Introduction to quantum electrodynamics (QED)

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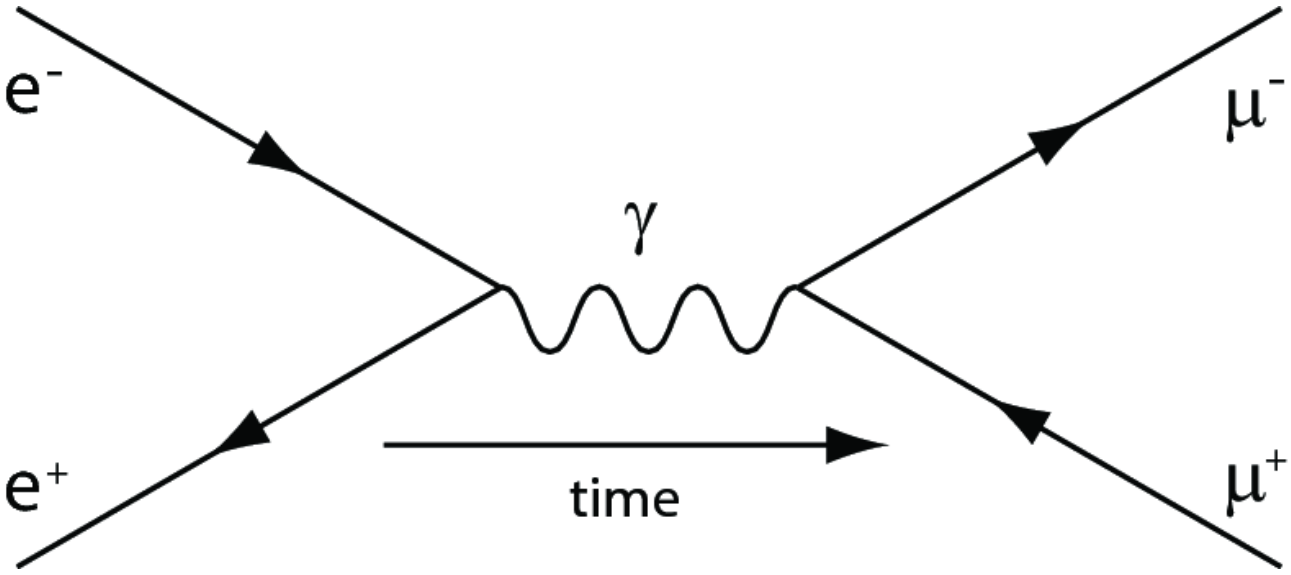

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Contents

Chapter 1
Introduction to Quantum Electro-Dynamics (QED)

In this chapter we are going to review the first gauge field theory of the Standard Model, the one that describes electromagnetic interactions, called Quantum Electro-Dynamics or QED. We will first derive the QED Lagrangian density. For this, we will profit from the knowledge we obtained from the chapter **??**. We will start from the Dirac Lagrangian that describes the dynamics of a free particle. We will then impose a transformation, similar to a difference in phase, described by the $\mathrm{U}(1)$ group. The requirement of local gauge invariance will automatically lead us to the term of the Lagrangian that reflects the interactions between the free particle and the field. The kinetic term of this field will be derived from the Proca Lagrangian. Requiring local gauge invariance also for this, will reveal the nature of the QED field: one massless field, the photon.

We will then reformulate the Maxwell equation in terms of the field tensor, thus making the connection with knowledge obtained in previous years. In addition, we will see what is the strength of the interaction between the particles and the fields, discussing about the fine structure constant whose value changes, but not drastically, as a function of the momentum transfer of a process. We will also see some QED processes, describing them with the help of Feynman diagrams.

The chapter will be concluded with the discussion over a specific type of QED process, the elastic scattering of electrons off stationary nucleons. These processes allow us to quantify the internal structure of nucleons, embodied in what is referred to as the form factors.

1.1 The free Dirac equation

We start with the free Dirac Lagrangian for a fermion with spin-$1/2$. These fermions can be either leptons or quarks for the case of QED. The Lagrangian is given by:

$$\mathscr{L} = i\overline{\Psi}\gamma_\mu \partial^\mu \Psi - m\overline{\Psi}\Psi \tag{1.1.1}$$

We now consider a transformation of the form

$$U = e^{igA}, \tag{1.1.2}$$

where g is a constant and A is for the time being another constant. This type of transformation can be seen as a phase transformation of the eigen-function given by $\Psi' = e^{i\xi}\Psi$, where $\xi = gA$ an 1×1 matrix described by the $\mathrm{U}(1)$ group.

1.1.1 Global gauge invariance

Let us first discuss the case where the transformation is global i.e. Λ is not a function of space and time but rather a constant. Does this global transformation change the Lagrangian density? Or in other words, is the Lagrangian density of Eq. 1.1.1 invariant under these types of transformations?

$$\mathscr{L}' = i\overline{\Psi}'\gamma_\mu\partial^\mu\Psi' - m\overline{\Psi}'\Psi'$$

$$= ie^{-ig\Lambda}\overline{\Psi}\gamma_\mu\partial^\mu\left(e^{ig\Lambda}\Psi\right) - me^{-ig\Lambda}\overline{\Psi}e^{ig\Lambda}\Psi$$

Since Λ does not depend on space and time, it can be taken out from the partial derivative in the first term, such that:

$$\mathscr{L}' = ie^{-ig\Lambda}e^{ig\Lambda}\overline{\Psi}\gamma_\mu\partial^\mu\Psi - me^{-ig\Lambda}e^{ig\Lambda}\overline{\Psi}\Psi$$

$$= i\overline{\Psi}\gamma_\mu\partial^\mu\Psi - m\overline{\Psi}\Psi \Rightarrow$$

$$\mathscr{L}' = \mathscr{L}$$

This is something that we could have expected: our system, described by the Lagrangian density of Eq. 1.1.1, is invariant under global $\mathrm{U}(1)$ (i.e. phase) transformations.

1.1.2 Local gauge invariance

Let us now promote the transformation of Eq. 1.1.2 from global to local. This can be done by adding a dependence of Λ on space and/or time such that $\Lambda = \Lambda(\mathbf{x},t) = \Lambda(x^\mu)$:

$$U = e^{ig\Lambda(x^\mu)}, \tag{1.1.3}$$

where g is still a constant and Λ this time is a scalar field. Let's now look at how the Lagrangian density of Eq. 1.1.1 transforms.

$$\mathscr{L}' = i\overline{\Psi}'\gamma_\mu\partial^\mu\Psi' - m\overline{\Psi}'\Psi'$$

$$= ie^{-ig\Lambda}\overline{\Psi}\gamma_\mu\partial^\mu\left(e^{ig\Lambda}\Psi\right) - me^{-ig\Lambda}\overline{\Psi}e^{ig\Lambda}\Psi$$

This time, Λ is not a constant but a scalar field i.e. a field whose value depends on space and time. That means that we can not just take it out from the partial derivative of the first term, but instead the derivative has to act on $\Lambda(x^\mu)$ (please note that in what follows the 4-vector in parenthesis will be suppressed, but that does not mean that it is not there!):

$$\mathscr{L}' = ie^{-ig\Lambda}\overline{\Psi}\gamma_\mu\left(\partial^\mu e^{ig\Lambda}\right)\Psi + ie^{-ig\Lambda}\overline{\Psi}\gamma_\mu e^{ig\Lambda}\left(\partial^\mu\Psi\right) - me^{-ig\Lambda}e^{ig\Lambda}\overline{\Psi}\Psi$$

$$= ie^{-ig\Lambda}\overline{\Psi}\gamma_\mu(ig)e^{ig\Lambda}\left(\partial^\mu\Lambda\right)\Psi + i\overline{\Psi}\gamma_\mu\partial^\mu\Psi - m\overline{\Psi}\Psi$$

$$= -g\overline{\Psi}\gamma_\mu\left(\partial^\mu\Lambda\right)\Psi + i\overline{\Psi}\gamma_\mu\partial^\mu\Psi - m\overline{\Psi}\Psi$$

$$= -g\overline{\Psi}\gamma_\mu\Big(\partial^\mu\Lambda\Big)\Psi + \mathscr{L} \neq \mathscr{L}$$

It is thus clear that the Lagrangian does not remain invariant under this local gauge transformation. The first term in the equation above is responsible for this, with the underlying reason being that the exponent that describes the transformation can not be taken out from the partial derivative: ∂^μ has to act on $e^{ig\Lambda(x^\mu)}$ since now Λ is not a constant but a scalar field that depends on x^μ.

So the first term of the Lagrangian density of Eq. 1.1.1, clearly introduces a problem and the Lagrangian is not anymore invariant under local gauge transformations. As illustrated above, the problem is related to the fact that the partial derivative has to act on the components of the transformation now, contrary to the case of the global gauge transformation, since this time $\Lambda = \Lambda(x^\mu)$. Note that in what follows, the dependence on x^μ will be suppressed but it is still there. Let's look a bit closer at the responsible term:

$$\partial^\mu\Psi' = \partial^\mu\Big(e^{ig\Lambda}\Psi\Big) = e^{ig\Lambda}\partial^\mu\Psi + ig\Big(\partial^\mu\Lambda\Big)\Psi = e^{ig\Lambda}\Big[\partial^\mu + ig\Big(\partial^\mu\Lambda\Big)\Big]\Psi \neq e^{ig\Lambda}\partial^\mu\Psi$$

So how about trying to absorb the term $\Big[\partial^\mu + ig\Big(\partial^\mu\Lambda\Big)\Big]$ which is responsible for breaking the invariance into a new quantity that will be built for the exact purpose of restoring the desired invariance of the Lagrangian density?

For this reason we introduce the covariant derivative d^{mu} constructed in a way to have the desired property $D^{\mu'}\Psi' = e^{ig\Lambda}D^\mu\Psi$. The term responsible for breaking the invariance of the Lagrangian density, $\Big[\partial^\mu + ig\Big(\partial^\mu\Lambda\Big)\Big]$, motivates the form of the covariant derivative:

$$D^\mu \equiv \partial^\mu + igA^\mu, \tag{1.1.4}$$

where A^μ is an external vector field. So from now on, we need to replace in the Lagrangian density the term $\partial^\mu\Psi$ with $D^\mu\Psi$. The latter provides us with the feature of $D^{\mu'}\Psi' = e^{ig\Lambda}D^\mu\Psi$.

$$D^{\mu'}\Psi' = (\partial^\mu + igA^{\mu'})(e^{ig\Lambda}\Psi) = \partial^\mu(e^{ig\Lambda}\Psi) + igA^{\mu'}e^{ig\Lambda}\Psi$$

$$= e^{ig\Lambda}\partial^\mu\Psi + ige^{ig\Lambda}\Big(\partial^\mu\Lambda\Big)\Psi + ige^{ig\Lambda}A^{\mu'}\Psi$$

The other part of the equality gives:

$$e^{ig\Lambda}D^\mu\Psi = e^{ig\Lambda}(\partial^\mu + igA^\mu)\Psi = e^{ig\Lambda}\partial^\mu\Psi + ige^{ig\Lambda}A^\mu\Psi$$

It is clear that the first terms of the two equations above are the same. That means that in order for the equality $D^{\mu'}\Psi' = e^{ig\Lambda}D^\mu\Psi$ to be valid, the external vector field needs to transform according to:

$$ige^{ig\Lambda}\Big(\partial^\mu\Lambda\Big)\Psi + ige^{ig\Lambda}A^{\mu'}\Psi = ige^{ig\Lambda}A^\mu\Psi \Rightarrow$$

$$A^{\mu'} = A^\mu - \partial^\mu\Lambda \tag{1.1.5}$$

1.1.3 The interaction term

The Dirac Lagrangian density should now be modified to account for the transition from the normal partial derivative to the covariant derivative that ensures the invariance under local gauge transformations.

$$\mathscr{L} = i\overline{\Psi}\gamma_\mu\partial^\mu\Psi - m\overline{\Psi}\Psi \rightarrow i\overline{\Psi}\gamma_\mu D^\mu\Psi - m\overline{\Psi}\Psi$$

$$\mathscr{L} = i\overline{\Psi}\gamma_\mu\Big(\partial^\mu + igA^\mu\Big)\Psi - m\overline{\Psi}\Psi$$

$$= i\overline{\Psi}\gamma_\mu\partial^\mu\Psi - g\overline{\Psi}\gamma_\mu A^\mu\Psi - m\overline{\Psi}\Psi$$

$$\mathscr{L}_{\text{QED}}^{\text{partial}} = \overline{\Psi}(i\gamma_\mu\partial^\mu - m)\Psi - g\overline{\Psi}\gamma_\mu A^\mu\Psi \tag{1.1.6}$$

The first term of Eq. 1.1.6 is the standard Lagrangian density of the free particle i.e. the Dirac Lagrangian density. The second term that was introduced, involves both the spinors $\overline{\Psi}$ and Ψ but also the external vector field A^μ. This is the interaction term! This term introduced by the requirement of local gauge invariance, reflects the interactions between the particles (leptons or quarks in this case) with the external field.

1.2 The field kinetic term

So far we have described our theory with a component of the Lagrangian density that describes the free particle (i.e. a lepton or a quark) and with the component that reflects the interaction between the particle and the external field. To make our theory complete, we need to involve the term that describes the field itself, also known as kinetic term of the theory. For this we have to go to the Lagrangian density that describes spin-1 fields i.e. the Proca Lagrangian described in Section **??**:

$$\mathscr{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2A_\mu A^\mu, \tag{1.2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor of the theory. It is also important to note that in this equation, M is the mass of the vector field (i.e. the gauge boson of our theory), whereas in Eq. 1.1.1 m corresponds to the mass of the particle that feels the field i.e. the lepton or the quark in this case.

Let us now apply the transformation of Eq. 1.1.3 and thus the transformation of the external field given by Eq. 1.1.5 in Eq. 1.2.1:

$$\mathscr{L}' = -\frac{1}{4}F'_{\mu\nu}F^{\mu\nu'} + \frac{1}{2}M^2A'_\mu A^{\mu'}$$

$$= -\frac{1}{4}(\partial_\mu A'_\nu - \partial_\nu A'_\mu)(\partial^\mu A^{\nu'} - \partial^\nu A^{\mu'}) + \frac{1}{2}M^2A'_\mu A^{\mu'}$$

$$= -\frac{1}{4}[\partial_\mu A_\nu - \partial_\mu(\partial_\nu\Lambda) - \partial_\nu A_\mu + \partial_\nu(\partial_\mu\Lambda)][\partial^\mu A^\nu - \partial^\mu(\partial^\nu\Lambda) - \partial^\nu A^\mu + \partial^\nu(\partial^\mu\Lambda)] + \frac{1}{2}M^2(A_\mu - \partial_\nu\Lambda)(A^\mu - \partial^\nu\Lambda)$$

$$= -\frac{1}{4}(\partial_\mu A_\nu)(\partial^\mu A^\nu) + \frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)(\partial^\mu A^\nu)] + \frac{1}{4}[(\partial_\nu A_\mu)(\partial^\mu A^\nu) - \frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)(\partial^\mu A^\nu)]$$

$$+\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu(\partial^\nu\Lambda)] - \frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)\partial_\mu(\partial_\nu\Lambda)] - \frac{1}{4}[\partial_\nu A_\mu\partial^\mu(\partial^\nu\Lambda)] + \frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)\partial^\mu(\partial^\nu\Lambda)]$$

$$+\frac{1}{4}(\partial_\mu A_\nu)(\partial^\nu A^\mu) - \frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)(\partial^\nu A^\mu)] - \frac{1}{4}[(\partial_\nu A_\mu)(\partial^\nu A^\mu) + \frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)(\partial^\nu A^\mu)]$$

$$-\frac{1}{4}[(\partial_\mu A_\nu)\partial^\nu(\partial^\mu\Lambda)]+\frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)\partial_\nu(\partial_\mu\Lambda)]+\frac{1}{4}[\partial_\nu A_\mu\partial^\nu(\partial^\mu\Lambda)]-\frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)\partial^\nu(\partial^\mu\Lambda)]$$

$$+\frac{1}{2}M^2[A_\mu A^\mu-(\partial_\nu\Lambda)A^\mu-A_\mu(\partial^\nu\Lambda)+(\partial_\nu\Lambda)(\partial^\nu\Lambda)]$$

$$=-\frac{1}{4}(\partial_\mu A_\nu)(\partial^\mu A^\nu)+\frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)(\partial^\mu A^\nu)]+\frac{1}{4}[(\partial_\nu A_\mu)(\partial^\mu A^\nu)-\frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)(\partial^\mu A^\nu)]$$

$$+\frac{1}{4}[(\partial_\mu A_\nu)\partial^\mu(\partial^\nu\Lambda)]-\frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)\partial_\mu(\partial_\nu\Lambda)]-\frac{1}{4}[\partial_\nu A_\mu\partial^\mu(\partial^\nu\Lambda)]+\frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)\partial^\mu(\partial^\nu\Lambda)]$$

$$+\frac{1}{4}(\partial_\mu A_\nu)(\partial^\nu A^\mu)-\frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)(\partial^\nu A^\mu)]-\frac{1}{4}[(\partial_\nu A_\mu)(\partial^\nu A^\mu)+\frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)(\partial^\nu A^\mu)]$$

$$-\frac{1}{4}[(\partial_\mu A_\nu)\partial^\nu(\partial^\mu\Lambda)]+\frac{1}{4}[\partial_\mu(\partial_\nu\Lambda)\partial_\nu(\partial_\mu\Lambda)]+\frac{1}{4}[\partial_\nu A_\mu\partial^\nu(\partial^\mu\Lambda)]-\frac{1}{4}[\partial_\nu(\partial_\mu\Lambda)\partial^\nu(\partial^\mu\Lambda)]$$

$$+\frac{1}{2}M^2[A_\mu A^\mu-(\partial_\nu\Lambda)A^\mu-A_\mu(\partial^\nu\Lambda)+(\partial_\nu\Lambda)(\partial^\nu\Lambda)]$$

$$=-\frac{1}{4}\Big[(\partial_\mu A_\nu)(\partial^\mu A^\nu)-(\partial_\nu A_\mu)(\partial^\mu A^\nu)-(\partial_\mu A_\nu)(\partial^\nu A^\mu)+(\partial_\nu A_\mu)(\partial^\nu A^\mu)\Big]$$

$$+\frac{1}{2}M^2A_\mu A^\mu+\frac{1}{2}M^2[-(\partial_\nu\Lambda)A^\mu-A_\mu(\partial^\nu\Lambda)+(\partial_\nu\Lambda)(\partial^\nu\Lambda)]$$

$$=-\frac{1}{4}\Big(\partial_\mu A_\nu-\partial_\nu A_\mu\Big)\Big(\partial^\mu A^\nu-\partial^\nu A^\mu\Big)+\frac{1}{2}M^2A_\mu A^\mu$$

$$+\frac{1}{2}M^2[-(\partial_\nu\Lambda)A^\mu-A_\mu(\partial^\nu\Lambda)+(\partial_\nu\Lambda)(\partial^\nu\Lambda)]$$

$$=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\frac{1}{2}M^2A_\mu A^\mu$$

$$+\frac{1}{2}M^2[-(\partial_\nu\Lambda)A^\mu-A_\mu(\partial^\nu\Lambda)+(\partial_\nu\Lambda)(\partial^\nu\Lambda)]$$

It is clear that the first term in the equation above gives the initial Lagrangian density. The second part is now responsible for breaking the invariance after the imposed transformation. Our goal though is to impose local gauge invariance also for the kinetic term, that means that we have to find a way to make this happen. The easiest way to do so, is to set the mass of the field M to zero. Doing so, the second part of the equation above, together with the term $\frac{1}{2}M^2A_\mu A^\mu$ vanish. Not only is the invariance recovered, but we also identified that our external field is massless, it's the photon!!!

$$\mathscr{L}'=-\frac{1}{4}F'_{\mu\nu}F^{\mu\nu'}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}=\mathscr{L}$$

Our theory is now complete! The complete Lagrangian density takes the following form:

$$\mathscr{L}_{\text{QED}}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\overline{\Psi}(i\gamma_\mu\partial^\mu-m)\Psi-g\overline{\Psi}\gamma_\mu A^\mu\Psi \tag{1.2.2}$$

The theory contains the term that describes the free Dirac particle, the second term in Eq. 1.2.2 and the term that describes the interactions between quarks or leptons and the external massless field, the third term in Eq. 1.2.2. Finally, it contains the kinetic term of the field, first term in Eq. 1.2.2, where $F_{\mu\nu}$ is the field tensor given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.2.3}$$

One important point worth making at this stage is that the definition of the field tensor given above is not gauge invariant. It works for QED, which is based on the U(1) Abelian group, but for other gauge theories of the Standard Model (e.g. QCD that is described by the SU(3) group) which are non-Abelian (i.e. the elements of the group do not commute) one has to derive the relevant field tensor using a definition which is gauge invariant. This can be done using the commutator of the covariant derivatives, defined in Eq. 1.1.4 as shown below:

$$[D_\mu, D_\nu]\Psi = (\partial_\mu + igA_\mu)(\partial_\nu + igA_\nu)\Psi - (\partial_\nu + igA_\nu)(\partial_\mu + igA_\mu)\Psi$$

$$= \partial_\mu(\partial_\nu\Psi) + ig\partial_\mu(A_\nu\Psi) + igA_\mu(\partial_\nu\Psi) - g^2A_\mu A_\nu\Psi$$

$$-\partial_\nu(\partial_\mu\Psi) - ig\partial_\nu(A_\mu\Psi) - igA_\nu(\partial_\mu\Psi) + g^2A_\nu A_\mu\Psi$$

$$= \partial_\mu(\partial_\nu\Psi) + ig\partial_\mu(A_\nu\Psi) + igA_\mu(\partial_\nu\Psi) - g^2A_\mu A_\nu\Psi$$

$$-\partial_\nu(\partial_\mu\Psi) - ig\partial_\nu(A_\mu\Psi) - igA_\nu(\partial_\mu\Psi) + g^2A_\nu A_\mu\Psi$$

$$= ig\Big(\partial_\mu A_\nu - \partial_\nu A_\mu\Big)\Psi \Rightarrow$$

$$[D_\mu, D_\nu] = igF_{\mu\nu} \tag{1.2.4}$$

1.2.1 Maxwell equations revisited

Doing all these exercises and in some cases lengthy calculations, we ended up deriving the QED Lagrangian density from first principles. Equation 1.2.2 describes electromagnetism and it is (to first order) all you need to know to explain the electromagnetic effects you witness in everyday's life. However, at first glance this is not the form of electromagnetism we were taught either in our high school or early university years. Back then, we used to deal with electromagnetic effects via the well known Maxwell equations:

- **Gauss law:** The Gauss law for the electric field reflects the fact that the electric charge leaving a volume is proportional to the total charge enclosed in a surface and it is expressed by

$$\nabla\mathbf{E} = \rho \tag{1.2.5}$$

- **Faraday-Lenz law:** The voltage induced in a closed circuit is proportional to the rate of change of the magnetic field and it is expressed by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.2.6}$$

- **Gauss law for B:** The total magnetic charge through a closed surface is zero (i.e. there are no magnetic monopoles) and it is expressed by

$$\nabla \mathbf{B} = 0 \tag{1.2.7}$$

- **Ampere law:** The magnetic field induced around a closed loop is proportional to the electric current and the rate of change of the electric field it encloses and it is expressed by

$$\nabla \times \mathbf{B} = \mathbf{J} + \frac{\partial \mathbf{E}}{\partial t} \tag{1.2.8}$$

Vector Calculus Identities Let us now open a small parenthesis and remind ourselves some of the vector calculus identities that we might find useful in our next steps:

- **Curl of a gradient:** $\nabla \times (\nabla \cdot \Lambda) = \mathbf{0}$
- **Divergence of curl:** $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- **Divergence of gradient:** $\nabla^2 \Lambda = \nabla \cdot (\nabla \Lambda)$
- **Curl of a curl:** $\nabla \times (\nabla \times \mathbf{A}) = \nabla \cdot (\nabla \cdot \mathbf{A}) = \nabla^2 \mathbf{A}$

Let us now go back to the Maxwell equations. If we take the divergence of the Ampere law then we get:

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \mathbf{J} + \nabla \Big(\frac{\partial \mathbf{E}}{\partial t} \Big)$$

and since the divergence of the curl is 0 i.e. $\nabla \cdot (\nabla \times \mathbf{B}) = 0$, then we have:

$$0 = \nabla \mathbf{J} + \frac{\partial}{\partial t} (\nabla \mathbf{E}) \Rightarrow$$

$$\frac{\partial \rho}{\partial t} + \nabla \mathbf{J} = 0 \tag{1.2.9}$$

where $\mathbf{J}$ and ρ are the current and charge densities, respectively. Both in the classical and in its quantum mechanical formulation, electromagnetism profits from the introduction of a vector field $A^\mu(x^\mu) = (A_0, \mathbf{A})$ that replaces the physical fields $\mathbf{E}$ and $\mathbf{B}$ according to:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\mathbf{E} = -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t}$$

It is important to note that the physical fields are always the $\mathbf{E}$ and $\mathbf{B}$ while the scalar and vector potentials A_0 and $\mathbf{A}$ respectively (i.e. the components of the vector field A^μ) help with the mathematical formulation of the theory.

Now that we have introduced this new vector field, let's go back to the Maxwell equations and see how we can rewrite them in such a way to eliminate the appearance of $\mathbf{E}$ and $\mathbf{B}$ which should now be replaced by A^μ. Let's first start with the Gauss law for the magnetic field and see if it needs to be rewritten:

$$\nabla \mathbf{B} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$$

since based on one of the identities above $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. That means that the Gauss law for $\mathbf{B}$ is automatically transformed to a form that does not include either $\mathbf{E}$ or $\mathbf{B}$ but only includes the external vector field A^μ without changing anything.

Let us now go to the Faraday-Lenz law that can be written according to:

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}\Rightarrow\nabla\times\Big(-\nabla A_0-\frac{\partial\mathbf{A}}{\partial t}\Big)=-\frac{\partial}{\partial t}\Big(\nabla\times\mathbf{A}\Big)\Rightarrow$$

$$\nabla\times(\nabla\cdot A_0)-\nabla\times\Big(\frac{\partial\mathbf{A}}{\partial t}\Big)=-\frac{\partial}{\partial t}\Big(\nabla\times\mathbf{A}\Big)\Rightarrow$$

$$0-\frac{\partial}{\partial t}\Big(\nabla\times\mathbf{A}\Big)=-\frac{\partial}{\partial t}\Big(\nabla\times\mathbf{A}\Big)$$

That means that both the Gauss low for the magnetic field and the Lenz-Faraday law are satisfied automatically, without us having to rewrite them in any given way! We will try to do the same with the other two equations as well i.e. the Gauss law for the electric field and the Ampere law. Before doing so, it is better if we introduce at this stage the current 4-vector, defined as $j_\mu=(\rho,-\mathbf{J})$.

The Gauss law for the electric field can now be written as:

$$\nabla\mathbf{E}=\rho\Rightarrow\nabla\Big(-\nabla A_0-\frac{\partial\mathbf{A}}{\partial t}\Big)=\rho\Rightarrow-\nabla^2A_0-\nabla\Big(\frac{\partial\mathbf{A}}{\partial t}\Big)=\rho\Rightarrow$$

$$-\nabla^2A_0+\frac{\partial^2A_0}{\partial t^2}-\frac{\partial^2A_0}{\partial t^2}-\nabla\Big(\frac{\partial\mathbf{A}}{\partial t}\Big)=\rho\Rightarrow\Big(\frac{\partial^2}{\partial t^2}-\nabla^2\Big)A_0-\frac{\partial}{\partial t}\Big(\frac{\partial A_0}{\partial t}\Big)-\nabla\Big(\frac{\partial\mathbf{A}}{\partial t}\Big)=\rho\Rightarrow$$

$$\Big(\frac{\partial^2}{\partial t^2}-\nabla^2\Big)A_0-\frac{\partial}{\partial t}\Big(\frac{\partial A_0}{\partial t}\Big)-\frac{\partial}{\partial t}\Big(\nabla\mathbf{A}\Big)=j^0\Rightarrow$$

$$\partial_\mu\partial^\mu A_0-\partial_0\Big(\partial_\mu A^\mu\Big)=j^0 \tag{1.2.10}$$

At this stage it is important to remind the reader that $\partial^\mu=(\partial_0,-\nabla)$ and $\partial_\mu=(\partial_0,\nabla)$.

Let's now look at the Ampere law:

$$\nabla\times\mathbf{B}=\mathbf{J}+\frac{\partial\mathbf{E}}{\partial t}\Rightarrow\nabla\times\Big(\nabla\times\mathbf{A}\Big)=\mathbf{J}+\frac{\partial}{\partial t}\Big(-\nabla A_0-\frac{\partial\mathbf{A}}{\partial t}\Big)$$

$$\nabla\cdot\Big(\nabla\mathbf{A}\Big)-\nabla^2\mathbf{A}+\frac{\partial}{\partial t}\Big(\nabla A_0\Big)+\frac{\partial^2\mathbf{A}}{\partial t^2}=\mathbf{J}$$

$$\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla^2\mathbf{A}+\nabla\Big(\frac{\partial A_0}{\partial t}+\nabla\mathbf{A}\Big)=\mathbf{J}\Rightarrow$$

$$\Big(\frac{\partial^2}{\partial t^2}-\nabla^2\Big)\mathbf{A}-(-\nabla)\Big(\frac{\partial A_0}{\partial t}+\nabla\mathbf{A}\Big)=\mathbf{J}\Rightarrow$$

$$\partial_\mu\partial^\mu A^{123}-\partial^{123}\Big(\partial_\mu A^\mu\Big)=j^{123} \tag{1.2.11}$$

where the indices 123 represent the "spacial" component of the relevant 4-vector. If one looks carefully at Eq. 1.2.10 and Eq. 1.2.11, then one can notice that one is supplementary of the other in terms of indices. If combined and written in a compact way, they take the following form:

$$\partial_\mu\partial^\mu A^\nu-\partial^\nu\Big(\partial_\mu A^\mu\Big)=j^\nu\Rightarrow\partial_\mu\Big(\partial^\mu A^\nu-\partial^\nu A^\mu\Big)=j^\nu\Rightarrow$$

$$\partial_\mu F^{\mu\nu} = j^\nu \tag{1.2.12}$$

Thus, the Maxwell equations can now be written in a rather compact and elegant way depicted by Eq. 1.2.12.

As a final comment on this section, it is instructive to make the connection between the electromagnetic tensor $F_{\mu\nu}$ and the physical fields **E** and **B**. This is given by the components of the representation of the tensor as a matrix according to:

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

The Lorentz invariant quantity is given by

$$F_{\mu\nu}F^{\mu\nu} = 2(E^2 - B^2)$$

1.3 Some comments

At this stage, one probably needs a break to reflect what we had done and what does it mean. Let's start by making some remarks:

- Every term of the QED Lagrangian density of Eq. 1.2.2 corresponds and is described by a part of a Feynman diagram. The first part of it, the kinetic term of the field, describes the photon propagator and is represented by the left diagram of fig. 1.1. The second part, the one that described the free Dirac particle with mass m, is reflected by the middle diagrams of fig. 1.1. These two different lines correspond to a fermion (upper line) and an anti-fermion (lower line) moving forward, backward in time, respectively. Finally, the interaction term is represented by the right diagram of fig. 1.1. It describes the interaction between the field (i.e. photon, the curly line of the diagram), the incoming (i.e. Ψ, the left, incoming line) and the outgoing (i.e. $\overline{\Psi}$, the right, outgoing line) fermion.
- The factor g introduced in the transformation $U = e^{ig\Lambda(x^\mu)}$ was initially considered to be just a constant. It now appears to be of fundamental importance and the reason is that it shows up in the third term of the Lagrangian density of QED of Eq. 1.2.2 as a multiplicative factor to the interaction term. This constant actually reflects the strength of the interaction between the field and the fermions. It is also connected to the charge of the particles: if you have a neutral fermion (e.g. a neutrino) it will not feel any electromagnetic interaction, so $g = 0$ in this case. It turns out that this factor g is connected with the fine structure constant, the coupling strength α of QED. The connection between α, g and the electric charge is given by:

$$g \equiv q = \sqrt{4\pi\alpha} \tag{1.3.1}$$

- Every interaction term of the form $g\overline{\Psi}\gamma_\mu A^\mu \Psi$ corresponds to a vertex as the one indicated in fig. 1.1-(c). The strength of each interaction, g or $\sqrt{\alpha}$, is depicted on each such vertex.
- The fact that we requested for a local gauge invariance implies the conservation of a quantity which in this case is the electric charge (to be more precise it is the hypercharge) and thus the existence of a symmetry. This symmetry does not involve any change in the space (**x** or time (t) coordinates. Let's see what this operation that we performed is actually doing. The transformation that we perform looks like:

$$\Psi' = e^{ig\Lambda(x^\mu)}\Psi = [\cos(g\Lambda) + i\sin(g\Lambda)]\Psi$$

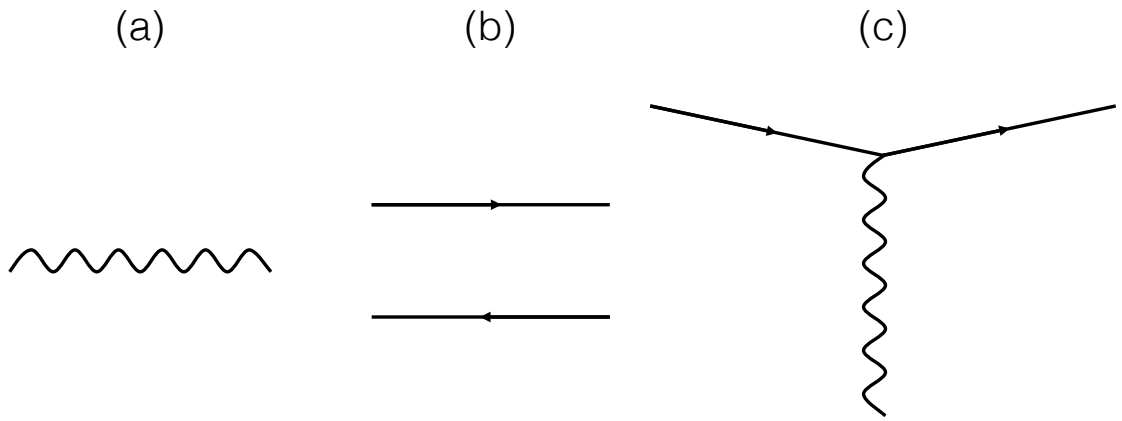

Fig. 1.1: The three basic parts of the QED Feynman diagrams as depicted in the QED Lagrangian of Eq. 1.2.2.

Both the new and the old eigen functions can be decomposed into their real and imaginary components, according to

$$\Psi^{'} = \Psi^{'}_{Re} + i\Psi^{'}_{Im}$$

$$\Psi = \Psi_{Re} + i\Psi_{Im}$$

The combination of these last three equations gives:

$$[\cos(g\Lambda) + i\sin(g\Lambda)](\Psi_{Re} + i\Psi_{Im}) = \Psi^{'}_{Re} + i\Psi^{'}_{Im} \Rightarrow$$

$$\Psi^{'}_{Re} + i\Psi^{'}_{Im} = [\Psi_{Re}\cos(g\Lambda) - \Psi_{Im}\sin(g\Lambda)] + i[\Psi_{Re}\sin(g\Lambda) + \Psi_{Im}\cos(g\Lambda)] \Rightarrow$$

$$\Psi^{'}_{Re} = \Psi_{Re}\cos(g\Lambda) - \Psi_{Im}\sin(g\Lambda)$$

$$\Psi^{'}_{Im} = \Psi_{Re}\sin(g\Lambda) + \Psi_{Im}\cos(g\Lambda)$$

The two equations above look like some kind of rotation. In fact they do describe rotations but not the ones we are used to! The equations describe rotations in the $\Psi_{Re} - \Psi_{|m}$ space as they act on the real and the imaginary parts of the wave function. This plane is called **internal space** and the symmetry is called **internal symmetry**.

- Let's do two successive transformation in the U(1) space that transform a wave function as $\Psi \to \Psi' \to \Psi''$, according to

$$\Psi' = e^{i\xi_1}\Psi$$

$$\Psi'' = e^{i\xi_2}\Psi' = e^{i\xi_2}e^{i\xi_1}\Psi = e^{i\xi_1}e^{i\xi_2}\Psi = e^{i(\xi_2+\xi_1)}\Psi = e^{i\xi}\Psi$$

with $\xi = \xi_1 + \xi_2$. These are transformations with 1×1 matrices (i.e. complex numbers) that form the U(1) group. The matrices are unitary since $UU^\dagger = U^\dagger U = I$. The electromagnetic gauge group is the one of U(1). The transformation of elements of U(1) have the property that the order of the transformation does not matter: $e^{i\xi_1}e^{i\xi_2} = e^{i(\xi_1+\xi_2)} = e^{i\xi_2}e^{i\xi_1}$ which is always true even if we apply a local gauge transformation i.e. $\xi_1(x^\mu$ and $\xi_2(x^\mu$. We say that U(1) is an Abelian group, with elements that commute.

- The origin of the gauge invariance in the case of the classical electromagnetic fields lies on the fact that the potentials A_0 and $\mathbf{A}$ are not unique for the physical fields $\mathbf{E}$ and $\mathbf{B}$. Let's illustrate that the two physical fields are indeed gauge invariant. We apply a gauge transformation in the external field of the form:

$$A^{\mu'} = A^\mu - \partial^\mu \Lambda$$

which means that

$$A^{0'} = A^0 - \frac{\partial \Lambda}{\partial t}$$

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda$$

The physical fields can now be written as

  - Electric field:

$$\mathbf{E}' = -\nabla A_0' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla\left(A^0 - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\mathbf{A} + \nabla\Lambda\right)$$

$$= -\nabla A_0 + \nabla\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\left(\nabla\Lambda\right) = -\nabla A_0 + \nabla\left(\frac{\partial \Lambda}{\partial t}\right) - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial}{\partial t}\left(\nabla\Lambda\right)$$

$$= -\nabla A_0 - \frac{\partial \mathbf{A}}{\partial t} \Rightarrow \mathbf{E}' = \mathbf{E}$$

  - Magnetic field:

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \left(\mathbf{A} + \nabla\Lambda\right) = \nabla \times \mathbf{A} + \nabla \times \left(\nabla\Lambda\right)$$

$$= \nabla \times \mathbf{A} + \nabla \times \left(\nabla\Lambda\right) = \nabla \times \mathbf{A} \Rightarrow \mathbf{B}' = \mathbf{B}$$

1.4 QED processes and Feynman diagrams

In Chapter **??**, we have discussed about the way one can calculate the cross-section of a process. We have also seen that this goes through the calculation of the so-called matrix element, M_{if}, that describes the transition from the initial to the

final state. We have also seen that the calculation of this matrix element is greatly facilitated if one follows a given set of rules, established by Feynman. These rules are based on a pictorial representation of a given interaction via the so-called Feynman diagrams. Below, we summarise the basic Feynman rules for QED for completeness and we then take a look at some of the basic elastic and inelastic processes. Note that the calculation of the matrix element of a process using the Feynman rules is beyond the scope of this course, this is why there is no such attempt here.

1.4.1 Feynman rules for QED processes

Let's now review how these general rules described before change in the case of an electromagnetic interaction.

- **Labeling:** We label every external line with the ingoing and outgoing momenta $\mathbf{P_1}$,..., $\mathbf{P_n}$, adding also an arrow indicating whether a particle is approaching or moving away from the vertex. If the diagram includes antiparticles, we still label them as particles but with the reverse direction of the arrow. We then label the 4-momenta for all internal lines $\mathbf{q_1}$,..., $\mathbf{q_j}$ and we give an arbitrary direction to the relevant arrow.
- **External lines:** Each external line contributes the following factors:

$$\text{Incoming electron} \rightarrow u$$
$$\text{Outgoing electron} \rightarrow \overline{u}$$

$$\text{Incoming positron} \rightarrow \overline{v}$$
$$\text{Outgoing positron} \rightarrow v$$

$$\text{Incoming photon} \rightarrow \varepsilon_\mu$$
$$\text{Outgoing photon} \rightarrow \varepsilon_\mu^*$$

  where u and v are the relevant Dirac spinors and ε_μ are the photon polarisation vectors which are orthogonal to the 4-momentum $\mathbf{P}_\nu \varepsilon^\nu = 0$.
- **Vertices:** For each vertex we note down in the diagram the coupling constant factor $-ig_\varepsilon$. This factor is connected to the coupling constant via the equation

$$g_\varepsilon = \sqrt{4\pi\alpha}$$

- **Propagators:** For each internal line, we give a factor of

$$e^\pm : \frac{i(\gamma_\mu \mathbf{q}^\mu + m)}{\mathbf{q}^2 - m^2}$$

$$\gamma : \frac{-ig_{\mu\nu}}{\mathbf{q}^2}$$

- **δ-functions and integration:** The remaining steps are identical as in the general rules described before.

Figure 1.2 presents the lines for the basic particles and anti-particles but also the propagators for the electromagnetic interactions.

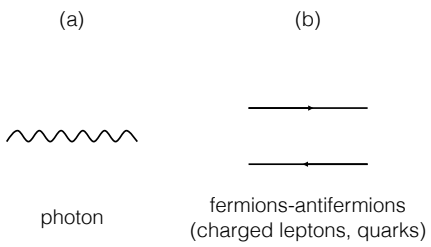

Fig. 1.2: The most characteristic lines for the Feynman diagrams in electromagnetic interactions.

1.4.2 Elastic processes

Let us now review some of the basic elastic QED processes. We start with the elastic electrom-muon scattering, depicted in fig. 1.3. The diagram indicates that an electron and a muon come in from left to right (i.e. along the time direction) with $\mathbf{P_1}$ and $\mathbf{P_2}$ 4-momenta. They interact by exchanging a photon. The outgoing particles are the same pair of electron and muon, that fly away from the vertices of the diagram with $\mathbf{P_3}$ and $\mathbf{P_4}$ 4-momenta.

Figure 1.4 presents the so-called Moeller scattering between two electrons. This process obviously involves two identical fermions, which means that one needs to consider not only the t- but also the u-channel that involves the crossed outgoing particles. The left diagram of fig. 1.4 presents the t-channel. In this case the two electrons, with 4-momenta $\mathbf{P_1}$ and $\mathbf{P_2}$, come in and they interact via the exchange of a photon. The same pair of electrons emerge with 4-momenta $\mathbf{P_3}$ and $\mathbf{P_4}$. In the u-channel, the same electron pair emerges in the final state, only this time their 4-momenta are crossed relative to the t-channel configuration.

Figure 1.5 presents the so-called Bhabha scattering between an electron and a positron. This process contains two contributions, one from the t-channel (i.e. scattering process - left diagram) and the other from the s-channel (i.e. annihilation process - right diagram). The left diagram of fig. 1.4 presents the t-channel. In this case the electron and the positron, with 4-momenta $\mathbf{P_1}$ and $\mathbf{P_2}$, come in and they interact via the exchange of a photon. The same pair of particles emerge with 4-momenta $\mathbf{P_3}$ and $\mathbf{P_4}$. In the s-channel, the same incoming electron-positron pair annihilate, giving rise to a photon, that in turns produces the outgoing electron-positron pair.

1.4.3 Inelastic processes

We now move to reviewing inelastic processes, such as the one depicted in fig. 1.6. The two diagrams present the annihilation process between an electron and a positron. In both diagrams, an electron and a positron pair annihilate via the exchange of a virtual electron. The final state particles, the two photons, can be seen emerging from the interaction, either on the t-channel diagram or in the u-one with 4-momenta $\mathbf{P_3}$ and $\mathbf{P_4}$. The latter, i.e. the u-channel diagram, is needed since we have two identical particles in the final state of the process.

The two diagrams of fig. 1.7 present the production of an electron and a positron from two photons. In both diagrams, the two photons interact via the exchange of a virtual electron. The final state particles, the electron and the positron, can be seen emerging from the interaction, either on the t-channel diagram or in the u-one with 4-momenta $\mathbf{P_3}$ and $\mathbf{P_4}$. The latter, i.e. the u-channel diagram, is needed since we have two identical particles in the initial state of the process.

The two diagrams of fig. 1.8 present the scattering of a photon off an electron, also known as Compton scattering. The process can be describes by either the t- or the s-channel diagrams, where in both cases the photon interacts with the electron via the exchange of a virtual electron. The emerging photon and electron have 4-momenta $\mathbf{P_3}$ and $\mathbf{P_4}$.

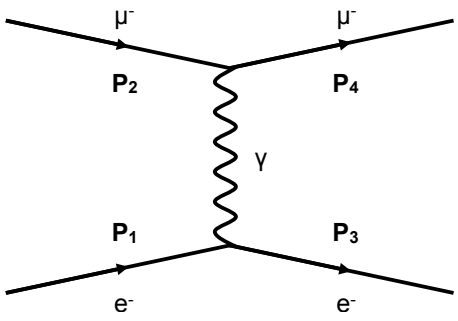

Fig. 1.3: The Feynman diagram describing the elastic scattering of electrons and muons.

1.5 The fine structure constant

In this section we will discuss in a bit more detail about the factor g introduced in the transformation $U = e^{ig\Lambda(x^\mu)}$. We have seen that this factor is of fundamental importance since it indicates the strength of the interaction. We also identified this factor as being related to the fine structure constant, the coupling strength α of QED. Before going into the details of the fine structure constant in terms of the field theory, let's open a small parenthesis and remind ourselves what is the reason we call α the fine structure constant. For this, we have to go back to atomic physics and spectroscopy.

1.5.1 Spectroscopy and the fine structure constant

The hydrogen atom consists of an electron and a proton. The dynamics of the system is described in it's non-relativistic regime by the Schroedinger equation, which gives us the wave function of the system. The proton, however, is so heavy (relatively) that it essentially just sits at the origin and the wave function in question is that of the electron. The potential energy, due to the electrical attraction of the nucleus, is (in Gaussian units)

$$V(r) = \frac{-e^2}{r}$$

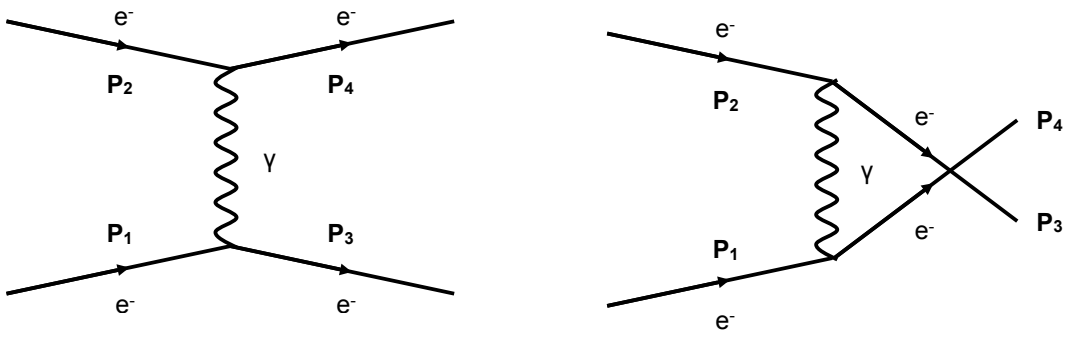

Fig. 1.4: The Feynman diagram describing the elastic Moeller scattering between two electrons.

When this potential is put into the radial Schroedinger equation, it is found that normalisable solutions occur only when E assumes one of the special values

$$E_n = -\frac{me^4}{2\hbar^2 n^2} = -\alpha^2 mc^2 \frac{1}{2n^2} = -\frac{13.6}{n^2}\ \text{eV}$$

In the previous equation, α is the fine structure constant and is given by

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}$$

The normalised wave function depends on three numbers:

- the principle quantum number n, which can be any positive integer and appears also in the quantised energy spectrum determining the energy of the state,
- the orbital angular momentum l, an integer which ranges from 0 up to $n-1$,
- and the z-component of the orbital angular momentum m_l, which can take values from $-l, -l+1, ...$ up to l.

Based on the previous, it is clear that there are $2l+1$ different m_l values for each I and n different l values for each n. The total number of distinct states that share the same principal quantum number n and hence the same energy is therefore

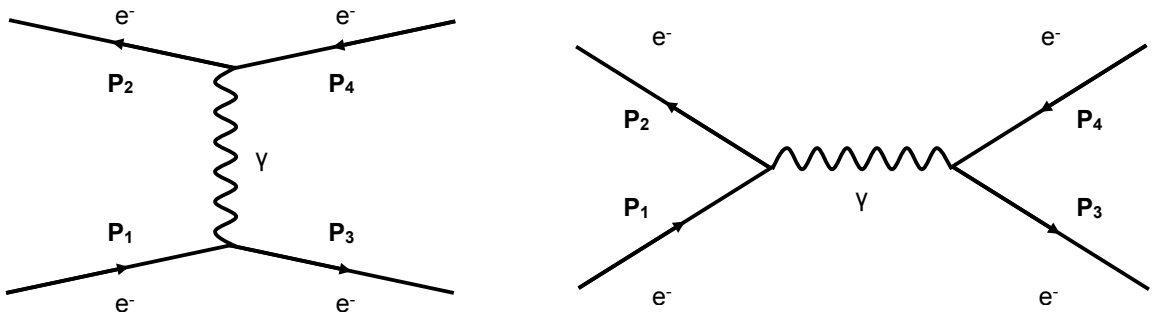

Fig. 1.5: The Feynman diagram describing the elastic Bhabha scattering between an electron and a positron.

$$\sum_{l=0}^{n-1}(2l+1)=n^2$$

This is called the degeneracy of the n^{th} energy level.

In the laboratory, we do not measure the energies themselves but rather the wavelength of the light emitted when the electron makes a transition from a higher level to a lower one. The photon carries the difference in energy between the initial and final states according to the Planck formula

$$E_{\text{photon}}=h\nu=E_{\text{initial}}-E_{\text{final}}=-\frac{me^4}{2\hbar^2}\Big(\frac{1}{n_i^2}-\frac{1}{n_f^2}\Big)$$

The emitted photon thus carries a wavelength given by the Rydberg formula

$$\frac{1}{\lambda}=R\Big(\frac{1}{n_i^2}-\frac{1}{n_f^2}\Big)$$

where $R=me^4c/(4\pi\hbar^3)$.

As the precision of experimental spectroscopy improved, small departures from the Rydberg formula were detected. Spectral lines were resolved into doublets, triplets, and even larger families of closely spaced peaks. This **fine structure** is actually attributable to two distinct mechanisms:

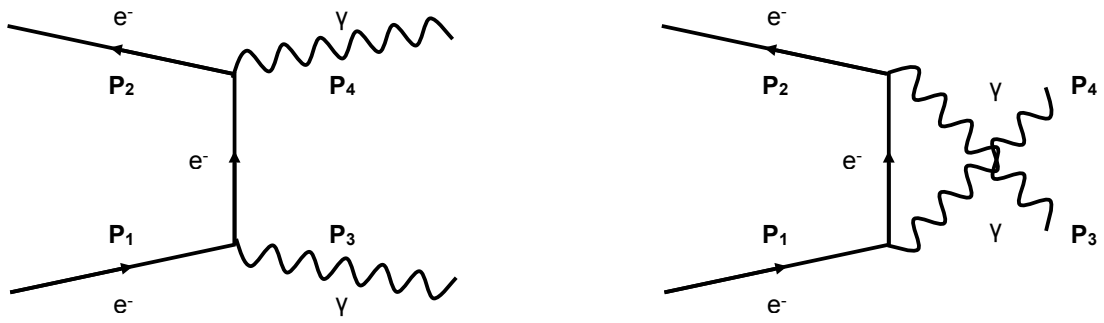

Fig. 1.6: The Feynman diagram describing the annihilation process between an electron and a positron.

- a small relativistic correction which results into a term that reads

$$\Delta E_{\text{relat. corr.}} = \alpha^4 mc^2 \frac{1}{4n^2}\left(\frac{2n}{l+1/2} - \frac{3}{2}\right)$$

- A magnetic coupling between the electron's spin and its orbital motion which also results into a correction of a similar form as before:

$$\Delta E_{\text{spin}} = \alpha^4 mc^2 \frac{1}{4n^2}\left(\frac{2n}{j+1/2} - \frac{3}{2}\right)$$

Both corrections are proportional to α^4 i.e. two more powers of α than the standard energy levels of the different n-states that are proportional to α^2. Thus, these corrections are smaller by a factor of about 10^{-4}. The fine structure constant, α, owes its name to the fact that it sets the relative scale of the fine structure in hydrogen.

1.5.2 The fine structure constant reconsidered

In QED, a charged particle like the electron is surrounded by a cloud of virtual photons and e^+e^- pairs continuously popping in and out of existence. Because of the attraction of opposite charges, the virtual positrons tend to be closer to

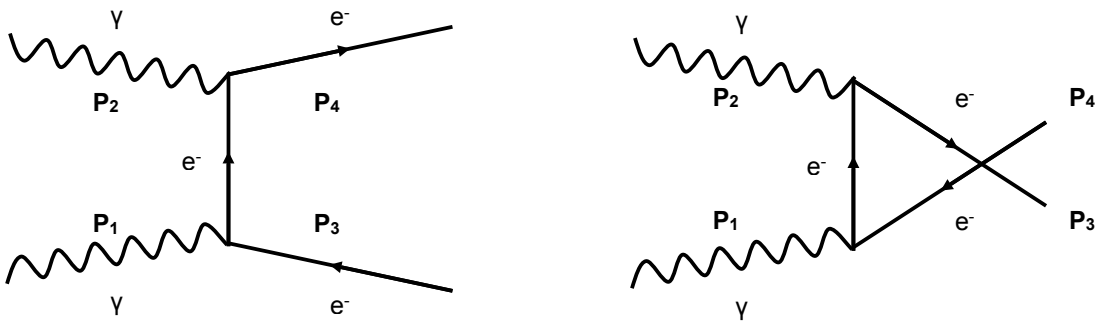

Fig. 1.7: The Feynman diagram describing the production of an electron and a positron from two photons.

the electron and screen the electron charge, as is indicated in fig. 1.9. This is analogous to the polarisation of a dielectric medium in the presence of a charge and is called **vacuum polarisation**.

This gives rise to the notion of an effective charge $e(r)$ that becomes smaller with larger distance. One says that the β-function is positive in QED:

$$\beta(r) \equiv -\frac{\mathrm{d}\, e(r)}{\mathrm{d}\ln r}$$

Let us first consider the process $e^- + \mu^- \to e^- + \mu^-$. The Feynman diagram for this process can be seen in the left plot of fig. 1.10, which actually constitutes the lower order diagram. The matrix element can be calculated using the Feynman rules of QED (we will not attempt to go through the derivation in this document since this goes beyond the scope of the lectures). This leads to the relevant cross-section whose accuracy is good up to $O(\alpha^2)$ and is therefore an approximate perturbative result. To improve further the accuracy though, one needs to include higher order diagrams, an example of which is given in the right plot of fig. 1.10. In this diagram, it is seen that the virtual photon spends some time as a virtual e^+e^- pair. Let us now try to evaluate the matrix element for this process, which is quite similar to the one for the lower order diagram indicated above, with the only part that changes being the one that is related to the loop. The calculation of the matrix element goes through similar QED Feynman rules and leads to the extraction of the cross-section of the relevant process with higher accuracy. This matrix element this time contains terms that diverge. This divergence is absorbed by imposing a cutoff M in a process which is called renormalisation.

The Q^2 evolution of the bare coupling constant is thus given by

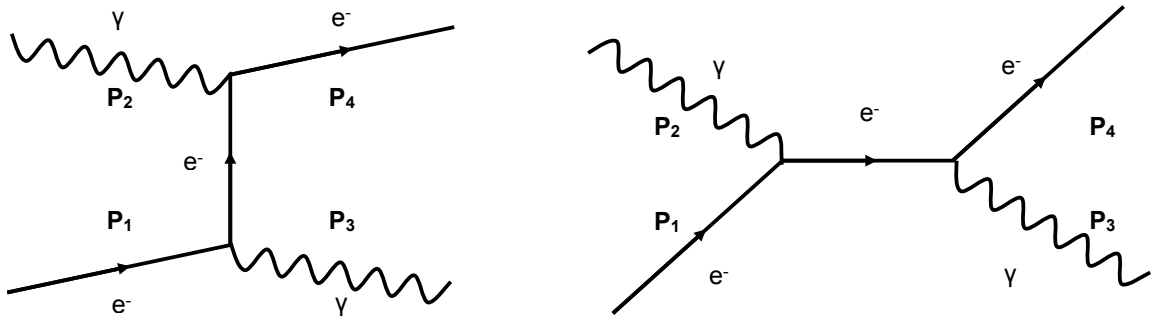

Fig. 1.8: The Feynman diagram describing the Compton scattering between an electron and a photon.

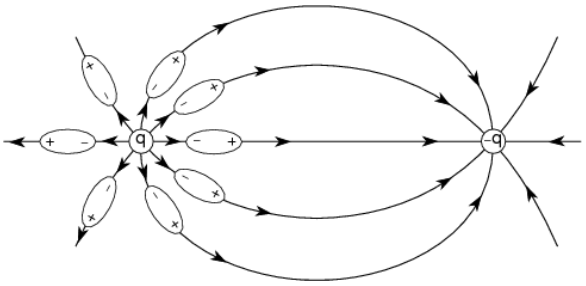

Fig. 1.9: The photon vacuum polarisation (left) generates a charge screening effect, making α smaller at larger distances.

$$g_{\mathrm{e}}^2 \to g_{\mathrm{e}}^2 \left\{ 1 - \frac{g_{\mathrm{e}}^2}{12\pi^2} \left[\ln\left(\frac{M^2}{m_{\mathrm{e}}^2}\right) - f\left(\frac{Q^2}{m_{\mathrm{e}}^2}\right) \right] \right\}$$

The first term is called the renormalised coupling constant

$$g_0^2 = g_{\mathrm{e}}^2 \left[1 - \frac{g_{\mathrm{e}}^2}{12\pi^2} \ln\left(\frac{M^2}{m_{\mathrm{e}}^2}\right) \right]$$

so that we may write

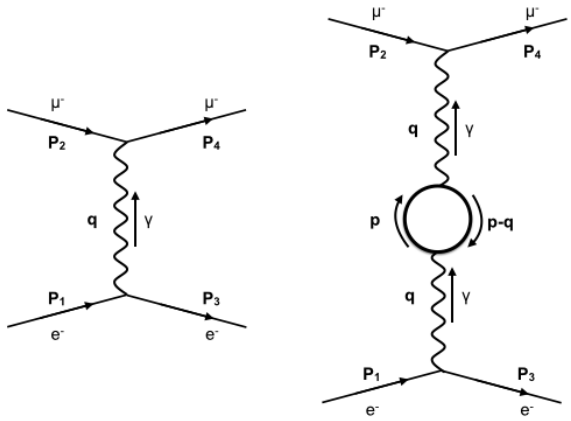

Fig. 1.10: The one loop correction to the Feynman diagram describing the scattering of an electron off a charge.

$$g_e^2 \to g_0^2 + \frac{g_e^4}{12\pi^2} f\left(\frac{Q^2}{m_e^2}\right) = g_0^2 \left\{ 1 + \frac{1}{12\pi^2} \frac{g_e^4}{g_0^2} f\left(\frac{Q^2}{m_e^2}\right) \right\}$$

Up to terms $O(g_e^4)$ we may set $g_e^4 = g_0^4$ inside the braces, so that

$$g_e^2 \to g_0^2 \left\{ 1 + \frac{g_0^2}{12\pi^2} f\left(\frac{Q^2}{m_e^2}\right) + O(g_0^4) \right\} \equiv g_R^2(Q^2)$$

Here $g_R^2(Q^2)$ is called the **running coupling constant**. Because $f(0) = 0$ we can set $g_0^2 = g_R^2(0)$ and thus:

$$g_R^2(Q^2) = g_R^2(0) \left\{ 1 + \frac{g_R^2(0)}{12\pi^2} f\left(\frac{Q^2}{m_e^2}\right) + O(g_R^4) \right\}$$

The cutoff M has now disappeared from view since it is absorbed in $g_R^2(0)$ which becomes infinitely large when we let $M \to \infty$. The mathematical technique to isolate the singularities in a perturbative calculation is called regularisation, cut-off regularisation in our case.

In terms of $\alpha = g_e^2/4\pi$, the running coupling becomes

$$\alpha(Q^2) = \alpha(0) \left\{ 1 + \frac{\alpha(0)}{3\pi} f\left(\frac{Q^2}{m_e^2}\right) + O(\alpha^2) \right\}$$

The next step is to admit that our theory cannot describe physics at asymptotically small distances so that we must replace the singular part of the calculation by measurement. This is called renormalisation. In fact 't Hooft and Veltman showed that this can be done consistently to all orders, without spoiling gauge invariance: they proved in general that gauge theories are renormalisable. They received for this work the Nobel prize in 1999. In QED it means that $\alpha(0)$ is replaced by the fine structure constant $\alpha = 1/137$, as measured at large distances of the order of the nuclear scale. There remains a finite correction term $f(Q^2)$ which causes the coupling to run with Q^2. This is a consequence of vacuum polarisation, as we have already discussed before. It turns out that the effect of the running QED coupling constant is really small and can safely be neglected at atomic or nuclear scales. Even at large momentum transfers of $Q^2 \sim 1000$ GeV^2 at the HERA collider, the correction to α is only about 1–2%.

Figure 1.11-a presents the vacuum polarisation graph discussed before. Apart from this graph there are three more divergent graphs to consider also shown in the same figure. The vertex correction (b) modifies the electron magnetic moment while the graphs (c) renormalise the electron mass. The three graphs (b) and (c) also contribute to the renormalisation of the electron charge. However, it turns out that these contributions cancel each other so that our previous calculation, based on diagram (a) alone, remains valid. This cancellation is called a Ward identity and is quite fortunate: without it, the graphs (c) would cause the coupling constant to be dependent on the lepton mass, and we would have different renormalisation for the electron and the muon electric charge.

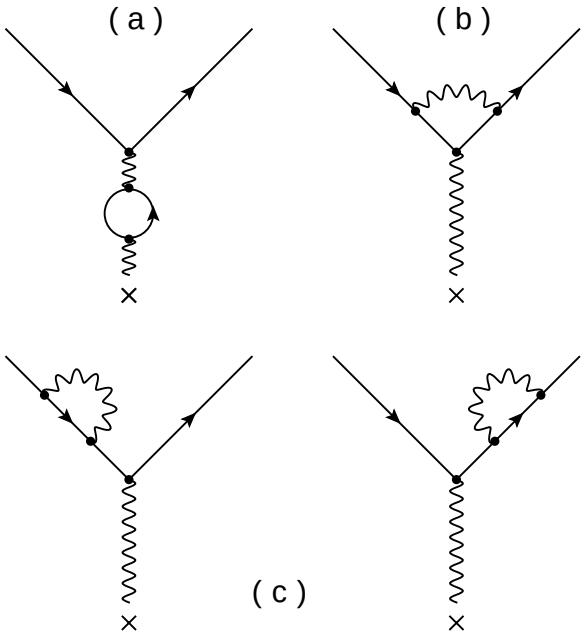

Fig. 1.11: A complete set of $O(\alpha^2)$ Feynman graphs.

For $Q^2 \gg m_e^2$ the one-loop corrected coupling constant is given by

$$\alpha(Q^2) = \alpha(0)\left\{1 + \frac{\alpha(0)}{3\pi}\ln\left(\frac{Q^2}{m_e^2}\right) + \mathrm{O}(\alpha^2)\right\}.$$

Because of the Ward identities, only propagator loops will contribute at higher orders. A set of them is presented in fig. 1.12:

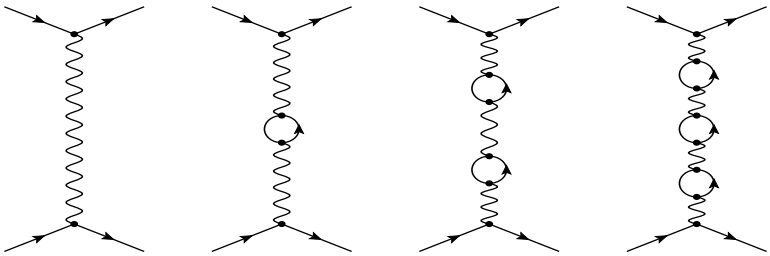

Fig. 1.12: A set of higher order correction diagrams.

This induces a series

$$1 + X + X^2 + X^3 + \cdots = \frac{1}{1-X}$$

and indeed, from a full calculation one gets[1]

$$\alpha(Q^2) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi]\ln(Q^2/m_e^2)} \quad \text{for} \quad m_e^2 \ll Q^2 < Q_{\max}^2$$

[1] This is an example of resummation where terms in a perturbative calculation are arranged in a geometric series which is then summed up to all orders.

The expression blows up when $\ln(Q^2/m_e^2) = 3\pi/\alpha(0)$, which occurs at an astronomical scale of $Q^2_{\max} = 10^{280}$ MeV^2. Although the loops are summed to all orders, there are still more complicated propagator diagrams (like multi-photon exchange between loops), which are ignored. The result given above is thus not exact, and is known as the leading log approximation.

We have seen that the running QED coupling constant decreases with decreasing Q^2 (increasing distance) to the asymptotic value $\alpha(0) = 1/137$ at $Q^2 = 0$. However, we could also have specified an input value $\alpha(\mu^2)$ at some arbitrary reference scale μ^2. We will now derive the formula for the coupling constant running from $Q^2 = \mu^2$, instead of from $Q^2 = 0$. This is useful because, as we will see, the reference scale $Q^2 = 0$ cannot be used in QCD.

From

$$\alpha(Q^2) = \frac{\alpha(0)}{1 - [\alpha(0)/3\pi]\ln(Q^2/m_e^2)}$$

we have

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi}\ln\left(\frac{Q^2}{m_e^2}\right)$$

and

$$\frac{1}{\alpha(\mu^2)} = \frac{1}{\alpha_0} - \frac{1}{3\pi}\ln\left(\frac{\mu^2}{m_e^2}\right).$$

A simple subtraction gives

$$\frac{1}{\alpha(Q^2)} - \frac{1}{\alpha(\mu^2)} = -\frac{1}{3\pi}\left[\ln\left(\frac{Q^2}{m_e^2}\right) - \ln\left(\frac{\mu^2}{m_e^2}\right)\right] = -\frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right)$$

and thus

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1 - [\alpha(\mu^2)/3\pi]\ln(Q^2/\mu^2)} \quad \text{for} \quad m_e^2 \ll Q^2 < Q^2_{\max}$$

The reference scale μ^2 where we wish to specify our input value of α is called the renormalisation scale. Obviously, the value of $\alpha(Q^2)$ does not depend on what renormalisation scale μ^2 we chose.

The running coupling can now be written as:

$$\frac{1}{\alpha(Q^2)} = \frac{1}{\alpha(\mu^2)} - \frac{1}{3\pi}\ln\left(\frac{Q^2}{\mu^2}\right)$$

Differentiation to $t = \ln Q^2$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{\alpha}\right) = -\frac{1}{\alpha^2}\frac{\mathrm{d}\alpha}{\mathrm{d}t} = -\frac{1}{3\pi} \quad \text{or} \quad \frac{\mathrm{d}\alpha}{\mathrm{d}t} \equiv \beta(\alpha) = \frac{1}{3\pi}\alpha^2$$

In the above, we have introduced the so-called β-function:

$$\frac{\mathrm{d}\alpha(Q^2)}{\mathrm{d}\ln(Q^2)} \equiv \beta(\alpha) = -(\beta_0\alpha^2 + \beta_1\alpha^3 + \beta_2\alpha^4 + \cdots)$$

Here we have written this function as a series expansion in powers of the coupling constant,[2] where the first term corresponds to the leading log approximation. It is an important task of perturbative QED to calculate the coefficients in this

[2] The minus sign in front of the series expansion is a matter of convention.

expansion. The QED one-loop beta function is $\beta = \alpha^2/3\pi > 0$. This means that the coupling constant increases with increasing Q^2 (decreasing distance). The one-loop QED coupling constant can now be written as:

$$\alpha(Q^2) = \frac{\alpha(\mu^2)}{1+\beta_0\,\alpha(\mu^2)\ln(Q^2/\mu^2)} \quad \text{with} \quad \beta_0 = -\frac{1}{3\pi}$$

Figure 1.13 presented the momentum transfer dependence of the fine structure constant. It is seen that over many orders of magnitude in terms of Q, the value of α does not change drastically.

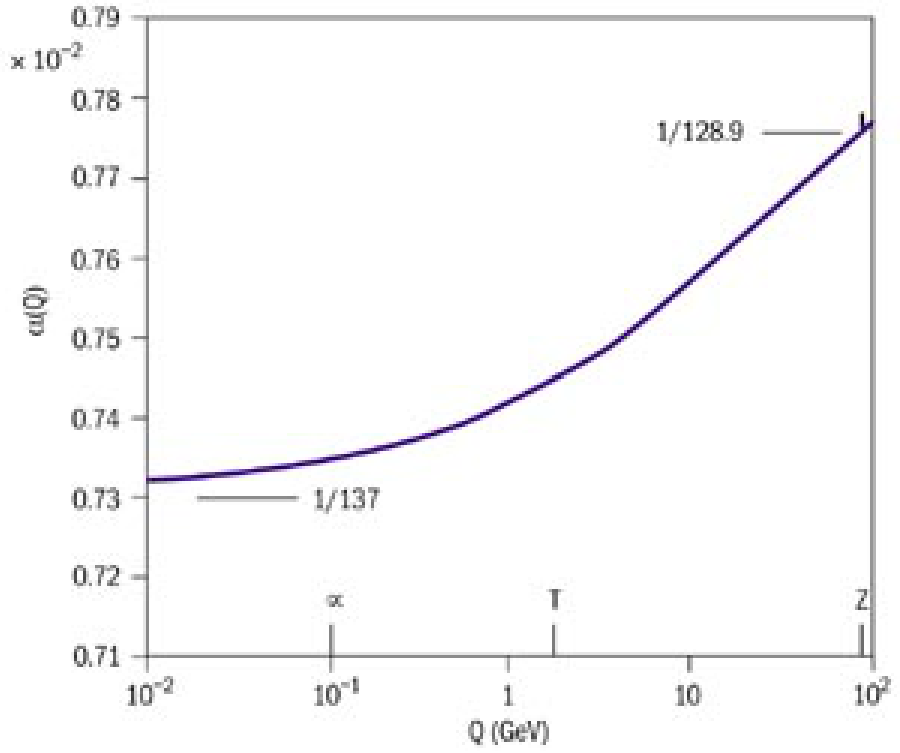

Fig. 1.13: The running of the QED coupling strength

1.6 Structure of subatomic particles

Structure of particles is studied via either elastic or inelastic interactions between the particles of study and probes (beams) of incoming particles. Elastic scattering experiments have provided a significant part of the information we now know about the structure of subatomic particles. These experiments are very similar to the spectroscopy ones. In spectroscopy, one angle is selected and the spectrum of the scattered particles is studied at this angle. In structure experiments, the detector configuration is very similar to the one used in spectroscopy experiments the detectors look at the elastic peak. The intensity of the elastic peak is measured as a function of the scattering angle the intensity changes with angle because of the different recoil of the target particles the observed intensity is translated into a differential cross-section ($d\sigma/d\Omega$). The information about the structure of a particle is then deduced from the cross-section.

1.6.1 Rutherford scattering

When talking about structure of particles, in case of atoms, then this means that we are after the spatial distribution of electrons around the nucleus. This is described by the ground state of the wave function. For the case of the hydrogen atom and neglecting for now the effects of spin the probability density ($\rho(\mathbf{x})$ is given by

$$\rho(\mathbf{x}) = \psi^*(\mathbf{x}) \cdot \psi(\mathbf{x})$$

where $\psi(\mathbf{x})$ is the wave function of the electron. The structure in this case includes all excited states. It is important to note that only if all excited states are known, is the structure of an atom fully known and determined. On the other hand, for nuclei charge and mass are not identical.

Let us now start with the known Rutherford scattering experiment which is nothing else than the elastic scattering of α-particles by the Coulomb field of the nucleus of charge Ze. The cross section can be easily computed, with the same results, either classically or quantum-mechanically. A similar experiment can be done with electrons being scattered off a stationary proton. The Rutherford scattering is the low-energy limit of $e--p$ scattering. In these cases the electron (or the α-particle in this case) energy is sufficiently low that the kinetic energy of the recoiling proton is negligible compared to its rest mass. The proton can be considered as a fixed, point-like source of $1/r$ electrostatic potential. The cross-sections are calculated from scattering theory by using the first order terms in the perturbation expansion.

Figure 1.14 illustrates how such an interaction can be depicted in momentum space. It is seen that the incoming particle (e.g. α-particle or electron) has a momentum $\mathbf{p}$ and is scattered off a static potential by an angle θ. The scattered particle comes out of the interaction region with momentum $\mathbf{p}^{'}$. On the other side, the target picks up a momentum transfer, represented by $\mathbf{q}$ in the figure. Consider energy and momentum conservation, one has:

$$\mathbf{q} = \mathbf{p} - \mathbf{p}^{'} \Rightarrow q^2 = p^2 + p^{'2} - 2pp^{'}\cos\theta$$

Considering that the momenta of the incoming and outgoing electron (or α-particle) are similar (i.e. $p \approx p^{'}$), one has:

$$q^2 = 2p^2 - 2p^2\cos\theta = 2p^2(1-\cos\theta)v \Rightarrow q^2 = 4p^2\sin^2(\theta/2) \Rightarrow$$

$$q \approx 2p\sin(\theta/2)$$

In general the differential cross-section can be written as

$$\frac{d\sigma}{d\Omega} = |g(\mathbf{q})|^2 \tag{1.6.1}$$

where $g(\mathbf{q})$ is the scattering amplitude that depends on the momentum transfer $\mathbf{q}$.

The regime, in terms of momenta values, that the Rutherford scattering is relevant is when the proton recoil can be neglected and the electron is non-relativistic. The differential cross-section is given then by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{4m^2(Z_1Ze^2)^2}{q^4} \tag{1.6.2}$$

where m is the mass of the incoming particle and Z, Z_1 are the atomic numbers of the target and the incoming particles, respectively.

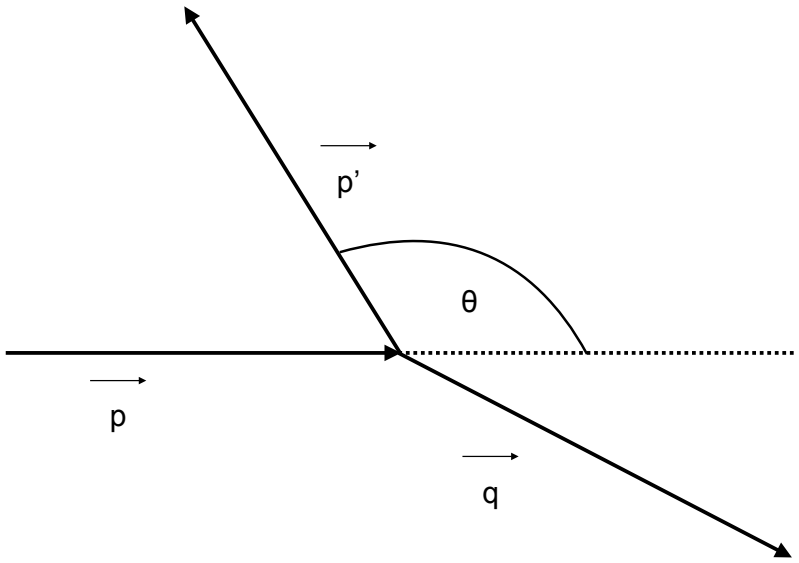

Fig. 1.14: The scattering of an electron off a proton in momentum space.

1.6.2 Mott scattering

For experiments, involving scattering of electrons i.e. particles with half-integer spin, the spin contribution needs to be taken into account. The Mott scattering is the limit where the electron is relativistic but the proton recoil can still be negligible. These conditions apply when $m_e << E << m_p$. The differential cross-section is given then by

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = 4(Ze^2)^2 \frac{E^2}{(qc)^4}\left(1-\beta^2\sin^2(\theta/2)\right) \tag{1.6.3}$$

where E is the energy of the incident electron. The term $\beta^2\sin^2(\theta/2)$ originates from the interaction of the electron's magnetic moment with the magnetic field of the target.

1.6.3 Form factors

Scattering of electrons off a proton is one of the cleanest probes of the internal structure of the latter. In order to study the relevant interaction, one simply has to go back to a well known process, the scattering of electrons off of muons since the diagrams are quite similar. This is indicated in the left diagram of fig. 1.15. However, we need to point out that we do not really know how the virtual photon interacts with the proton which, depending on the momentum transfer we probe it

with, might not be seen as an elementary particle anymore. As a result we have the right diagram of fig. 1.15, where this unknown interaction is covered by the red blob.

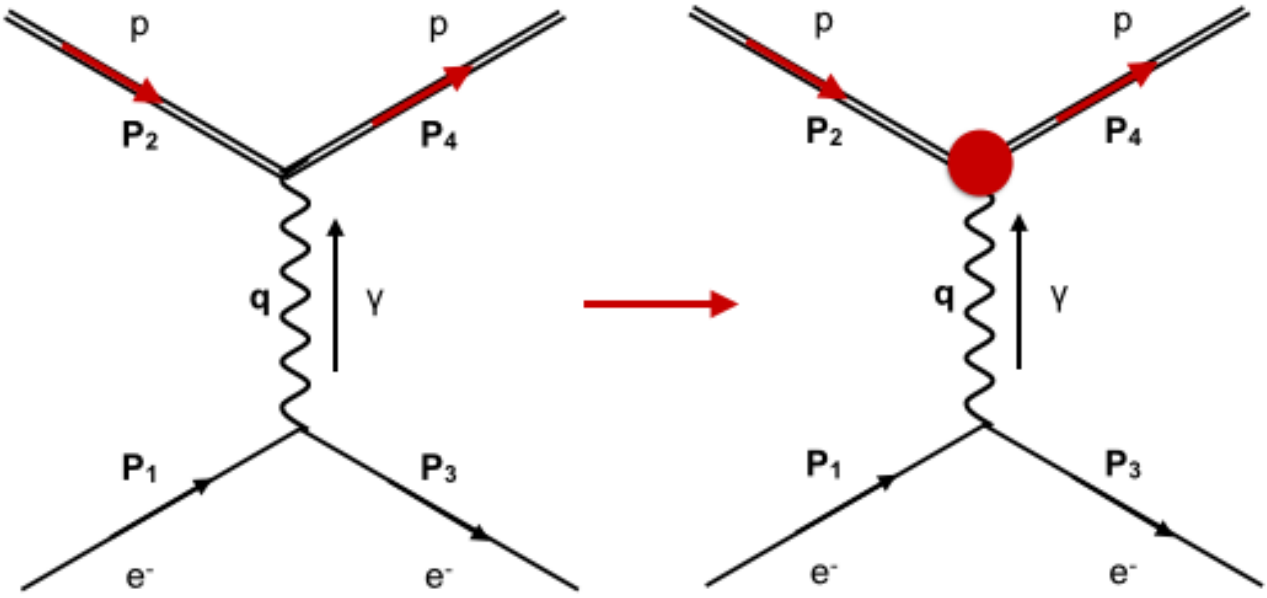

Fig. 1.15: The scattering of an electron off a proton. The blob on the right hand side diagram indicates that we do not really know how the virtual photon interacts with a complex, non-elementary object as the proton.

Once you crank up the energy of the process, then you start probing the internal structure of non-elementary particles. Form factors are thus introduced to account for the fact that some particles are not elementary but have internal structure. The electrons and in general the whole lepton family do not exhibit any internal structure, at least at the energy range we are able to reach and are thus ideal probes to study the internal structure of particles. The differential cross-section of Eq. 1.6.3 is modified and takes the following form:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} |F(q^2)|^2 \tag{1.6.4}$$

where $F(q^2)$ is the form factor which is a function of the square of the momentum transfer, q^2. Let's briefly review what form factors represent by looking at an example.

Let's assume that an elastic scattering process takes place between an electron and a spherically symmetric nucleus. Figure 1.16 presents a schematic view of such a scattering process between a spinless electron and a spinless nucleus with extended charge distribution. The charge density of total charge Q is given by $Q\rho(r)$, where $\rho(r)$ is the normalised probability density given by

$$\int d^3r\rho(r) = 1$$

The scattering potential $V(x)$ at the position of the electron (see fig. 1.16) consists of contributions from the entire nucleus (e.g. the entire proton). Each volume element d^3r contributes to the potential $Ze\rho(r)d^3r$ so that

$$dV(x) = -\frac{Ze^2}{z} e^{-z/\xi} \rho(r) d^3r$$

where ξ is a length characteristic of atomic dimension. Integrating the above gives the potential:

$$V(x) = -Ze^2 \int d^3r \frac{\rho(r)}{z} e^{-z/\xi}$$

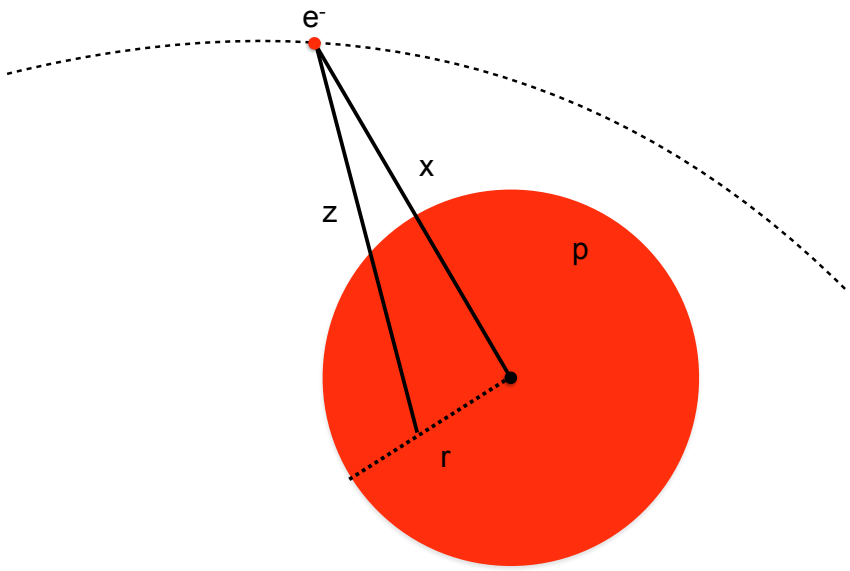

Fig. 1.16: Schematic view of the scattering of a spinless electron by a spinless nucleus with extended charge distribution.

The scattering amplitude introduced in Eq. 1.6.1 takes the form:

$$g(\mathbf{q}) = -\frac{m}{2\pi\hbar^2}\int V(\mathbf{x})e^{i\mathbf{q}\mathbf{x}/\hbar}d^3x$$

For a spherically symmetric potential, one can perform the integration over angles. The scattering amplitude then becomes

$$g(q^2) = -\frac{2m}{q\hbar}\int_0^\infty dx x\sin(qx/\hbar)V(x)$$

Let's now go back to our problem! If we introduce the formula that gives the potential to the one that gives the scattering amplitude, then

$$g(q^2) = \frac{mZe^2}{2\pi\hbar^2}\int d^3r e^{i\mathbf{q}\mathbf{r}/\hbar}\rho(r)\int d^3x\frac{e^{-z/\xi}}{z}e^{i\mathbf{q}\mathbf{z}/\hbar}$$

For a given, fixed $\mathbf{r}$, d^3x can be replaced by d^3z, such that

$$\int d^3x\frac{e^{-z/\xi}}{z}e^{i\mathbf{q}\mathbf{z}/\hbar} = \int d^3z\frac{e^{-z/\xi}}{z}e^{i\mathbf{q}\mathbf{z}/\hbar} =$$

$$\frac{4\pi\hbar^2}{q^2+(\hbar/\xi)^2} \approx \frac{4\pi\hbar^2}{q^2}$$

for $(\hbar/\xi)^2 << q^2$. The integral over d^3r is the form factor of Eq. 1.6.4. The previous equation gives an idea how form factors are constrained experimentally:

- the differential cross section, $d\sigma/d\Omega$ is measured experimentally at different angles,
- the Mott cross section is computed,
- the ratio of the two cross-sections is plotted as a function of q^2. This ratio gives the form factor.

The form factor is related to the charge density via a Fourier transform:

$$F(q^2) = \int d^3r\rho(\mathbf{r})e^{i\mathbf{qr}/\hbar} \Rightarrow \rho(r) = \frac{1}{(2\pi)^3}\int d^3qF(q^2)e^{-i\mathbf{qr}/\hbar}$$

The connection is materialised by measuring $F(q^2)$ for various values of q^2 and we fit, extracting $\rho(r)$ from the fit.