Convex Analysis for Optimization

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Lecture 1

Organization

- Format: weekly lectures for 9 weeks
- Obligatory attendance of at least 7 lectures (Sept 9 to Nov 4)
- Grade: take-home assignment, groups of up to two students
- Weekly exercises, not graded, published on the course website
- Office hours or mistakes in the course material: contact us during the lecture or via email

Prerequisites

- Real analysis and linear algebra at bachelor's level

Literature

- D. Bertsekas, **Convex Optimization Theory**, Athena Scientific, 2009 (main book), online version
- S. Boyd and L. Vandenberghe, **Convex Optimization**, Cambridge University Press, 2004 (for more applications and details), online version
- R. T. Rockafellar. **Convex analysis**. Princeton University Press, 1970 or later editions (for somewhat more theory), online version

Course plan

- Week 1: Introduction to convexity
- Week 2: More on convex sets
- Week 3: More on convex functions
- Week 4: Dual description of convex functions
- Week 5: Duality and optimization
- Week 6: Introduction to algorithms, descend methods
- Week 7: Proximal methods, projected gradients
- Weeks 8 - 9: Fix point approach, averaged operators

On which sets we work

- Usually we just use $\mathbb{R}^n$
- Sometimes extended reals: $\overline{\mathbb{R}}^n \cup \{\infty\} \cup \{-\infty\}$
- All we do is generalizable to topological vector spaces

Convex set

Line L between points $x, y \in \mathbb{R}^n$ is

$$L := \{z \in \mathbb{R}^n : z = \alpha x + (1 - \alpha)y,\ \alpha \in \mathbb{R}\}$$

Line segment LS between points $x, y \in \mathbb{R}^n$ is

$$LS := \{z \in \mathbb{R}^n : z = \alpha x + (1 - \alpha)y,\ 1 \geq \alpha \geq 0\}$$

Def: convex set contains the line segment between its any two points

Convex function

Epigraph of a function $f : S \to \overline{\mathbb{R}}$ is

$$\text{epi}(f) := \{(x, t) \in S \times \mathbb{R} : x \in S, t \geq f(x)\}$$

- Def: a function is convex if it lies below the line segment between any two points in its domain

Convex function

Epigraph of a function $f : S \to \overline{\mathbb{R}}$ is

$$\text{epi}(f) := \{(x, t) \in S \times \mathbb{R} : x \in S, t \geq f(x)\}$$

- Def: a function is convex if it lies below the line segment between any two points in its domain

- Another def: a function is convex if its epigraph is a convex set

Functions onto extended line

Domain of a function is the set where it is defined

Effective domain of a function $f : S \to \overline{\mathbb{R}}$ is

$$\mathrm{dom}(f) := \{x \in \mathbb{R}^n : f(x) < \infty\}$$

Def: f is proper if $f(x) < \infty$ for some $x \in S$ and $f(x) > -\infty$ for all $x \in S$ (i.e., its epigraph is non-empty and contains no vertical lines)

Convex optimization problem

A problem

$$\begin{aligned} \min_x \ & f_0(x) \\ \text{s.t. } & f_i(x) \leq 0, \qquad i = 1, \ldots, m, \end{aligned}$$

where all functions are convex.

Why convexity?

Global minima of convex functions

Separation theorems for convex sets

Duality for convex problems

Usage of convexity

- Convexity is a basis for more complex problems

- Many data science problems (e.g., most regressions, SVM, PCA)
- Problems in physics (e.g., power, water, gas, signal processing)
- Other problems, e.g., neural networks, are not convex, but algorithms from this course help to find local optima
- Can also use convex approximations (e.g., McCormick envelopes, difference-of-convex algorithms, high-dimensional liftings)

Combinations

Def: Convex combination of $x_1, \ldots, x_n$ is $\sum_{i=1}^{n} \alpha_i x_i$ for some $\alpha_1, \ldots, \alpha_n$ where $\alpha_1, \ldots, \alpha_n \geq 0$ (*) and $\sum_{i=1}^{n} \alpha_i = 1$ (**) [DCC]

Conic combination: remove (**) from [DCC]

Affine combination: remove (*) from [DCC]

Linear combination: remove both (*) and (**) from [DCC]

Convexifying sets

Convex hull of set S:
$\text{conv}(S) = \{\sum_{i=1}^{n} \alpha_i x_i : x \in S, \alpha_1, \ldots, \alpha_n \geq 0, \sum_{i=1}^{n} \alpha_i = 1\}$

Conic hull of set S:
$\text{cone}(S) = \{\sum_{i=1}^{n} \alpha_i x_i : x \in S, \alpha_1, \ldots, \alpha_n \geq 0\}$

Affine hull of set S:
$\text{aff}(S) = \{\sum_{i=1}^{n} \alpha_i x_i : x \in S, \sum_{i=1}^{n} \alpha_i = 1\}$

Dimension of a convex set

Dimension of a convex set is equal to the dimension of its affine hull

Caratheodory's Theorem

Let S be a nonempty subset of $\mathbb{R}^n$. Then

(a) Every $y \in \text{cone}(S), y \neq 0$ can be written as $\sum_{i=1}^{n} \alpha_i x_i$, where $x_1, \ldots, x_n \in S$ are linearly independent and $\alpha_1, \ldots, \alpha_n$ are positive.

(b) Every $y \in \text{conv}(S)$ is a convex combination of no more than $n + 1$ elements from S.

Proof of Caratheodory's Theorem

Affine transformation

An affine transformation L from vector space X to vector space Y:

$$L(x \in X) = Ax + b \in Y, \text{ for some linear operator } A \text{ and } b \in Y.$$

When $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$, A is a matrix in $\mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$.

Frequently used convex sets

- Hyperplane for some given $a \in \mathbb{R}^n$, $b \in \mathbb{R}$:

$$HP := \{x \in \mathbb{R}^n : a^\top x = b\}$$

- Half-space for some given $a \in \mathbb{R}^n$, $b \in \mathbb{R}$:

$$HS := \{x \in \mathbb{R}^n : a^\top x \leq b\}$$

- Polyhedron for some given $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^m$:

$$P := \{x \in \mathbb{R}^n : A^\top x \leq b\}$$

More of frequently used sets

- Ball B for some given norm $\|\cdot\|$, center y, and ϵ:

$$B(y, \epsilon) = \{x \in \mathbb{R}^n : \|x - y\| \leq \epsilon\}$$

- Ellipsoid: affine transformation of a ball

- Cone C: for all $x \in C$ we have $\alpha x \in S$ if $\alpha > 0$. Most popular convex cones: second-order, positive semidefinite, exponential.

Closure of a set

Closure of a set S is the set together with all its limit points (aka points that are limits of sequences belonging to S), denoted by $\text{cl}(S)$.

Convexity preserving operations on sets

- Intersection of any number of convex sets
- Cartesian product of convex sets
- Closure of a convex set
- Affine transformation (including projection onto some coordinates)
- Sum of elements of convex sets:
  $S = \{\sum_i x_i,\ x_i \in A_i,\ A_i \text{ are convex for all } i\}$
- Perspective mapping $S = \{x/t : [x, t] \in A,\ A \text{ is convex}\}$
- Linear-fractional mapping $S = \{\frac{Ax+b}{c^\top x+d} : x \in A,\ A \text{ is convex}\}$
- These are the main ones but not the only

Counterexample: union of two convex sets can be non-convex

How to show a set is convex

- Apply definition
- Show the set is defined by convex functions
- Show the set is obtained from other convex sets via convexity preserving operations

Proof that linear-fractional map preserves convexity

Concepts of interior

Let $S \subseteq \mathbb{R}^n$

- Interior:
  $\text{int}(S) := \{x \in S : \exists \text{ open ball } A \text{ such that } x \in A \subseteq S\}$

- Algebraic interior:
  $\text{core}(S) := \{x \in S : \forall z \in \mathbb{R}^n \ \exists \delta > 0 \text{ such that } [x, x + \delta z] \subseteq S\}$

- Relative interior:
  $\text{ri}(S) := \{x \in S : \exists \text{ open ball } A \text{ such that } x \in A \cap \text{aff}(S) \subseteq S\}$

Line segment principle

Let $S \subseteq \mathbb{R}^n$ be a convex set. If $x \in \text{int}(S)$ (resp. $\text{ri}(S)$) and $y \in \text{cl}(S)$, then $[x, y) \subset \text{int}(S)$ (resp. $\text{ri}(S)$). In particular, $\text{int}(S)$ (resp. $\text{ri}(S)$) is a convex set. This is called "Line segment principle".

Algebraic interior of convex sets

For convex sets, the definition of algebraic interior reduces to:
$\text{core}(S) := \{x \in S : \forall z \in \mathbb{R}^n \; \exists \delta > 0 \text{ such that } x + \delta z \in S\}$

$\text{core}(S) = \text{int}(S)$ for convex $S \subseteq \mathbb{R}^n$: can use them interchangeably in proofs. Can show using the Line Segment Principle for $\text{int}(S)$.