Lag Length Selection for Unit Root Tests in the Presence of Nonstationary Volatility*

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Abstract

A number of recently published papers have focused on the problem of testing for a unit root in the case where the driving shocks may be unconditionally heteroskedastic. These papers have, however, assumed that the lag length in the unit root test regression is a deterministic function of the sample size, rather than data-determined, the latter being standard empirical practice. In this paper we investigate the finite sample impact of unconditional heteroskedasticity on conventional data-dependent methods of lag selection in augmented Dickey-Fuller type unit root test regressions and propose new lag selection criteria which allow for the presence of heteroskedasticity in the shocks. We show that standard lag selection methods show a tendency to over-fit the lag order under heteroskedasticity, which results in significant power losses in the (wild bootstrap implementation of the) augmented Dickey-Fuller tests under the alternative. The new lag selection criteria we propose are shown to avoid this problem yet deliver unit root tests with almost identical finite sample size and power properties as the corresponding tests based on conventional lag selection methods when the shocks are homoskedastic.

Keywords: Unit root test; lag selection; information criteria; wild bootstrap; nonstationary volatility.

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1 Introduction

Applied researchers have recently focused attention on the question of whether or not the variability in the shocks driving macroeconomic time series has changed over time; see, e.g.,

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the literature review in Busetti and Taylor (2003). The empirical evidence has suggested that time-varying behaviour (specifically, a general decline) in unconditional volatility in the shocks driving macroeconomic time series over the past two decades or so is a relatively common phenomena, consonant with the so-called great moderation; see, *inter alia*, Kim and Nelson (1999), McConnell and Perez Quiros (2000), Van Dijk, Osborn, and Sensier (2002), Sensier and Van Dijk (2004) and references therein.[1] Sensier and Van Dijk (2004), eg, report that over 80% of the real and price variables in the Stock and Watson (1999) data-set reject the null of constant unconditional innovation variance. Empirical evidence also suggests that data are often characterized by smooth volatility changes rather than by abrupt changes (see, inter alia, Van Dijk et al., 2002).

Such, nonstationary volatility, effects can significantly impact on the size of standard unit root tests, even asymptotically, as has been shown by Cavaliere and Taylor (2007, 2008), among others. A solution to this problem is analyzed by Cavaliere and Taylor (2008, 2009b), who employ the wild bootstrap to capture the nonstationary volatility within the re-sampled data. They show that the wild bootstrap correctly reproduces the first-order limiting null distribution under nonstationary volatility, thereby allowing for the construction of asymptotically valid bootstrap tests.

The analysis in Cavaliere and Taylor (2008, 2009b) is based on the use of a lag length in the augmented Dickey-Fuller [ADF] test regression which is a deterministic function of the sample size. In practice, however, applied researchers usually choose the lag length by data-dependent methods. Often this is done using standard information criteria or by sequential t-testing (using conventional critical values) on the significance on the highest lag. However, both of these approaches are misspecified in the presence of nonstationary volatility: standard information criteria are based on the assumption of constant volatility, while the limit distributions used in sequential t-testing are affected by the presence of nonstationary volatility. As such, if nonstationary volatility is present in the data, the lag length selected by the applied researcher may not be appropriate. While not necessarily invalidating the asymptotic properties of the unit root test, this may nonetheless have a significant impact on finite sample performance.

In this paper we analyze the finite sample effects of nonstationary volatility on the selection of the lag order in (bootstrap) unit root testing. Using Monte Carlo simulation methods we will show that, under certain time-varying volatility specifications, standard information criteria select too many lags and that this has a significant negative effect on the power of the resulting unit root test. As a consequence, we also propose a modification of the standard information criteria, based on the approach of Beare (2008) which re-scales the data by an estimate of the underlying volatility process. Again using Monte Carlo methods, we show that

[1]The recent financial turmoil and disruption of economic activity associated with the onset of the 2008 credit crisis has undoubtedly reversed this decline and produced a corresponding rise in unconditional volatility. Such changes reinforce the need to allow for the possibility of non-constancy in unconditional volatility.

these new criteria are considerably more robust, in terms of the lag length they select, than the standard criteria in the presence of nonstationary volatility and perform very similarly to the standard criteria when volatility is constant. We show that this results in unit root tests which display significantly more power than those based on the standard lag selection criteria under nonstationary volatility yet do not lose power relative to these tests when volatility is constant. Moreover, the sizes of the tests based on the standard and new criteria are shown to be broadly the same under both constant and nonconstant volatility environments.

The structure of the paper is as follows. In Section 2 we introduce our reference data generating process (DGP) and detail the class of heteroskedastic volatility processes under which we will work. The (wild bootstrap) unit root tests, and associated lag selection criteria with the new heteroskedasticity-robust modification thereof, are discussed in Section 3. The finite-sample properties of the standard and new lag selection criteria, along with the size and power properties of the associated (wild bootstrap) unit root tests, are explored through Monte Carlo simulation in Section 4. Section 5 concludes. The Appendix contains additional simulation results.

2 The Heteroskedastic Model

Consider the case where we have $T+1$ observations generated according to the DGP,

$$y_t = x_t + \beta' z_t, \qquad t = 0, 1, \ldots, T, \tag{1a}$$

$$x_t = \rho x_{t-1} + u_t, \tag{1b}$$

$$u_t = \varepsilon_t + \sum_{j=1}^{\infty} \psi_j \varepsilon_{t-j} =: \psi(L)\varepsilon_t, \tag{1c}$$

$$\varepsilon_t = \sigma_t e_t \tag{1d}$$

with $E(x_0^2) < \infty$. Our focus in this paper is on tests for whether or not y_t contains a unit root; that is, on testing $H_0 : \rho = 1$ against $H_1 : |\rho| < 1$ in (1).

In (1a), z_t is a vector of deterministic components. As in Ng and Perron (2001) we focus on the κth-order trend function, $z_t := (1, t, \ldots, t^\kappa)'$, with special focus on the leading cases of a constant ($\kappa = 0$) and linear trend ($\kappa = 1$). We also make the following assumptions on the shocks u_t, where $\mathcal{D} := D[0,1]$ denotes the space of right continuous with left limit (càdlàg) processes:

Assumption 1. (i) $\psi(z) \neq 0$ for all $|z| \leq 1$, and $\sum_{j=1}^{\infty} j|\psi_j| < \infty$. (ii) e_t is i.i.d. with $\mathrm{E}\, e_t = 0$, $\mathrm{E}\, e_t^2 = 1$ and $\mathrm{E}\, |e_t|^4 < \infty$. (iii) The volatility term σ_t satisfies $\sigma_{\lfloor Tr \rfloor} = \omega(r)$ for all $r \in [0,1]$, where $\omega(\cdot) \in \mathcal{D}$ is nonstochastic, twice-differentiable and strictly positive.

Remark 1. Assumption 1 corresponds to the set of conditions imposed on the shocks in

Cavaliere and Taylor (2008) and Smeekes and Taylor (2012), strengthened by the addition of condition (iii). This additional condition ensures that the new heteroskedasticity-robust information criteria, which we propose in section 3 below, are based on a consistent estimate of the volatility process; see Beare (2008) who shows that (iii) suffices for this purpose when using a nonparametric kernel estimator. As the conditions in Assumption 1 are stronger than those in Cavaliere and Taylor (2008) and Smeekes and Taylor (2012), the large sample validity of the bootstrap unit root tests discussed in the next section is guaranteed. The reader is directed to Cavaliere and Taylor (2008) and Beare (2008) for further discussion of the conditions imposed by Assumption 1. Notice that Assumption 1 contains unconditional homoskedasticity as a special case, but does not allow for models of conditional heteroskedasticity.

3 Unit Root Testing and Information Criteria

3.1 Bootstrap Unit Root Tests

In this paper we focus attention on wild bootstrap implementations of the ADF tests because of their enduring popularity with practitioners. However, the analysis provided in this paper is also valid for any unit root test that requires an autoregressive lag order to be selected. The ADF t-statistic is the usual regression t-statistic of significance on γ, denoted t_γ^d in what follows, in the ADF regression

$$\Delta y_t^d = \gamma y_{t-1}^d + \sum_{j=1}^{p} \phi_{p,j} \Delta y_{t-j}^d + \varepsilon_{p,t}^d, \qquad t = p+1, \dots, T. \tag{2}$$

where $y_t^d := y_t - \hat{\beta}' z_t$ is the de-trended analogue of y_t, where the parameter estimate $\hat{\beta}$ can be obtained either by the OLS or the quasi-difference (QD) regression of y_t on z_t; see, eg, Elliott, Rothenberg, and Stock (1996). In the context of (2), p is the lag truncation order. We defer a discussion of the criteria that will be used to estimate p until sections 3.2 and 3.3.

Under nonstationary volatility, the t_γ^d statistic is not asymptotically pivotal and the associated ADF test can display very large size distortions; see Cavaliere and Taylor (2008, 2009b). One solution to this problem, studied by Cavaliere and Taylor (2008, 2009b) and Smeekes and Taylor (2012) among others, is to apply the wild bootstrap principle. Cavaliere and Taylor (2008, 2009b) demonstrate the asymptotic validity of this approach, for the case of a deterministic lag length satisfying Assumption 2 below, and give simulation results which show that the method works well in finite samples. Hence, our focus in what follows will be on wild bootstrap implementations of the ADF test where data-dependent methods are used to select the lag length in (2). We now outline the wild bootstrap algorithm we will use.

Algorithm 1.

1. Calculate $y_t^d := y_t - \hat{\beta}' z_t$, where $\hat{\beta}$ is obtained either by the OLS or QD regression of y_t on z_t, $t = 0, ..., T$.

2. Estimate by OLS the ADF regression in (2) using a lag order, q, to obtain the ADF residuals

$$\check{\varepsilon}_{q,t}^d := \Delta y_t^d - \check{\gamma} y_{t-1}^d - \sum_{j=1}^{q} \check{\phi}_{q,j} \Delta y_{t-j}^d, \qquad t = 1, \ldots, T, \tag{3}$$

by defining $y_{-1}^d, \ldots, y_{-q}^d := 0$.[2]

3. Construct (wild) bootstrap errors ε_t^* according to the device $\varepsilon_t^* := \xi_t \check{\varepsilon}_{q,t}^d$, where ξ_t satisfies $\mathrm{E}(\xi_t) = 0$ and $\mathrm{E}(\xi_t^2) = 1$.[3]

4. Build u_t^* recursively as $u_t^* = \sum_{j=1}^{q} \check{\phi}_{q,j} u_{t-j}^* + \varepsilon_t^*$, using the estimated parameters $\check{\phi}_{q,j}$ from Step 2 (initialised at $u_0^*, \ldots, u_{1-q}^* = 0$), and build y_t^* as $y_t^* = y_{t-1}^* + u_t^*$, $t = 1, ..., T$, initialised at $y_0^* = 0$.

5. Using the bootstrap sample y_t^*, apply the same method of detrending as applied to the original sample in step 1 to obtain the detrended bootstrap series $y_t^{d*} := y_t^* - \hat{\beta}^{*\prime} z_t$, where $\hat{\beta}^*$ is defined analogously as in step 1, but with the bootstrap data. Calculate the bootstrap augmented ADF statistic, denoted t_γ^{d*}, from the bootstrap analogue of the ADF regression, with lag truncation p^*,

$$\Delta y_t^{d*} = \gamma^* y_{t-1}^{d*} + \sum_{j=1}^{p^*} \phi_{p^*,j} \Delta y_{t-j}^{d*} + \varepsilon_{p^*,t}^{d*}, \qquad t = p^* + 1, \ldots, T. \tag{4}$$

6. Repeat Steps 3 to 5 N times, obtaining bootstrap test statistics, $t_{\gamma,b}^{d*}$ say, for $b = 1, \ldots, N$, and calculate the bootstrap critical value

$$cv^{d*}(\pi) := \max\{x : N^{-1} \sum_{b=1}^{N} I(t_{\gamma,b}^{d*} < x) \leq \pi\}$$

or, equivalently, as the π-quantile of the ordered $\{t_{\gamma,b}^{d*}\}_{b=1}^N$ statistics. Reject the null of a unit root if t_γ^d is smaller than $cv^{d*}(\pi)$, where π is the nominal level of the test. □

Remark 2. In this paper we do not consider the question of whether OLS or QD detrending should be preferred in unit root testing. This depends critically on the initial condition, as

[2] This initialisation ensures that we obtain T residuals and, hence, T bootstrap errors in Step 3. An asymptotically equivalent alternative is to omit this initialisation thereby yielding only $T - q$ residuals, but to then initialise the recursion in Step 4 with the first q detrended sample values. We found virtually no differences between the two schemes for the sample sizes considered.

[3] In this paper we take ξ_t to be standard normal. Other choices are also possible, although Cavaliere and Taylor (2008, Remark 6) mention that this has almost no impact on finite sample behaviour.

is now well known in the unit root literature, see e.g. Müller and Elliott (2003). Harvey, Leybourne, and Taylor (2009) propose a union of OLS and QD detrended tests if there is uncertainty about the initial condition. This approach is extended to allow for nonstationary volatility using the wild bootstrap by Smeekes and Taylor (2012). However, using such a union-based approach still requires one to select lag lengths for use in ADF regressions. As such the problem and remedies considered in the current paper directly extend to the union tests, which is why we do not treat them explicitly in this paper. A different question is whether OLS or QD detrending should be used in the lag length selection procedure itself; as elaborated on below, Perron and Qu (2007) find that OLS is superior in this context.

3.2 Standard Lag Selection Criteria

While the wild bootstrap procedure outlined in Algorithm 1 takes account of any possible nonstationary volatility in the shocks without the need to parametrically model the volatility process, the presence of the lagged dependent variables in (2) is required to parametrically account for any serial correlation in the shocks. Consequently, in order to implement the ADF test, the selection of an appropriate lag length in (2), and indeed in (3) and (4), is required.

It is unrealistic to assume that the true value of p, p_0 say, in (2) is known to the practitioner, since the nature of the serial correlation in u_t cannot be reasonably assumed known. Indeed, p_0 may be infinite, as is the case, for example, if u_t is a finite-order moving average (MA) process. In such cases, it is well known, see for example Chang and Park (2002), that if the lag truncation order in (2) satisfies the following deterministic rate condition:

Assumption 2. Let $p \to \infty$ and $p = o(T^{1/3})$ as $T \to \infty$.

then, provided ε_t in (1c) is either homoskedastic or conditionally heteroskedastic (but unconditionally homoskedastic), the resulting ADF statistic, t_γ^d, will have the usual Dickey-Fuller limiting null distribution free of serial correlation nuisance parameters; as tabulated for the case of OLS detrending in Fuller (1996, p. 642) and for QD detrending in Elliott et al. (1996, p. 825). As noted in section 3.1, Cavaliere and Taylor (2008, 2009b) demonstrate a corresponding result for the case where ε_t is unconditionally heteroskedastic; here the limiting null distribution of t_γ^d remains free of serial correlation nuisance parameters but does now depend on the form of the underlying volatility process.

As pointed out by Cavaliere and Taylor (2009a, Section 3.3), the sieve, or re-colouring, device in step 4 of Algorithm 1 is motivated purely by finite sample concerns, and q does not therefore have to increase to infinity with the sample size.[4] Also, although p^* is not required to diverge with T, we do require that $q \leq p^*$ for large T.[5] Specifically, we make the following

[4] This differs from the approach taken by Smeekes and Taylor (2012, Assumption 5) for reasons explained in their Remark 15.

[5] Cavaliere and Taylor (2009a) assume that $q \leq p^*$ for all T but this is not necessary for the validity of the bootstrap. By allowing p^* to be smaller than q one can replicate the effect of under-fitting the lag length in the bootstrap, which may improve finite sample performance (cf. Richard, 2009).

assumptions on q and p^*:

Assumption 3. (i) Let $p^* = o(T^{1/3})$; (ii) there is a T^* such that $q \leq p^*$ for all $T > T^*$.

For a given sample size, Assumptions 2 and 3 give no practical guidance on how to select the lag length in (2), (3) and (4). A popular choice, which permits a trade-off between the size distortions that result from including too few lags and the power losses that obtain when too many lags are included, is to base it on an information criterion (see also Remark 3). This estimates the lag length as follows

$$\hat{p} := \underset{p_{\min} \leq k \leq p_{\max}}{\arg\min}\ IC(k), \qquad IC(k) := \ln \hat{\sigma}_k^2 + k\frac{C_T}{T}, \tag{5}$$

where $\hat{\sigma}_k := (T - p_{\max})^{-1} \sum_{t=p_{\max}+1}^{T} (\hat{\varepsilon}_{k,t}^d)^2$ with $\hat{\varepsilon}_{k,t}^d$ the OLS residuals from the k-th order ADF regression for y_t^d in (2); that is, $\hat{\varepsilon}_{k,t}^d := \Delta y_t^d - \hat{\gamma} y_{t-1}^d - \sum_{j=1}^{k} \hat{\phi}_{k,j} \Delta y_{t-j}^d$, and where $p_{\min} \leq p_{\max}$ are selected such that $p_{\min}, p_{\max} \to \infty$ as $T \to \infty$ with $p_{\max}$ satisfying the rate condition in Assumption 2. In (5), C_T is a penalty function that differs according to the specific information criterion to be used; for AIC $C_T := 2$, for BIC $C_T := \ln T$. Tsay (1984) shows that for finite p, the properties of AIC and BIC in the stationary case remain the same in the presence of unit roots; ie, BIC is consistent while AIC is not (it overestimates with a positive probability). Pötscher (1989) extends these results to allow for nonconstant volatility in the errors, and finds that BIC is still consistent for the setting considered in this paper.

Ng and Perron (2001) propose a class of modified information criteria (MIC), motivated specifically for selecting the lag length in the ADF regression, (2), of the form

$$MIC(k) := \ln \hat{\sigma}_k^2 + k\frac{C_T + \tau_T(k)}{T},$$

where $\tau_T(k) := (\hat{\sigma}_k^2)^{-1} \hat{\gamma}^2 \sum_{p_{\max}+1}^{T} (y_{t-1}^d)^2$. The associated lag length estimate is then defined as in (5) but replacing $IC(k)$ by $MIC(k)$ in the definition of $\hat{p}$. The penalty function C_T is selected as for the original criteria; eg, $C_T := 2$ and $C_T := \ln T$ yield the modified AIC (MAIC) and criterion and the modified BIC (MBIC) criterion respectively. Although asymptotically the properties of the original criteria will be maintained, Ng and Perron (2001) show that these modified criteria yield large improvements over the standard criteria for the purpose of unit root testing, in particular if a negative moving average parameter is present in the short-run dynamics. Perron and Qu (2007) propose a further modification of these criteria, by suggesting that they should always be applied to OLS rather than QD detrended data even if the unit root test itself is based on QD detrended data. This will improve the power properties of the test, in particular for alternatives further from the null.

Note that provided $p_{\min}$ and $p_{\max}$ satisfy the conditions stated above, the limiting null distributions of t_γ^d and t_γ^{d*} will not be affected by the short-run dynamics irrespective of the asymptotic properties of the selected information criterion. Our investigation is therefore purely related to the performance of the (wild bootstrap) ADF test in finite samples, since in finite samples the lag selection criteria are misspecified if the volatility process is time-varying and cannot necessarily be relied upon to yield an appropriate estimate of the required lag lengths. This is confirmed by the simulation results we present in Section 4.

Remark 3. We focus here on lag length selection through information criteria rather than through sequential t-testing, as this approach has proven to be more popular and also more successful; unreported simulations show that sequential t-testing tends to select too many lags on average. Sequential t-testing can be adapted to the setting of nonstationary volatility by either using heteroskedasticity-robust standard errors, or by again applying the wild bootstrap principle.

3.3 Heteroskedasticity-Robust Lag Selection Criteria

In this subsection we propose a method for lag length selection based on information criteria that is designed to be robust to heteroskedasticity. Rather than modifying the information criteria themselves, we modify the series that is the input to the information criteria. We adapt the idea proposed in Beare (2008) to lag length selection; that is, we estimate the volatility nonparametrically and then re-scale the series with the estimated volatility.

To estimate the volatility nonparametrically we use the local constant, or Nadaraya-Watson, estimator used by Beare (2008).[6] The volatility estimator at time t is defined as

$$\hat{\sigma}_{m,t} := \sqrt{\hat{\omega}_m^2(t/T)}, \qquad \hat{\omega}_k^2(r) := \frac{\sum_{t=1}^{T} K\left(\frac{t/T-r}{h}\right)(\breve{\varepsilon}_{m,t}^d)^2}{\sum_{t=1}^{T} K\left(\frac{t/T-r}{h}\right)} \tag{6}$$

where $\{\breve{\varepsilon}_{m,t}^d\}$ are defined in (3) with a lag truncation of m, $K(\cdot)$ is a kernel function and h is a bandwidth parameter. As in Beare (2008), the following assumption is needed on the kernel $K(\cdot)$ and the bandwidth h in order to ensure that (6) consistently estimates the volatility:

Assumption 4. (i) $K(\cdot)$ is continuously differentiable and satisfies $\int K(x)dx > 0$, $\int |xK(x)|\, dx < \infty$, and $\int |xK'(x)|\, dx < \infty$. Moreover, the Fourier transform of $K(\cdot)$, denoted $\psi(\cdot)$, satisfies $\int |x\psi(x)|\, dx < \infty$. (ii) $h \to 0$ and $Th^4 \to \infty$ as $T \to \infty$.

The volatility estimates from (6) are then used to re-scale the series of interest as follows:

$$\tilde{y}_t := \sum_{s=1}^{t} \frac{\Delta y_s^d}{\hat{\sigma}_{m,s}}, \qquad \tilde{y}_0 := 0. \tag{7}$$

[6] We also considered the re-weighted local constant estimator proposed by Xu and Phillips (2011). As discussed by Xu and Phillips (2011), this estimator shares all the advantages of the local linear estimator. However, unlike the local linear estimator (but like the local constant estimator), it cannot be negative. The simulation results with this estimator were virtually identical to the results reported here with the local constant estimator and, hence, are omitted in the interests of space.

The idea behind (7) is that $\tilde{y}_t$ will be rendered (approximately) homoskedastic. The re-scaled series $\tilde{y}_t$ is then used as input to the information criteria. The corresponding re-scaled (modified) information criteria, denoted RS(M)IC in what follows, are then calculated as

$$RSIC(k) := \ln \tilde{\sigma}_k^2 + k\frac{C_T}{T}, \qquad RSMIC(k) := \ln \tilde{\sigma}_k^2 + k\frac{C_T + \tilde{\tau}_T(k)}{T},$$

where $\tilde{\tau}_T(k) := (\tilde{\sigma}_k^2)^{-1}\tilde{\gamma}^2 \sum_{p_{\max}+1}^{T}(\tilde{y}_{t-1}^d)^2$, $\tilde{\sigma}_k = (T - p_{\max})^{-1}\sum_{t=p_{\max}+1}^{T}(\tilde{\varepsilon}_{k,t}^d)^2$, and where $\tilde{\varepsilon}_{k,t}^d$ is the OLS residual from a k-th order ADF regression on $\tilde{y}_t^d$, which is either the OLS or QD detrended analogue of $\tilde{y}_t$.

In practice one must also select a value for the lag truncation m used in the construction of the volatility estimator in (6). The choice $m = 0$ corresponds to Beare (2008), while taking $m = p_{\max}$ would also seem to be a sensible choice in the lag selection framework. In this paper we will follow Beare (2008) and set $m = 0$, but unreported simulations showed that setting $m = p_{\max}$ gave virtually identical results.[7]

4 Monte Carlo Simulations

In this section we will use Monte Carlo simulation methods to investigate the finite sample performance of the standard information criteria and their new heteroskedasticity-robust analogues developed in the previous section. Comparison is made both of the lag order selected by these criteria and of the size and power properties of the associated wild bootstrap ADF tests for a variety of homoskedastic and heteroskedastic ARMA models.

4.1 The Monte Carlo Design

In the simulation study we use the following DGP:

$$y_t = x_t + \beta' z_t, \qquad t = 0, 1, \ldots, T, \tag{8a}$$

$$x_t = \rho_T x_{t-1} + u_t, \qquad x_0 = 0 \tag{8b}$$

$$u_t = \phi_1 u_{t-1} + \phi_2 u_{t-2} + \phi_3 u_{t-3} + \varepsilon_t + \theta\varepsilon_{t-1} \tag{8c}$$

$$\varepsilon_t = \sigma_t e_t, \qquad e_t \sim i.i.d.\ N(0,1), \tag{8d}$$

for the local-to-unity setting where $\rho_T = 1 - c/T$, such that $c = 0$ corresponds to the unit root null hypothesis and $c > 0$ to local alternatives. Without loss of generality we set $\beta = 0$.

We report results for the combinations of the AR and MA parameters ϕ_1, ϕ_2, ϕ_3 and θ in (8c) given in Table 1. Table 1 also reports for each model the true value, p_0, of the associated lag augmentation in (2). These ARMA parameters allow for a range of different dynamic

[7]Similarly, it is possible to use the residuals which are obtained when imposing the unit root null hypothesis. Unreported simulation results indicated that the results do not change in this case either.

models, ranging from near I(2) data (models 8 and, to a lesser extent, 5 and 10, with $\rho_T = 1$) to near over-differenced data (model 11 with $\rho_T = 1$). The range of models is very similar to that considered by Ng and Perron (2005) and allows both finite AR (of orders 1,2 and 3) and MA(1) models.

Insert Table 1 about here

Results are reported for the following three volatility models:

1. Smooth transition: $\sigma_t^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)\mathbb{S}_t$, where $\mathbb{S}_t = (1 + \exp(-\gamma(t - \lfloor \tau T \rfloor)/T))^{-1}$ with $\sigma_0 = 1$. We consider parameters $\delta = 1/3, 3$ and $\tau = 0.2, 0.8$ with $\gamma = 25$.

2. Single break in volatility: $\sigma_t^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)I(t > \lfloor \tau T \rfloor)$, where we set $\sigma_0 = 1$. Defining $\delta = \sigma_0/\sigma_1$, we consider parameters $\delta = 1/3, 3$ and $\tau = 0.2, 0.8$.

3. Stochastic volatility: $\sigma_t^2 = \omega^2(t/T)$ where $\omega^2(s) = \sigma_0^2 \exp(\nu J_{\tilde{c}}(s))$ and $J_{\tilde{c}}$ is an Ornstein-Uhlenbeck process. Again we set $\sigma_0 = 1$, and we consider parameters $\tilde{c} = 0, 10$ and $\nu = 4, 9$.

In the first model a smooth (logistic) transition occurs in the variance from σ_0^2 to σ_1^2 centred at time $\lfloor T\tau \rfloor$, $\lfloor \cdot \rfloor$ denoting the integer part of its argument, with γ the speed of transition. The second model can be seen as a limiting case of the first, obtained by setting $\gamma = \infty$. Notice that neither the single break nor the stochastic volatility models are formally allowed under the assumptions needed on the kernel estimation. We chose these models of volatility as they are popular choices in the literature and appear to describe empirically observed patterns well. Moreover, good performance by the new lag selection criteria for models such as these which fall outside the class of models they are intended for can be argued to reinforce their potential. The smooth transition models, which are smoothed versions of the single break models considered, are included to evaluate the importance of the smoothness assumption for the volatility estimation. Also observe that the homoskedastic case is contained in the first two models when $\delta = 1$, and in the third model when $\nu = 0$.

In the appendix we also give results for the single-break model with a mid-sample break, $\tau = 0.5$, and results obtained from a model with two abrupt breaks in volatility when the dominant break of the two coincided with that in the single break model. These results are qualitatively similar to the single break results reported here.

In this analysis we present results only for the MAIC criterion of Ng and Perron (2001) and the heteroskedasticity-robust analogue thereof, RSMAIC, from section 3.3. We do so because MAIC is the most popular and successful criterion used in unit root testing. However, a summary of the corresponding results for other popular lag selection methods is given at the end of this section. In the context of the MAIC and RSMAIC criteria the minimum lag length, $p_{\min}$ was set to zero throughout, while the maximum lag length was set to $p_{\max} =$

$\lfloor A(T/100)^{1/4} \rfloor$, with the choice of the constant A specified in the subsections which follow. We report results for the sample sizes $T = 150$ and $T = 250$.[8] Throughout this section we will only report results for the specification where a constant is included in z_t in (8a). Results for the constant and trend case are very similar, and are available on request. As recommended by Perron and Qu (2007), we apply the information criteria to OLS detrended data. As mentioned before, the volatility estimator used in the RSMAIC is $\hat{\sigma}_{0,t}$, with the kernel $K(\cdot)$ taken as the Gaussian kernel and the bandwidth set equal to $h = 0.1$, a value that produced good results in Beare (2008).[9]

4.2 Selected Lag Lengths

We first focus on the lag lengths selected by the standard MAIC and the new heteroskedasticity-robust RSMAIC criteria. As part of our analysis we vary the maximum lag length, $p_{\max}$, by considering results for both $A = 6$ and $A = 12$. In large samples and for the (low-order) autoregressive models the lag selection should not be significantly affected by changing the upper bound. If, however, a criterion is seriously affected by the choice of $p_{\max}$ then this provides clear evidence that the criterion is not selecting the lag length appropriately for the sample sizes considered. All results are based on 5000 simulations.

Insert Table 2 about here

Table 2 reports the average (taken across the Monte Carlo replications) selected lag lengths obtained under homoskedasticity. It can be seen from these results that the MAIC and RSMAIC criteria perform very similarly to one another here for all of the AR and MA models considered. These results suggest that the re-scaling approach used in calculating the RSMAIC criterion does not fundamentally change its properties from those of the MAIC criterion under homoskedasticity, which is a necessary condition to apply it successfully. It can also be seen that for the AR models considered, other things being equal, changing the maximum lag length (through the choice of the constant A) has only a minor impact on the average lag length selected for both criteria, as expected.

Insert Tables 3-6 about here

Tables 3 to 6 present the corresponding results for the case of a smooth transition break in volatility. From these results we can see that both the direction and timing of the break have a considerable impact on the lag length selected by the standard MAIC criterion. In particular, while late negative or early positive breaks do not appear to have a significant impact on the

[8] For smaller sample sizes the differences between the regular and re-scaled IC are less obviously seen, at least in part because the maximum lag length parameters will be smaller; see, for example, the additional results for $T = 50$ reported in the appendix.

[9] Different specifications again gave very similar results.

lag length selected by MAIC, the effect of either a late positive or early negative break is, on the other hand, substantial. For these volatility models MAIC selects considerably higher lag lengths lags than it does under homoskedasticity. This effect can be seen for all of the ARMA models considered. Moreover, in these cases changing the maximum lag length now has a major impact on the performance of MAIC, which is again a clear indication that the standard MAIC criterion selects too many lags, approaching the upper bound as a result. In contrast, the RSMAIC criterion appears to select roughly the same number of lags in the smooth transition break models as it does under homoskedasticity; there only appears to be a minimal increase in some of the cases considered. Also, RSMAIC is far less affected by varying $p_{\max}$ than MAIC is, which again confirms the robustness of the lag length selected by RSMAIC in this case.

Insert Tables 7-10 about here

Tables 3 to 6 present the average selected lags for the single break model. MAIC shows a similar tendency to select too many lags as for the corresponding smooth transition models. Interestingly, RSMAIC does very similar, but not better, as for the single break case. Hence there is no evidence from these results that violating the continuity assumption on the volatility process - in the form of a single abrupt break - has a negative effect on the performance of the RSMAIC.

Insert Tables 11-14 about here

Tables 11 to 14 present the average selected lags under stochastic volatility. Compared with the results in Table 2, it can again be seen that the standard MAIC criterion selects a higher lag length on average than it does under homoskedasticity, most notably when $\tilde{c} = 0$. We now also see an increase in the average lag length selected by RSMAIC, although it still selects a considerably lower average lag length than MAIC. Hence, even though RSMAIC is affected to some degree by stochastic volatility, it remains considerably more reliable than MAIC in this setting.

To summarise, our simulation results have shown that lag length selection by MAIC is affected by the presence of nonstationary volatility in the errors. As such it cannot be reliably used to select an appropriate lag length for a unit root test in this setting. The simulation results also show that RSMAIC appears to be significantly more robust to the presence of nonstationary volatility, while its performance under homoskedasticity is almost identical to MAIC. In the context of unit root testing, it is arguably the performance of the unit root test for which lag orders are selected, rather than the actual selected lag order, which is of primary importance. If the lag selection has no effect on the size or power properties of the resulting unit root test, then there is no problem in using a potentially misspecified method such as MAIC. Therefore we will now investigate the impact of nonstationary volatility on the finite sample size and power properties of the wild bootstrap ADF unit root test, when the lag length in the ADF regression has been selected by either MAIC or RSMAIC.

4.3 Rejection Frequencies of Bootstrap Unit Root Tests

In this subsection we investigate the performance of the wild bootstrap ADF unit root test from Algorithm 1, using QD detrending, and where the lag truncation order in the original ADF regression (2), the sieve regression (3), and the bootstrap ADF regression (4), were selected by either MAIC or RSMAIC, using the same tuning parameters as outlined in section 4.1, with results reported for $A = 12$. All results in this subsection are based on 5000 simulations and 199 bootstrap replications.

Insert Tables 15-18 about here

We first report, in Tables 15 to 18, the size properties of the wild bootstrap ADF tests based on MAIC and RSMAIC lag selection for the same set of ARMA and volatility models as were used in the previous subsection. Sizes for MAIC and RSMAIC seem to be comparable across the different models; both give sizes close to the nominal level of 5% except for model 11 (which has a large negative MA parameter), where there is some oversize (of roughly the same degree) seen for both methods. Overall it does not appear that the choice between using MAIC or RSMAIC when choosing the lag length has a significant impact on the size of the resulting unit root test, regardless of whether the errors are homoskedastic or heteroskedastic.

We next present finite sample local power curves for the bootstrap ADF tests. In order to keep the number of graphs manageable, we need to make a selection of the ARMA models considered. To this end we report results for the i.i.d. model (model 1), the AR(1) model with $\phi_1 = 0.5$ (model 4) and the MA(1) model with $\theta = -0.5$ (model 12). We consider the same type of volatility models as before but focus on the cases where MAIC is most affected by the volatility process. The simulation results in Section 4.2 showed that for the volatility model with a single break or a smooth transition break, MAIC was most affected by a late positive or early negative shift, while for the stochastic volatility model, MAIC was most affected if a unit root was present in the volatility ($\tilde{c} = 0$). These cases together with the benchmark of homoskedasticity will therefore be considered in the power analysis.

Insert Figure 1 about here

In Figure 1 we first present the finite sample local power curves of the wild bootstrap ADF tests based on MAIC and RSMAIC lag selection for the homoskedastic model. In the homoskedastic case the power of the tests using MAIC and RSMAIC are almost identical to one another, which is again as expected given the results from section 4.2. This shows that the power losses incurred by using the RSMAIC criterion to select the lag length when in fact the MAIC criterion is correctly specified are negligible even for $T = 150$.

Inserts Figures 2-7 about here

Figures 2 and 3 give the corresponding local power curves for the smooth transition variance break model with a late positive break and an early negative break, respectively.[10] For these models, the bootstrap ADF test based on the use of RSMAIC is clearly more powerful than the corresponding test based on MAIC. This is a direct consequence of the results reported in section 4.2 which showed that the MAIC criterion significantly over-fits the lag order relative to the RSMAIC criterion for these designs. It is clear that in these cases there are considerable finite sample power gains available by using RSMAIC. Moreover, the power differences between using MAIC and RSMAIC lag selection even increase slightly between $T = 150$ and $T = 250$, which appears to be related to the associated increase in the maximum lag length, $p_{\max}$, between the two sample sizes.

The power curves for the single break models are given in Figures 4 and 5. As expected from the lag length selection results in Section 4.2, the power curves are very similar to those observed for the smooth transition model.

Figures 6 and 7 graph the finite sample local power curves for the stochastic volatility models with $\tilde{c} = 0$ and $\nu = 4, 9$. While the bootstrap ADF test based on RSMAIC is still more powerful than the corresponding test based on MAIC, the difference between the two is now rather smaller than was seen for the break in volatility models. This is to be expected from the results on the average lag length selected by these two criteria in section 4.2, which showed that RSMAIC has a tendency to over-fit the lag length in this case, although not to the same extent as is seen with MAIC. While the gains of using RSMAIC may be smaller for the stochastic volatility case, it is nonetheless important to note that there is never a loss in power when using RSMAIC rather than MAIC to select the lag length.

We can summarise the results in this subsection by observing that lag order selection based on MAIC has a negative impact on the finite sample power of the resulting wild bootstrap ADF unit root test if nonstationary volatility is present, with the extent of this effect depending on the specific volatility model. Based on our results, we recommend the use of the RSMAIC lag selection criterion for selecting the lag length in the context of ADF unit root testing, given its greater degree of robustness to nonstationary volatility than the standard MAIC lag selection criterion, and the resulting higher finite sample power which is achievable when using RSMAIC over MAIC. These power gains are most strongly seen for the break in volatility models. Moreover, under homoskedasticity we found almost no differences in power between the unit root tests which use RSMAIC and MAIC to select the lag order. Under all of the

[10] Notice that the local power curves for these models are quite different from the corresponding local power curves seen in Figure 1 under homoskedasticity, even for $T = 250$. This is not an effect of the lag order selection method but rather a consequence of the result that if nonstationary volatility is present, then the limiting distributions of the ADF statistic, t_{γ}^{d}, under both the null hypothesis and local alternatives, and hence the asymptotic local power function of the associated bootstrap test, are functions of the underlying volatility process (cf. Cavaliere and Taylor, 2008, p. 8).

volatility and ARMA models considered the finite sample size properties of the unit root tests based on MAIC and RSMAIC were virtually identical. As such we believe it provides a reliable practical alternative to MAIC.

We conclude this section by noting that the conclusions drawn above concerning wild bootstrap ADF tests based on the MAIC lag selection method and its re-scaled analogue, RSMAIC, all carry through qualitatively to the corresponding ADF tests based other information criteria such as AIC and BIC (where the re-scaling in computing their heteroskedasticity-robust analogues is done identically). We also considered sequential t-tests for specifying the lag truncation order, as in Ng and Perron (1995), comparing their standard approach with modifications thereof based on either the use of White (1980) heteroskedasticity-robust standard errors or the wild bootstrap. Simulations indicated that sequential t-testing is affected by nonstationary volatility in much the same way as the information criteria reported here. Using White standard errors helps to alleviate the problems, but does not erase them. Wild bootstrap ADF tests using lag selection based on wild bootstrap sequential t-tests, like the tests based on the RSMAIC method, achieve higher power than the tests based on the standard sequential t-tests but have the considerable drawback that they take a very long time to compute. Moreover, we found them to be generally inferior than the tests based on RSMAIC, and so we do not report these results in detail. They are, however, available on request.

5 Conclusion

We have investigated the effect of nonstationary volatility on lag length selection in the context of unit root testing, proposing a modification of the popular information criteria used for lag length selection, designed to be robust against nonstationary volatility. The modification consisted of re-scaling the data by a nonparametric estimate of the volatility process before computing the information criterion of interest.

Focusing on the popular MAIC criterion, we found that nonstationary volatility can have a significant impact on lag length selection in finite samples. Simulations for several volatility models showed that the lag order was often over-fitted, with the selected lag length being highly dependent on the maximum lag length allowed in certain cases. Our proposed re-scaled MAIC, labeled RSMAIC, criterion did not demonstrate this feature and was shown to be robust to nonstationary volatility, most notably a break in volatility. Moreover, the RSMAIC criterion was shown to perform almost identically to the MAIC criterion in terms of the lag order selected under homoskedasticity.

We then investigated the relative behaviour of the wild bootstrap ADF unit root tests obtained for these two different lag selection criteria. It was found that using MAIC in the presence of nonstationary volatility leads to a loss of finite sample power in the associated unit root test, caused by the tendency of MAIC to fit significantly more lags than RSMAIC.

This despite the fact that size properties of the unit root tests based on MAIC and RSMAIC lag selection were shown to be broadly comparable. Moreover, under homoskedasticity no significant losses in power were observed for the unit root tests based on RSMAIC relative to those based on MAIC.

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Table 1: ARMA models considered.

Model	p_0	ϕ_1	ϕ_2	ϕ_3	θ
1	0	0.00	0.00	0.00	0.00
2	1	-0.80	0.00	0.00	0.00
3	1	-0.50	0.00	0.00	0.00
4	1	0.50	0.00	0.00	0.00
5	1	0.80	0.00	0.00	0.00
6	2	0.40	0.20	0.00	0.00
7	2	1.10	-0.35	0.00	0.00
8	2	1.30	-0.35	0.00	0.00
9	3	0.30	0.20	0.10	0.00
10	3	0.10	0.20	0.30	0.00
11	∞	0.00	0.00	0.00	-0.80
12	∞	0.00	0.00	0.00	-0.50
13	∞	0.00	0.00	0.00	0.50
14	∞	0.00	0.00	0.00	0.80

Table 2: Average lags lengths selected by MAIC and RSMAIC. Homoskedastic errors. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.57	0.54	0.67	0.69	0.76	0.69	0.85	0.82
2		1.59	1.63	1.65	1.69	1.85	1.95	1.90	1.98
3		1.57	1.54	1.61	1.60	1.82	1.79	1.84	1.81
4		1.57	1.62	1.62	1.75	1.79	1.86	1.82	1.99
5		1.55	1.70	1.62	1.86	1.76	2.03	1.83	2.25
6		2.20	2.23	2.49	2.56	2.47	2.52	2.76	2.89
7		2.49	2.58	2.56	2.64	2.80	2.96	2.82	3.06
8		2.50	2.79	2.56	2.96	2.78	3.50	2.82	3.81
9		2.56	2.53	2.94	2.98	2.83	2.85	3.20	3.36
10		3.34	3.35	3.49	3.55	3.66	3.72	3.82	3.94
11		5.21	5.21	6.21	6.22	7.54	7.56	8.62	8.61
12		3.00	2.98	3.40	3.38	3.37	3.32	3.75	3.67
13		2.70	2.69	3.18	3.17	3.02	3.00	3.47	3.52
14		4.82	4.79	5.75	5.74	5.94	5.91	7.20	7.23
1	7	0.75	0.71	0.76	0.71	0.96	0.91	1.01	0.93
2		1.85	1.92	1.86	1.89	2.33	2.43	2.18	2.27
3		1.73	1.72	1.77	1.75	2.06	2.05	2.07	2.05
4		1.64	1.61	1.71	1.68	1.92	1.83	1.97	1.89
5		1.63	1.61	1.68	1.69	1.94	1.90	1.94	1.95
6		2.03	1.97	2.43	2.39	2.37	2.25	2.72	2.65
7		2.59	2.59	2.66	2.65	2.97	2.94	2.98	2.96
8		2.54	2.62	2.63	2.75	2.88	3.02	2.93	3.16
9		2.13	2.08	2.66	2.63	2.40	2.31	3.02	2.95
10		3.15	3.07	3.54	3.52	3.60	3.42	3.93	3.84
11		5.06	5.05	6.34	6.34	8.35	8.34	9.97	10.01
12		3.45	3.43	3.86	3.86	4.12	4.09	4.40	4.40
13		2.67	2.62	3.05	3.02	3.03	2.96	3.42	3.33
14		4.69	4.66	5.47	5.46	5.74	5.66	7.04	6.98

Table 3: Average lags lengths selected by MAIC and RSMAIC. Smooth transition volatility model: $\delta = 1/3$, $\tau = 0.2$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.76	0.63	0.86	0.75	1.05	0.83	1.21	1.01
2		1.74	1.68	1.83	1.75	2.01	1.94	2.18	2.03
3		1.72	1.60	1.81	1.68	2.05	1.87	2.16	1.91
4		1.68	1.63	1.83	1.77	2.00	1.89	2.17	2.06
5		1.72	1.80	1.81	1.93	2.02	2.14	2.15	2.34
6		2.32	2.27	2.64	2.62	2.66	2.61	3.06	3.01
7		2.61	2.62	2.73	2.75	2.96	2.99	3.12	3.18
8		2.60	2.79	2.73	3.02	2.98	3.63	3.14	3.75
9		2.66	2.60	3.01	3.00	2.96	2.92	3.46	3.46
10		3.38	3.35	3.61	3.60	3.88	3.78	4.09	4.10
11		5.19	5.19	6.19	6.18	7.60	7.52	8.69	8.58
12		3.07	2.98	3.49	3.40	3.50	3.31	4.00	3.74
13		2.81	2.74	3.24	3.18	3.19	3.02	3.73	3.58
14		4.82	4.79	5.77	5.74	6.05	5.93	7.38	7.23
1	7	0.95	0.78	0.97	0.79	1.23	0.95	1.37	1.04
2		1.97	1.96	2.04	1.97	2.51	2.45	2.45	2.30
3		1.88	1.80	1.91	1.79	2.38	2.19	2.41	2.14
4		1.81	1.68	1.93	1.78	2.19	1.95	2.32	2.03
5		1.80	1.73	1.87	1.76	2.10	1.93	2.26	2.07
6		2.09	1.96	2.56	2.44	2.44	2.20	3.04	2.78
7		2.72	2.63	2.81	2.71	3.13	2.95	3.37	3.15
8		2.67	2.65	2.77	2.79	3.02	3.07	3.20	3.31
9		2.22	2.07	2.80	2.67	2.60	2.33	3.26	3.01
10		3.17	3.00	3.65	3.56	3.59	3.30	4.17	3.93
11		5.05	5.05	6.29	6.30	8.30	8.21	10.00	9.97
12		3.53	3.50	3.95	3.90	4.28	4.10	4.66	4.44
13		2.74	2.60	3.14	3.00	3.23	2.94	3.71	3.39
14		4.64	4.57	5.48	5.46	5.87	5.63	7.28	6.98

Table 4: Average lags lengths selected by MAIC and RSMAIC. Smooth transition volatility model: $\delta = 1/3$, $\tau = 0.8$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.06	0.64	2.57	0.73	4.12	0.81	5.46	0.96
2		2.71	1.69	3.15	1.72	4.74	2.06	6.04	2.02
3		2.72	1.63	3.16	1.66	4.74	1.80	5.98	1.92
4		2.66	1.66	3.16	1.77	4.88	1.93	5.95	2.03
5		2.71	1.83	3.06	1.91	4.78	2.29	5.96	2.36
6		2.97	2.24	3.55	2.61	5.03	2.62	6.37	2.99
7		3.30	2.61	3.75	2.74	5.43	3.07	6.55	3.22
8		3.30	2.89	3.79	3.06	5.50	3.92	6.57	4.24
9		3.20	2.54	3.89	2.99	5.28	2.87	6.59	3.38
10		3.79	3.35	4.35	3.58	5.87	3.78	7.14	4.11
11		5.01	5.19	6.08	6.18	8.32	7.46	9.94	8.61
12		3.53	2.96	4.16	3.35	5.71	3.41	6.96	3.73
13		3.32	2.71	4.01	3.15	5.50	2.98	6.83	3.51
14		4.79	4.77	5.75	5.70	7.39	5.84	9.23	7.20
1	7	1.95	0.76	2.43	0.81	3.81	1.02	5.02	1.05
2		2.67	1.98	3.07	1.96	4.71	2.55	5.63	2.49
3		2.65	1.80	3.09	1.85	4.54	2.20	5.69	2.24
4		2.59	1.68	3.09	1.73	4.54	1.99	5.71	2.06
5		2.63	1.71	3.06	1.79	4.57	2.04	5.77	2.11
6		2.70	1.97	3.36	2.45	4.48	2.25	6.06	2.80
7		3.28	2.63	3.73	2.70	5.18	3.02	6.20	3.11
8		3.27	2.67	3.72	2.83	5.30	3.27	6.16	3.38
9		2.69	2.06	3.57	2.69	4.65	2.39	6.10	3.00
10		3.40	3.03	4.25	3.58	5.29	3.31	6.85	3.97
11		4.77	5.07	6.09	6.35	8.26	8.35	10.48	10.07
12		3.75	3.48	4.36	3.85	5.86	4.18	7.13	4.54
13		3.22	2.64	3.83	3.02	5.22	3.09	6.56	3.46
14		4.66	4.64	5.49	5.46	7.13	5.76	8.74	6.98

Table 5: Average lags lengths selected by MAIC and RSMAIC. Smooth transition volatility model: $\delta = 3$, $\tau = 0.2$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.06	0.61	2.58	0.67	4.01	0.79	5.27	0.89
2		2.67	1.69	3.15	1.72	4.71	2.05	5.97	2.06
3		2.72	1.61	3.11	1.62	4.67	1.85	5.93	1.85
4		2.67	1.62	3.16	1.70	4.66	1.87	5.96	2.01
5		2.67	1.76	3.18	1.90	4.63	2.13	5.92	2.24
6		3.03	2.25	3.63	2.55	4.98	2.51	6.39	2.90
7		3.29	2.60	3.76	2.71	5.31	2.96	6.54	3.06
8		3.28	2.83	3.72	2.97	5.34	3.58	6.49	3.90
9		3.24	2.55	3.96	2.99	5.27	2.90	6.68	3.38
10		3.84	3.40	4.36	3.58	5.91	3.76	7.11	3.94
11		5.04	5.24	6.10	6.22	8.54	7.79	10.24	8.90
12		3.52	3.01	4.22	3.43	5.69	3.44	6.98	3.80
13		3.38	2.73	4.06	3.16	5.49	3.01	6.82	3.44
14		4.87	4.87	5.81	5.74	7.67	6.02	9.29	7.26
1	7	1.96	0.73	2.50	0.77	3.69	0.91	5.06	0.93
2		2.67	1.95	3.10	1.91	4.51	2.41	5.85	2.37
3		2.64	1.75	3.10	1.78	4.44	2.05	5.65	2.07
4		2.63	1.63	3.13	1.74	4.46	1.94	5.65	1.96
5		2.57	1.65	3.06	1.75	4.37	1.96	5.55	2.05
6		2.77	2.09	3.48	2.49	4.56	2.36	6.14	2.80
7		3.27	2.59	3.69	2.63	5.15	2.99	6.32	3.03
8		3.27	2.67	3.74	2.77	4.99	3.17	6.26	3.30
9		2.85	2.24	3.64	2.78	4.70	2.56	6.22	3.08
10		3.57	3.18	4.27	3.51	5.54	3.55	6.89	3.90
11		4.84	5.18	6.10	6.32	8.55	8.48	10.61	9.89
12		3.78	3.43	4.41	3.77	5.80	4.04	7.08	4.34
13		3.29	2.66	3.91	3.06	5.20	3.03	6.57	3.46
14		4.78	4.79	5.54	5.59	7.22	5.87	9.05	7.18

Table 6: Average lags lengths selected by MAIC and RSMAIC. Smooth transition volatility model: $\delta = 3$, $\tau = 0.8$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.84	0.66	0.93	0.73	1.18	0.82	1.27	0.95
2		1.77	1.72	1.82	1.72	2.21	2.06	2.29	2.10
3		1.72	1.59	1.88	1.69	2.11	1.81	2.24	1.91
4		1.75	1.66	1.81	1.73	2.14	1.89	2.29	2.07
5		1.71	1.74	1.80	1.85	2.16	2.12	2.27	2.36
6		2.31	2.27	2.63	2.58	2.74	2.61	3.17	3.00
7		2.60	2.61	2.73	2.69	3.09	3.02	3.21	3.12
8		2.60	2.77	2.75	2.98	3.12	3.48	3.26	3.78
9		2.68	2.62	3.05	3.01	3.11	2.93	3.59	3.46
10		3.41	3.38	3.61	3.58	3.90	3.76	4.19	4.07
11		5.20	5.23	6.17	6.20	7.65	7.64	8.85	8.74
12		3.08	3.00	3.52	3.44	3.63	3.41	4.10	3.83
13		2.82	2.73	3.24	3.17	3.27	3.03	3.78	3.54
14		4.88	4.88	5.78	5.75	6.27	6.09	7.49	7.28
1	7	0.88	0.66	0.97	0.74	1.25	0.84	1.40	0.95
2		1.94	1.90	2.00	1.91	2.52	2.40	2.51	2.31
3		1.88	1.75	1.90	1.71	2.28	2.00	2.37	2.04
4		1.79	1.61	1.89	1.69	2.29	1.89	2.45	2.01
5		1.77	1.63	1.86	1.74	2.16	1.90	2.39	2.03
6		2.13	1.98	2.52	2.41	2.58	2.26	3.19	2.77
7		2.67	2.55	2.79	2.65	3.25	2.95	3.34	3.01
8		2.66	2.64	2.78	2.76	3.18	3.06	3.31	3.24
9		2.28	2.15	2.85	2.72	2.75	2.43	3.31	2.98
10		3.20	3.10	3.65	3.56	3.78	3.45	4.25	3.92
11		5.02	5.07	6.30	6.33	8.27	8.28	10.04	10.03
12		3.50	3.43	3.89	3.81	4.22	4.00	4.67	4.38
13		2.76	2.63	3.14	2.99	3.33	2.97	3.74	3.42
14		4.70	4.67	5.50	5.50	5.99	5.74	7.37	7.10

Table 7: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/3$ and $\tau = 0.2$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.82	0.70	0.95	0.84	1.08	0.86	1.29	1.08
2		1.76	1.75	1.86	1.80	2.12	2.11	2.24	2.14
3		1.76	1.66	1.89	1.77	2.11	1.91	2.28	2.07
4		1.74	1.70	1.86	1.81	2.05	1.98	2.25	2.14
5		1.76	1.81	1.88	1.92	2.10	2.20	2.23	2.36
6		2.33	2.30	2.68	2.66	2.73	2.63	3.17	3.11
7		2.64	2.64	2.72	2.74	3.05	3.06	3.15	3.22
8		2.63	2.79	2.81	2.99	3.02	3.59	3.26	3.80
9		2.68	2.61	3.09	3.04	3.13	3.00	3.57	3.52
10		3.43	3.40	3.66	3.64	3.89	3.81	4.24	4.17
11		5.17	5.17	6.15	6.16	7.64	7.53	8.74	8.57
12		3.08	2.98	3.53	3.44	3.59	3.36	4.11	3.88
13		2.83	2.76	3.29	3.22	3.29	3.14	3.78	3.62
14		4.84	4.80	5.73	5.72	6.17	5.98	7.42	7.29
1	7	0.93	0.78	1.09	0.92	1.27	1.02	1.53	1.20
2		1.98	1.99	2.10	2.03	2.54	2.49	2.67	2.55
3		1.92	1.84	2.04	1.90	2.42	2.24	2.54	2.28
4		1.86	1.73	1.89	1.78	2.26	2.05	2.40	2.15
5		1.78	1.72	1.99	1.91	2.19	2.02	2.46	2.29
6		2.15	2.01	2.62	2.50	2.59	2.34	3.15	2.88
7		2.75	2.67	2.84	2.75	3.22	3.08	3.44	3.24
8		2.68	2.70	2.84	2.86	3.18	3.18	3.38	3.49
9		2.17	2.04	2.84	2.71	2.55	2.31	3.41	3.12
10		3.13	2.96	3.67	3.58	3.60	3.28	4.30	4.03
11		4.99	4.98	6.28	6.29	8.31	8.20	10.03	10.00
12		3.47	3.42	3.96	3.90	4.33	4.13	4.77	4.54
13		2.79	2.66	3.22	3.07	3.34	3.06	3.87	3.54
14		4.70	4.64	5.48	5.43	5.89	5.62	7.29	7.02

Table 8: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/3$ and $\tau = 0.8$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.37	0.71	2.99	0.84	4.75	0.91	6.31	1.04
2		2.92	1.73	3.53	1.81	5.24	2.13	6.75	2.16
3		2.89	1.61	3.52	1.74	5.31	1.88	6.95	2.04
4		2.92	1.69	3.50	1.82	5.33	1.97	6.80	2.19
5		2.90	1.88	3.49	2.04	5.25	2.34	6.80	2.63
6		3.18	2.28	3.84	2.66	5.50	2.62	7.06	3.15
7		3.40	2.67	4.03	2.80	5.65	3.11	7.33	3.32
8		3.51	2.99	4.05	3.22	5.98	4.06	7.45	4.42
9		3.32	2.57	4.09	3.04	5.70	2.95	7.41	3.49
10		3.82	3.34	4.54	3.64	6.24	3.81	7.79	4.19
11		4.98	5.19	6.04	6.20	8.39	7.47	10.26	8.70
12		3.64	3.02	4.34	3.44	6.03	3.45	7.67	3.85
13		3.42	2.70	4.23	3.21	5.75	3.06	7.55	3.64
14		4.78	4.75	5.76	5.70	7.59	5.87	9.51	7.24
1	7	2.22	0.82	2.83	0.91	4.37	1.11	5.81	1.18
2		2.88	2.10	3.35	2.06	5.09	2.76	6.36	2.60
3		2.82	1.88	3.36	1.87	5.04	2.32	6.44	2.28
4		2.75	1.71	3.33	1.86	4.92	2.03	6.32	2.22
5		2.80	1.73	3.36	1.86	4.89	2.05	6.45	2.25
6		2.86	2.01	3.63	2.50	5.03	2.33	6.64	2.88
7		3.49	2.67	3.96	2.78	5.65	3.16	7.04	3.19
8		3.43	2.76	3.97	2.95	5.64	3.34	6.93	3.61
9		2.86	2.11	3.76	2.74	5.12	2.45	6.79	3.14
10		3.44	2.99	4.36	3.58	5.55	3.39	7.41	4.03
11		4.70	5.06	6.02	6.32	8.34	8.32	10.51	10.10
12		3.80	3.51	4.55	3.90	6.09	4.24	7.70	4.65
13		3.37	2.66	4.05	3.08	5.44	3.09	7.17	3.59
14		4.63	4.62	5.54	5.49	7.19	5.74	9.16	7.10

Table 9: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 3$ and $\tau = 0.2$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.38	0.65	3.07	0.76	4.84	0.86	6.46	0.97
2		2.93	1.76	3.53	1.78	5.23	2.13	6.82	2.18
3		2.95	1.60	3.50	1.70	5.29	1.88	6.87	1.97
4		2.95	1.63	3.53	1.76	5.33	1.91	6.95	2.09
5		2.89	1.76	3.58	1.93	5.42	2.19	6.89	2.36
6		3.15	2.27	3.90	2.61	5.60	2.62	7.26	3.01
7		3.45	2.60	3.99	2.73	5.88	3.02	7.26	3.18
8		3.42	2.81	4.01	3.06	5.88	3.66	7.33	4.00
9		3.33	2.55	4.10	2.98	5.67	2.94	7.33	3.40
10		3.87	3.35	4.59	3.62	6.24	3.77	7.91	4.10
11		4.99	5.22	6.10	6.24	8.50	7.78	10.41	8.93
12		3.67	3.04	4.35	3.46	6.15	3.42	7.71	3.83
13		3.52	2.77	4.24	3.18	5.86	3.05	7.61	3.57
14		4.89	4.84	5.83	5.75	7.86	6.02	9.71	7.30
1	7	2.25	0.81	2.88	0.85	4.42	1.01	6.14	1.19
2		2.83	1.98	3.45	2.04	5.11	2.50	6.60	2.52
3		2.83	1.79	3.39	1.83	5.01	2.16	6.47	2.17
4		2.79	1.70	3.42	1.79	4.88	1.95	6.64	2.13
5		2.84	1.70	3.41	1.83	5.03	2.06	6.55	2.15
6		2.94	2.09	3.71	2.53	5.05	2.41	6.78	2.86
7		3.41	2.64	3.94	2.73	5.64	3.08	7.07	3.16
8		3.43	2.70	3.95	2.83	5.59	3.27	6.96	3.45
9		2.96	2.29	3.77	2.79	5.17	2.58	6.94	3.17
10		3.63	3.18	4.46	3.56	5.83	3.59	7.57	3.96
11		4.82	5.21	6.08	6.33	8.57	8.49	10.57	9.85
12		3.82	3.42	4.59	3.82	6.02	4.04	7.74	4.40
13		3.38	2.70	4.10	3.14	5.53	3.06	7.39	3.56
14		4.78	4.78	5.59	5.59	7.45	5.96	9.44	7.27

Table 10: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 3$ and $\tau = 0.8$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.87	0.71	1.00	0.82	1.31	0.93	1.43	1.05
2		1.81	1.75	1.92	1.83	2.27	2.13	2.41	2.21
3		1.76	1.64	1.87	1.74	2.23	1.95	2.34	1.99
4		1.75	1.69	1.87	1.80	2.21	1.97	2.38	2.15
5		1.77	1.83	1.88	1.99	2.19	2.18	2.41	2.50
6		2.34	2.30	2.67	2.64	2.80	2.62	3.20	3.02
7		2.64	2.66	2.77	2.79	3.17	3.09	3.32	3.23
8		2.62	2.83	2.76	3.03	3.12	3.53	3.34	3.91
9		2.68	2.62	3.12	3.09	3.11	2.94	3.72	3.58
10		3.43	3.40	3.65	3.62	4.03	3.82	4.25	4.12
11		5.19	5.21	6.19	6.21	7.64	7.64	8.90	8.79
12		3.10	3.04	3.55	3.48	3.71	3.44	4.21	3.92
13		2.83	2.73	3.29	3.20	3.38	3.07	3.88	3.62
14		4.85	4.83	5.78	5.74	6.25	6.04	7.56	7.38
1	7	0.93	0.71	1.04	0.81	1.39	0.94	1.50	1.05
2		1.99	1.93	2.01	1.93	2.56	2.42	2.62	2.39
3		1.90	1.74	1.99	1.80	2.43	2.13	2.51	2.12
4		1.85	1.63	1.99	1.79	2.27	1.90	2.46	2.03
5		1.83	1.66	1.96	1.80	2.32	1.96	2.50	2.11
6		2.12	1.99	2.64	2.48	2.62	2.29	3.24	2.86
7		2.74	2.60	2.82	2.70	3.31	3.01	3.42	3.08
8		2.70	2.67	2.83	2.82	3.25	3.15	3.44	3.36
9		2.25	2.15	2.81	2.70	2.74	2.44	3.33	3.01
10		3.21	3.11	3.69	3.56	3.82	3.48	4.35	3.94
11		5.05	5.06	6.27	6.30	8.24	8.23	10.04	10.01
12		3.47	3.39	3.92	3.82	4.29	4.02	4.74	4.43
13		2.79	2.65	3.22	3.08	3.40	3.01	3.89	3.48
14		4.71	4.67	5.54	5.52	5.93	5.63	7.45	7.11

Table 11: Average lags lengths selected by MAIC and RSMAIC. Stochastic volatility model: $\tilde{c} = 0$ and $\nu = 4$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.27	0.98	2.96	1.21	4.28	1.32	5.98	1.66
2		2.86	2.04	3.44	2.26	4.90	2.63	6.27	2.94
3		2.81	1.89	3.48	2.13	4.80	2.30	6.38	2.69
4		2.78	1.92	3.50	2.22	4.90	2.30	6.38	2.80
5		2.83	2.08	3.47	2.43	4.93	2.71	6.37	3.31
6		3.06	2.42	3.89	2.90	5.17	2.92	6.90	3.61
7		3.39	2.83	3.97	3.12	5.44	3.40	6.95	3.92
8		3.42	3.10	4.00	3.46	5.55	4.12	7.07	4.99
9		3.34	2.74	4.09	3.25	5.34	3.23	6.98	3.98
10		3.86	3.45	4.51	3.86	5.95	4.05	7.57	4.73
11		4.93	5.14	6.02	6.14	8.21	7.67	10.02	8.90
12		3.63	3.13	4.35	3.63	5.72	3.66	7.33	4.29
13		3.50	2.88	4.26	3.44	5.53	3.35	7.25	4.12
14		4.81	4.80	5.81	5.75	7.50	6.07	9.38	7.57
1	7	2.15	1.06	2.85	1.29	4.12	1.42	5.56	1.75
2		2.81	2.21	3.37	2.40	4.80	2.97	6.22	3.30
3		2.77	1.98	3.33	2.16	4.74	2.48	6.12	2.79
4		2.79	1.88	3.36	2.09	4.67	2.25	6.17	2.55
5		2.77	1.95	3.36	2.21	4.68	2.43	6.02	2.79
6		2.86	2.20	3.64	2.73	4.84	2.63	6.45	3.32
7		3.38	2.79	3.92	3.00	5.35	3.31	6.66	3.63
8		3.38	2.89	3.99	3.20	5.32	3.70	6.73	4.23
9		2.90	2.33	3.88	3.01	4.81	2.70	6.63	3.60
10		3.53	3.15	4.38	3.72	5.51	3.61	7.17	4.33
11		4.70	5.04	6.02	6.27	8.30	8.27	10.42	9.95
12		3.79	3.47	4.50	3.96	5.84	4.24	7.29	4.85
13		3.30	2.79	4.10	3.30	5.27	3.29	6.85	3.93
14		4.75	4.71	5.54	5.55	7.14	5.89	9.02	7.34

Table 12: Average lags lengths selected by MAIC and RSMAIC. Stochastic volatility model: $\tilde{c} = 10$ and $\nu = 4$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.23	0.81	1.63	1.00	1.75	1.02	2.41	1.25
2		2.10	1.81	2.39	1.97	2.69	2.18	3.22	2.33
3		2.08	1.69	2.39	1.88	2.71	1.96	3.19	2.17
4		2.04	1.76	2.30	1.96	2.67	2.03	3.17	2.31
5		2.05	1.93	2.35	2.14	2.59	2.28	3.15	2.71
6		2.51	2.33	2.98	2.75	3.10	2.64	3.79	3.19
7		2.83	2.67	3.11	2.92	3.45	3.10	3.98	3.41
8		2.82	2.95	3.16	3.24	3.46	3.78	4.12	4.31
9		2.81	2.62	3.32	3.12	3.44	2.98	4.21	3.58
10		3.51	3.38	3.92	3.72	4.17	3.78	4.90	4.24
11		5.12	5.18	6.15	6.18	7.71	7.55	8.96	8.64
12		3.21	2.99	3.74	3.48	3.87	3.31	4.72	3.89
13		2.96	2.73	3.52	3.24	3.53	3.01	4.49	3.65
14		4.84	4.83	5.78	5.74	6.27	5.93	7.80	7.27
1	7	1.32	0.86	1.60	0.99	1.92	1.05	2.44	1.29
2		2.17	2.03	2.43	2.10	2.92	2.56	3.41	2.65
3		2.13	1.83	2.37	1.96	2.85	2.22	3.23	2.30
4		2.10	1.73	2.41	1.93	2.72	1.99	3.27	2.26
5		2.09	1.76	2.36	1.94	2.65	2.07	3.19	2.28
6		2.24	2.02	2.87	2.53	2.88	2.31	3.80	2.91
7		2.91	2.70	3.18	2.86	3.58	3.08	4.11	3.34
8		2.83	2.72	3.14	2.96	3.46	3.22	4.09	3.59
9		2.37	2.14	3.10	2.81	3.01	2.43	4.02	3.17
10		3.25	3.07	3.90	3.63	3.94	3.38	4.87	4.01
11		4.97	5.06	6.22	6.31	8.19	8.20	10.07	9.97
12		3.56	3.45	4.07	3.88	4.58	4.15	5.19	4.50
13		2.90	2.67	3.43	3.15	3.61	3.01	4.47	3.58
14		4.67	4.64	5.50	5.47	6.00	5.64	7.55	7.02

Table 13: Average lags lengths selected by MAIC and RSMAIC. Stochastic volatility model: $\tilde{c} = 0$ and $\nu = 9$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	3.33	2.19	4.44	3.06	6.41	3.29	8.87	5.01
2		3.58	2.99	4.64	3.72	6.61	4.43	9.18	6.12
3		3.66	2.83	4.68	3.60	6.64	4.13	9.15	5.75
4		3.63	2.81	4.66	3.59	6.69	4.11	9.23	5.72
5		3.72	3.01	4.75	3.86	6.78	4.40	9.32	6.31
6		3.72	3.07	4.84	3.96	6.78	4.34	9.29	6.14
7		3.95	3.40	4.94	4.20	7.13	4.81	9.46	6.47
8		4.06	3.69	5.00	4.53	7.27	5.61	9.76	7.55
9		3.84	3.24	4.92	4.12	6.95	4.52	9.48	6.34
10		4.18	3.77	5.12	4.54	7.25	5.08	9.54	6.77
11		4.54	4.93	5.72	6.02	8.14	7.84	10.45	9.65
12		3.97	3.53	4.97	4.35	7.07	4.92	9.35	6.53
13		3.95	3.36	4.92	4.14	6.94	4.55	9.36	6.25
14		4.82	4.82	5.81	5.76	8.33	6.74	10.42	8.57
1	7	3.19	2.23	4.34	3.05	6.13	3.48	8.64	5.12
2		3.47	3.08	4.46	3.83	6.39	4.78	8.82	6.42
3		3.54	2.88	4.50	3.57	6.47	4.26	8.81	5.77
4		3.50	2.71	4.50	3.48	6.41	3.88	8.83	5.59
5		3.61	2.86	4.59	3.62	6.54	4.19	9.09	5.77
6		3.54	2.91	4.63	3.80	6.56	4.11	9.01	5.83
7		3.89	3.39	4.77	4.08	6.81	4.68	9.07	6.21
8		4.02	3.54	4.90	4.31	7.10	5.29	9.38	7.06
9		3.56	3.02	4.67	3.96	6.61	4.26	8.99	5.97
10		3.91	3.56	4.99	4.43	6.87	4.78	9.31	6.53
11		4.37	4.93	5.70	6.08	8.11	8.21	10.62	10.21
12		3.98	3.69	4.90	4.43	6.93	5.26	9.07	6.71
13		3.79	3.33	4.81	4.10	6.74	4.55	9.11	6.21
14		4.75	4.78	5.61	5.66	7.97	6.64	10.32	8.56

Table 14: Average lags lengths selected by MAIC and RSMAIC. Stochastic volatility model: $\tilde{c} = 10$ and $\nu = 9$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.51	1.33	3.46	2.00	3.80	1.59	5.66	2.55
2		2.97	2.31	3.82	2.87	4.31	2.75	6.27	3.71
3		2.94	2.08	3.81	2.68	4.24	2.41	6.22	3.28
4		2.93	2.14	3.88	2.75	4.20	2.48	6.24	3.40
5		3.00	2.37	3.85	3.06	4.38	2.85	6.25	4.09
6		3.17	2.55	4.08	3.28	4.41	2.91	6.46	4.05
7		3.48	2.99	4.27	3.54	4.79	3.47	6.77	4.49
8		3.48	3.26	4.32	3.96	4.78	4.15	6.80	5.53
9		3.31	2.75	4.27	3.56	4.61	3.09	6.65	4.34
10		3.85	3.49	4.68	4.09	5.20	3.89	7.06	4.95
11		5.01	5.14	6.05	6.12	7.88	7.44	9.58	8.77
12		3.62	3.15	4.52	3.81	5.03	3.61	6.86	4.55
13		3.47	2.95	4.37	3.62	4.76	3.32	6.76	4.38
14		4.86	4.80	5.80	5.72	6.85	5.91	8.95	7.53
1	7	2.26	1.26	3.21	1.87	3.42	1.52	5.49	2.40
2		2.91	2.37	3.70	2.92	4.25	2.90	5.92	3.75
3		2.79	2.14	3.60	2.59	4.04	2.50	5.86	3.24
4		2.80	2.01	3.67	2.56	4.05	2.29	5.95	3.11
5		2.89	2.18	3.75	2.77	4.18	2.56	6.05	3.51
6		2.87	2.25	3.87	3.02	4.02	2.50	6.19	3.69
7		3.37	2.90	4.11	3.33	4.62	3.29	6.36	4.09
8		3.46	3.04	4.25	3.68	4.84	3.66	6.53	4.80
9		2.87	2.36	3.99	3.22	4.13	2.63	6.40	3.92
10		3.53	3.19	4.53	3.90	4.79	3.54	6.84	4.60
11		4.66	5.01	5.95	6.19	7.87	8.07	10.16	9.74
12		3.75	3.50	4.52	4.04	5.12	4.08	6.85	4.91
13		3.33	2.79	4.23	3.49	4.54	3.12	6.53	4.17
14		4.67	4.63	5.52	5.50	6.46	5.55	8.55	7.21

Table 15: Empirical rejection frequencies of the wild bootstrap ADF test with QD demeaning. Homoskedastic errors

	$T = 150$		$T = 250$	
Model	MAIC	RSMAIC	MAIC	RSMAIC
1	0.046	0.045	0.053	0.051
2	0.047	0.041	0.047	0.045
3	0.044	0.044	0.050	0.049
4	0.049	0.057	0.053	0.052
5	0.051	0.051	0.051	0.050
6	0.043	0.042	0.049	0.047
7	0.056	0.051	0.049	0.048
8	0.051	0.052	0.047	0.046
9	0.041	0.035	0.040	0.039
10	0.054	0.052	0.053	0.053
11	0.101	0.098	0.089	0.090
12	0.041	0.059	0.056	0.054
13	0.049	0.046	0.048	0.046
14	0.041	0.040	0.039	0.040

Table 16: Empirical rejection frequencies of the wild bootstrap ADF test with QD demeaning. Smooth transition volatility model

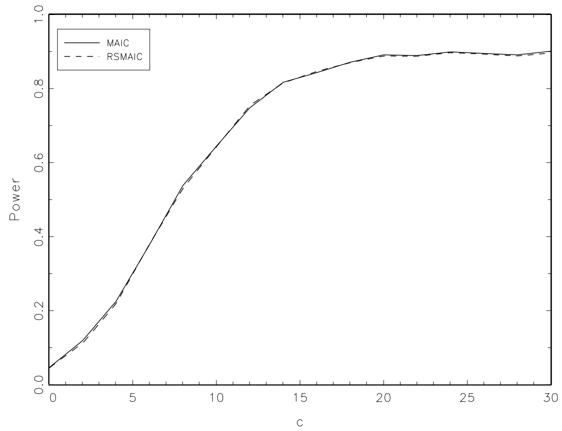
		$\delta = 1/3$				$\delta = 3$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	τ	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0.2	0.051	0.049	0.053	0.048	0.051	0.050	0.046	0.052
2		0.054	0.048	0.054	0.051	0.043	0.047	0.046	0.057
3		0.054	0.053	0.050	0.047	0.047	0.051	0.041	0.049
4		0.045	0.042	0.052	0.051	0.054	0.054	0.053	0.058
5		0.049	0.053	0.051	0.052	0.051	0.058	0.053	0.054
6		0.047	0.045	0.046	0.042	0.050	0.055	0.050	0.057
7		0.050	0.049	0.048	0.051	0.063	0.058	0.052	0.052
8		0.051	0.052	0.056	0.052	0.046	0.049	0.045	0.049
9		0.040	0.036	0.042	0.040	0.049	0.047	0.051	0.049
10		0.044	0.045	0.049	0.047	0.046	0.056	0.046	0.050
11		0.096	0.088	0.095	0.087	0.103	0.107	0.057	0.068
12		0.065	0.062	0.059	0.059	0.061	0.066	0.049	0.056
13		0.042	0.039	0.050	0.046	0.049	0.048	0.047	0.049
14		0.045	0.040	0.047	0.044	0.060	0.060	0.054	0.059
1	0.8	0.046	0.046	0.045	0.047	0.046	0.048	0.050	0.050
2		0.044	0.047	0.051	0.051	0.050	0.051	0.048	0.047
3		0.049	0.052	0.049	0.050	0.048	0.048	0.045	0.046
4		0.049	0.055	0.045	0.050	0.054	0.053	0.050	0.049
5		0.044	0.053	0.047	0.050	0.053	0.049	0.052	0.053
6		0.047	0.048	0.049	0.050	0.045	0.041	0.052	0.050
7		0.049	0.054	0.046	0.049	0.047	0.049	0.050	0.048
8		0.044	0.051	0.049	0.050	0.045	0.045	0.051	0.046
9		0.046	0.044	0.044	0.044	0.041	0.038	0.044	0.042
10		0.043	0.050	0.046	0.052	0.046	0.048	0.049	0.048
11		0.111	0.113	0.093	0.103	0.096	0.091	0.082	0.078
12		0.058	0.064	0.054	0.064	0.065	0.061	0.055	0.057
13		0.047	0.047	0.046	0.048	0.049	0.045	0.044	0.044
14		0.043	0.043	0.048	0.053	0.046	0.041	0.046	0.043

Table 17: Empirical rejection frequencies of the wild bootstrap ADF test with QD demeaning. Single break volatility model

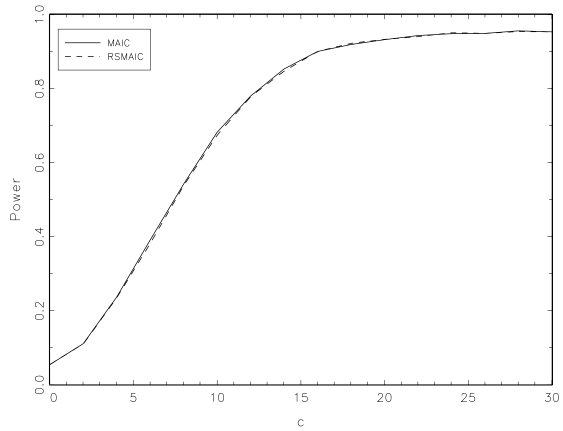
		$\delta = 1/3$				$\delta = 3$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	τ	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0.2	0.056	0.052	0.052	0.049	0.052	0.053	0.046	0.053
2		0.046	0.047	0.052	0.048	0.045	0.050	0.039	0.045
3		0.049	0.046	0.050	0.047	0.045	0.058	0.044	0.051
4		0.050	0.049	0.052	0.053	0.041	0.063	0.055	0.053
5		0.054	0.051	0.051	0.050	0.047	0.054	0.054	0.059
6		0.041	0.038	0.045	0.042	0.051	0.053	0.051	0.052
7		0.053	0.050	0.052	0.052	0.052	0.053	0.054	0.056
8		0.052	0.049	0.052	0.051	0.042	0.047	0.052	0.056
9		0.036	0.034	0.045	0.042	0.045	0.041	0.050	0.052
10		0.049	0.044	0.049	0.048	0.048	0.053	0.058	0.061
11		0.103	0.102	0.089	0.078	0.098	0.106	0.059	0.071
12		0.067	0.061	0.056	0.053	0.054	0.061	0.046	0.061
13		0.041	0.038	0.045	0.042	0.054	0.053	0.052	0.054
14		0.040	0.039	0.043	0.036	0.057	0.060	0.056	0.057
1	0.8	0.048	0.074	0.051	0.051	0.047	0.045	0.047	0.047
2		0.046	0.049	0.051	0.053	0.046	0.045	0.050	0.049
3		0.050	0.050	0.047	0.055	0.045	0.042	0.052	0.052
4		0.054	0.067	0.052	0.061	0.048	0.047	0.054	0.052
5		0.049	0.051	0.046	0.049	0.050	0.047	0.053	0.052
6		0.051	0.053	0.050	0.054	0.045	0.044	0.047	0.049
7		0.052	0.060	0.047	0.054	0.057	0.054	0.047	0.048
8		0.038	0.041	0.051	0.054	0.046	0.048	0.049	0.049
9		0.040	0.043	0.041	0.045	0.040	0.039	0.047	0.048
10		0.047	0.054	0.049	0.051	0.052	0.049	0.050	0.049
11		0.118	0.117	0.087	0.098	0.096	0.093	0.079	0.077
12		0.075	0.085	0.053	0.062	0.064	0.061	0.055	0.057
13		0.048	0.046	0.050	0.048	0.040	0.043	0.043	0.043
14		0.046	0.053	0.044	0.047	0.048	0.047	0.052	0.048

Table 18: Empirical rejection frequencies of the wild bootstrap ADF test with QD demeaning. Stochastic volatility model

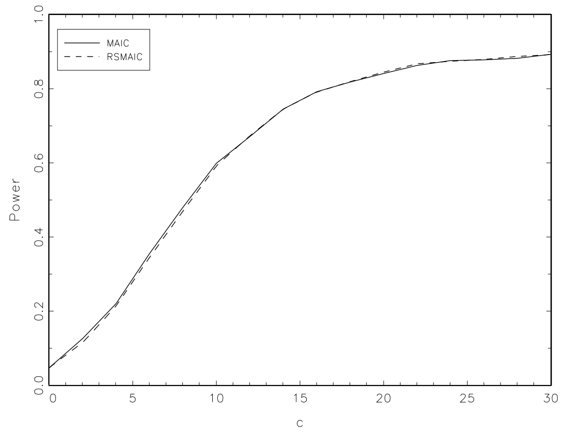
		$\tilde{c} = 0$				$\tilde{c} = 10$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	ν	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	4	0.050	0.049	0.051	0.052	0.048	0.045	0.053	0.051
2		0.046	0.047	0.046	0.047	0.052	0.048	0.046	0.045
3		0.056	0.054	0.051	0.049	0.051	0.048	0.045	0.046
4		0.048	0.055	0.052	0.055	0.050	0.049	0.051	0.050
5		0.054	0.059	0.057	0.059	0.049	0.049	0.044	0.042
6		0.048	0.050	0.051	0.057	0.044	0.047	0.050	0.046
7		0.050	0.050	0.047	0.048	0.048	0.047	0.050	0.047
8		0.048	0.052	0.047	0.050	0.047	0.045	0.046	0.044
9		0.046	0.043	0.050	0.046	0.038	0.036	0.043	0.044
10		0.048	0.048	0.055	0.053	0.046	0.048	0.052	0.055
11		0.117	0.111	0.083	0.084	0.106	0.106	0.086	0.085
12		0.065	0.067	0.056	0.060	0.059	0.062	0.065	0.069
13		0.050	0.050	0.049	0.048	0.046	0.044	0.044	0.046
14		0.049	0.048	0.050	0.051	0.039	0.040	0.050	0.046
1	9	0.057	0.057	0.051	0.050	0.048	0.050	0.046	0.050
2		0.057	0.055	0.050	0.051	0.049	0.049	0.044	0.044
3		0.051	0.052	0.052	0.054	0.053	0.049	0.048	0.049
4		0.050	0.053	0.053	0.059	0.044	0.051	0.043	0.048
5		0.052	0.050	0.042	0.050	0.045	0.049	0.046	0.046
6		0.048	0.045	0.050	0.054	0.042	0.046	0.051	0.054
7		0.052	0.048	0.055	0.061	0.049	0.049	0.047	0.049
8		0.044	0.049	0.046	0.047	0.041	0.042	0.040	0.041
9		0.038	0.046	0.044	0.046	0.039	0.043	0.042	0.048
10		0.043	0.052	0.049	0.056	0.047	0.048	0.043	0.046
11		0.136	0.106	0.088	0.084	0.107	0.099	0.084	0.084
12		0.065	0.059	0.049	0.056	0.060	0.061	0.054	0.056
13		0.049	0.054	0.046	0.045	0.049	0.048	0.047	0.050
14		0.050	0.052	0.053	0.055	0.045	0.041	0.049	0.049



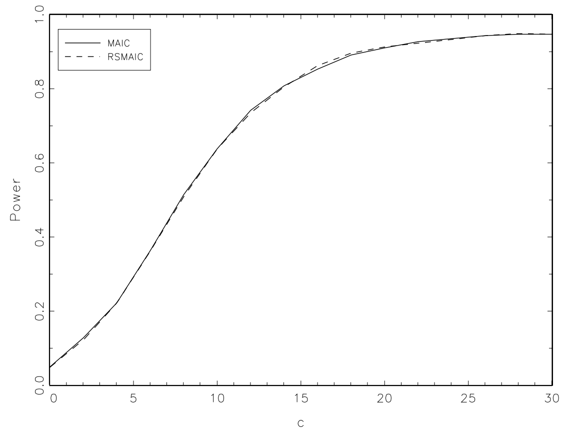
(a) $T = 150$, ARMA model 1



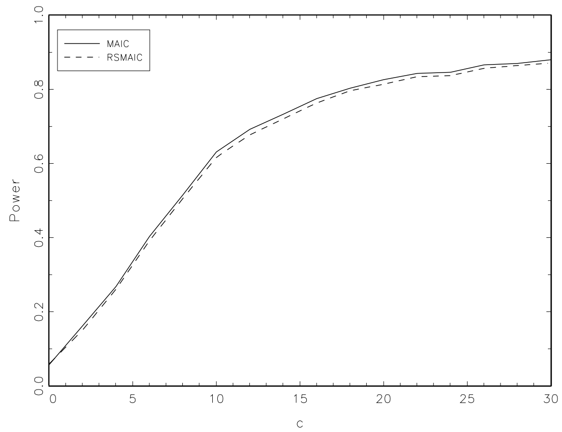
(b) $T = 250$, ARMA model 1



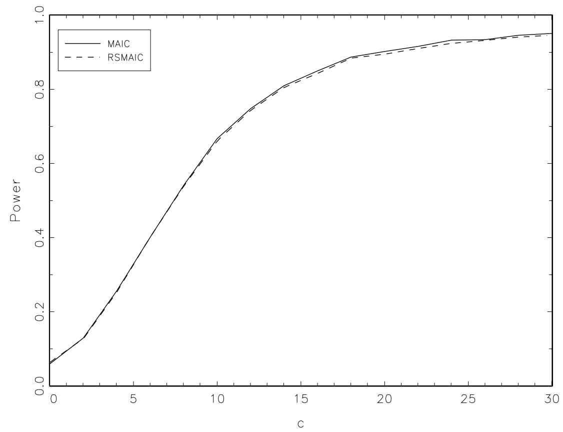
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

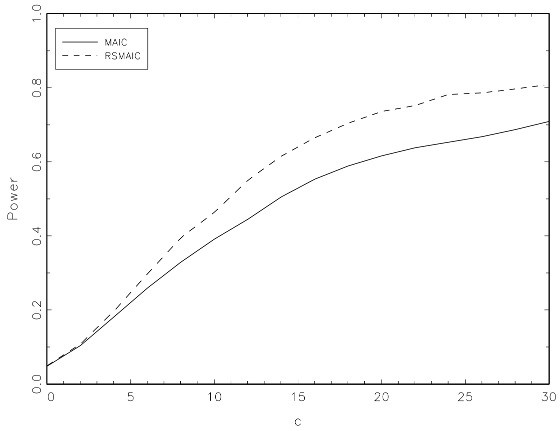


(e) $T = 150$, ARMA model 12

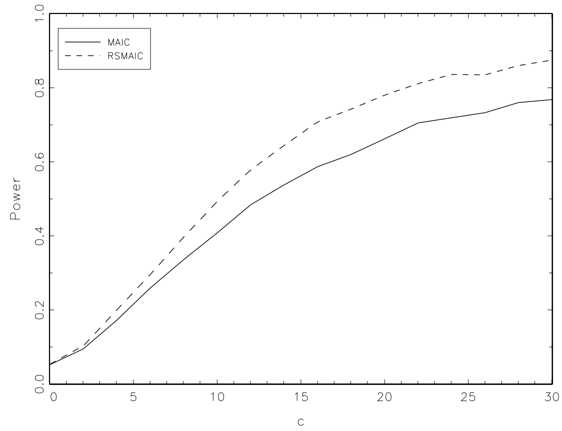


(f) $T = 250$, ARMA model 12

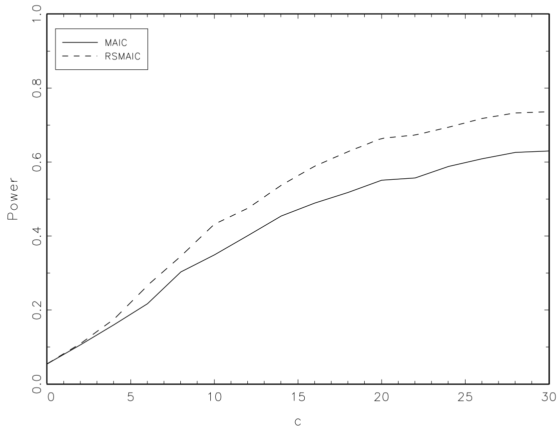
Figure 1: Power wild bootstrap ADF test with QD demeaning. Homoskedastic errors.



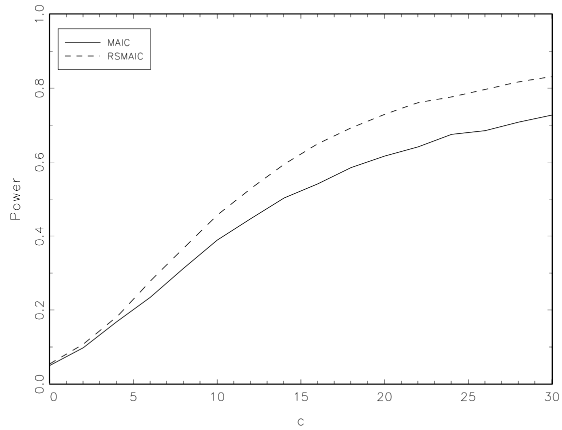
(a) $T = 150$, ARMA model 1



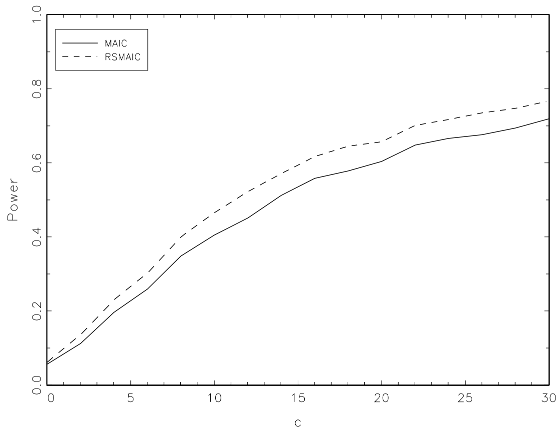
(b) $T = 250$, ARMA model 1



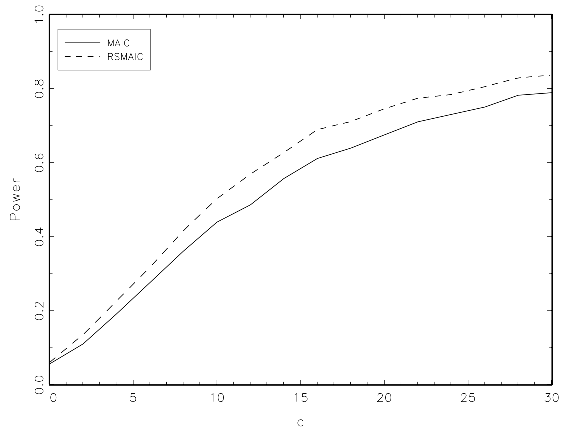
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

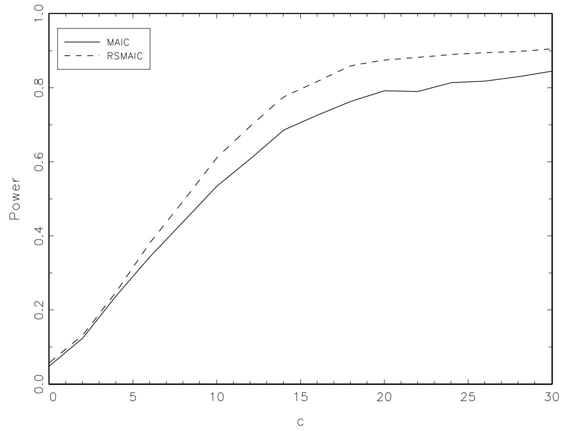


(e) $T = 150$, ARMA model 12

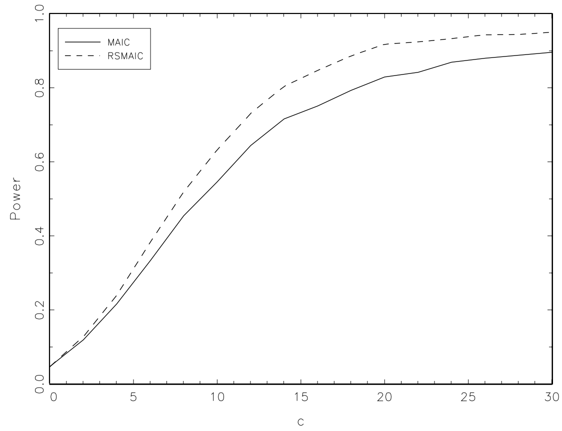


(f) $T = 250$, ARMA model 12

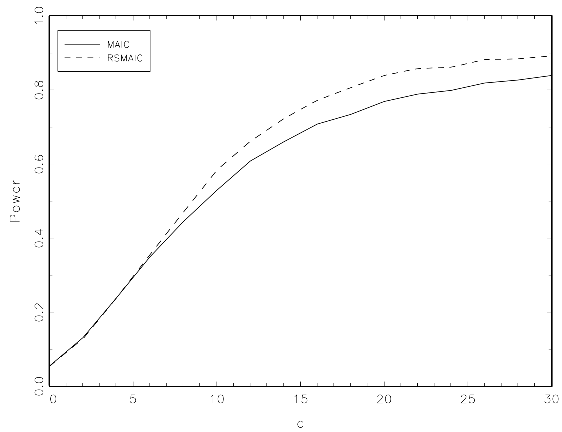
Figure 2: Power wild bootstrap ADF test with QD demeaning. Smooth transition volatility model: $\delta = 1/3$, $\tau = 0.8$.



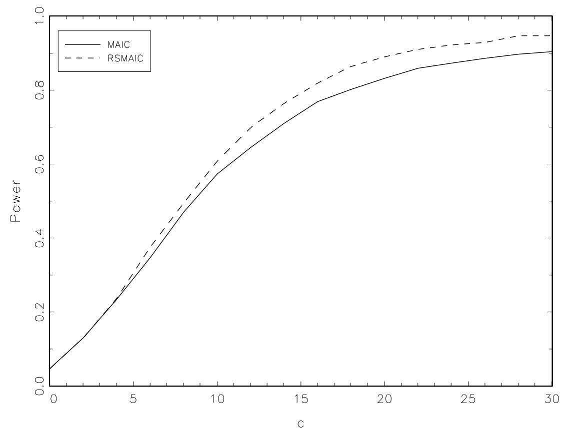
(a) $T = 150$, ARMA model 1



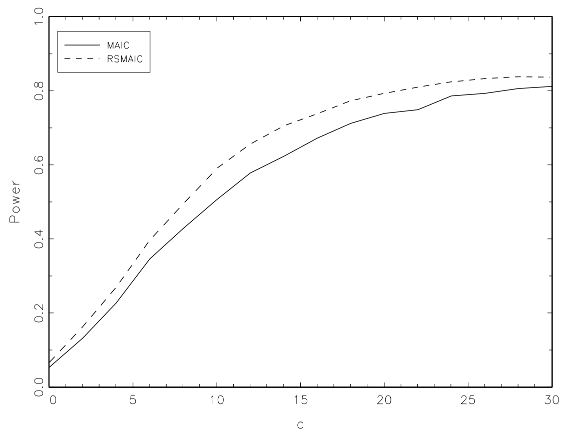
(b) $T = 250$, ARMA model 1



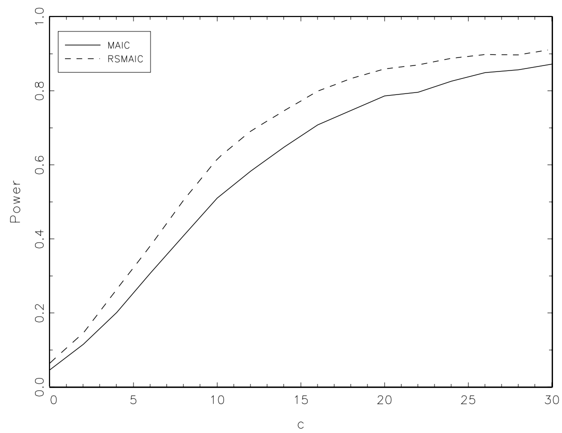
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

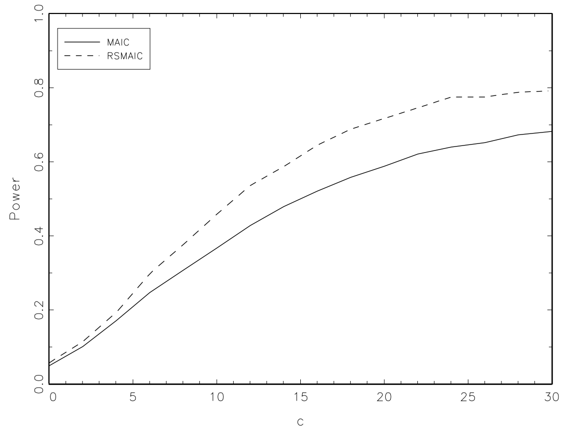


(e) $T = 150$, ARMA model 12

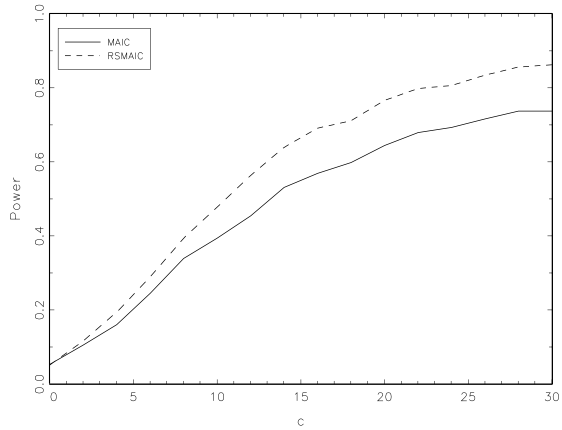


(f) $T = 250$, ARMA model 12

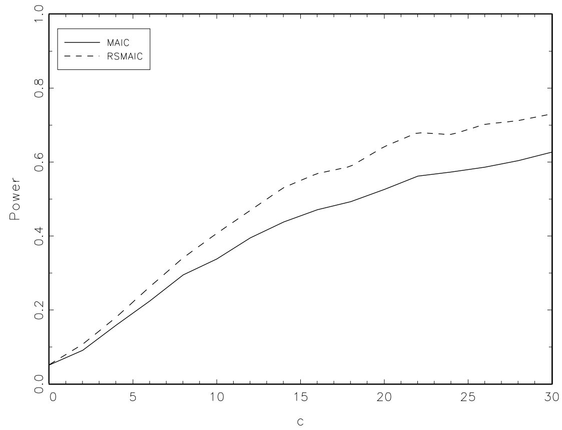
Figure 3: Power wild bootstrap ADF test with QD demeaning. Smooth transition volatility model: $\delta = 3$, $\tau = 0.2$.



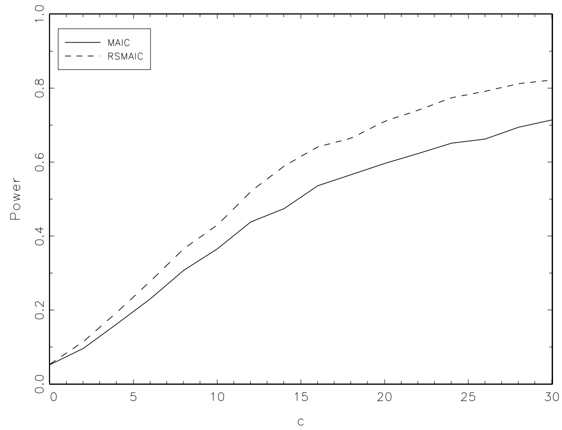
(a) $T = 150$, ARMA model 1



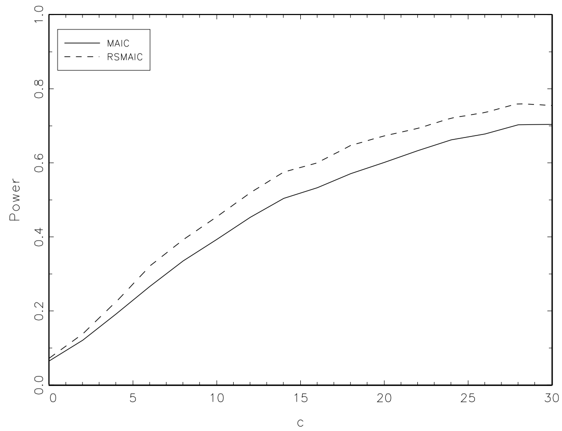
(b) $T = 250$, ARMA model 1



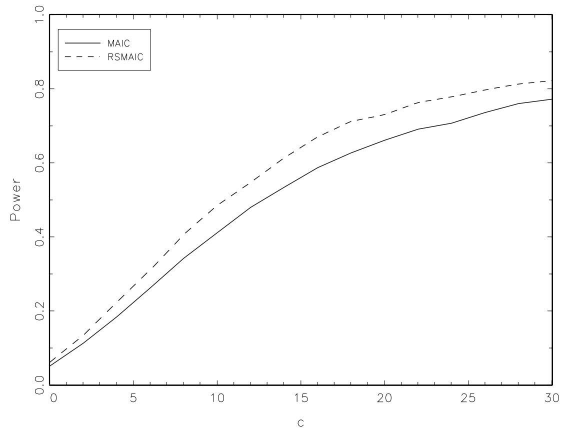
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

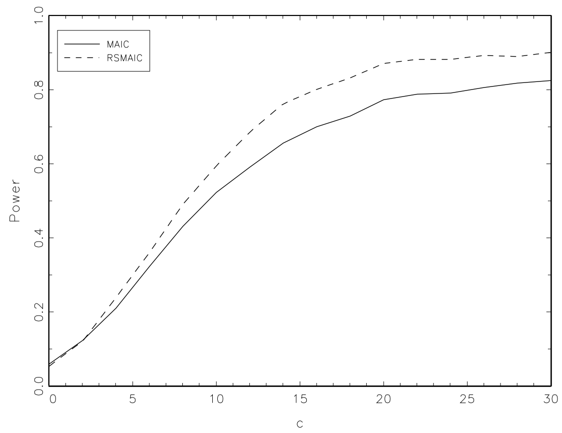


(e) $T = 150$, ARMA model 12

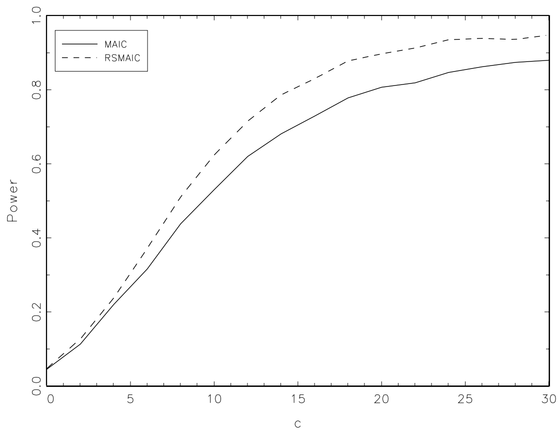


(f) $T = 250$, ARMA model 12

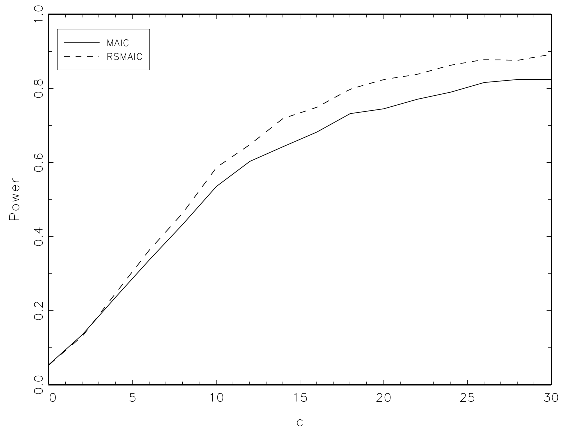
Figure 4: Power wild bootstrap ADF test with QD demeaning. Single break volatility model: $\delta = 1/3$, $\tau = 0.8$.



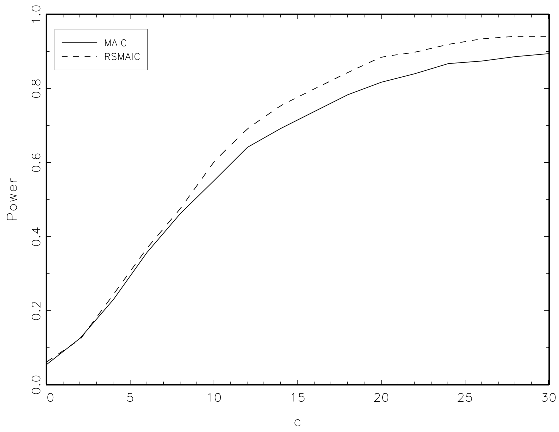
(a) $T = 150$, ARMA model 1



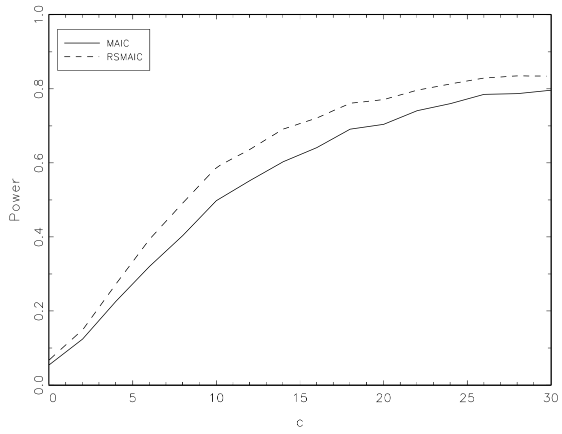
(b) $T = 250$, ARMA model 1



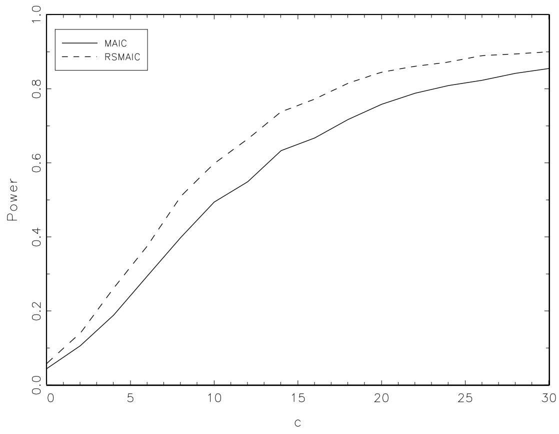
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

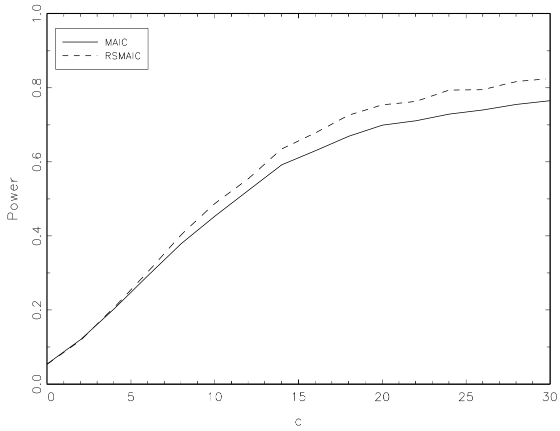


(e) $T = 150$, ARMA model 12

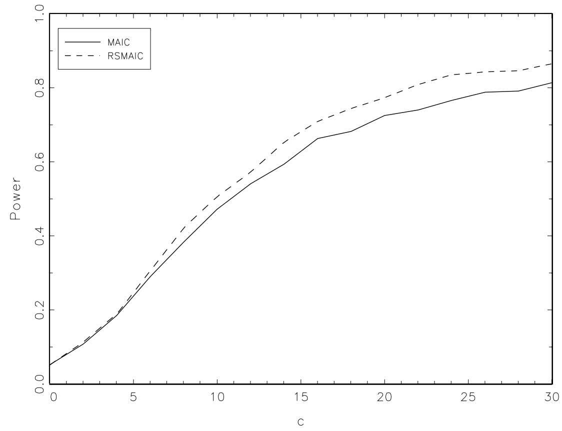


(f) $T = 250$, ARMA model 12

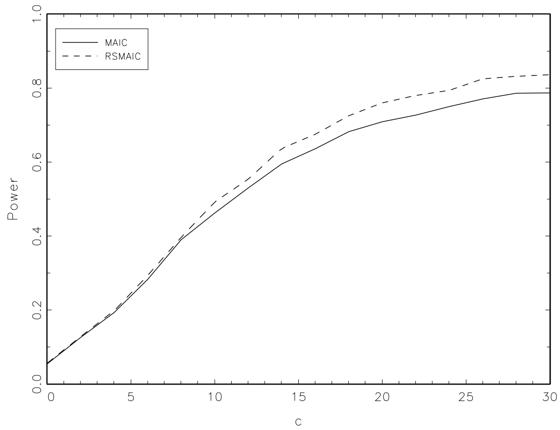
Figure 5: Power wild bootstrap ADF test with QD demeaning. Single break volatility model: $\delta = 3$, $\tau = 0.2$.



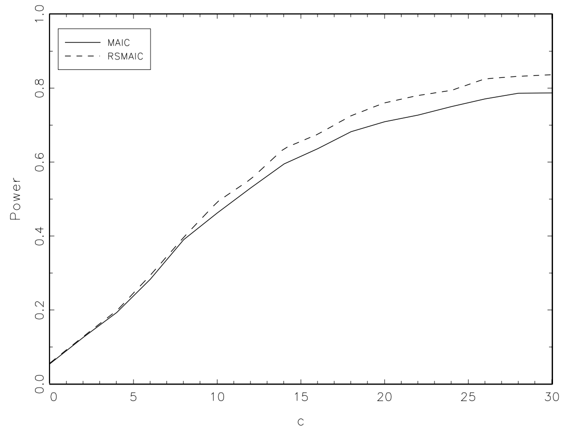
(a) $T = 150$, ARMA model 1



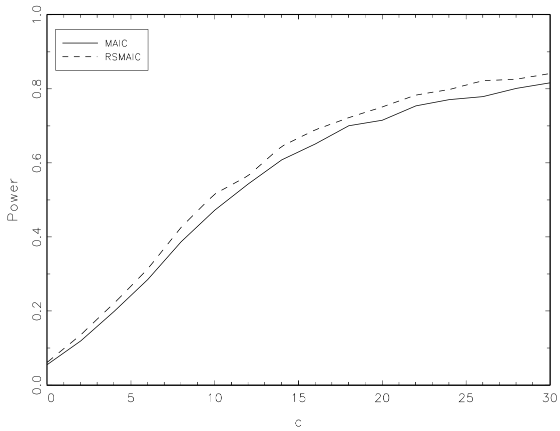
(b) $T = 250$, ARMA model 1



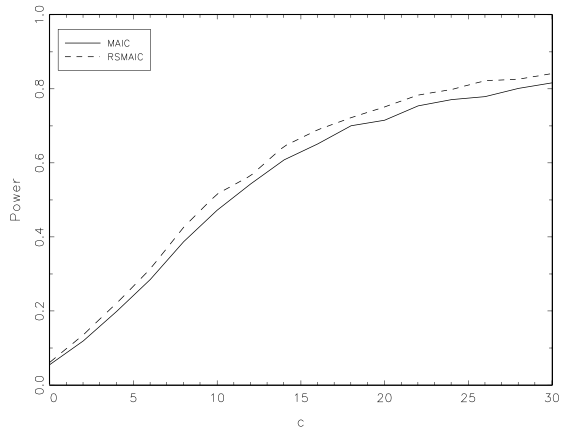
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4

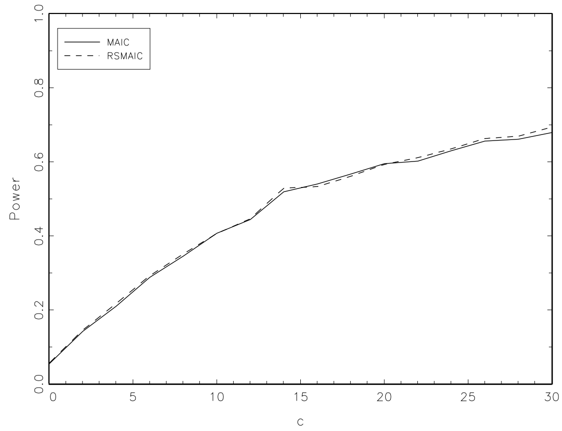


(e) $T = 150$, ARMA model 12

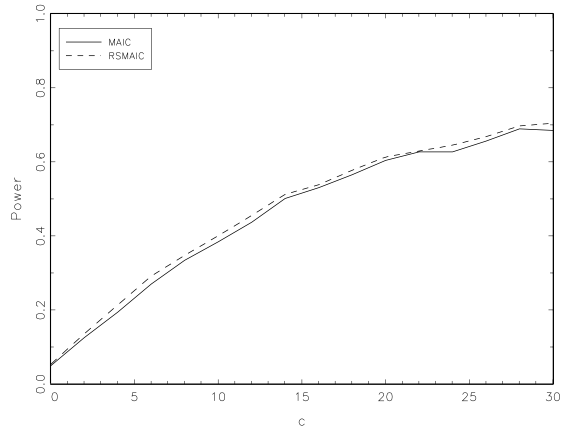


(f) $T = 250$, ARMA model 12

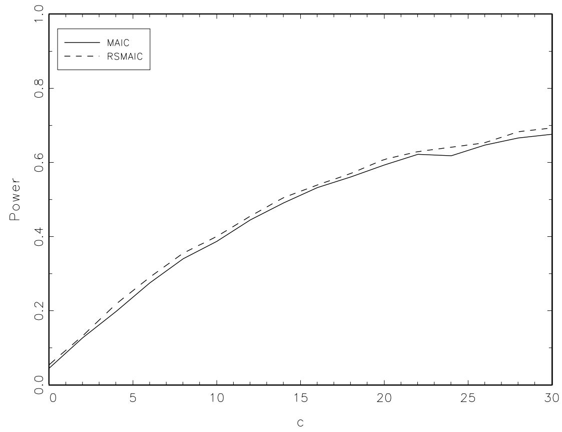
Figure 6: Power wild bootstrap ADF test with QD demeaning. Stochastic volatility model: $\tilde{c} = 0$, $\nu = 4$.



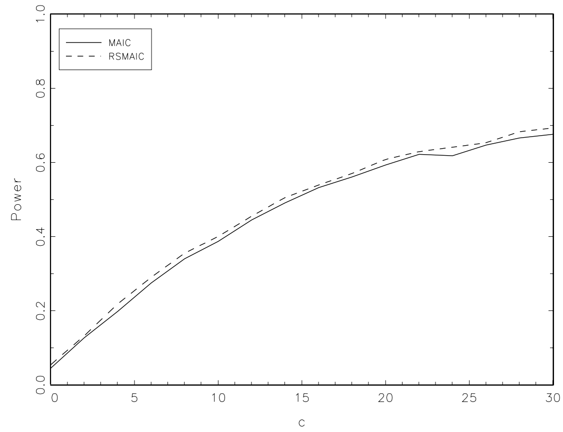
(a) $T = 150$, ARMA model 1



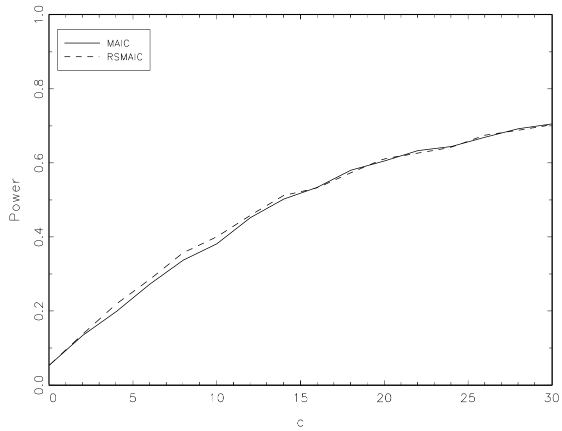
(b) $T = 250$, ARMA model 1



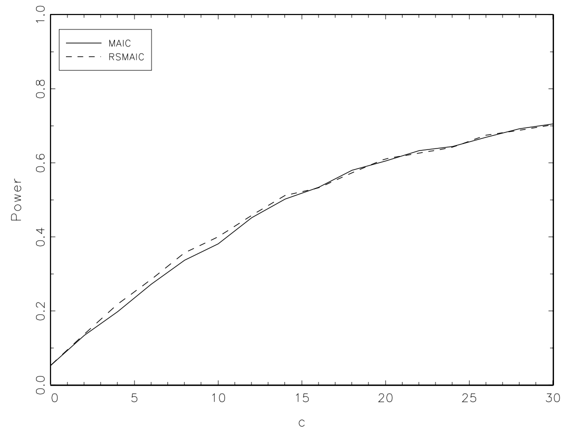
(c) $T = 150$, ARMA model 4



(d) $T = 250$, ARMA model 4



(e) $T = 150$, ARMA model 12



(f) $T = 250$, ARMA model 12

Figure 7: Power wild bootstrap ADF test with QD demeaning. Stochastic volatility model: $\tilde{c} = 0$, $\nu = 9$.

A Appendix: Additional simulation results

The appendix contains a variety of additional simulation results performed for different volatility models and different sample sizes.

A.1 Single break

This section contains additional simulation results for the single break model with a break halfway through the sample. Specifically, we use the model: $\sigma_t^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)I(\lfloor \tau T \rfloor < t)$, where we set $\sigma_0 = 1$. Defining $\delta = \sigma_0/\sigma_1$, we consider parameters $\delta = 1/3, 3; 1/5, 5$ and $\tau = 0.5$. Lag length selection results are based on 5000 simulations, power curves on 2000 simulations.

Table 19: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/3$ and $\tau = 0.5$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.464	0.682	1.790	0.849	2.545	0.907	3.197	1.077
2		2.332	1.734	2.610	1.830	3.430	2.058	4.130	2.113
3		2.311	1.641	2.554	1.744	3.508	1.934	4.173	2.034
4		2.243	1.711	2.565	1.902	3.478	2.074	3.962	2.245
5		2.306	1.917	2.564	2.065	3.512	2.390	4.094	2.684
6		2.684	2.344	3.213	2.738	3.842	2.680	4.785	3.191
7		3.017	2.657	3.316	2.862	4.332	3.135	4.865	3.401
8		2.997	2.903	3.345	3.130	4.211	3.893	4.810	4.319
9		2.947	2.632	3.486	3.072	4.249	3.042	5.196	3.688
10		3.595	3.373	4.048	3.688	4.940	3.842	5.678	4.230
11		5.088	5.171	6.101	6.164	8.075	7.490	9.554	8.615
12		3.343	2.997	3.845	3.401	4.667	3.360	5.576	3.817
13		3.122	2.735	3.687	3.236	4.440	3.143	5.316	3.756
14		4.835	4.779	5.758	5.706	6.907	5.957	8.407	7.316
1	7	1.534	0.831	1.817	0.868	2.600	0.982	3.111	1.106
2		2.383	2.015	2.677	2.033	3.838	2.654	4.213	2.553
3		2.341	1.866	2.563	1.857	3.562	2.229	4.200	2.342
4		2.308	1.717	2.604	1.817	3.534	2.072	3.987	2.164
5		2.283	1.796	2.548	1.869	3.427	2.030	4.048	2.302
6		2.459	2.014	3.065	2.516	3.681	2.293	4.607	2.846
7		3.031	2.665	3.330	2.793	4.385	3.135	4.846	3.267
8		3.030	2.753	3.276	2.905	4.224	3.233	4.901	3.561
9		2.525	2.088	3.219	2.738	3.711	2.406	4.851	3.134
10		3.299	3.022	4.017	3.586	4.578	3.357	5.642	4.034
11		4.849	5.061	6.169	6.323	8.345	8.286	10.301	10.012
12		3.641	3.455	4.192	3.883	5.288	4.255	5.988	4.580
13		3.066	2.693	3.556	3.060	4.445	3.111	5.221	3.468
14		4.688	4.670	5.491	5.468	6.584	5.634	8.183	7.053

Table 20: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/3$ and $\tau = 0.5$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.303	0.655	1.650	0.764	2.024	0.775	2.806	0.988
2		2.164	1.730	2.478	1.831	3.066	2.039	3.653	2.123
3		2.112	1.598	2.429	1.714	2.975	1.842	3.573	1.930
4		2.099	1.624	2.481	1.834	2.941	1.834	3.596	2.074
5		2.114	1.772	2.453	1.936	2.929	2.048	3.662	2.381
6		2.435	2.146	3.016	2.558	3.248	2.317	4.244	2.977
7		2.892	2.611	3.198	2.740	3.806	2.925	4.454	3.231
8		2.889	2.746	3.189	2.994	3.805	3.331	4.482	3.817
9		2.623	2.360	3.274	2.895	3.501	2.600	4.547	3.316
10		3.354	3.175	3.949	3.607	4.321	3.464	5.333	4.041
11		4.903	5.056	6.118	6.198	7.790	7.500	9.706	8.999
12		3.385	3.085	3.936	3.482	4.381	3.448	5.345	3.868
13		2.927	2.616	3.532	3.080	3.852	2.876	4.866	3.459
14		4.619	4.617	5.564	5.559	6.196	5.515	8.048	7.055
1	13.5	1.414	0.925	1.692	0.976	2.139	1.100	2.805	1.213
2		2.401	2.213	2.616	2.230	3.486	2.826	3.994	2.865
3		2.284	1.943	2.545	2.031	3.251	2.435	3.821	2.455
4		2.173	1.761	2.519	1.959	3.071	2.018	3.671	2.189
5		2.106	1.707	2.437	1.880	3.043	1.980	3.631	2.145
6		2.066	1.669	2.761	2.305	2.909	1.905	3.911	2.626
7		2.964	2.677	3.230	2.803	3.922	3.036	4.533	3.225
8		2.911	2.650	3.228	2.839	3.839	3.007	4.515	3.300
9		1.822	1.524	2.830	2.389	2.613	1.742	4.000	2.759
10		2.497	2.108	3.790	3.489	3.378	2.447	5.068	3.931
11		3.942	4.208	5.776	6.022	6.641	6.929	9.562	9.766
12		3.574	3.551	4.248	4.107	5.034	4.531	5.912	5.051
13		2.811	2.548	3.335	2.925	3.781	2.931	4.690	3.415
14		4.277	4.272	5.156	5.135	5.695	5.062	7.518	6.615

Table 21: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/5$ and $\tau = 0.5$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.836	0.907	2.179	1.063	3.453	1.223	4.454	1.549
2		2.496	1.873	2.863	1.937	4.172	2.366	4.982	2.481
3		2.479	1.742	2.907	1.871	4.270	2.168	4.961	2.299
4		2.556	1.898	2.836	2.061	4.196	2.359	4.978	2.731
5		2.457	2.048	2.862	2.295	4.159	2.818	5.085	3.225
6		2.896	2.449	3.397	2.888	4.645	3.067	5.642	3.660
7		3.181	2.784	3.573	3.063	4.918	3.516	5.846	3.876
8		3.174	3.016	3.563	3.350	4.851	4.179	5.790	4.830
9		3.059	2.709	3.751	3.300	4.745	3.317	6.050	4.125
10		3.724	3.485	4.257	3.851	5.471	4.106	6.522	4.694
11		5.033	5.136	6.090	6.146	8.161	7.487	9.804	8.745
12		3.466	3.041	4.066	3.537	5.349	3.621	6.450	4.108
13		3.242	2.833	3.873	3.346	5.041	3.279	6.207	4.064
14		4.817	4.755	5.719	5.696	7.317	6.080	8.863	7.477
1	7	1.758	0.895	2.103	1.014	3.260	1.257	4.214	1.444
2		2.538	2.104	2.838	2.169	4.241	2.866	4.860	2.908
3		2.512	1.909	2.820	1.977	4.176	2.443	4.849	2.496
4		2.459	1.841	2.782	1.955	4.140	2.207	4.977	2.469
5		2.501	1.909	2.822	2.087	4.101	2.384	4.910	2.649
6		2.603	2.083	3.264	2.629	4.210	2.492	5.357	3.162
7		3.181	2.746	3.531	2.906	4.924	3.328	5.672	3.520
8		3.174	2.830	3.519	3.071	4.784	3.525	5.716	4.051
9		2.628	2.165	3.412	2.876	4.310	2.571	5.520	3.471
10		3.384	3.046	4.119	3.680	5.055	3.505	6.331	4.357
11		4.774	5.007	6.111	6.274	8.365	8.316	10.419	10.142
12		3.689	3.496	4.285	3.962	5.634	4.349	6.594	4.828
13		3.171	2.699	3.760	3.153	4.902	3.243	5.951	3.792
14		4.657	4.599	5.518	5.479	6.940	5.795	8.611	7.187

Table 22: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 1/5$ and $\tau = 0.5$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.526	0.825	2.025	1.041	2.686	1.036	3.722	1.411
2		2.372	1.913	2.739	2.022	3.513	2.336	4.460	2.482
3		2.349	1.773	2.630	1.876	3.557	2.053	4.470	2.330
4		2.295	1.793	2.664	2.027	3.553	2.179	4.393	2.561
5		2.332	1.951	2.721	2.204	3.347	2.347	4.425	2.998
6		2.606	2.279	3.173	2.745	3.769	2.651	4.948	3.409
7		3.052	2.706	3.435	2.963	4.332	3.177	5.265	3.733
8		3.065	2.931	3.439	3.232	4.318	3.638	5.180	4.341
9		2.697	2.497	3.475	3.074	4.007	2.946	5.254	3.751
10		3.477	3.295	4.104	3.771	4.741	3.730	6.007	4.563
11		4.863	5.064	6.085	6.147	7.908	7.372	9.860	8.784
12		3.464	3.050	4.060	3.455	4.888	3.544	6.094	4.098
13		3.058	2.725	3.722	3.237	4.316	3.047	5.628	3.791
14		4.695	4.678	5.545	5.592	6.556	5.727	8.454	7.349
1	13.5	1.475	0.967	2.004	1.180	2.511	1.338	3.529	1.525
2		2.440	2.227	2.747	2.310	3.785	3.076	4.474	3.092
3		2.412	2.074	2.648	2.111	3.634	2.540	4.371	2.723
4		2.258	1.822	2.629	1.982	3.412	2.171	4.362	2.435
5		2.301	1.817	2.672	2.017	3.463	2.124	4.307	2.438
6		2.126	1.756	2.913	2.421	3.249	2.058	4.492	2.816
7		3.071	2.704	3.390	2.886	4.412	3.250	5.182	3.527
8		3.020	2.724	3.371	2.990	4.222	3.221	5.097	3.616
9		1.951	1.621	2.949	2.487	2.980	1.867	4.650	2.888
10		2.575	2.249	3.841	3.568	3.702	2.611	5.645	4.166
11		3.915	4.367	5.723	6.061	6.570	7.224	9.563	9.895
12		3.606	3.613	4.315	4.191	5.202	4.643	6.237	5.249
13		2.864	2.621	3.463	3.021	4.213	3.132	5.139	3.557
14		4.263	4.319	5.122	5.190	5.927	5.257	7.767	6.795

Table 23: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 3$ and $\tau = 0.5$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.548	0.696	1.852	0.814	2.980	0.878	3.425	1.046
2		2.304	1.750	2.614	1.803	3.704	2.188	4.433	2.243
3		2.317	1.651	2.666	1.728	3.775	1.955	4.337	1.971
4		2.263	1.678	2.620	1.812	3.697	1.941	4.291	2.125
5		2.276	1.810	2.639	1.998	3.642	2.222	4.317	2.423
6		2.680	2.242	3.224	2.666	4.108	2.601	4.953	3.060
7		3.033	2.644	3.345	2.796	4.345	3.029	5.029	3.226
8		3.015	2.876	3.364	3.082	4.446	3.680	5.071	4.105
9		2.998	2.610	3.560	3.063	4.304	2.965	5.271	3.447
10		3.668	3.386	4.067	3.641	5.143	3.862	5.806	4.119
11		5.091	5.225	6.137	6.228	8.139	7.708	9.628	8.842
12		3.393	3.047	3.916	3.444	4.900	3.470	5.842	3.840
13		3.158	2.733	3.724	3.212	4.678	3.063	5.494	3.622
14		4.852	4.855	5.790	5.759	7.028	6.027	8.623	7.431
1	7	1.551	0.778	1.837	0.850	2.715	0.986	3.465	1.060
2		2.365	1.968	2.604	1.976	3.743	2.521	4.308	2.503
3		2.315	1.758	2.597	1.810	3.555	2.140	4.174	2.130
4		2.309	1.682	2.605	1.781	3.594	1.932	4.200	2.049
5		2.209	1.666	2.539	1.783	3.588	2.022	4.046	2.067
6		2.413	2.055	3.073	2.528	3.767	2.409	4.632	2.755
7		3.011	2.603	3.308	2.693	4.353	3.027	5.013	3.133
8		2.979	2.644	3.300	2.843	4.260	3.112	4.995	3.399
9		2.577	2.268	3.238	2.792	3.944	2.570	4.938	3.178
10		3.386	3.161	4.036	3.585	4.785	3.588	5.742	3.957
11		4.921	5.149	6.191	6.350	8.365	8.460	10.234	9.869
12		3.648	3.405	4.131	3.768	5.158	4.045	6.017	4.355
13		3.050	2.673	3.589	3.115	4.467	3.108	5.249	3.462
14		4.752	4.767	5.513	5.561	6.696	5.840	8.326	7.179

Table 24: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 3$ and $\tau = 0.5$. OLS detrending.

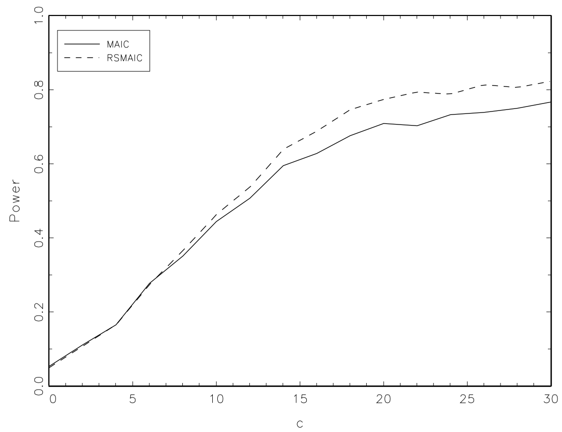
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.307	0.595	1.648	0.718	2.333	0.752	3.120	0.945
2		2.178	1.730	2.488	1.820	3.205	2.095	3.891	2.211
3		2.109	1.589	2.465	1.724	3.207	1.852	3.977	1.994
4		2.128	1.576	2.402	1.734	3.168	1.775	3.856	1.951
5		2.113	1.695	2.420	1.849	3.177	1.938	3.785	2.159
6		2.449	2.106	2.998	2.536	3.531	2.298	4.502	2.793
7		2.934	2.562	3.226	2.686	3.984	2.817	4.683	3.021
8		2.877	2.709	3.210	2.916	3.981	3.195	4.620	3.650
9		2.666	2.395	3.319	2.888	3.709	2.639	4.767	3.224
10		3.421	3.237	3.970	3.588	4.515	3.503	5.403	3.870
11		4.845	5.060	6.131	6.239	7.847	7.678	9.786	9.163
12		3.373	3.097	3.932	3.461	4.610	3.503	5.592	3.943
13		2.943	2.601	3.533	3.085	4.082	2.846	5.066	3.435
14		4.694	4.687	5.595	5.638	6.437	5.562	8.054	7.016
1	13.5	1.419	0.884	1.672	0.915	2.375	1.060	2.950	1.229
2		2.392	2.158	2.675	2.251	3.483	2.796	4.015	2.701
3		2.290	1.903	2.533	1.967	3.267	2.299	3.963	2.364
4		2.162	1.738	2.480	1.809	3.218	2.020	3.793	2.076
5		2.166	1.681	2.458	1.841	3.219	1.923	3.782	2.102
6		2.121	1.750	2.783	2.324	3.169	2.033	4.104	2.609
7		2.998	2.681	3.243	2.763	4.118	3.051	4.762	3.168
8		2.945	2.639	3.221	2.805	3.947	3.029	4.655	3.249
9		2.011	1.705	2.899	2.507	3.078	1.909	4.315	2.856
10		2.696	2.473	3.859	3.564	3.872	2.839	5.392	3.931
11		4.049	4.350	5.841	6.097	6.784	7.192	9.786	10.005
12		3.603	3.590	4.232	4.107	4.973	4.410	5.921	4.915
13		2.878	2.563	3.362	2.932	3.952	2.938	4.878	3.373
14		4.374	4.363	5.237	5.256	5.985	5.284	7.697	6.800

Table 25: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 5$ and $\tau = 0.5$. OLS demeaning.

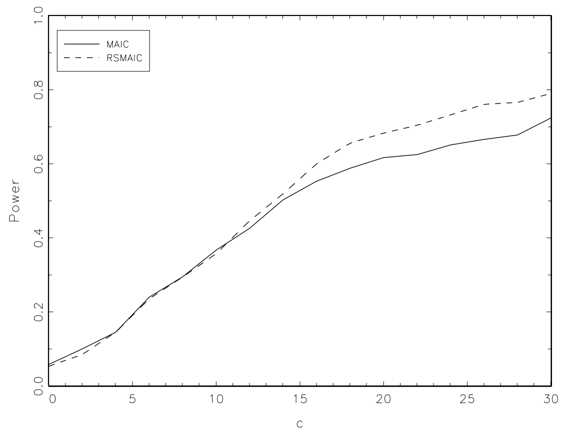
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.848	0.782	2.234	0.917	3.957	1.089	4.706	1.243
2		2.541	1.868	2.940	2.018	4.536	2.425	5.297	2.437
3		2.545	1.753	2.948	1.825	4.539	2.115	5.382	2.267
4		2.555	1.721	2.943	1.959	4.600	2.148	5.255	2.335
5		2.560	1.926	2.875	2.072	4.558	2.425	5.457	2.858
6		2.898	2.353	3.484	2.736	4.846	2.747	5.838	3.263
7		3.168	2.725	3.555	2.877	5.268	3.222	5.976	3.483
8		3.192	2.990	3.581	3.219	5.212	3.911	6.065	4.466
9		3.127	2.656	3.705	3.103	5.040	3.085	6.294	3.718
10		3.730	3.405	4.237	3.692	5.727	3.941	6.822	4.394
11		5.019	5.214	6.103	6.209	8.231	7.846	9.901	9.024
12		3.487	3.111	4.050	3.507	5.507	3.606	6.652	4.118
13		3.321	2.841	3.897	3.271	5.376	3.196	6.539	3.783
14		4.867	4.844	5.814	5.784	7.494	6.094	9.127	7.528
1	7	1.791	0.854	2.147	0.926	3.434	1.110	4.456	1.333
2		2.537	2.058	2.924	2.183	4.511	2.814	5.150	2.792
3		2.471	1.858	2.848	1.948	4.379	2.329	5.239	2.469
4		2.450	1.709	2.939	1.893	4.192	2.067	5.116	2.218
5		2.448	1.741	2.777	1.893	4.279	2.200	5.213	2.332
6		2.641	2.152	3.281	2.601	4.424	2.555	5.590	3.032
7		3.185	2.654	3.500	2.779	5.040	3.143	5.776	3.308
8		3.204	2.806	3.530	2.947	5.012	3.474	5.974	3.775
9		2.711	2.406	3.443	2.907	4.553	2.799	5.730	3.353
10		3.459	3.265	4.139	3.603	5.308	3.717	6.395	4.067
11		4.820	5.145	6.164	6.339	8.434	8.458	10.477	9.906
12		3.680	3.383	4.298	3.784	5.641	4.154	6.721	4.527
13		3.205	2.743	3.738	3.182	5.073	3.191	6.074	3.651
14		4.755	4.796	5.557	5.646	7.118	6.003	8.738	7.304

Table 26: Average lags lengths selected by MAIC and RSMAIC. Single break volatility model: $\delta = 5$ and $\tau = 0.5$. QD detrending

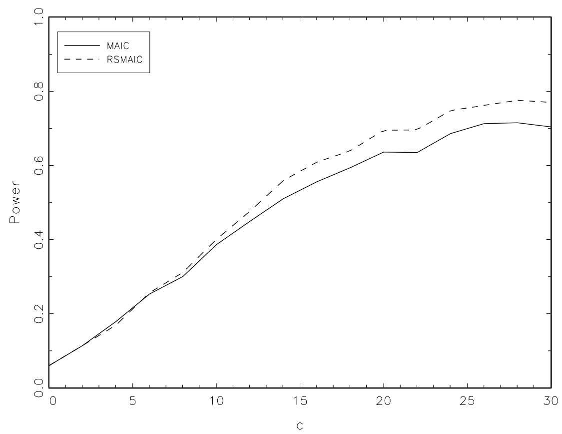
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.623	0.841	2.048	1.003	3.110	1.129	4.029	1.383
2		2.336	1.922	2.706	2.041	3.959	2.472	4.828	2.698
3		2.343	1.799	2.712	1.916	3.989	2.206	4.872	2.342
4		2.319	1.717	2.758	1.917	3.924	2.013	4.866	2.224
5		2.331	1.878	2.792	2.039	3.923	2.252	4.782	2.545
6		2.571	2.255	3.237	2.664	4.089	2.545	5.299	3.153
7		3.017	2.633	3.460	2.842	4.705	3.086	5.507	3.342
8		3.035	2.889	3.440	3.167	4.515	3.622	5.496	4.236
9		2.765	2.567	3.530	3.102	4.294	2.930	5.592	3.528
10		3.489	3.336	4.094	3.663	5.086	3.716	6.245	4.182
11		4.798	5.085	6.062	6.166	7.921	7.546	9.992	9.101
12		3.450	3.020	4.148	3.536	5.111	3.561	6.204	4.040
13		3.088	2.751	3.735	3.264	4.741	3.103	5.832	3.670
14		4.751	4.738	5.603	5.707	6.837	5.897	8.724	7.408
1	13.5	1.611	0.931	1.962	1.031	2.862	1.221	3.906	1.414
2		2.510	2.275	2.752	2.209	3.911	2.965	4.627	2.896
3		2.402	1.971	2.742	2.067	3.823	2.491	4.644	2.573
4		2.334	1.758	2.732	1.961	3.764	2.084	4.472	2.311
5		2.336	1.762	2.698	1.912	3.727	2.043	4.624	2.337
6		2.327	1.899	2.953	2.393	3.694	2.194	4.881	2.815
7		3.120	2.712	3.433	2.849	4.583	3.173	5.406	3.382
8		3.043	2.720	3.480	2.919	4.573	3.172	5.400	3.493
9		2.167	1.863	3.094	2.583	3.590	2.137	4.968	3.096
10		2.806	2.652	3.947	3.604	4.374	3.082	5.924	4.136
11		3.980	4.479	5.760	6.135	6.751	7.410	9.892	10.203
12		3.620	3.594	4.271	4.070	5.285	4.569	6.434	5.060
13		2.940	2.645	3.564	3.090	4.341	3.100	5.627	3.557
14		4.355	4.469	5.238	5.314	6.282	5.462	8.060	6.937



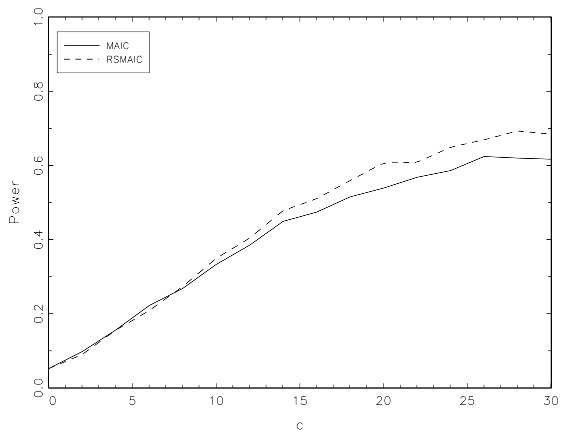
(a) $T = 150$, ARMA model 1, $\delta = 1/3$



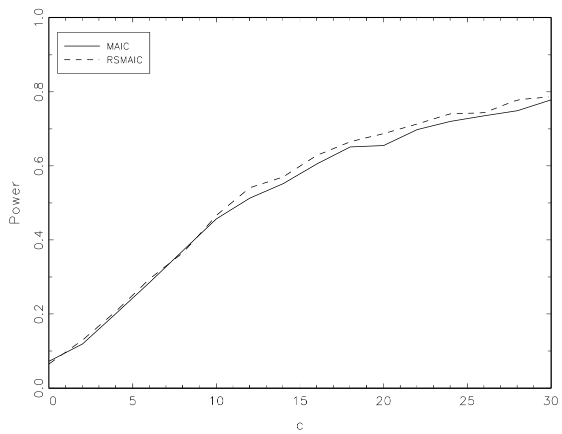
(b) $T = 150$, ARMA model 1, $\delta = 1/5$



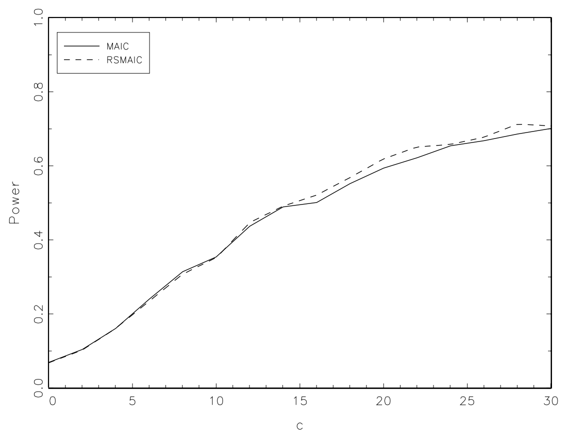
(c) $T = 150$, ARMA model 4, $\delta = 1/3$



(d) $T = 150$, ARMA model 4, $\delta = 1/5$

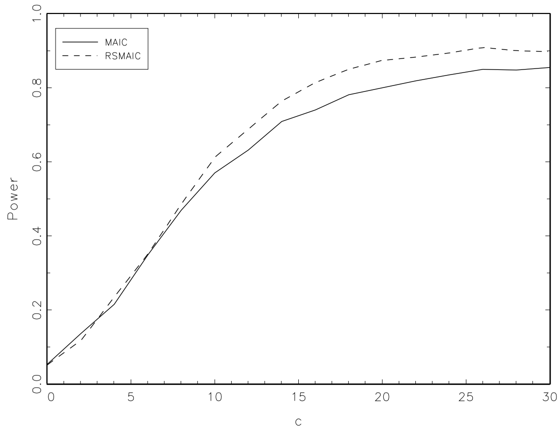


(e) $T = 150$, ARMA model 12, $\delta = 1/3$

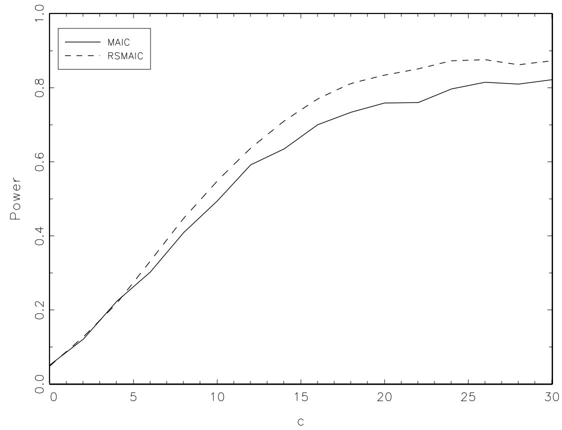


(f) $T = 150$, ARMA model 12, $\delta = 1/5$

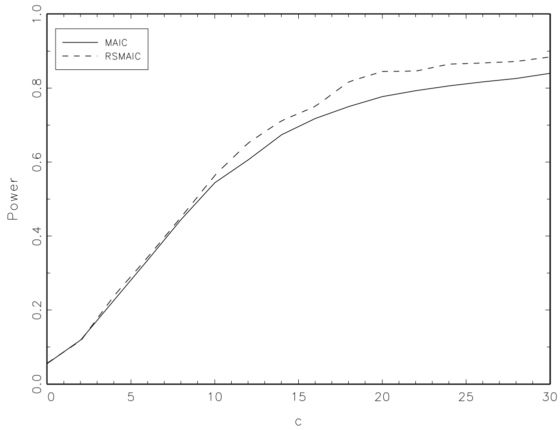
Figure 8: Power wild bootstrap ADF test with QD demeaning. Single break volatility model: $\tau = 0.5$.



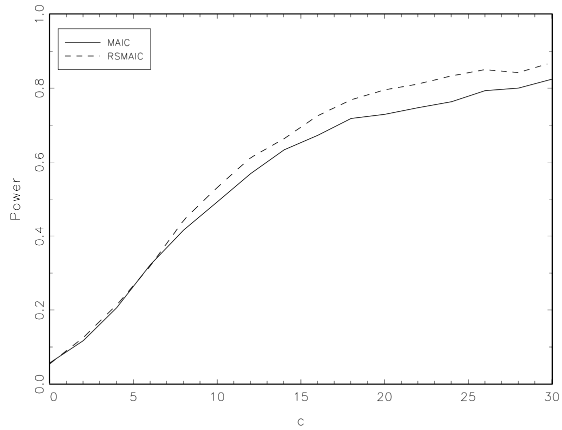
(a) $T = 150$, ARMA model 1, $\delta = 3$



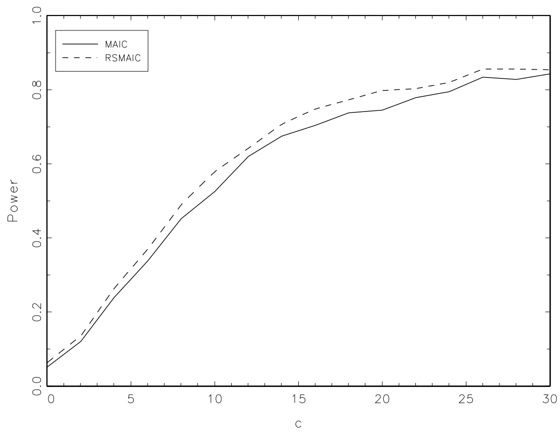
(b) $T = 150$, ARMA model 1, $\delta = 5$



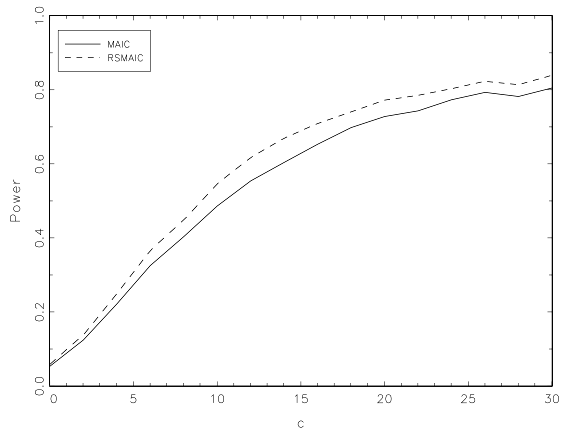
(c) $T = 150$, ARMA model 4, $\delta = 3$



(d) $T = 150$, ARMA model 4, $\delta = 5$



(e) $T = 150$, ARMA model 12, $\delta = 3$



(f) $T = 150$, ARMA model 12, $\delta = 5$

Figure 9: Power wild bootstrap ADF test with QD demeaning. Single break volatility model: $\tau = 0.5$.

A.2 Double break

This section contains simulation results for the double break model. Specifically, we use the model: $\sigma_t^2 = \sigma_0^2 + (\sigma_1^2 - \sigma_0^2)I(\lfloor \tau T \rfloor < t \leq \lfloor (1-\tau)T \rfloor)$, where we set $\sigma_0 = 1$. Defining $\delta = \sigma_0/\sigma_1$, we consider parameters $\delta = 1/3, 3; 1/5, 5$ and $\tau = 0.1, 0.4; 0.05, 0.45$. Lag length selection results are based on 5000 simulations, power curves on 2000 simulations.

Table 27: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/3$ and $\tau = 0.1$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.757	0.681	0.955	0.886	0.996	0.844	1.167	1.028
2		1.746	1.763	1.876	1.889	2.024	2.058	2.127	2.109
3		1.718	1.649	1.873	1.782	1.951	1.827	2.134	2.021
4		1.723	1.687	1.845	1.837	1.959	1.933	2.112	2.123
5		1.722	1.811	1.850	1.974	1.913	2.071	2.144	2.420
6		2.342	2.291	2.634	2.649	2.603	2.562	3.020	3.000
7		2.617	2.653	2.742	2.778	2.904	2.977	3.147	3.237
8		2.610	2.784	2.752	3.018	2.846	3.355	3.121	3.856
9		2.643	2.611	3.060	3.067	2.912	2.858	3.503	3.489
10		3.411	3.375	3.622	3.611	3.772	3.754	4.077	4.074
11		5.202	5.205	6.169	6.186	7.568	7.540	8.703	8.672
12		3.072	3.015	3.501	3.460	3.537	3.387	3.988	3.848
13		2.817	2.761	3.282	3.240	3.079	2.995	3.646	3.614
14		4.839	4.834	5.770	5.767	5.973	5.889	7.341	7.265
1	7	0.934	0.813	1.017	0.881	1.120	0.900	1.243	1.072
2		1.914	1.952	2.018	1.998	2.372	2.400	2.429	2.459
3		1.839	1.791	1.958	1.896	2.257	2.132	2.374	2.216
4		1.819	1.695	1.908	1.809	2.068	1.906	2.283	2.089
5		1.764	1.683	1.873	1.826	2.141	1.984	2.283	2.160
6		2.111	2.008	2.579	2.489	2.360	2.173	2.977	2.819
7		2.687	2.625	2.791	2.741	3.048	2.940	3.291	3.138
8		2.664	2.686	2.814	2.849	3.018	3.033	3.240	3.307
9		2.198	2.103	2.870	2.789	2.573	2.360	3.230	3.063
10		3.217	3.070	3.665	3.596	3.548	3.319	4.190	4.007
11		5.001	4.976	6.299	6.309	8.244	8.191	10.039	10.045
12		3.485	3.433	3.910	3.870	4.178	4.054	4.606	4.438
13		2.749	2.649	3.180	3.074	3.185	2.974	3.692	3.466
14		4.705	4.630	5.484	5.442	5.759	5.548	7.186	6.945

Table 28: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/3$ and $\tau = 0.1$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.684	0.576	0.810	0.719	0.792	0.673	1.018	0.876
2		1.714	1.725	1.850	1.840	1.956	1.969	2.090	2.082
3		1.688	1.618	1.817	1.737	1.840	1.741	2.149	1.982
4		1.649	1.580	1.793	1.722	1.798	1.697	1.985	1.897
5		1.628	1.621	1.805	1.814	1.820	1.827	2.064	2.085
6		2.072	2.027	2.519	2.481	2.259	2.178	2.830	2.777
7		2.553	2.541	2.703	2.701	2.786	2.755	2.948	2.932
8		2.560	2.653	2.682	2.840	2.760	2.916	2.935	3.251
9		2.326	2.267	2.893	2.842	2.545	2.461	3.226	3.132
10		3.204	3.159	3.583	3.565	3.457	3.362	3.974	3.897
11		5.061	5.064	6.235	6.244	7.617	7.593	9.354	9.305
12		3.181	3.120	3.684	3.603	3.606	3.499	4.115	3.991
13		2.663	2.596	3.140	3.067	2.875	2.749	3.460	3.357
14		4.683	4.648	5.571	5.550	5.490	5.417	6.980	6.915
1	13.5	0.997	0.890	1.089	0.967	1.189	1.004	1.415	1.215
2		2.054	2.106	2.198	2.229	2.576	2.604	2.726	2.747
3		1.946	1.907	2.040	1.995	2.330	2.245	2.423	2.300
4		1.817	1.701	1.996	1.875	2.102	1.925	2.387	2.183
5		1.785	1.688	1.909	1.812	1.988	1.839	2.203	2.074
6		1.804	1.682	2.416	2.325	2.042	1.851	2.781	2.572
7		2.707	2.651	2.866	2.810	3.118	2.961	3.302	3.149
8		2.585	2.569	2.755	2.768	2.829	2.833	3.155	3.191
9		1.704	1.589	2.537	2.445	1.869	1.690	2.871	2.693
10		2.483	2.310	3.604	3.533	2.755	2.462	4.119	3.903
11		4.244	4.229	6.020	6.010	6.670	6.559	9.838	9.829
12		3.522	3.481	4.124	4.099	4.349	4.233	4.958	4.853
13		2.620	2.528	3.015	2.930	2.995	2.816	3.494	3.270
14		4.305	4.230	5.229	5.204	5.147	4.939	6.784	6.617

Table 29: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/5$ and $\tau = 0.05$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.674	0.643	0.796	0.777	0.855	0.759	0.962	0.933
2		1.640	1.668	1.763	1.792	1.835	1.902	1.927	1.952
3		1.602	1.565	1.711	1.661	1.842	1.812	1.902	1.883
4		1.625	1.649	1.689	1.760	1.840	1.859	1.898	2.002
5		1.569	1.724	1.691	1.881	1.853	2.098	1.933	2.256
6		2.267	2.247	2.545	2.601	2.501	2.496	2.770	2.871
7		2.473	2.567	2.606	2.702	2.776	2.913	2.914	3.112
8		2.486	2.750	2.615	2.941	2.796	3.400	2.886	3.742
9		2.549	2.553	2.970	3.001	2.796	2.813	3.238	3.361
10		3.359	3.365	3.532	3.585	3.652	3.680	3.825	3.961
11		5.200	5.199	6.189	6.206	7.563	7.583	8.564	8.588
12		3.000	2.972	3.424	3.383	3.401	3.333	3.816	3.754
13		2.746	2.707	3.141	3.135	3.005	2.965	3.498	3.488
14		4.803	4.788	5.792	5.791	6.021	5.973	7.114	7.159
1	7	0.784	0.707	0.855	0.783	1.051	0.911	1.056	0.934
2		1.825	1.883	1.889	1.956	2.265	2.369	2.206	2.291
3		1.760	1.734	1.832	1.796	2.148	2.095	2.210	2.123
4		1.741	1.665	1.819	1.766	2.012	1.899	2.046	1.956
5		1.666	1.634	1.786	1.785	1.995	1.938	2.061	2.017
6		2.005	1.950	2.440	2.391	2.288	2.169	2.820	2.692
7		2.619	2.601	2.708	2.692	2.980	2.912	3.019	2.966
8		2.559	2.619	2.675	2.782	2.961	3.079	2.977	3.158
9		2.124	2.063	2.748	2.718	2.458	2.343	3.104	3.006
10		3.118	3.009	3.590	3.553	3.485	3.331	4.009	3.921
11		5.054	5.031	6.341	6.336	8.377	8.374	9.927	9.982
12		3.453	3.444	3.874	3.858	4.205	4.139	4.403	4.318
13		2.667	2.592	3.109	3.050	3.122	2.998	3.442	3.330
14		4.663	4.616	5.500	5.482	5.730	5.584	7.129	7.020

Table 30: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/5$ and $\tau = 0.05$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.602	0.548	0.724	0.671	0.699	0.628	0.820	0.774
2		1.618	1.690	1.691	1.736	1.828	1.947	1.907	1.948
3		1.564	1.542	1.659	1.624	1.805	1.757	1.837	1.809
4		1.575	1.548	1.637	1.655	1.736	1.676	1.829	1.822
5		1.501	1.562	1.650	1.761	1.696	1.742	1.818	1.933
6		1.996	1.976	2.447	2.446	2.221	2.170	2.655	2.663
7		2.462	2.483	2.596	2.649	2.726	2.707	2.818	2.858
8		2.458	2.609	2.558	2.753	2.695	2.907	2.783	3.221
9		2.265	2.209	2.802	2.773	2.450	2.383	2.963	2.939
10		3.153	3.111	3.485	3.501	3.420	3.339	3.772	3.778
11		5.075	5.064	6.250	6.264	7.710	7.726	9.191	9.259
12		3.157	3.142	3.568	3.544	3.589	3.543	4.012	3.978
13		2.565	2.523	3.029	2.988	2.850	2.743	3.326	3.285
14		4.636	4.616	5.527	5.510	5.445	5.354	6.840	6.803
1	13.5	0.818	0.783	0.959	0.879	1.091	0.998	1.220	1.117
2		2.000	2.047	2.050	2.111	2.575	2.662	2.515	2.599
3		1.857	1.838	1.933	1.924	2.217	2.193	2.311	2.272
4		1.735	1.661	1.908	1.839	2.045	1.915	2.146	2.045
5		1.701	1.647	1.866	1.813	1.982	1.904	2.061	1.962
6		1.672	1.603	2.337	2.257	1.968	1.825	2.636	2.512
7		2.696	2.638	2.788	2.741	3.036	2.952	3.145	3.089
8		2.556	2.564	2.704	2.719	2.885	2.919	2.944	3.095
9		1.587	1.499	2.449	2.390	1.806	1.670	2.700	2.603
10		2.335	2.183	3.566	3.497	2.642	2.402	3.930	3.824
11		4.197	4.130	6.022	6.012	6.745	6.680	9.776	9.736
12		3.542	3.524	4.084	4.069	4.348	4.316	4.847	4.837
13		2.576	2.499	2.970	2.913	2.919	2.814	3.298	3.195
14		4.260	4.230	5.193	5.187	5.149	5.020	6.717	6.603

Table 31: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/3$ and $\tau = 0.4$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.250	0.894	2.846	1.097	3.735	1.112	5.550	1.471
2		2.826	1.876	3.444	2.034	4.411	2.319	6.157	2.511
3		2.812	1.784	3.415	1.967	4.432	2.102	6.082	2.305
4		2.842	1.860	3.413	2.097	4.375	2.308	5.966	2.678
5		2.851	2.046	3.446	2.298	4.448	2.686	5.965	3.145
6		3.054	2.399	3.769	2.822	4.651	2.872	6.454	3.522
7		3.379	2.779	3.934	2.982	4.872	3.354	6.535	3.761
8		3.447	3.084	3.958	3.374	5.114	4.126	6.628	4.772
9		3.231	2.685	4.051	3.242	4.802	3.154	6.611	3.945
10		3.864	3.456	4.510	3.799	5.569	4.062	7.048	4.587
11		5.057	5.180	6.087	6.173	8.134	7.576	9.971	8.807
12		3.594	3.082	4.357	3.536	5.178	3.486	7.100	4.119
13		3.444	2.824	4.147	3.346	5.031	3.223	6.754	3.928
14		4.886	4.803	5.802	5.737	7.181	6.103	9.115	7.520
1	7	2.002	0.835	2.649	0.938	3.357	1.160	4.989	1.340
2		2.710	2.053	3.242	2.109	4.171	2.626	5.780	2.755
3		2.712	1.889	3.278	1.969	4.082	2.257	5.643	2.490
4		2.679	1.759	3.344	1.949	4.005	2.069	5.667	2.296
5		2.718	1.782	3.259	1.997	4.093	2.136	5.718	2.483
6		2.766	2.039	3.552	2.554	4.256	2.384	6.049	3.057
7		3.308	2.685	3.824	2.855	4.804	3.165	6.249	3.370
8		3.340	2.785	3.868	3.081	4.769	3.281	6.338	3.835
9		2.845	2.228	3.753	2.864	4.265	2.494	6.058	3.310
10		3.485	3.066	4.350	3.632	4.925	3.495	6.738	4.181
11		4.759	5.013	6.084	6.271	8.152	8.146	10.332	10.008
12		3.710	3.433	4.451	3.904	5.409	4.122	6.931	4.622
13		3.293	2.683	3.997	3.155	4.740	3.093	6.412	3.669
14		4.716	4.657	5.542	5.496	6.675	5.586	8.778	7.040

Table 32: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/3$ and $\tau = 0.4$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.903	0.720	2.628	0.964	2.825	0.878	4.674	1.213
2		2.596	1.831	3.215	1.998	3.690	2.177	5.391	2.417
3		2.546	1.710	3.204	1.847	3.600	1.944	5.316	2.209
4		2.583	1.697	3.157	1.913	3.632	1.958	5.218	2.239
5		2.532	1.778	3.184	2.069	3.669	2.128	5.352	2.586
6		2.741	2.143	3.531	2.651	3.807	2.356	5.715	3.100
7		3.181	2.651	3.749	2.850	4.306	2.986	5.830	3.324
8		3.223	2.821	3.836	3.188	4.273	3.414	5.864	4.097
9		2.903	2.376	3.802	3.000	3.911	2.664	5.931	3.397
10		3.588	3.228	4.330	3.662	4.768	3.523	6.468	4.132
11		4.806	5.011	6.043	6.223	7.766	7.545	9.843	9.162
12		3.507	3.109	4.275	3.597	4.723	3.521	6.438	4.031
13		3.242	2.669	3.957	3.168	4.312	2.868	6.146	3.602
14		4.716	4.612	5.609	5.570	6.407	5.560	8.616	7.102
1	13.5	1.878	0.977	2.468	1.121	2.955	1.247	4.630	1.488
2		2.718	2.281	3.138	2.347	3.934	2.889	5.301	3.020
3		2.610	1.994	3.053	2.103	3.805	2.486	5.198	2.616
4		2.537	1.796	3.132	1.991	3.661	2.108	5.142	2.368
5		2.585	1.798	3.147	1.973	3.774	2.101	5.246	2.422
6		2.464	1.768	3.344	2.448	3.618	2.114	5.360	2.824
7		3.250	2.716	3.713	2.883	4.380	3.189	5.782	3.430
8		3.198	2.734	3.750	2.953	4.444	3.167	5.940	3.549
9		2.303	1.680	3.417	2.553	3.388	1.969	5.450	2.978
10		2.944	2.396	4.143	3.569	4.120	2.764	6.298	4.110
11		4.013	4.284	5.714	6.027	6.820	7.064	9.869	9.906
12		3.663	3.577	4.419	4.095	5.291	4.506	6.790	5.085
13		3.100	2.628	3.793	3.045	4.222	3.062	5.805	3.539
14		4.454	4.335	5.270	5.239	6.166	5.223	8.231	6.788

Table 33: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/5$ and $\tau = 0.45$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	3.458	1.387	4.662	1.779	5.145	1.634	8.738	2.507
2		3.752	2.304	4.792	2.620	5.520	2.782	8.946	3.511
3		3.704	2.132	4.770	2.503	5.441	2.510	8.990	3.217
4		3.697	2.186	4.812	2.671	5.477	2.655	8.898	3.676
5		3.779	2.482	4.799	3.034	5.535	3.255	9.022	4.358
6		3.805	2.643	4.885	3.274	5.510	3.168	9.030	4.405
7		4.041	3.041	5.031	3.508	5.761	3.732	9.062	4.830
8		4.095	3.416	5.089	3.960	6.005	4.663	9.285	6.139
9		3.871	2.907	5.018	3.622	5.689	3.464	9.056	4.720
10		4.203	3.561	5.241	4.134	5.970	4.150	9.324	5.278
11		4.951	5.140	6.040	6.151	7.874	7.449	10.670	9.010
12		4.049	3.192	5.071	3.738	5.875	3.753	9.088	4.640
13		3.970	2.977	5.060	3.639	5.665	3.497	9.098	4.625
14		4.928	4.791	5.923	5.749	7.248	6.135	10.149	7.778
1	7	3.141	1.163	4.245	1.510	4.611	1.579	7.942	2.061
2		3.381	2.339	4.452	2.689	4.897	3.008	8.203	3.599
3		3.435	2.115	4.463	2.391	4.824	2.518	8.266	3.135
4		3.471	1.970	4.504	2.316	4.998	2.355	8.264	2.931
5		3.560	2.144	4.644	2.587	5.216	2.550	8.440	3.365
6		3.500	2.235	4.593	2.879	5.023	2.556	8.476	3.678
7		3.870	2.911	4.814	3.206	5.486	3.408	8.545	3.989
8		3.960	3.081	4.896	3.585	5.767	3.717	8.944	4.925
9		3.503	2.400	4.709	3.151	5.034	2.702	8.517	3.768
10		3.903	3.171	4.999	3.884	5.470	3.582	8.769	4.617
11		4.469	4.893	5.839	6.198	7.440	7.913	10.312	9.843
12		3.981	3.501	4.925	4.013	5.612	4.225	8.546	5.039
13		3.778	2.822	4.766	3.405	5.313	3.306	8.337	4.184
14		4.819	4.640	5.683	5.528	6.784	5.671	9.727	7.383

Table 34: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 1/5$ and $\tau = 0.45$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	3.081	1.148	4.205	1.514	4.196	1.284	7.800	1.990
2		3.346	2.201	4.472	2.595	4.648	2.622	7.923	3.310
3		3.355	1.994	4.500	2.361	4.640	2.351	7.951	2.876
4		3.414	1.967	4.467	2.430	4.567	2.257	8.068	3.112
5		3.444	2.180	4.594	2.760	4.768	2.506	8.256	3.639
6		3.491	2.399	4.648	3.035	4.680	2.629	8.178	3.748
7		3.790	2.891	4.773	3.274	5.058	3.231	8.331	4.135
8		3.802	3.188	4.858	3.801	5.152	3.776	8.575	5.362
9		3.527	2.556	4.661	3.310	4.842	2.844	8.227	4.003
10		3.917	3.342	4.998	3.946	5.232	3.589	8.547	4.796
11		4.549	4.954	5.841	6.144	7.269	7.342	10.141	9.150
12		3.851	3.191	4.885	3.786	5.157	3.646	8.281	4.582
13		3.735	2.815	4.863	3.520	4.916	3.052	8.254	4.117
14		4.739	4.639	5.716	5.590	6.514	5.550	9.563	7.293
1	13.5	2.648	1.172	3.892	1.510	3.866	1.516	7.042	2.084
2		3.106	2.420	4.029	2.696	4.446	3.095	7.027	3.708
3		3.050	2.186	4.105	2.433	4.370	2.696	7.259	3.171
4		3.114	1.941	4.176	2.318	4.396	2.273	7.528	2.954
5		3.347	1.986	4.395	2.408	4.715	2.384	7.801	3.097
6		3.150	1.937	4.289	2.653	4.416	2.244	7.666	3.346
7		3.701	2.861	4.603	3.212	5.045	3.320	7.922	3.921
8		3.813	2.945	4.755	3.370	5.183	3.467	8.300	4.338
9		2.962	1.827	4.294	2.799	4.315	2.153	7.765	3.419
10		3.435	2.555	4.671	3.735	4.844	2.936	8.084	4.403
11		3.602	4.132	5.375	5.937	5.966	6.696	9.423	9.763
12		3.728	3.602	4.694	4.174	5.143	4.462	7.805	5.353
13		3.448	2.734	4.508	3.281	4.659	3.086	7.665	3.993
14		4.516	4.337	5.407	5.252	6.243	5.254	9.027	7.022

Table 35: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 3$ and $\tau = 0.1$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.269	0.838	2.941	0.935	3.556	1.007	5.826	1.253
2		2.801	1.919	3.362	2.019	4.206	2.280	6.224	2.578
3		2.825	1.730	3.405	1.868	4.217	2.041	6.234	2.247
4		2.805	1.799	3.431	1.976	4.117	2.143	6.163	2.412
5		2.809	2.030	3.394	2.257	4.246	2.597	6.209	2.983
6		3.078	2.343	3.730	2.725	4.530	2.773	6.530	3.321
7		3.343	2.763	3.936	2.960	4.775	3.317	6.728	3.629
8		3.405	3.169	3.948	3.406	4.831	4.459	6.801	5.153
9		3.289	2.640	4.025	3.081	4.703	3.077	6.891	3.713
10		3.859	3.413	4.526	3.710	5.270	3.930	7.304	4.396
11		4.949	5.189	6.052	6.202	7.995	7.575	9.921	8.790
12		3.531	3.036	4.305	3.491	4.892	3.464	7.105	4.064
13		3.380	2.753	4.158	3.269	4.753	3.141	6.964	3.803
14		4.828	4.792	5.751	5.700	6.897	5.953	9.249	7.365
1	7	2.162	0.919	2.795	1.043	3.434	1.106	5.429	1.328
2		2.814	2.182	3.373	2.318	4.261	2.737	6.046	2.950
3		2.736	1.887	3.352	2.050	4.091	2.220	5.856	2.457
4		2.768	1.781	3.341	1.973	4.100	2.145	5.966	2.377
5		2.796	1.839	3.326	2.015	4.102	2.274	6.030	2.508
6		2.806	2.125	3.617	2.595	4.215	2.477	6.287	3.131
7		3.398	2.760	3.878	2.898	4.627	3.180	6.504	3.459
8		3.342	2.896	3.911	3.152	4.643	3.764	6.552	4.142
9		2.866	2.293	3.783	2.875	4.308	2.643	6.394	3.336
10		3.536	3.167	4.396	3.642	4.952	3.608	6.947	4.128
11		4.790	5.170	6.059	6.312	8.175	8.373	10.475	9.943
12		3.779	3.470	4.531	3.925	5.258	4.091	7.154	4.601
13		3.326	2.708	4.030	3.192	4.670	3.087	6.631	3.757
14		4.717	4.752	5.507	5.542	6.711	5.835	8.982	7.274

Table 36: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 3$ and $\tau = 0.1$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	1.874	0.601	2.533	0.866	2.955	0.839	4.824	1.046
2		2.524	1.888	3.122	2.016	3.671	2.274	5.350	2.439
3		2.488	1.678	3.174	1.796	3.545	1.861	5.335	2.092
4		2.523	1.630	3.125	1.820	3.661	1.899	5.372	2.100
5		2.616	1.821	3.199	2.067	3.696	2.114	5.482	2.486
6		2.797	2.064	3.533	2.574	3.864	2.308	5.805	2.889
7		3.190	2.618	3.724	2.838	4.336	3.003	6.134	3.340
8		3.287	2.856	3.807	3.158	4.475	3.532	6.239	4.167
9		2.919	2.231	3.776	2.856	3.972	2.478	5.971	3.271
10		3.568	3.149	4.294	3.581	4.768	3.436	6.709	4.075
11		4.580	4.956	5.950	6.209	7.515	7.581	9.713	9.370
12		3.505	3.198	4.222	3.647	4.632	3.595	6.466	4.147
13		3.159	2.590	3.906	3.096	4.261	2.860	6.168	3.447
14		4.599	4.576	5.555	5.503	6.311	5.426	8.598	6.951
1	13.5	1.840	0.978	2.423	1.187	2.879	1.269	4.438	1.468
2		2.613	2.345	3.043	2.384	3.744	2.970	5.202	3.265
3		2.603	2.111	3.090	2.201	3.614	2.476	5.220	2.659
4		2.431	1.817	3.034	2.011	3.462	2.079	5.125	2.417
5		2.505	1.774	3.029	1.982	3.547	2.048	5.160	2.393
6		2.338	1.776	3.196	2.412	3.365	2.014	5.147	2.738
7		3.188	2.749	3.656	2.921	4.394	3.185	5.737	3.419
8		3.189	2.724	3.671	2.986	4.318	3.290	5.889	3.906
9		2.147	1.607	3.254	2.526	3.289	1.942	5.490	2.978
10		2.763	2.271	4.018	3.527	3.925	2.677	6.235	4.101
11		3.725	4.302	5.565	6.046	6.549	7.196	9.618	9.916
12		3.596	3.605	4.406	4.180	5.069	4.473	6.623	5.153
13		2.991	2.621	3.712	3.053	4.007	2.896	5.846	3.565
14		4.241	4.282	5.134	5.203	5.912	5.259	8.117	6.812

Table 37: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 5$ and $\tau = 0.05$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	3.157	0.953	4.654	1.276	3.944	1.274	6.275	1.621
2		3.444	2.093	4.738	2.515	4.419	2.496	6.760	2.938
3		3.406	1.805	4.767	2.169	4.329	2.203	6.548	2.518
4		3.468	1.944	4.813	2.324	4.283	2.320	6.613	2.819
5		3.553	2.249	4.872	2.729	4.497	2.781	6.788	3.649
6		3.637	2.432	4.939	3.022	4.418	2.856	6.836	3.612
7		3.769	2.909	5.027	3.292	4.810	3.468	7.028	4.120
8		3.911	3.469	5.039	3.995	5.135	4.557	7.420	5.873
9		3.747	2.697	5.030	3.301	4.518	3.086	6.907	4.015
10		4.081	3.388	5.285	3.855	4.866	3.862	7.199	4.599
11		4.784	5.189	5.882	6.164	6.870	7.175	9.031	8.523
12		3.818	3.088	5.028	3.659	4.644	3.503	6.857	3.998
13		3.770	2.803	5.098	3.425	4.497	3.194	6.813	3.897
14		4.811	4.733	5.851	5.691	5.865	5.793	8.252	7.289
1	7	3.070	1.018	4.495	1.439	3.868	1.150	6.255	1.483
2		3.362	2.267	4.592	2.701	4.312	2.694	6.775	3.163
3		3.338	1.970	4.593	2.331	4.205	2.249	6.519	2.573
4		3.362	1.888	4.722	2.230	4.290	2.110	6.610	2.488
5		3.375	1.978	4.700	2.447	4.403	2.386	6.747	2.931
6		3.472	2.212	4.777	2.848	4.432	2.564	6.701	3.204
7		3.720	2.843	4.904	3.225	4.817	3.267	6.892	3.718
8		3.803	3.120	4.940	3.653	4.884	3.819	7.068	4.685
9		3.408	2.350	4.775	3.087	4.405	2.665	6.751	3.453
10		3.865	3.202	5.097	3.761	4.706	3.577	6.962	4.164
11		4.619	5.176	5.857	6.282	6.919	7.915	9.415	9.464
12		3.861	3.406	5.036	3.979	4.646	3.870	6.841	4.458
13		3.694	2.771	4.909	3.332	4.454	3.099	6.699	3.685
14		4.836	4.748	5.710	5.588	5.793	5.539	8.168	7.010

Table 38: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 5$ and $\tau = 0.05$. OLS detrending.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	2.858	0.806	4.157	1.140	3.463	1.005	5.937	1.475
2		3.112	2.069	4.328	2.484	3.931	2.344	6.139	2.865
3		3.132	1.796	4.418	2.125	3.836	2.060	6.124	2.379
4		3.215	1.770	4.517	2.140	3.915	2.042	6.196	2.446
5		3.415	2.062	4.697	2.534	4.215	2.424	6.481	3.104
6		3.379	2.134	4.663	2.776	4.089	2.390	6.385	3.228
7		3.644	2.788	4.821	3.157	4.476	3.181	6.658	3.791
8		3.819	3.138	4.923	3.721	4.883	3.812	7.098	4.958
9		3.469	2.309	4.743	3.049	4.183	2.555	6.534	3.548
10		3.917	3.162	5.103	3.758	4.541	3.391	6.843	4.265
11		4.289	4.948	5.641	6.166	6.421	7.244	8.969	9.159
12		3.635	3.231	4.802	3.809	4.346	3.507	6.439	4.195
13		3.530	2.634	4.837	3.225	4.153	2.912	6.420	3.683
14		4.647	4.552	5.647	5.464	5.435	5.297	7.885	6.864
1	13.5	2.499	1.052	3.868	1.408	3.160	1.244	5.583	1.670
2		2.968	2.430	4.040	2.791	3.768	2.872	5.823	3.354
3		2.928	2.081	4.047	2.370	3.702	2.422	5.849	2.711
4		3.023	1.838	4.181	2.200	3.740	2.056	5.966	2.555
5		3.120	1.907	4.295	2.316	4.094	2.182	6.249	2.888
6		2.977	1.833	4.278	2.545	3.957	2.148	6.084	2.966
7		3.549	2.863	4.533	3.178	4.487	3.261	6.490	3.624
8		3.614	2.944	4.732	3.469	4.714	3.513	6.817	4.457
9		2.857	1.745	4.272	2.674	3.836	2.036	6.159	3.093
10		3.281	2.465	4.700	3.634	4.133	2.748	6.598	4.134
11		3.645	4.446	5.291	6.035	5.845	7.145	8.796	9.751
12		3.660	3.639	4.694	4.242	4.358	4.223	6.351	4.963
13		3.316	2.633	4.463	3.191	4.046	2.949	6.224	3.589
14		4.459	4.375	5.308	5.239	5.388	5.123	7.669	6.719

Table 39: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 3$ and $\tau = 0.4$. OLS demeaning.

		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.902	0.784	0.954	0.874	1.259	1.033	1.412	1.161
2		1.810	1.798	1.891	1.889	2.264	2.165	2.383	2.274
3		1.765	1.690	1.910	1.813	2.253	2.005	2.408	2.183
4		1.785	1.755	1.911	1.929	2.184	2.056	2.366	2.352
5		1.713	1.897	1.877	2.141	2.172	2.360	2.367	2.647
6		2.357	2.341	2.689	2.703	2.850	2.665	3.197	3.225
7		2.625	2.687	2.788	2.862	3.182	3.108	3.296	3.412
8		2.607	2.900	2.776	3.172	3.124	3.814	3.286	4.340
9		2.692	2.667	3.097	3.126	3.120	3.001	3.614	3.572
10		3.394	3.387	3.660	3.683	3.988	3.863	4.290	4.243
11		5.195	5.206	6.178	6.191	7.696	7.639	8.837	8.765
12		3.094	3.003	3.536	3.469	3.719	3.460	4.186	3.939
13		2.836	2.775	3.277	3.244	3.326	3.123	3.891	3.697
14		4.854	4.820	5.734	5.709	6.170	5.992	7.482	7.356
1	7	0.989	0.844	1.089	0.946	1.405	1.129	1.531	1.243
2		2.012	2.037	2.074	2.098	2.605	2.553	2.666	2.599
3		1.960	1.868	1.978	1.909	2.495	2.245	2.568	2.327
4		1.864	1.740	2.000	1.861	2.370	2.058	2.542	2.234
5		1.789	1.732	1.925	1.888	2.324	2.095	2.521	2.358
6		2.136	2.045	2.600	2.527	2.633	2.389	3.180	2.961
7		2.709	2.656	2.849	2.775	3.281	3.116	3.460	3.280
8		2.660	2.695	2.816	2.916	3.174	3.271	3.444	3.627
9		2.269	2.182	2.829	2.780	2.795	2.583	3.447	3.235
10		3.194	3.099	3.668	3.604	3.789	3.534	4.291	4.054
11		5.031	5.092	6.288	6.314	8.308	8.335	10.017	9.993
12		3.469	3.443	3.925	3.859	4.397	4.214	4.828	4.568
13		2.735	2.665	3.203	3.107	3.370	3.135	3.809	3.557
14		4.671	4.681	5.528	5.538	5.953	5.815	7.364	7.144

Table 40: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 3$ and $\tau = 0.4$. OLS detrending.

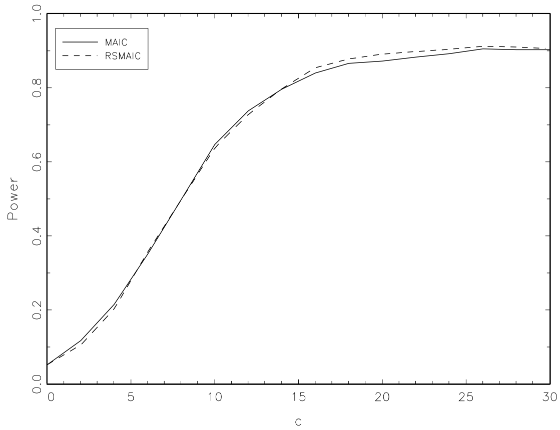
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.731	0.633	0.898	0.803	1.037	0.820	1.245	1.032
2		1.759	1.759	1.847	1.859	2.151	2.130	2.291	2.232
3		1.709	1.641	1.808	1.745	2.040	1.892	2.196	2.003
4		1.692	1.615	1.817	1.778	2.004	1.830	2.172	2.062
5		1.658	1.693	1.836	1.914	1.944	1.889	2.150	2.256
6		2.128	2.073	2.545	2.519	2.406	2.255	3.019	2.858
7		2.563	2.577	2.729	2.763	2.952	2.880	3.174	3.099
8		2.600	2.686	2.697	2.931	2.927	3.050	3.173	3.512
9		2.338	2.259	2.891	2.857	2.675	2.472	3.286	3.195
10		3.226	3.154	3.585	3.576	3.670	3.435	4.108	3.967
11		5.005	5.028	6.199	6.238	7.631	7.652	9.354	9.353
12		3.175	3.118	3.654	3.597	3.777	3.590	4.286	4.097
13		2.661	2.578	3.152	3.091	3.086	2.888	3.613	3.425
14		4.640	4.611	5.561	5.548	5.633	5.469	7.135	6.944
1	13.5	1.015	0.892	1.082	0.986	1.292	1.095	1.529	1.281
2		2.055	2.127	2.153	2.222	2.675	2.754	2.834	2.892
3		1.907	1.874	2.094	2.022	2.443	2.311	2.632	2.459
4		1.879	1.741	1.993	1.907	2.192	1.991	2.503	2.202
5		1.792	1.704	1.956	1.859	2.143	1.941	2.443	2.266
6		1.769	1.695	2.393	2.319	2.152	1.923	2.889	2.724
7		2.750	2.702	2.861	2.802	3.205	3.025	3.464	3.225
8		2.620	2.614	2.808	2.858	3.060	2.996	3.265	3.342
9		1.637	1.575	2.467	2.411	1.997	1.801	2.967	2.763
10		2.286	2.204	3.584	3.525	2.859	2.556	4.216	3.973
11		4.061	4.135	5.922	5.977	6.695	6.788	9.648	9.772
12		3.513	3.532	4.113	4.106	4.414	4.375	5.160	5.043
13		2.623	2.548	3.037	2.978	3.119	2.903	3.656	3.436
14		4.238	4.220	5.158	5.170	5.180	4.998	6.902	6.704

Table 41: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 5$ and $\tau = 0.45$. OLS demeaning.

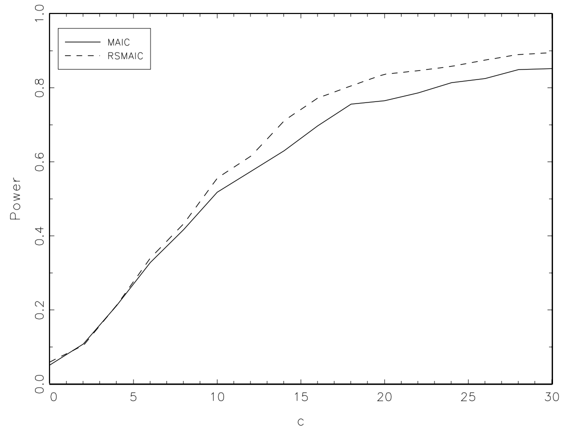
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.755	0.711	0.826	0.855	0.997	0.874	1.147	1.075
2		1.668	1.702	1.752	1.791	1.991	2.080	2.108	2.138
3		1.668	1.630	1.801	1.789	2.005	1.887	2.079	2.028
4		1.633	1.681	1.797	1.878	1.967	1.917	2.130	2.187
5		1.625	1.814	1.722	2.002	1.957	2.200	2.040	2.490
6		2.262	2.270	2.567	2.657	2.579	2.559	2.948	3.123
7		2.558	2.636	2.640	2.789	2.889	3.052	3.040	3.273
8		2.584	2.865	2.666	3.091	2.936	3.565	2.983	4.011
9		2.591	2.572	3.040	3.084	2.924	2.898	3.397	3.481
10		3.376	3.355	3.571	3.618	3.719	3.704	3.978	4.102
11		5.186	5.178	6.187	6.198	7.514	7.489	8.722	8.693
12		3.008	2.977	3.438	3.410	3.520	3.399	3.894	3.809
13		2.743	2.709	3.217	3.196	3.143	3.063	3.616	3.593
14		4.816	4.771	5.732	5.713	5.982	5.907	7.279	7.281
1	7	0.846	0.780	0.912	0.854	1.142	0.980	1.278	1.133
2		1.941	2.004	1.916	1.979	2.456	2.502	2.416	2.430
3		1.813	1.803	1.868	1.848	2.210	2.139	2.338	2.231
4		1.773	1.702	1.857	1.801	2.176	2.007	2.239	2.115
5		1.770	1.710	1.857	1.842	2.029	1.939	2.242	2.173
6		2.060	1.984	2.511	2.451	2.450	2.327	2.936	2.807
7		2.662	2.639	2.737	2.723	3.123	3.041	3.224	3.149
8		2.599	2.658	2.719	2.831	3.019	3.123	3.123	3.354
9		2.225	2.172	2.751	2.727	2.535	2.427	3.171	3.065
10		3.185	3.088	3.658	3.607	3.614	3.446	4.129	4.035
11		5.058	5.077	6.332	6.339	8.316	8.329	9.980	9.978
12		3.530	3.507	3.872	3.848	4.265	4.178	4.503	4.453
13		2.706	2.641	3.117	3.085	3.192	2.994	3.628	3.491
14		4.715	4.692	5.495	5.495	5.812	5.667	7.174	7.083

Table 42: Average lags lengths selected by MAIC and RSMAIC. Double break volatility model: $\delta = 5$ and $\tau = 0.45$. OLS detrending.

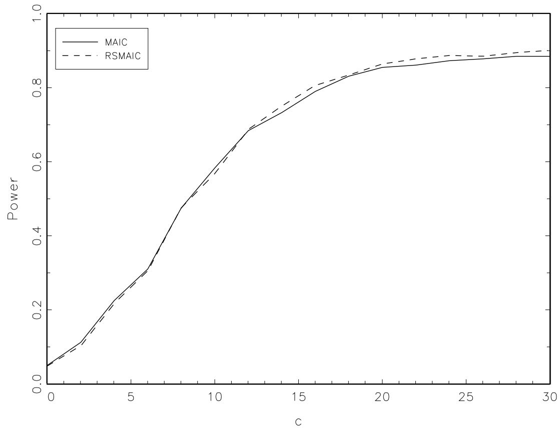
		$A = 6$				$A = 12$			
		$T = 150$		$T = 250$		$T = 150$		$T = 250$	
Model	c	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC	MAIC	RSMAIC
1	0	0.635	0.576	0.758	0.736	0.845	0.721	0.975	0.898
2		1.663	1.699	1.736	1.774	1.927	1.996	2.048	2.112
3		1.609	1.584	1.691	1.690	1.902	1.846	1.940	1.886
4		1.580	1.573	1.681	1.720	1.769	1.712	1.928	1.984
5		1.585	1.665	1.693	1.860	1.795	1.832	1.912	2.108
6		2.042	2.023	2.456	2.496	2.193	2.132	2.775	2.764
7		2.524	2.563	2.641	2.731	2.750	2.774	2.990	3.038
8		2.510	2.645	2.620	2.885	2.773	3.016	2.912	3.398
9		2.312	2.240	2.796	2.809	2.532	2.402	3.093	3.109
10		3.186	3.112	3.526	3.547	3.503	3.354	3.850	3.836
11		5.046	5.061	6.238	6.250	7.587	7.576	9.163	9.221
12		3.128	3.114	3.579	3.568	3.518	3.478	4.081	4.018
13		2.657	2.590	3.036	3.020	2.885	2.771	3.394	3.317
14		4.615	4.589	5.531	5.503	5.508	5.387	6.962	6.884
1	13.5	0.891	0.824	0.986	0.926	1.176	1.056	1.281	1.158
2		2.042	2.131	2.072	2.109	2.583	2.705	2.638	2.756
3		1.890	1.903	2.008	1.986	2.308	2.265	2.439	2.391
4		1.755	1.676	1.916	1.865	2.093	1.946	2.300	2.144
5		1.726	1.675	1.878	1.874	2.046	1.925	2.188	2.119
6		1.694	1.625	2.331	2.277	2.001	1.892	2.727	2.600
7		2.723	2.690	2.814	2.802	3.109	2.985	3.251	3.165
8		2.573	2.612	2.698	2.786	2.854	2.873	3.094	3.210
9		1.611	1.522	2.436	2.361	1.833	1.707	2.780	2.669
10		2.301	2.184	3.534	3.499	2.652	2.411	4.054	3.934
11		4.167	4.164	5.979	5.992	6.773	6.762	9.649	9.705
12		3.520	3.503	4.132	4.131	4.353	4.311	4.912	4.883
13		2.582	2.507	2.946	2.907	2.973	2.793	3.482	3.348
14		4.245	4.210	5.163	5.158	5.130	4.997	6.720	6.603



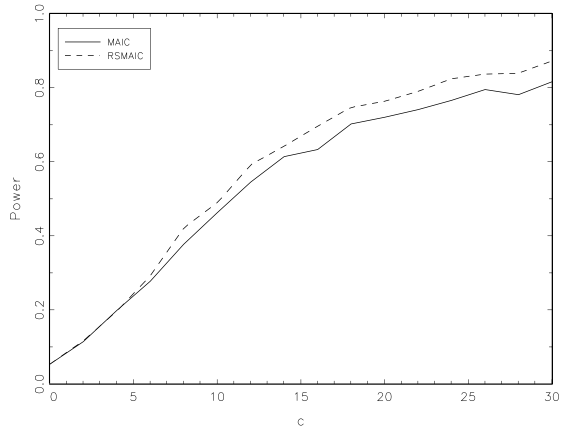
(a) $T = 150$, ARMA model 1, $\tau = 0.1$



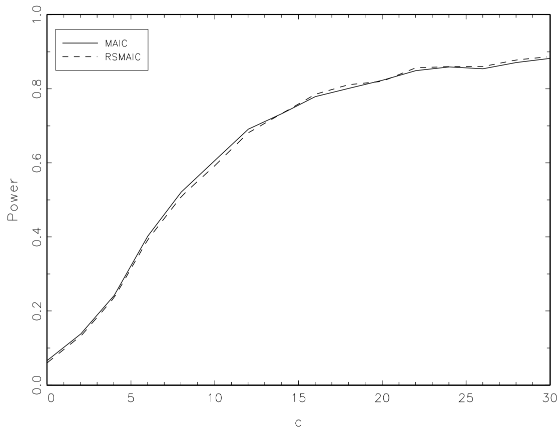
(b) $T = 150$, ARMA model 1, $\tau = 0.4$



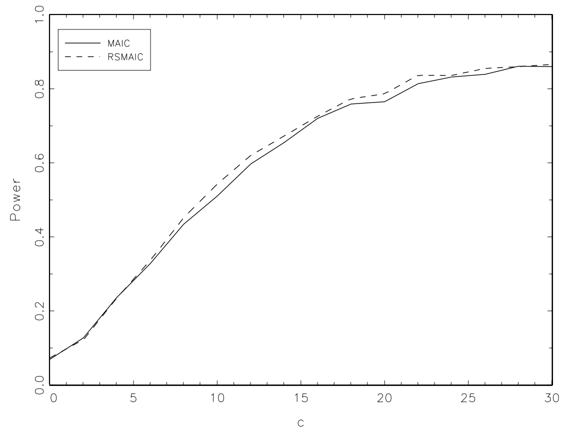
(c) $T = 150$, ARMA model 4, $\tau = 0.1$



(d) $T = 150$, ARMA model 4, $\tau = 0.4$

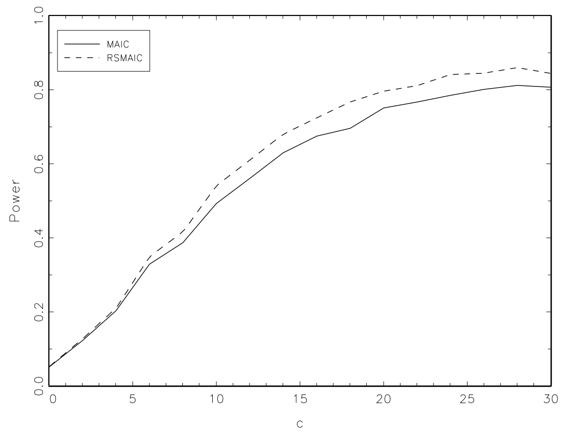


(e) $T = 150$, ARMA model 12, $\tau = 0.1$

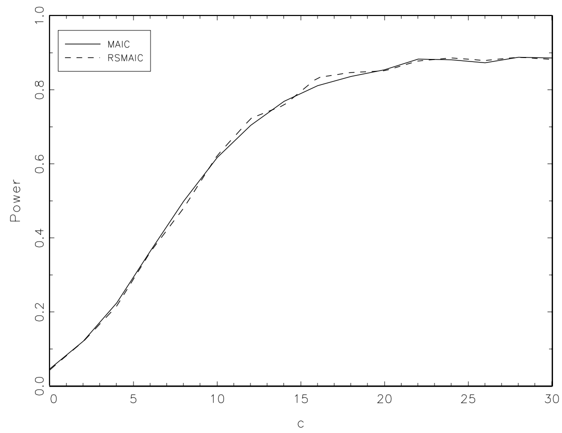


(f) $T = 150$, ARMA model 12, $\tau = 0.4$

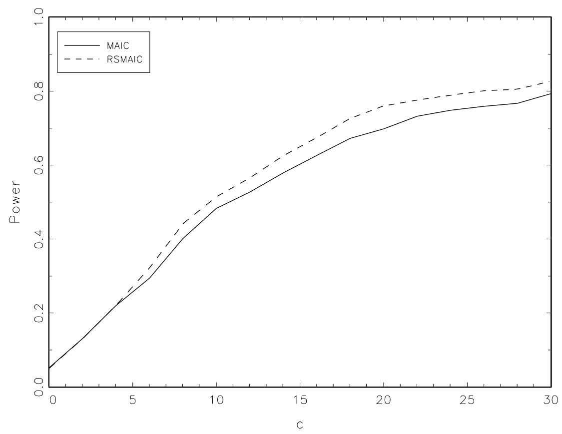
Figure 10: Power wild bootstrap ADF test with QD demeaning. Double break volatility model: $\delta = 1/3$.



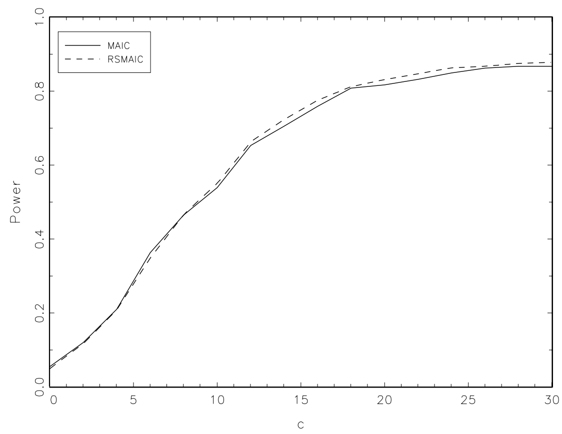
(a) $T = 150$, ARMA model 1, $\tau = 0.1$



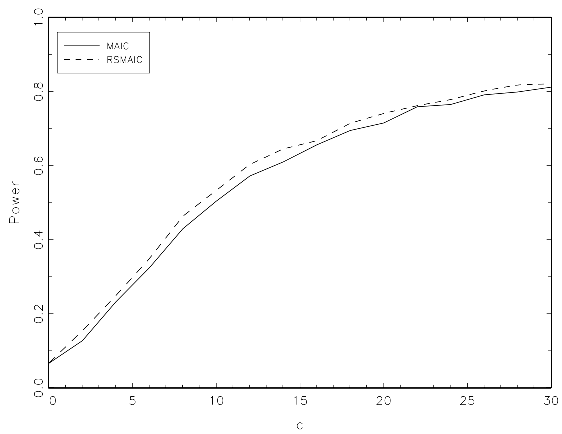
(b) $T = 150$, ARMA model 1, $\tau = 0.4$



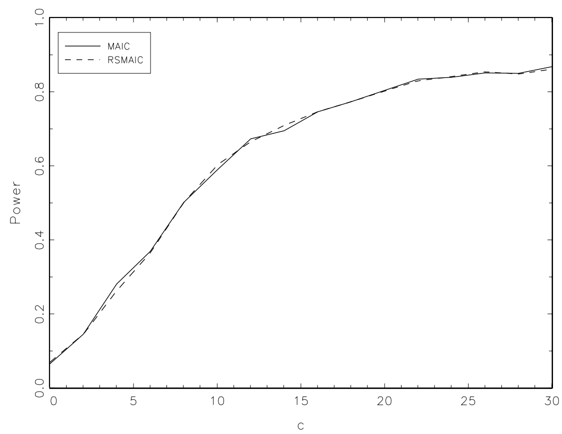
(c) $T = 150$, ARMA model 4, $\tau = 0.1$



(d) $T = 150$, ARMA model 4, $\tau = 0.4$

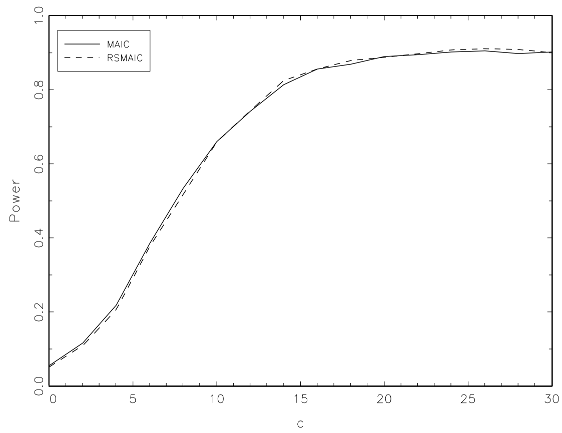


(e) $T = 150$, ARMA model 12, $\tau = 0.1$

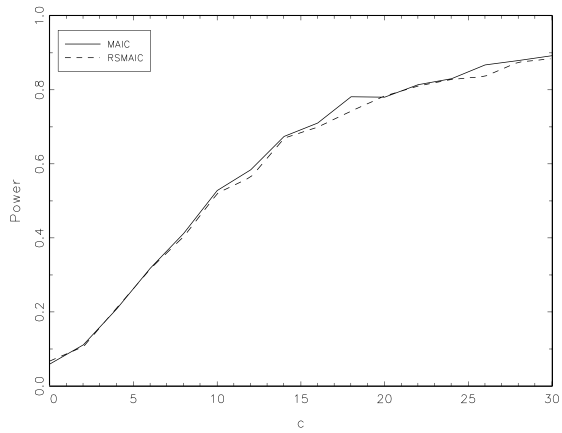


(f) $T = 150$, ARMA model 12, $\tau = 0.4$

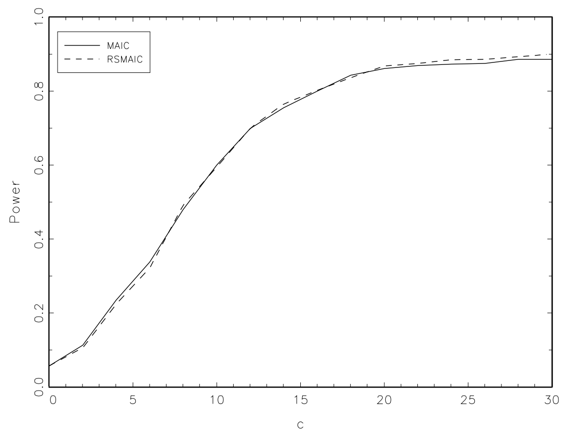
Figure 11: Power wild bootstrap ADF test with QD demeaning. Double break volatility model: $\delta = 3$.



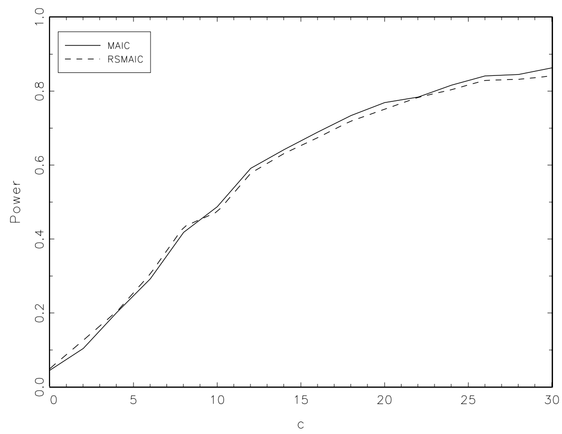
(a) $T = 150$, ARMA model 1, $\tau = 0.05$



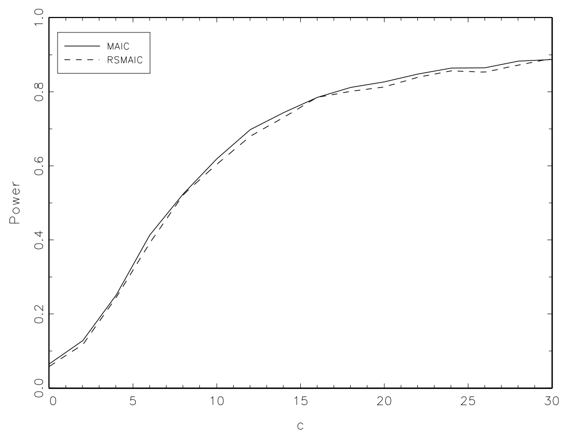
(b) $T = 150$, ARMA model 1, $\tau = 0.45$



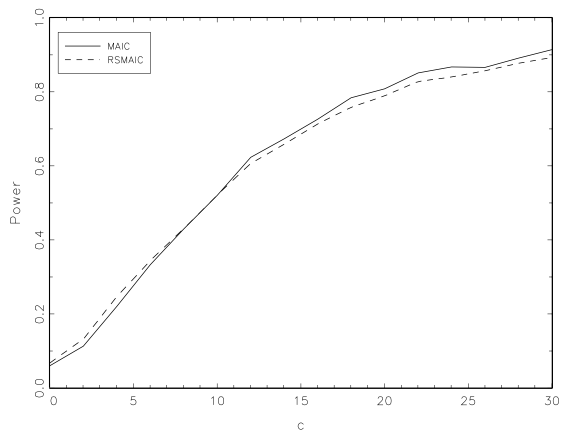
(c) $T = 150$, ARMA model 4, $\tau = 0.05$



(d) $T = 150$, ARMA model 4, $\tau = 0.45$

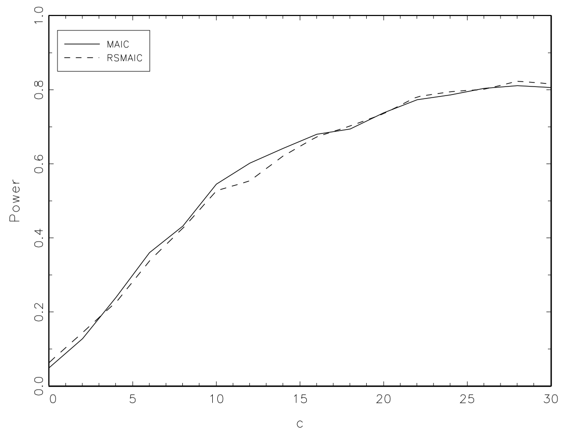


(e) $T = 150$, ARMA model 12, $\tau = 0.05$

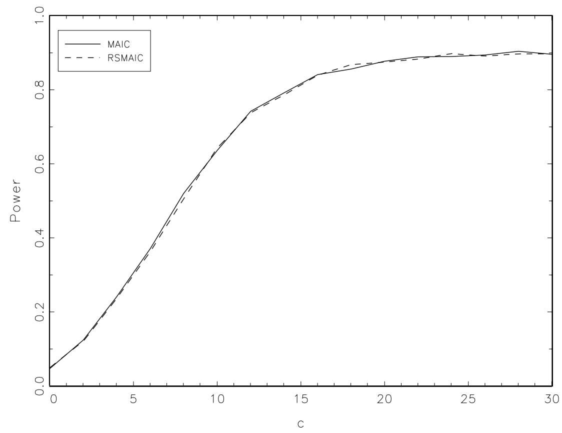


(f) $T = 150$, ARMA model 12, $\tau = 0.45$

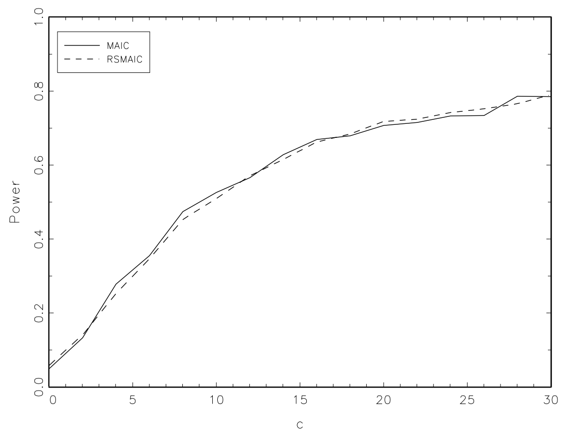
Figure 12: Power wild bootstrap ADF test with QD demeaning. Double break volatility model: $\delta = 1/5$.



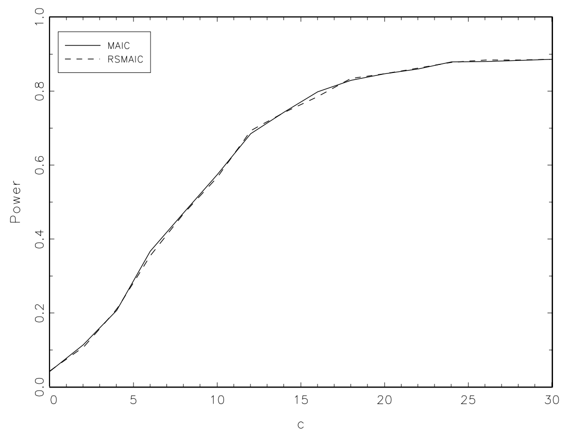
(a) $T = 150$, ARMA model 1, $\tau = 0.05$



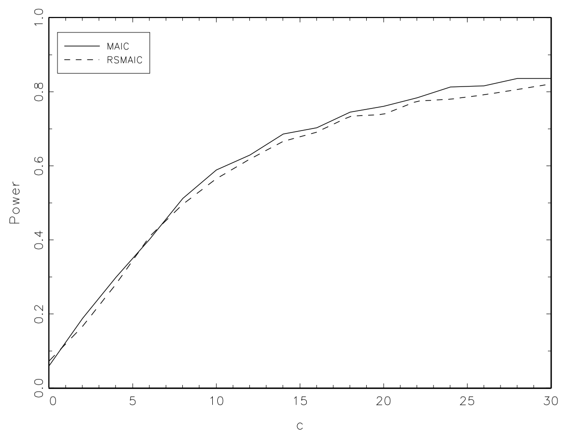
(b) $T = 150$, ARMA model 1, $\tau = 0.45$



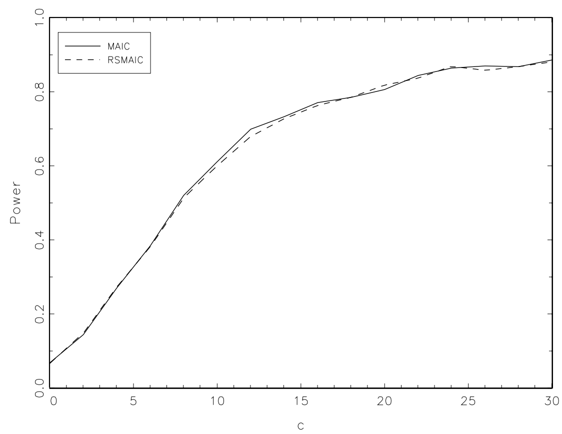
(c) $T = 150$, ARMA model 4, $\tau = 0.05$



(d) $T = 150$, ARMA model 4, $\tau = 0.45$



(e) $T = 150$, ARMA model 12, $\tau = 0.05$

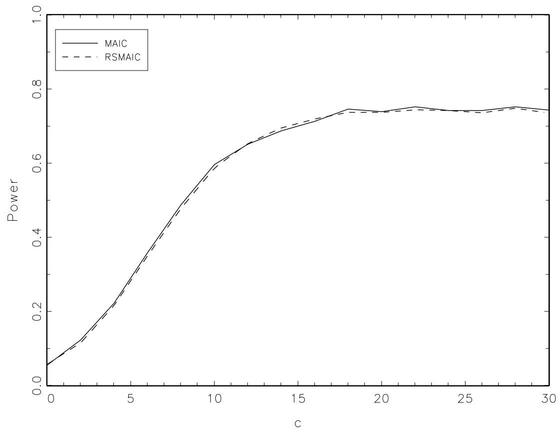


(f) $T = 150$, ARMA model 12, $\tau = 0.45$

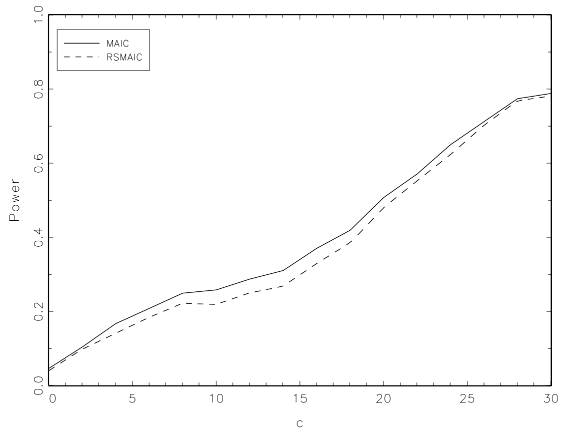
Figure 13: Power wild bootstrap ADF test with QD demeaning. Double break volatility model: $\delta = 5$.

A.3 Power curves for the models used in the paper with $T = 50$

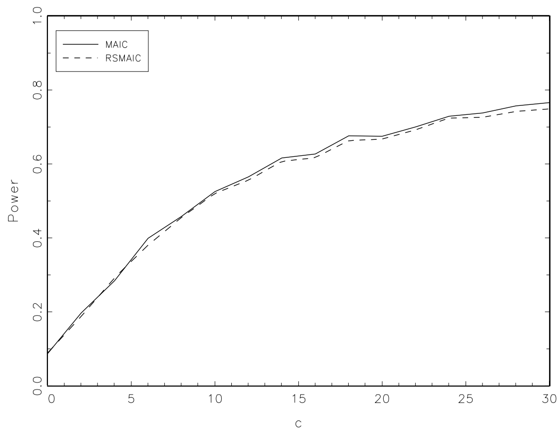
In this section we consider the same models as used in the main body of the paper, but with sample size $T = 50$. The power curves are based on 5000 simulations.



(a) ARMA model 1

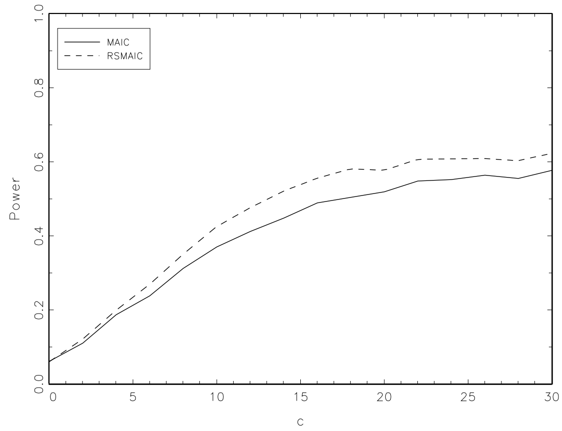


(b) ARMA model 4

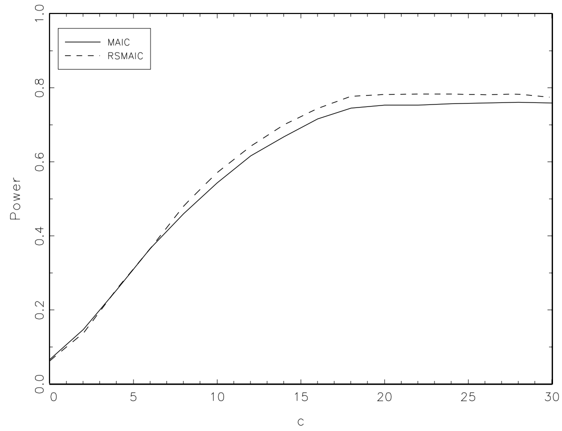


(c) ARMA model 12

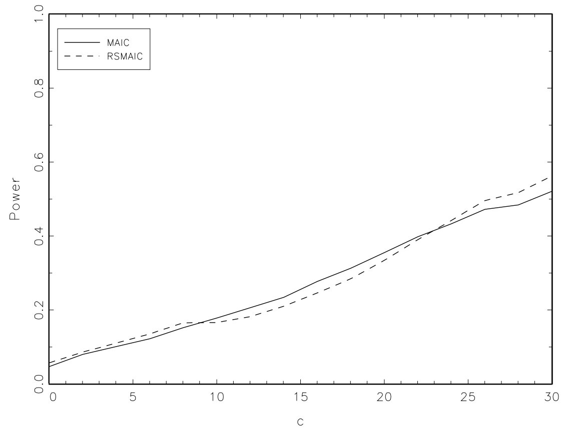
Figure 14: Power wild bootstrap ADF test with QD demeaning. Homoskedastic errors. $T = 50$.



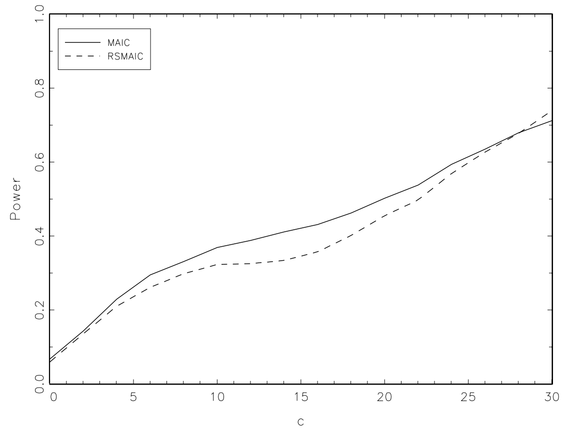
(a) $\delta = 1/3$, $\tau = 0.8$, ARMA model 1



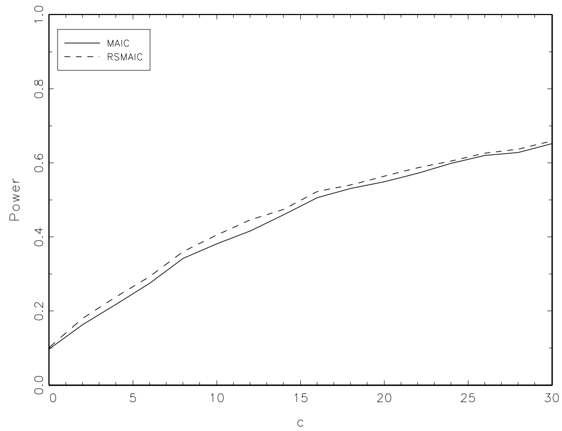
(b) $\delta = 3$, $\tau = 0.2$, ARMA model 1



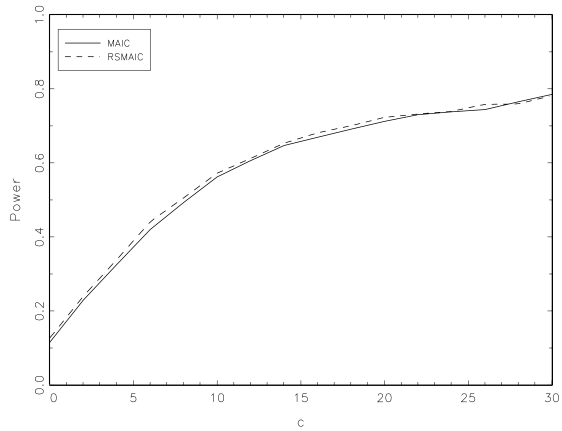
(c) $\delta = 1/3$, $\tau = 0.8$, ARMA model 4



(d) $\delta = 3$, $\tau = 0.2$, ARMA model 4

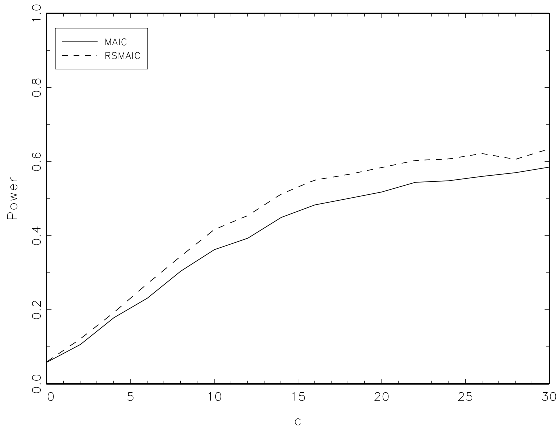


(e) $\delta = 1/3$, $\tau = 0.8$, ARMA model 12

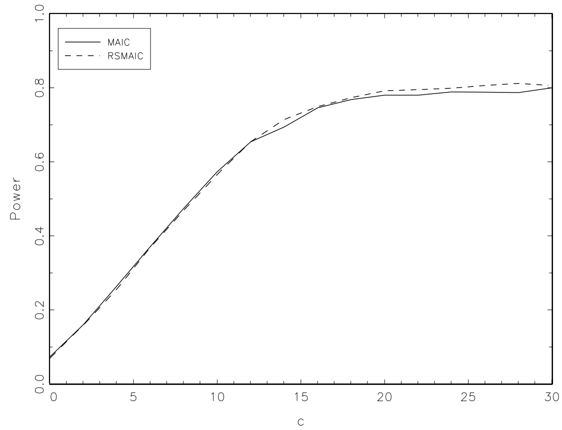


(f) $\delta = 3$, $\tau = 0.2$, ARMA model 12

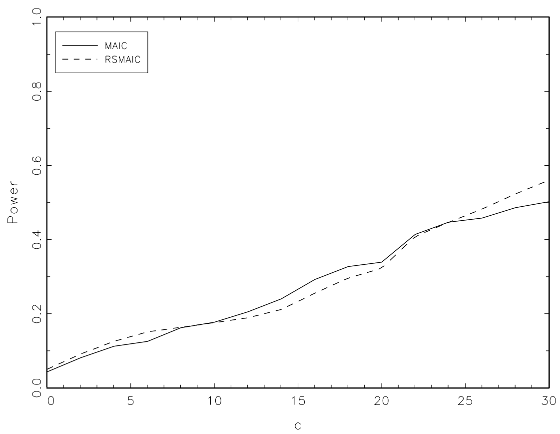
Figure 15: Power wild bootstrap ADF test with QD demeaning. Smooth transition volatility model. $T = 50$.



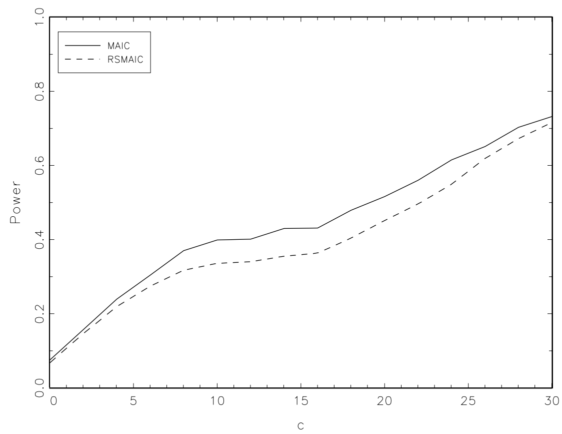
(a) $\delta = 1/3$, $\tau = 0.8$, ARMA model 1



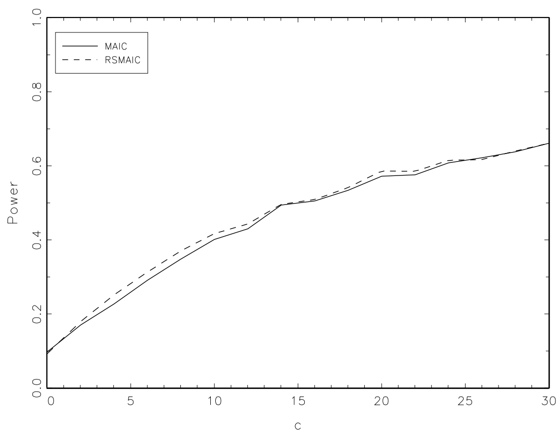
(b) $\delta = 3$, $\tau = 0.2$, ARMA model 1



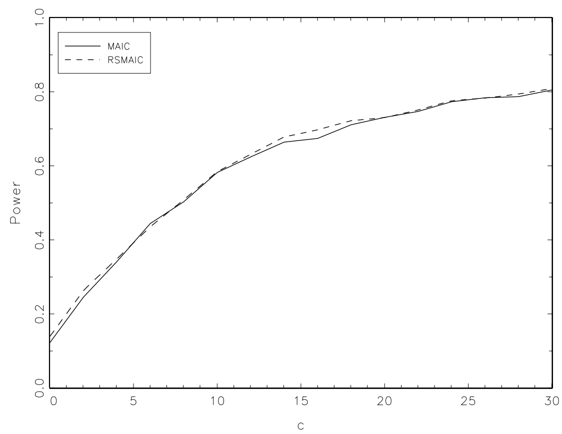
(c) $\delta = 1/3$, $\tau = 0.8$, ARMA model 4



(d) $\delta = 3$, $\tau = 0.2$, ARMA model 4

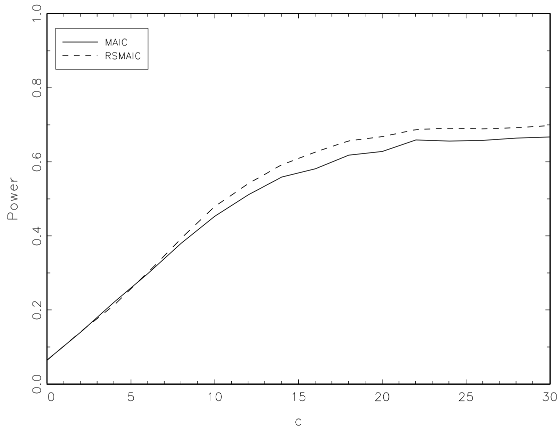


(e) $\delta = 1/3$, $\tau = 0.8$, ARMA model 12

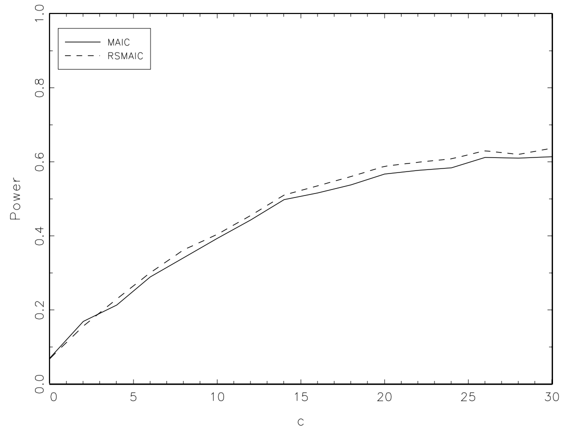


(f) $\delta = 3$, $\tau = 0.2$, ARMA model 12

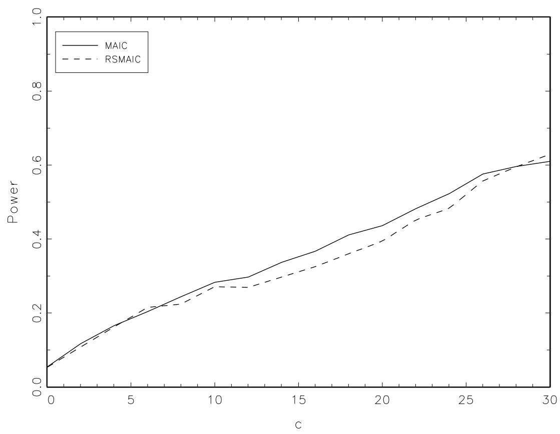
Figure 16: Power wild bootstrap ADF test with QD demeaning. Single break volatility model. $T = 50$.



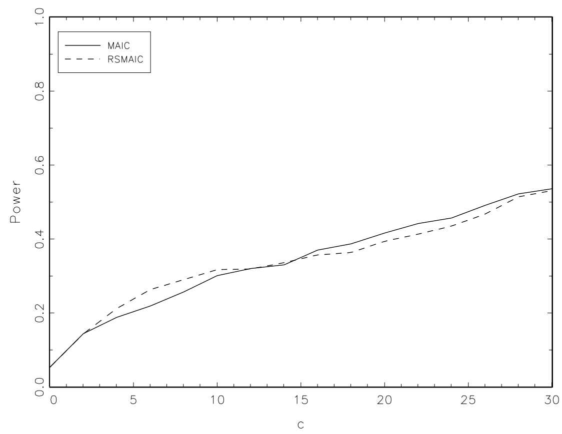
(a) $\tilde{c} = 0$, $\nu = 4$, ARMA model 1



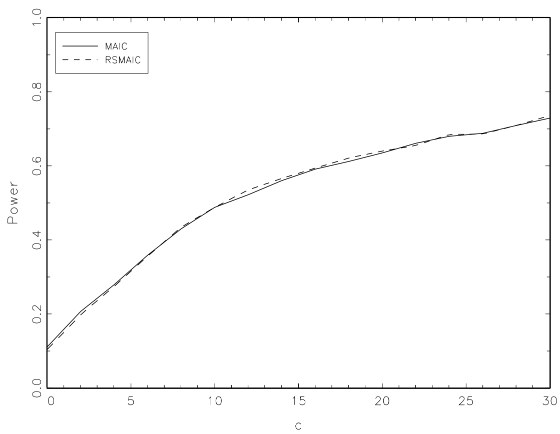
(b) $\tilde{c} = 0$, $\nu = 9$, ARMA model 1



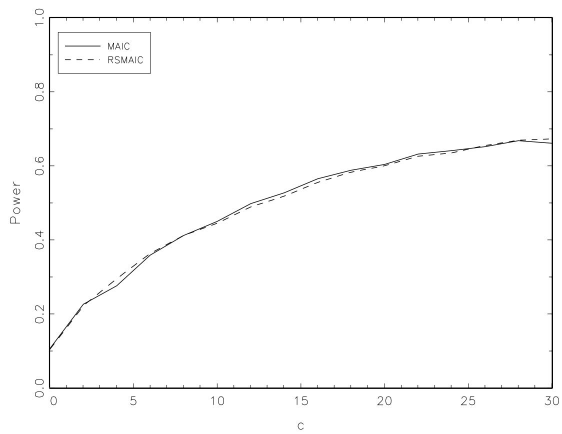
(c) $\tilde{c} = 0$, $\nu = 4$, ARMA model 4



(d) $\tilde{c} = 0$, $\nu = 9$, ARMA model 4



(e) $\tilde{c} = 0$, $\nu = 4$, ARMA model 12



(f) $\tilde{c} = 0$, $\nu = 9$, ARMA model 12

Figure 17: Power wild bootstrap ADF test with QD demeaning. Stochastic volatility model. $T = 50$.