^2}{2} \cdot \frac{\sin(\theta)/2}{\sin(\theta)}|$$

$$\Leftrightarrow \frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$

The total integrated cross-section is then given by

$$\sigma = \int \frac{R^2}{4} d\Omega = \frac{R^2}{4} \int_0^{\pi} \sin(\theta) d\theta \int_0^{2\pi} d\varphi = \pi R^2$$

Finally, let's define as luminosity the number of particles per unit time and per unit area, such that:

$$dN = L \cdot d\sigma$$

1.1 Golden rule for decays and scattering

To calculate the previous quantities one uses the golden rule for decays and scattering that states that a transition rate is given by the product of the amplitude or matrix element M_{if} squared and the phase space factor, the latter being purely kinematic:

$$(\text{transition rate}) = \frac{2\pi}{\hbar} |M_{if}|^2 \times (\text{phase space factor})$$

Decays: Assume that we have the following decay mode $|1\rangle \to |2\rangle + |3\rangle + ... + |n\rangle$. The decay rate is then given by

$$\Gamma = \frac{S}{2 \cdot \hbar \cdot m_1} \int |M_{if}|^2 (2\pi)^4 \delta^4(\mathbf{P_1} - \mathbf{P_2} - \mathbf{P_3} - ... - \mathbf{P_n}) \times \prod_{j=2}^{n} 2\pi\delta(p_j^2 - m_j^2 c^2) \theta(p_j) \frac{d^4\mathbf{P_j}}{(2\pi)^4}, \tag{1.1.1}$$

where S is a statistical factor that reflects the number of states accessible to distinguishable particles. The $(1/k!)$ (for each group go k identical particles in the final state) factor divides out the number of equivalent states for identical particles.

Scattering: Assume that we have the following scattering process mode $|1\rangle + |2\rangle \to |3\rangle + ... + |n\rangle$. The interaction rate is then given by

$$\Gamma = \frac{S}{2 \cdot \hbar \cdot m_1} \int |M_{if}|^2 (2\pi)^4 \delta^4(\mathbf{P_1} + \mathbf{P_2} - \mathbf{P_3} - ... - \mathbf{P_n}) \times \prod_{j=3}^{n} 2\pi\delta(p_j^2 - m_j^2 c^2) \theta(p_j) \frac{d^4\mathbf{P_j}}{(2\pi)^4}, \tag{1.1.2}$$

1.2 General Feynman rules

We will first start by giving the general Feynman rules, no matter what the underlying theory is. In what follows we will be working in natural units i.e. $\hbar = c = 1$.

A characteristic diagram describing any kind of interaction of the type $A + B \to C + D$ is given in fig. 1.2.

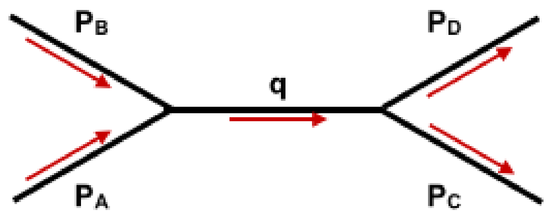

Fig. 1.2: A diagram describing the process $A + B \to C + D$.

The general rules that one needs to follow in order to calculate the matrix element M_{if} are given below:

- **Labeling:** We first label the incoming and outgoing four-momenta $\mathbf{P_A}$, $\mathbf{P_B}$, $\mathbf{P_C}$ and $\mathbf{P_D}$. For each external line we add an arrow which indicates the positive (i.e. along the time axis) direction for particles. We then label all internal lines e.g. in fig. 1.2 we label the four-momentum of the propagator as $\mathbf{q}$ and give an arbitrary direction to the relevant arrow.
- **Vertices:** For each vertex we note down in the diagram the coupling constant factor $-ig$.
- **Propagators:** For each internal line, we give a factor of

$$\frac{i}{\mathbf{q}^2 - m^2}$$

  where $\mathbf{q^2} = q_\nu q^\nu$ the invariant mass of the propagator.
- **δ-functions:** For each vertex we should add a δ-function of the form

$$(2\pi)^4 \delta^4(\mathbf{P_A} + \mathbf{P_B} - \mathbf{q})$$

  where depending on the direction of the arrow of the relevant line in the diagram, we give a positive or negative sign to the 4-momentum symbol. This function guarantees the conservation of energy and momentum at the vertex.
- **Integrate:** For each internal line we should be writing down a factor

$$\frac{1}{(2\pi)^4} d^4\mathbf{q}$$

  and we should integrate over all internal momenta.
- **Cancelation of δ-functions:** The result will contain a factor of

$$(2\pi)^4 \delta^4(\mathbf{P_A} + \mathbf{P_B} - \mathbf{P_C} - \mathbf{P_D})$$

  If this factor is cancelled, what remains is $-\mathbf{iM_{if}}$.