Matrix Models, Group Singlets, and Scars

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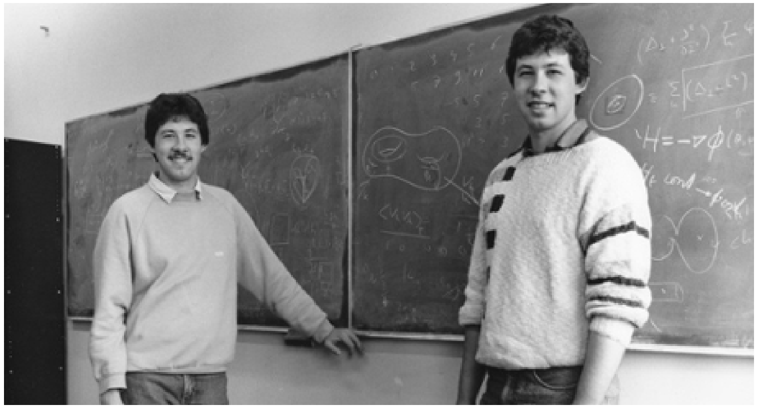


The Early 1990s

- Flourishing of 2D Quantum Gravity: Continuum Path Integral, Matrix Models, Topological Recursion Relations, 2D Black Holes
- Herman and I were very close!
- Jadwin 324 and 326
- Also, Hibben and Magie

- I could not help overhearing many hours of conversations between Erik, Herman and Robbert emanating from Jadwin 324, but I understood nothing.

ICTP Spring School 1991

- Herman was an organizer and very good host
- Lectures on large N matrix models and Liouville gravity.
- IRK, hep-th/9108019

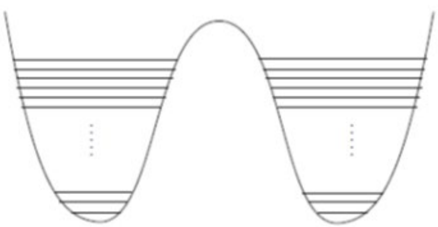


Fig. 3a) N fermions in a double well potential.

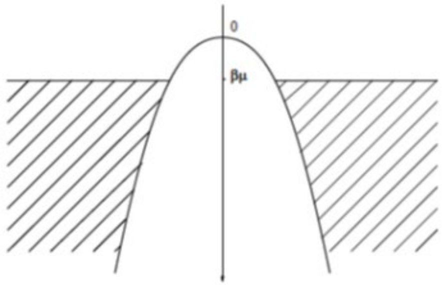
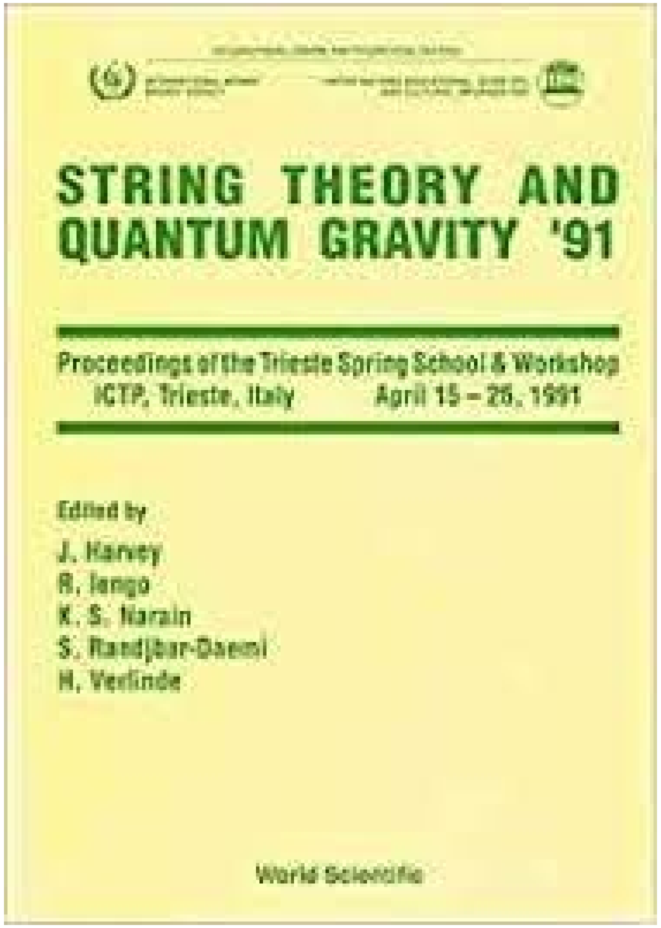


Fig. 3b) The system relevant to the double-scaling limit.



Singlet Sector Simplification

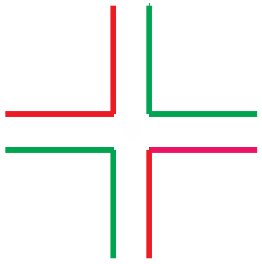
- The SU(N) invariant sector of the Hermitian matrix quantum mechanics is described by the free fermions. Brezin, Itzykson, Parisi and Zuber
- This is the secret of the exact solvability of the model.
- The non-singlet sectors are related to the BKT vortices or, alternatively, long strings. Gross, IRK; Maldacena

The c=1 Matrix Reloaded

- A dozen years later, in 2003, the Hermitian matrix quantum mechanics was given a new interpretation as describing N unstable D0-branes. J. McGreevy, H. Verlinde; Polyakov
- Linear dilaton holography: String theory in two dimensions is dual to the large N matrix quantum mechanics on ZZ D0-branes.
- A new hat for the c=1 matrix model. The double-scaled, double-well matrix quantum mechanics is dual to the type 0B NSR string theory in 2D, which has worldsheet fermions. Takayanagi, Toumbas; Douglas, IRK, Kutasov, Maldacena, Martinec, Seiberg

O(N) x O(N) Matrix Model

- Theory of real matrices ϕ^{ab} with distinguishable indices, i.e. in the bi-fundamental representation of $O(N)_a x O(N)_b$ symmetry.
- The interaction is at least quartic: $g \, \mathrm{tr} \, \phi\phi^T\phi\phi^T$
- Propagators are represented by colored double lines, and the interaction vertex is
- In d=0 or 1 special limits describe two-dimensional quantum gravity.



From Bi- to Tri-Fundamentals

- For a 3-tensor with distinguishable indices the propagator has index structure

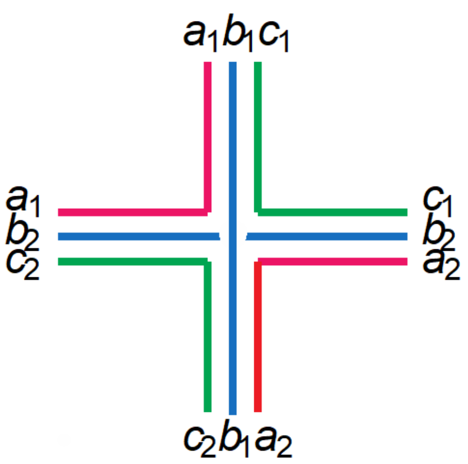
$$\langle \phi^{abc} \phi^{a'b'c'} \rangle = \delta^{aa'} \delta^{bb'} \delta^{cc'}$$

- It may be represented graphically by 3 colored wires 

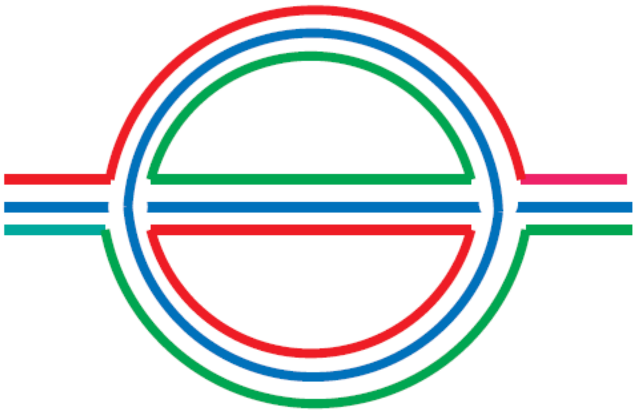
- Tetrahedral interaction with $O(N)_a \times O(N)_b \times O(N)_c$ symmetry

Carrozza, Tanasa; IK, Tarnopolsky

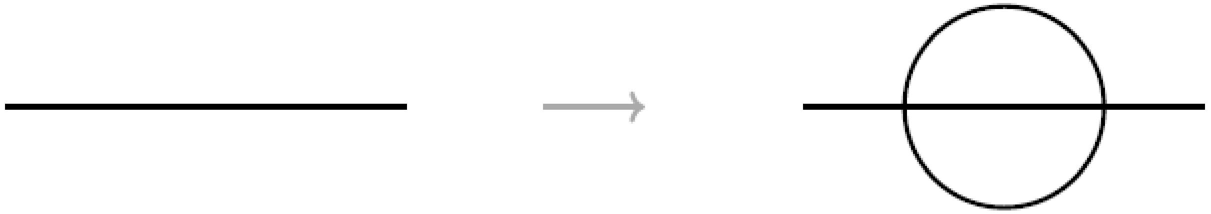
$$\frac{1}{4} g \phi^{a_1 b_1 c_1} \phi^{a_1 b_2 c_2} \phi^{a_2 b_1 c_2} \phi^{a_2 b_2 c_1}$$



- Leading correction to the propagator has 3 index loops



- Requiring that this “melon” insertion is of order 1 means that $\lambda = gN^{3/2}$ must be held fixed in the large N limit.
- Melonic graphs obtained by iterating



$O(N)^3$ Tensor QM

- Quantum Mechanics of N^3 Majorana fermions
  IRK, Tarnopolsky

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N^4$$

- Has $O(N)_a$x$O(N)_b$x$O(N)_c$ symmetry under

$$\psi^{abc} \to M_1^{aa'} M_2^{bb'} M_3^{cc'} \psi^{a'b'c'}, \quad M_1, M_2, M_3 \in O(N)$$

- The SO(N) symmetry charges are

$$Q_1^{aa'} = \frac{i}{2}[\psi^{abc}, \psi^{a'bc}], \quad Q_2^{bb'} = \frac{i}{2}[\psi^{abc}, \psi^{ab'c}], \quad Q_3^{cc'} = \frac{i}{2}[\psi^{abc}, \psi^{abc'}]$$

$O(N)^3$ vs. SYK Model

- Using composite indices $I_k = (a_k b_k c_k)$

$$H = \frac{1}{4!} J_{I_1 I_2 I_3 I_4} \psi^{I_1} \psi^{I_2} \psi^{I_3} \psi^{I_4}$$

The couplings take values $0, \pm 1$

$$J_{I_1 I_2 I_3 I_4} = \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_1 b_3} \delta_{b_2 b_4} \delta_{c_1 c_4} \delta_{c_2 c_3} - \delta_{a_1 a_2} \delta_{a_3 a_4} \delta_{b_2 b_3} \delta_{b_1 b_4} \delta_{c_2 c_4} \delta_{c_1 c_3} + 22 \text{ terms}$$

- The number of distinct terms is

$$\frac{1}{4!} \sum_{\{I_k\}} J^2_{I_1 I_2 I_3 I_4} = \frac{1}{4} N^3 (N-1)^2 (N+2)$$

- Much smaller than in SYK model with $N_{\mathrm{SYK}} = N^3$

$$\frac{1}{24} N^3 (N^3 - 1)(N^3 - 2)(N^3 - 3)$$

Gauged Model

- To eliminate large degeneracies, gauge the symmetry. Witten
- Focus on the states invariant under $SO(N)^3$.
- Their number can be found by gauging the free theory IRK, Milekhin, Popov, Tarnopolsky

$$\#\text{singlet states} = \int d\lambda_G^N \prod_{a=1}^{M/2} 2\cos(\lambda_a/2)$$

$$d\lambda_{SO(2n)} = \prod_{i<j}^{n} \sin\left(\frac{x_i - x_j}{2}\right)^2 \sin\left(\frac{x_i + x_j}{2}\right)^2 dx_1 \ldots dx_n$$

- No singlets for odd N due to a QM anomaly.
- The number grows very rapidly for even N

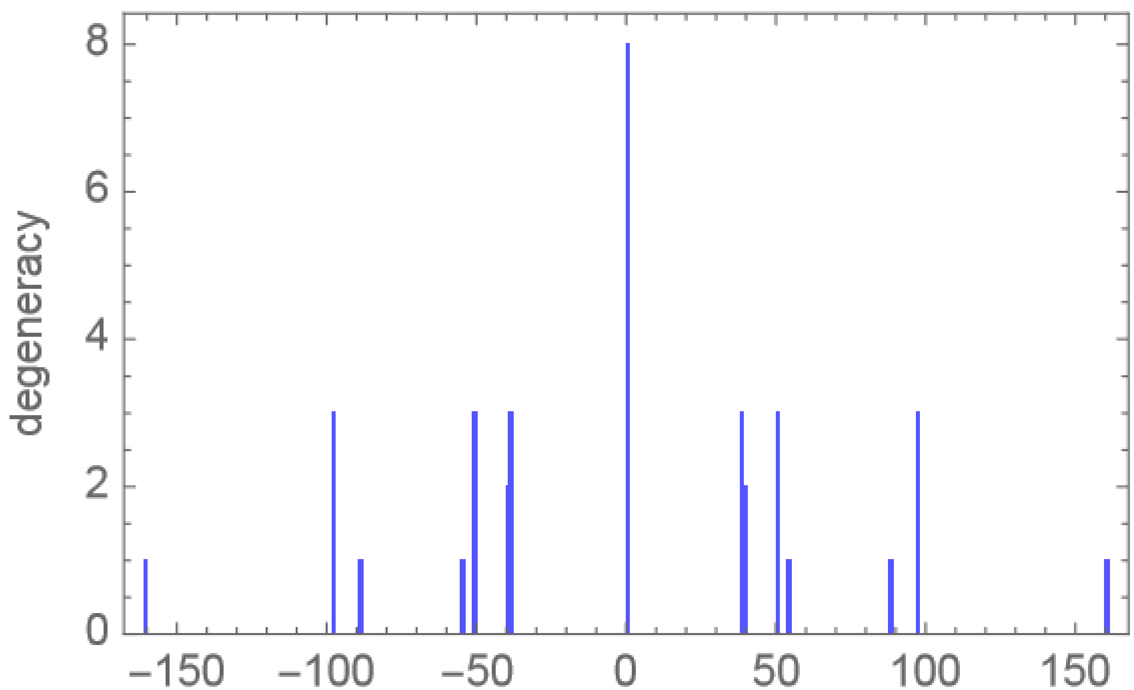
N	# singlet states
2	2
4	36
6	595354780

Table 1: Number of singlet states in the $O(N)^3$ model

$$\#\text{singlet states} \sim \exp\left(\frac{N^3}{2}\log 2 - \frac{3N^2}{2}\log N + O(N^2)\right)$$

- Large N dynamics in the singlet sector is similar to SYK. Same melonic Dyson-Schwinger eqns.
- The large low-temperature entropy suggests tiny gaps for singlet excitations ~ c^{-N^3}

Singlet Energies for N=4



- For N=6 there will be over 595 million states packed into energy interval <1932. So, the gaps should be tiny. Pakrouski, IRK, Popov, Tarnopolsky

Spectra of Energy Eigenstates

- Generalize the Majorana tensor model to have $O(N_1) \times O(N_2) \times O(N_3)$ symmetry
- The traceless Hamiltonian is

$$H = \frac{g}{4}\psi^{abc}\psi^{ab'c'}\psi^{a'bc'}\psi^{a'b'c} - \frac{g}{16}N_1N_2N_3(N_1 - N_2 + N_3)$$

$$\{\psi^{abc}, \psi^{a'b'c'}\} = \delta^{aa'}\delta^{bb'}\delta^{cc'}$$

$$a = 1, \ldots, N_1;\ b = 1, \ldots, N_2;\ c = 1, \ldots, N_3$$

- The Hilbert space has dimension $2^{[N_1N_2N_3/2]}$
- The eigenstates of H form irreducible representations of the symmetry.

Energy Bounds

- The bound on the singlet ground state energy
  IRK, Milekhin, Popov, Tarnopolsky

$$|E| \leq E_{bound} = \frac{g}{16} N^3 (N+2)\sqrt{N-1}$$

- In the melonic limit, this correctly scales as N^3.
- The gap to the lowest non-singlet state scales as 1/N.
- For unequal ranks the bound is

$$|E| \leq \frac{g}{16} N_1 N_2 N_3 (N_1 N_2 N_3 + N_1^2 + N_2^2 + N_3^2 - 4)^{1/2}$$

A Fermionic Matrix Model

- For N_3=2 the bound simplifies to

$$|E|_{N_3=2} \leq \frac{g}{8} N_1 N_2 (N_1 + N_2)$$

- Saturated by the ground state.
- This is a fermionic matrix model with symmetry

$$O(N_1) \times O(N_2) \times U(1)$$

$$\bar{\psi}_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} + i \psi^{ab2} \right) , \qquad \psi_{ab} = \frac{1}{\sqrt{2}} \left(\psi^{ab1} - i \psi^{ab2} \right)$$

$$\{ \bar{\psi}_{ab}, \bar{\psi}_{a'b'} \} = \{ \psi_{ab}, \psi_{a'b'} \} = 0, \qquad \{ \bar{\psi}_{ab}, \psi_{a'b'} \} = \delta_{aa'} \delta_{bb'}$$

- The traceless Hamiltonian is

$$H = \frac{g}{2}\left(\bar{\psi}_{ab}\bar{\psi}_{ab'}\psi_{a'b}\psi_{a'b'} - \bar{\psi}_{ab}\bar{\psi}_{a'b}\psi_{ab'}\psi_{a'b'}\right) + \frac{g}{8}N_1 N_2(N_2 - N_1)$$

- Describes qubits on N_1 x N_2 lattice with non-local couplings.
- May be expressed in terms of quadratic Casimirs

$$-\frac{g}{2}\left(4C_2^{SU(N_1)} - C_2^{SO(N_1)} + C_2^{SO(N_2)} + \frac{2}{N_1}Q^2 + (N_2 - N_1)Q - \frac{1}{4}N_1 N_2(N_1 + N_2)\right)$$

- $SU(N_1) \times SU(N_2)$ is not a symmetry here but an enveloping algebra. Some states have enhanced symmetry: they are SU(N) invariant.

Complete Spectrum

- The singlets "scar" the energy distribution.

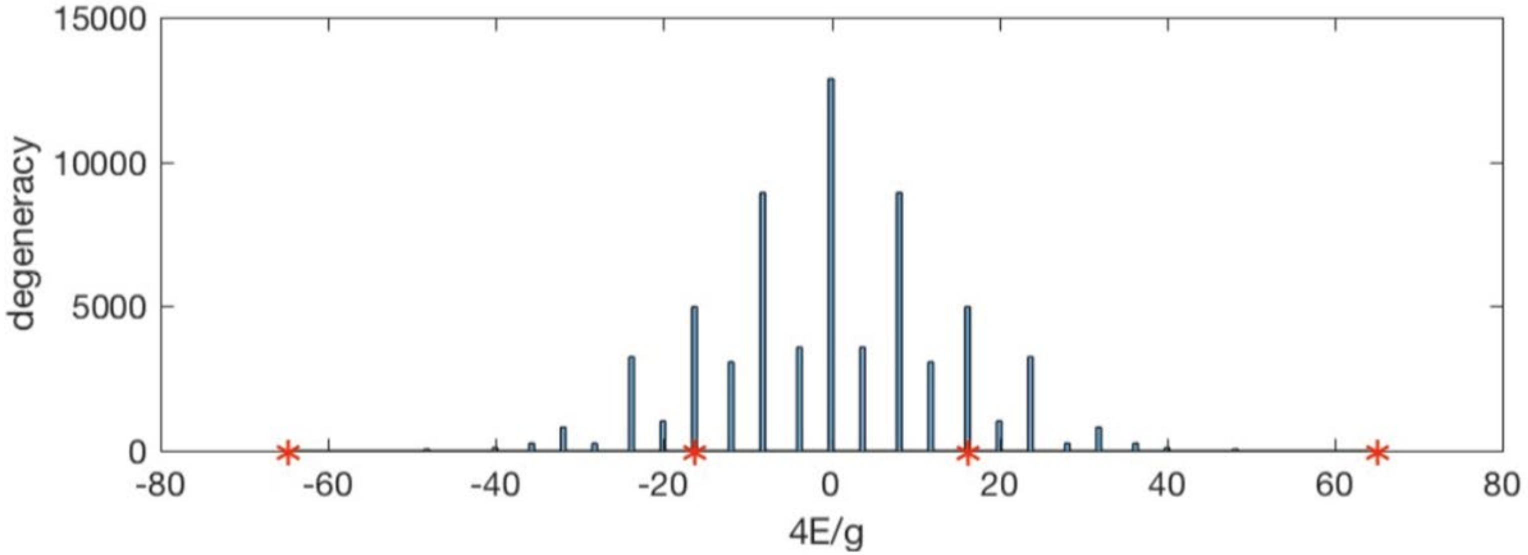


Figure 1: Spectrum of the $O(4)^2 \times O(2)$ model. There are four singlet states, and the stars mark their energies.

Towards Hubbard Model

- Can also think of the first index as labeling the lattice site, and the second as labeling spin. When $N_2=2$, there are two spin states, up and down. The model is beginning to resemble a non-local Hubbard model, but need to add quadratic hopping terms. Pakrouski, Pallegar, Popov, IRK
- Imaginary hopping terms are SO(N) generators

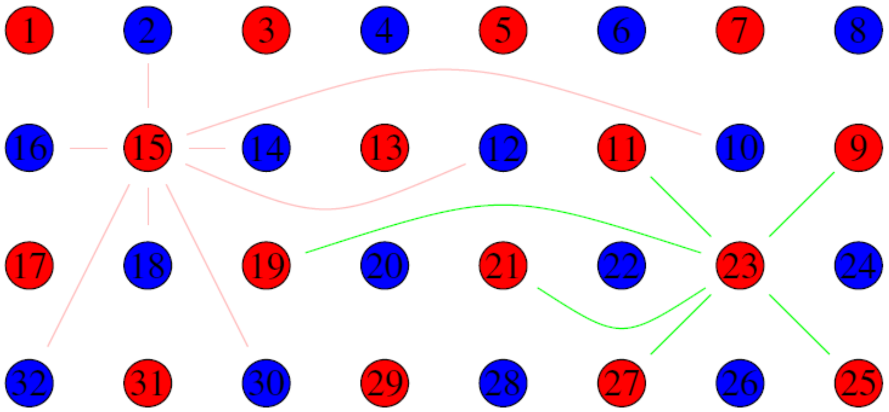
$$T^{O}_{kl} = i\sum_{\sigma}(c^{\dagger}_{k\sigma}c_{l\sigma} - c^{\dagger}_{l\sigma}c_{k\sigma}) \qquad \sigma = \uparrow, \downarrow$$

- Adding them to H keeps singlets as eigenstates but mixes up the non-singlets.

- A simple transformation leads to a model with a real nearest neighbor hopping parameter:

$$H_{nn} = t \sum_{\langle ij \rangle \sigma} (c^{\dagger}_{i\sigma} c_{j\sigma} + h.c.)$$

- This transformation is possible on a bi-partite lattice



SO(N) Singlets as Scars

- For the eigenstates that are SO(N) singlets, the energy is independent of the hopping parameter t. Pakrouski, Pallegar, Popov, IRK
- This is the kind of simplification characteristic of the Quantum Many-Body Scars, that have been an active area in Condensed Matter Physics and Quantum Information.
- Scars form an "integrable subsector" in a Hamiltonian that is in general not integrable.

Group Theoretic Approach to Scars

- There are Hamiltonians that are not symmetric under a Lie group G, yet some of their eigenstates are invariant. These are the quantum many-body scars!
- A class of such Hamiltonians is

$$H = H_0 + \sum_j O_j T_j^G$$

- Includes local lattice systems like the tJU model

$$H^{tJU} = \sum_{\langle ij \rangle \sigma} (t c^\dagger_{i\sigma} c_{j\sigma} + h.c.) + J \sum_{\langle ij \rangle} \vec{S}_i \cdot \vec{S}_j + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu Q$$

Pseudospin

- An example is provided by C.N. Yang's eta-pairing states in the Hubbard model.
- The pseudospin group $\mathrm{SU}(2)_\eta$ is generated by

$$\eta^+ = \sum_j c^\dagger_{j\uparrow} c^\dagger_{j\downarrow} = \frac{1}{2} \sum_{j,\sigma,\sigma'} c^\dagger_{j\sigma} c^\dagger_{j\sigma'} \epsilon_{\sigma\sigma'}$$

$$\eta^- = (\eta^+)^\dagger, \quad \eta^3 = \frac{1}{2}(Q - N) \qquad Q = \sum_{i=1}^{N} n_i$$

$$n_{i\uparrow} = c^\dagger_{i\uparrow} c_{i\uparrow}\ , \quad n_{i\downarrow} = c^\dagger_{i\downarrow} c_{i\downarrow}\ , \quad n_i = n_{i\uparrow} + n_{i\downarrow}$$

- It commutes with the rotation group and with the O(N) that acts on the lattice index.

Eta-pairing states

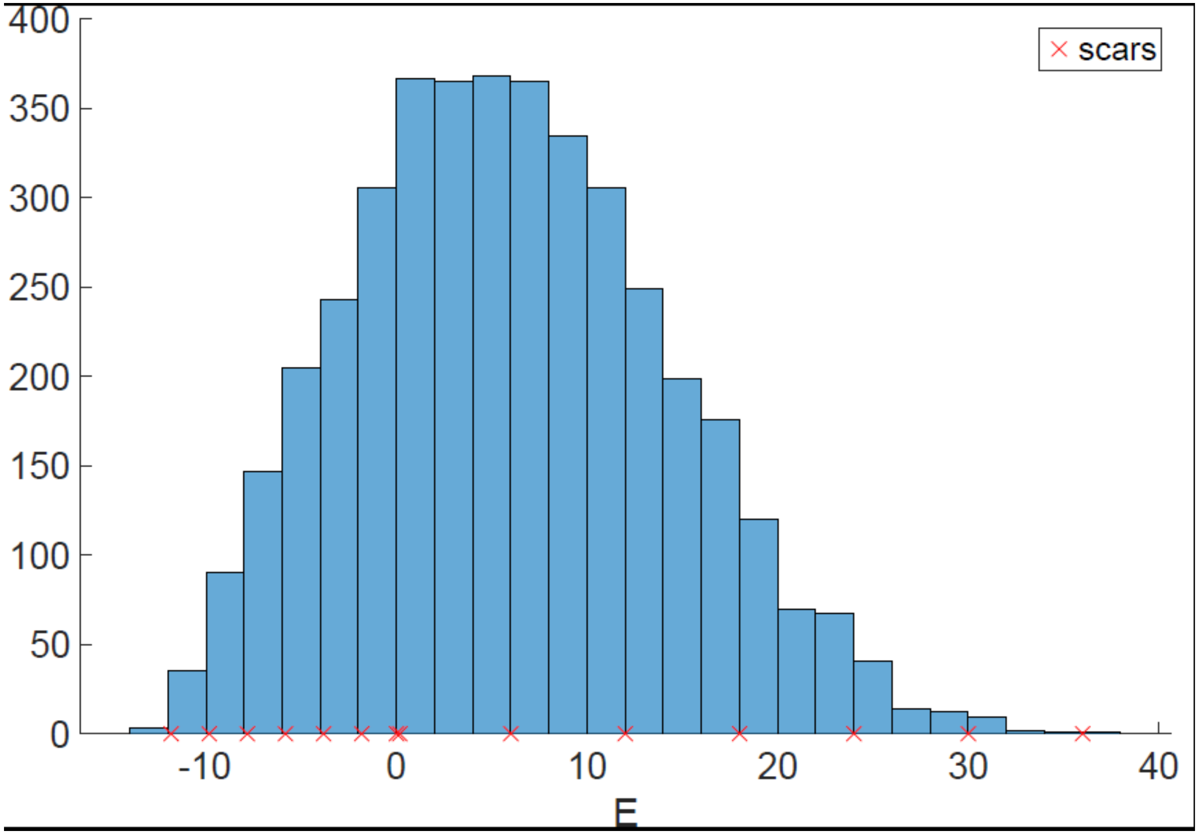
- There are N+1 SO(N) invariant states that form a multiplet of pseudospin N/2

$$|n^\eta\rangle = \frac{(\eta)^n}{\sqrt{\frac{N!n!}{(N-n)!}}}|0\rangle \ , \qquad n = 0, \dots, N$$

- They are the highly excited, equally spaced states that play the role of scars in the Hubbard model and its deformations. Moudgalya et al.; Mark, Motrunich; Pakrouski et al.

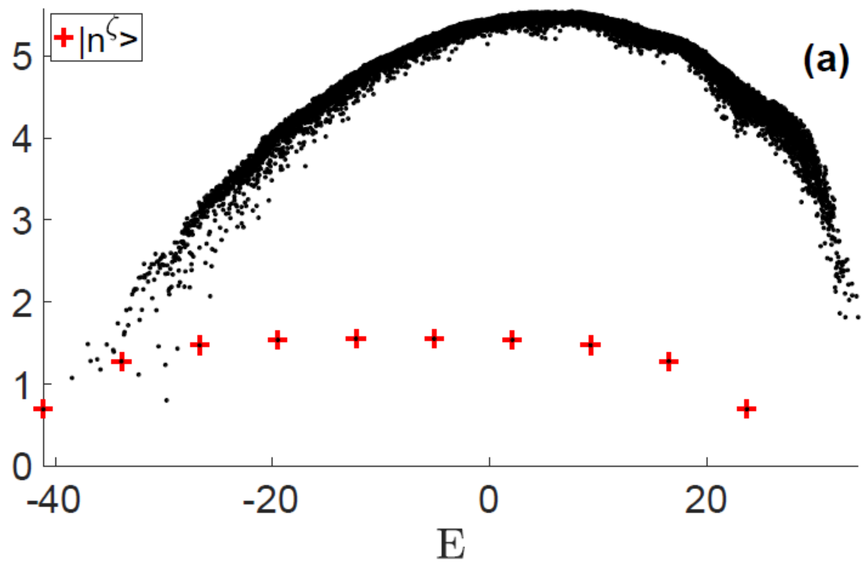
Histogram of the Energy Levels

- A typical distribution for the tJU model deformed by OT terms



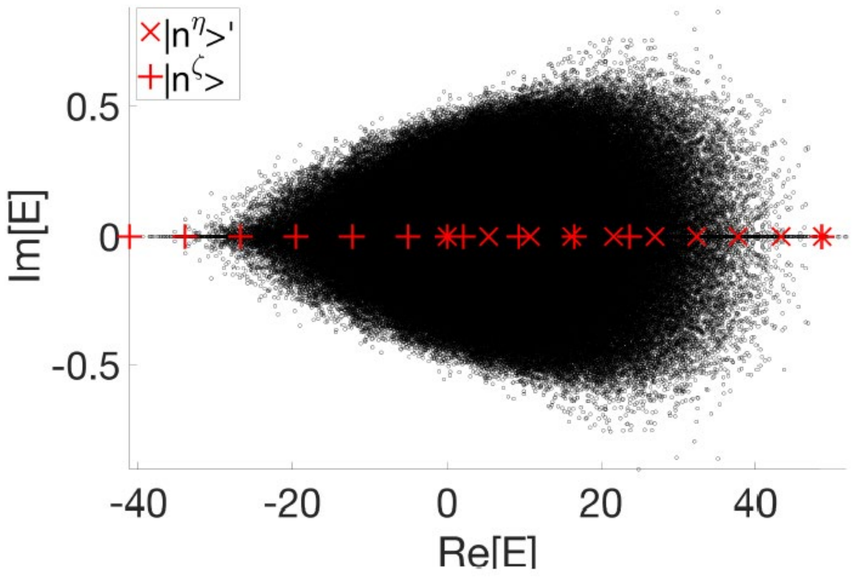
Low Entanglement

- The scar states are distinguished by their low entanglement entropy when the system is divided into two parts:



Non-Hermitian Hamiltonians

- The group theoretic approach to scars continues to work when non-Hermitian terms are added to the Hamiltonians, e.g. the tJU model.
- The energies of scars continue to be real



Comments

- The scar states, which are invariant under the large Lie group acting on the lattice sites, are decoupled from all the non-singlet states. Only the latter thermalize.
- This decoupling survives the OT perturbations and may approximately survive some other perturbations.
- The Group theoretic approach to scars applies to non-Hermitian Hamiltonians.
- Scar states in QFT? In AdS/CFT?

- Happy Birthday, Erik and Herman!
- Gelukkige verjaardag!