1 Fenchel duality

Overview of Fenchel Duality

- Key concepts: Convex conjugates and the Fenchel-Young inequality.
- The goal: To derive a dual problem from a primal problem using convex conjugates.

Blanket assumption

- Our focus is on functions $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ that are convex and proper.

Motivation Fenchel conjugates

Theorem 1 (Existence of the Subdifferential)**.** *Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ be a convex function.*

$$x_0 \in \operatorname{int}\operatorname{dom}(f) \text{ implies } \partial f(x_0) \neq \emptyset.$$

Pf. Apply separ. hyperplane thm. to $(x_0, f(x_0))$ and $\operatorname{dom}(f)$. □

Given $x \in \operatorname{int}\operatorname{dom}(f)$ there is always $y \in \partial f(x)$

What about opposite: Given y, find x such that $y \in \partial f(x)$?

Fenchel conjugates

Definition: Conjugate The Fenchel conjugate of $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, denoted $f^* : \mathbb{R}^n \to \overline{\mathbb{R}}$, is:

$$f^*(y) := \sup_{x \in \mathbb{R}^n} \{\langle y, x\rangle - f(x)\}.$$

Fenchel-Young Inequality

Let $f : \mathbb{R}^n \to \overline{\mathbb{R}}$,

$$f(x) + f^*(y) \geq \langle x, y\rangle, \quad \forall x, y \in \mathbb{R}^n.$$

Equality holds iff $y \in \partial f(x)$.

Primal and Dual Problems

Given $f : \mathbb{R}^n \to \overline{\mathbb{R}}$, $g : \mathbb{R}^m \to \overline{\mathbb{R}}$, and $A \in \mathbb{R}^{m \times n}$.
This **primal problem** is:

$$p^* = \inf_x f(x) + g(Ax)$$

And the **dual problem** is:

$$d^* = \sup_y -f^*(A^T y) - g^*(-y).$$

Fenchel Duality - weak

Weak Duality

For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$,

$$f(x) + g(Ax) \geq -f^*(A^T y) - g^*(-y).$$

Fenchel Duality - strong

Strong Duality

If $f : \mathbb{R}^n \to \overline{\mathbb{R}}$ and $g : \mathbb{R}^m \to \overline{\mathbb{R}}$ are convex functions, and $0 \in \operatorname{int}(\operatorname{dom} g - A \operatorname{dom} f)$ then

$$\inf_x f(x) + g(Ax) =: \mathbf{p}^* = \mathbf{d}^* := \sup_y -f^*(A^T y) - g^*(-y).$$

And dual problem is attained if finite.

Example Semidefinite Programming and Fenchel Duality

Semidefinite Program (SDP): Consider the primal form of an SDP:

$$\begin{aligned} \min_X \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i \qquad \forall i = 1, \ldots, m, \\ & X \succeq 0. \end{aligned}$$

where $C \in \mathbb{R}^{n \times n}$ and $A_i \in \mathbb{R}^{n \times n}$

Convex subgradient calculus

Corollary

Under the assumptions of strong duality,

$$\partial(f + g \circ A)(x) = \partial f(x) + A^T \partial g(Ax).$$

Duality: Switching roles between solutions and (sub)gradients

Lagrangean

Primal Problem:

$$p* := \inf_x \{f(x) : g_i(x) \le 0, \, i = 1, \ldots, m\}.$$

Lagrangean Function: The Lagrangean function $\mathcal{L}(x, \lambda)$ for inequality constraints is:

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i=1}^m \lambda_i g_i(x),$$

where $\lambda_i \ge 0$ are the Lagrange multipliers associated with the inequality constraints.

Saddle-Point Condition: The solution to the primal problem is found by solving a *saddle-point problem*:

$$p^* = \inf_x \sup_{\lambda \ge 0} \mathcal{L}(x, \lambda).$$

Dual Problem and Saddle-Point Formulation

Dual Function:

$$q(\lambda) = \inf_x \mathcal{L}(x, \lambda) = \inf_x \left(f(x) + \sum_{i=1}^m \lambda_i g_i(x) \right).$$

Dual Problem:

$$d^* := \sup_{\lambda \ge 0} q(\lambda) = \sup_{\lambda \ge 0} \inf_x \mathcal{L}(x, \lambda).$$

Lagrangean Duality

Question

What is the relation between

$$p^* = \inf_x \sup_{\lambda \geq 0} \mathcal{L}(x, \lambda) \text{ and } d^* = \sup_{\lambda \geq 0} \inf_x \mathcal{L}(x, \lambda)?$$

Super Lagrangian Duality

Let $f, g_i : \mathbb{R}^n \to \overline{\mathbb{R}}$, (not necessarily convex). Let $v : \mathbb{R}^m \to \overline{\mathbb{R}}$:

$$v(b) := \inf_x \{f(x) : g_i(x) \leq b, \, i = 1, \dots, m\}.$$

Then

1. $p^* = v(0)$.
2. $v^*(-\lambda) = \begin{cases} -g(\lambda) & \text{if } \lambda \geq 0 \\ \infty & \text{o.w.} \end{cases}$.
3. $d^* = v^{**}(0)$

From Super-lagrangian Duality we obtain:

1. [Weak Lagrangian duality] $p^* \leq d^*$
2. [Strong Lagrangian duality] Assume f and g_i are convex, and there is $\tilde{x}$ such that $g_i(\tilde{x}) < 0$ for all i (Slater condition). Then $p^* = d^*$, and if d^* is finite, it is attained.

Proof. For any function v we have $v(x) \geq v^{**}(x)$ for any x. Then Weak duality follows. Now we prove strong duality. From weak duality, if $p^* = -\infty$, we have then $d^* = -\infty$. Then we assume $p^* > -\infty$. From Slater condition, the primal is feasible, and thus p^* is finite. That is $0 \in \operatorname{dom} v$. Now notice (exercise) that f and g_i convex implies v is convex. Therefore $v(x) = v^{**}(x)$ if and only if v is lower semicontinuous at x. We claim that Slater condition implies $0 \in \operatorname{int}(\operatorname{dom}(v))$, which from convexity implies that v is continuous at 0. To prove the claim, Let $\tilde{b} = g(\tilde{x}) > 0$. For any $b \leq \tilde{b}$ the primal is feasible, and thus $v(b) < +\infty$. Let $r = \min\{\tilde{b}_i : i = 1, \dots, m\} > 0$. As $0 \in \operatorname{dom}(v)$ and v is convex we get $\{b : \|b\| \leq r \subseteq \operatorname{dom}(v)\}$, that is $0 \in \operatorname{int} \operatorname{dom}(v)$. Attainment follows as λ is dual optimum if and only if $0 \in \partial v^*(-\lambda)$. But, this condition is equivalent to $-\lambda \in \partial v^{**}(0)$. □