Estimating the Hidden Overheads in the BDGL Lattice Sieving Algorithm

Léo Ducas

Centrum Wiskunde & Informatica, Amsterdam
Leiden University, Mathematical Institute

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Lattice Sieving

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A core algorithm for lattice cryptanalysis

- Lattice Sieving solves the Shortest Vector Problem (SVP)
- It is used inside BKZ for strong lattice reduction

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The Central task

- Given $N = 2^{.2075d+o(d)}$ random points on the sphere of $\mathbb{R}^d$
- Find all close-by pairs: $\vec{v}, \vec{w}$ s.t. $\langle \vec{v}, \vec{w} \rangle \leq \frac{1}{2}$

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Complexity

- Naively: [AKS01, NV08, MV10]

$$O(N^2) = 2^{.415d+o(d)}$$

- Locality Sensitive Hashing (LSH): [Laarhoven, BDGL16]

$$2^{.292d+o(d)}$$

Diving in the small $o(d)$

Concrete security estimation requires more than asymptotics

e.g. for PQC NIST standardization

The concrete analysis of [AGPS20]

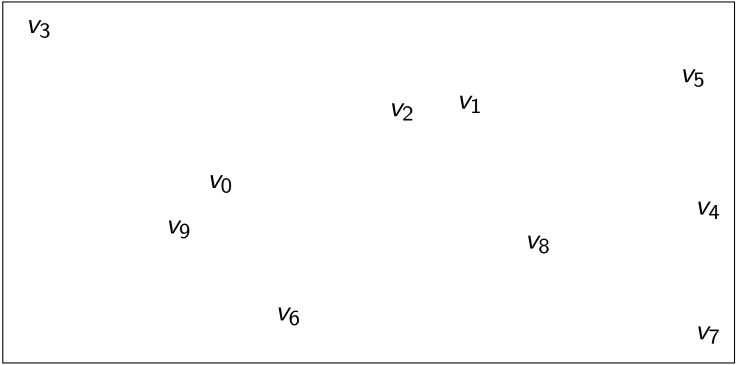
- Replaced asymptotic by exact formula
- Costed each elementary step in term of gates
- Used by NIST candidates to substanciate security claims

Remaining Innacuracies

- Overcosted some steps of BDGL, as shown by [Mat22]
- **Idealized Quality of List-Decodable Codes** (this talk) — Q2 of Kyber Spec, Sec 5.3
- And others... — Q1,3,4,5 of Kyber Spec, Sec 5.3

Naive Close Pair Search

Compute all distances and check



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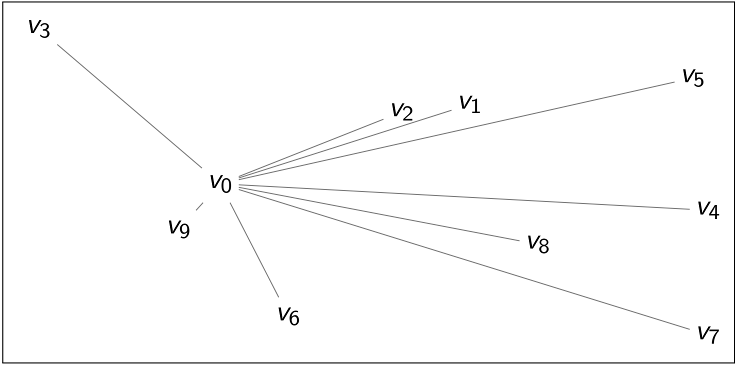

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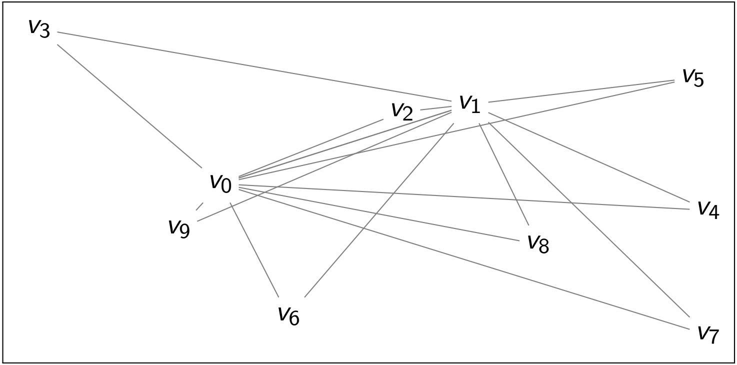

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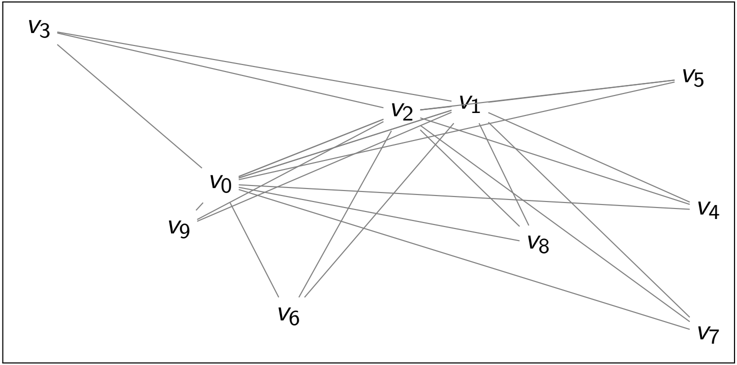

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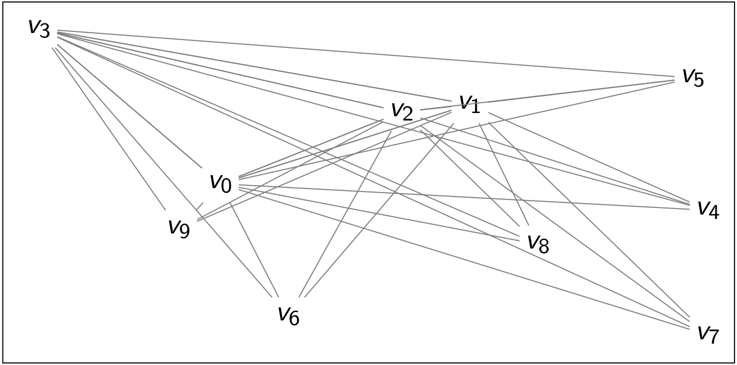

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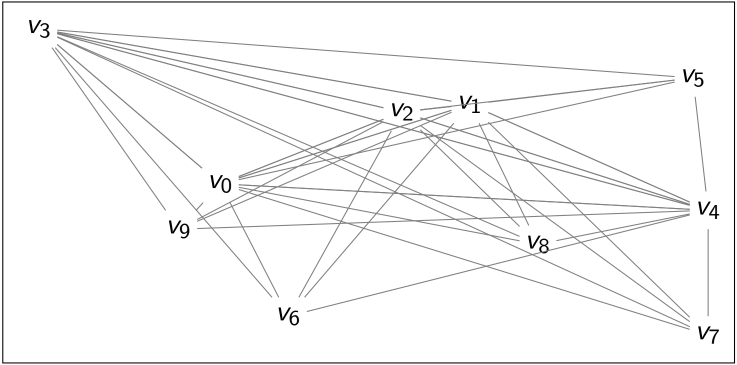

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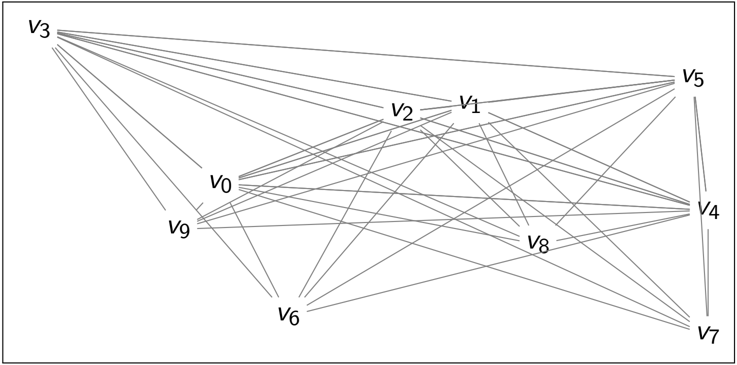

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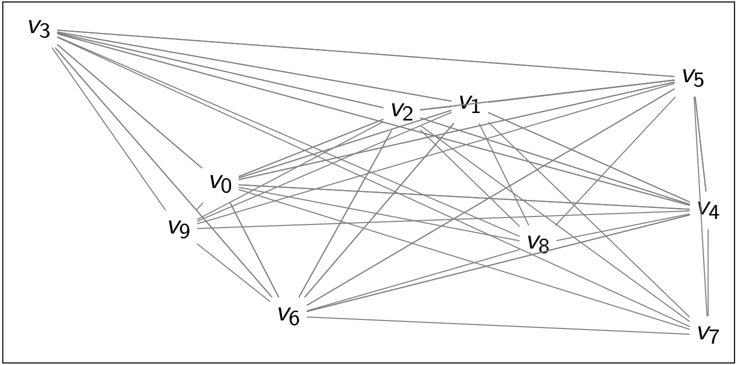

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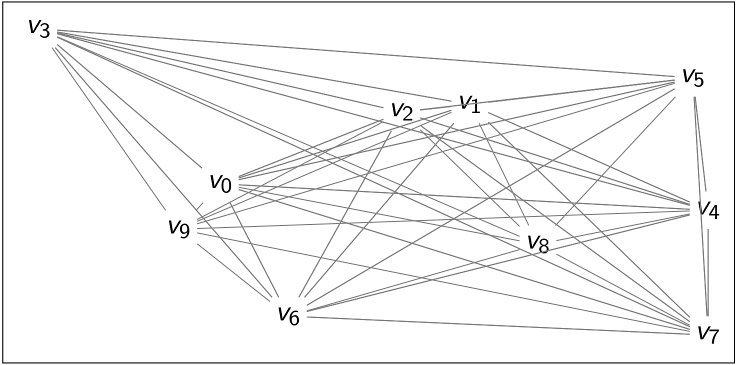

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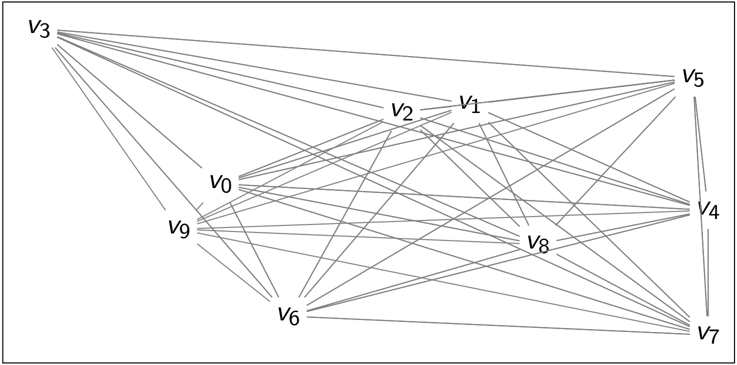

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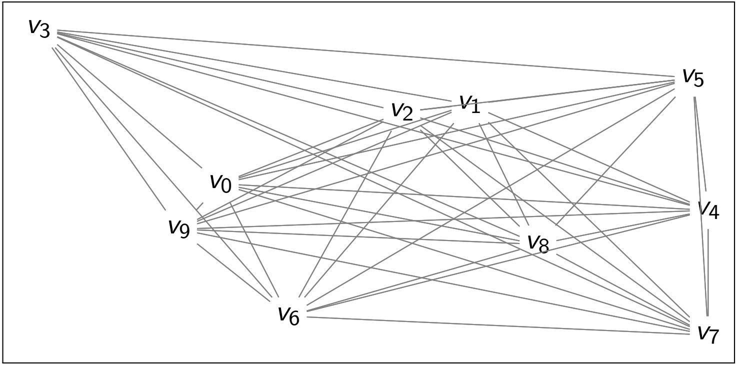


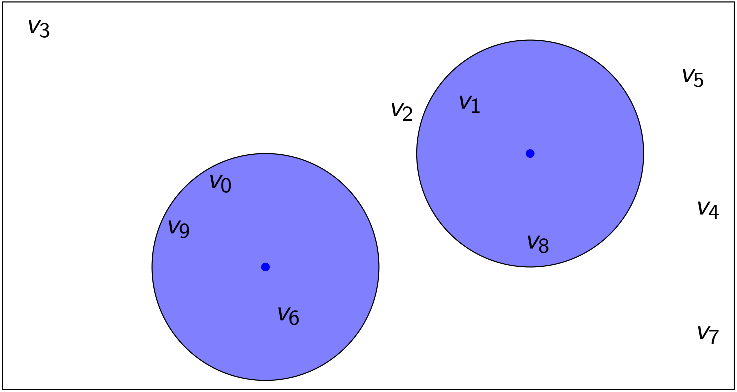
for a cost of $O(N^2)$.

BDGL Lattice Sieving

(Becker-D.-Gama-Laarhoven, SODA 2016)

Locality Sensitive Filtering

Make local buckets, and search within them



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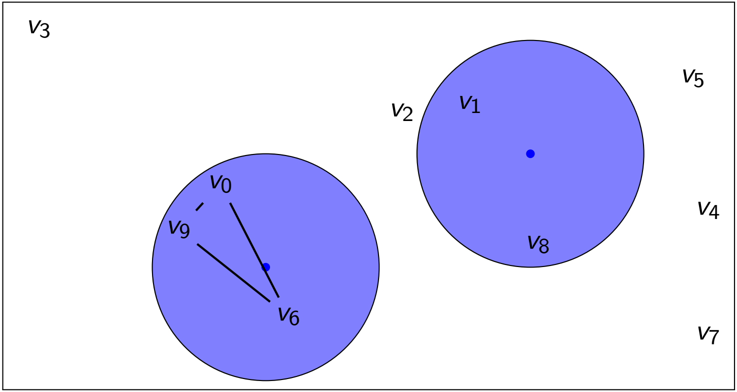

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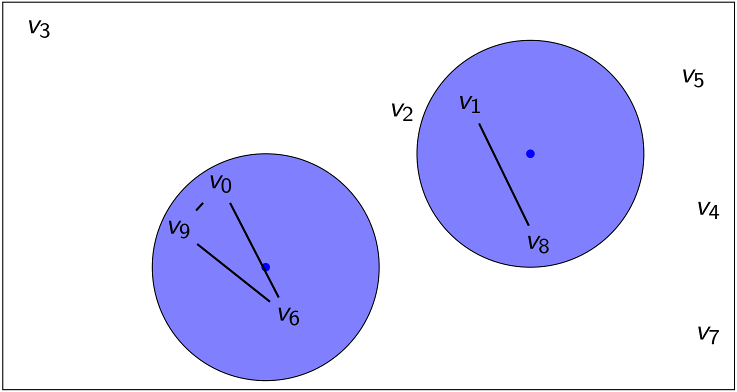

All kinds of Shapes, and Time-Memory-Success trade-offs.
[Laarhoven, BGJ15, BDGL16]

BDGL Idealized

1. Filters are spherical caps
2. Filter centers are **efficiently list-decodable**
3. Filter centers are uniform and **independent**

Complexity:

$$T = \sqrt{3/2}^{\,n+o(n)} \qquad M = \sqrt{4/3}^{\,n+o(n)}$$

Requirements 2 and 3 in tensions

To be efficiently list-decodable, the centers need some structure...

BDGL, instantiated with Random Product Codes

The set of Filters centers F is a direct product:

$$F = F_1 \times F_2 \times \cdots \times F_k$$

where each F_i has uniform independent elements in dimension n/k.

Cost Overhead for List-Decoding

$$|F|^{1/k} + \text{number of passing filters}$$

Probability Overhead Lack of Independance [BDGL16, Thm. 5.1]

For $k = O(\log n)$ the success probability loss compared to idealized is at most $2^{\tilde{O}(\sqrt{n})}$

This Work

The Problem Considered in This Work

Concrete Estimation of the Probability Overhead

- A concrete analytic approach ? No clue how . . .
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- Conditional Sampling

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Concrete Results

- Time overhead of about 2^6 in dimension $n = 380$
- Costly and limited trade-off with Memory Overhead

What We Want to Measure ?

- For a random close pair v, w, and a random product code F
- What is the probability that $\exists f \in F$ close to both v and w ?

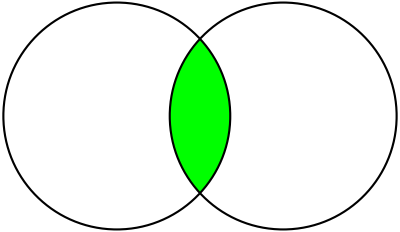
Naive Solution

- Sample a close pair v, w and an RPC F
- List-decode F around v and w
- Check if intersection is non-empty

Techniques

Trick 1: Intersected List-Decoding

Is there a point $f \in F$ in this intersection ?



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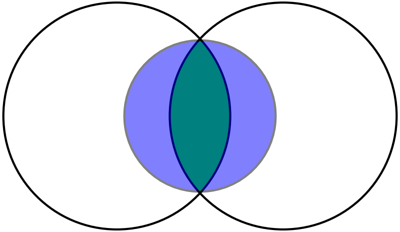


Faster Solution

- Sample a close pair v, w and an RPC F
- List decode around $m = (v + w)/2$, **with a smaller radius**
- In that list, check if a point is in the intersection

Toward Trick 2: A Needle in a Haystack

For each $f \in F$ consider the event that f leads to finding v, w

$$E_f : w \in f \wedge v \in f$$

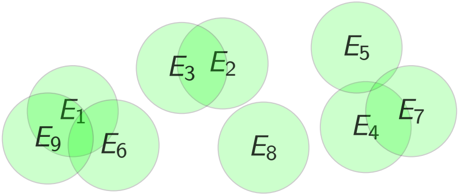
We want to estimate $S = \mathbb{P}[\bigcup_{f \in F} E_f]$.

Note: All event E_f have the same and known probability, but are not independent.

S is very small in the regime of interest

- Number of samples for a fixed relative error is $\Theta(1/S)$.
- In the low memory regime, $S = 2^{-0.084n+o(n)}$

Trick 2: Focus on Successes



A conditioned Experiment

- Sample F
- Choose a uniform $f \in F$
- Sample close pair v, w conditioned on E_f
- Return $i \geq 1$, the number of successful events E_f

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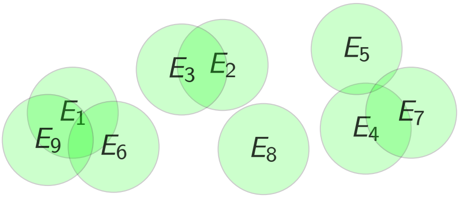

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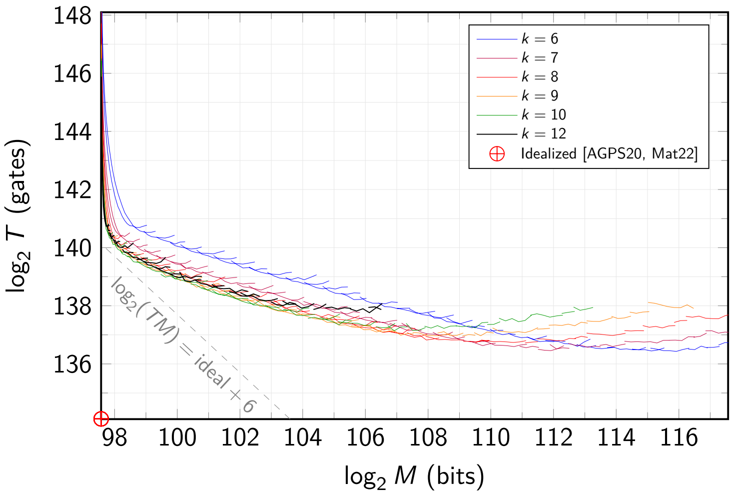
Lemma (Conditioned Estimation)

$$\frac{\mathbb{P}[\bigcup E_f]}{\sum \mathbb{P}[E_f]} = \mathbb{E}\left[\frac{1}{i}\right]$$

Convergence is much faster !

Results

Results in dimension 384



- Sieve Time-Memory cost 2^6 more than previously estimated

Mitigating the Overheads

This is an overhead on a subroutine

- In full attacks: many BDGL calls, with various dimension
- One can play with more memory in smaller dimensions as

$$T_{\text{total}} = \sum T_i \quad M_{\text{total}} = \max M_i$$

E.g. inside progressive-BKZ, one may save back a factor $2^{0.9}$.

Closing Remarks

+5 bits for lattice schemes targetting NIST level 1 ?

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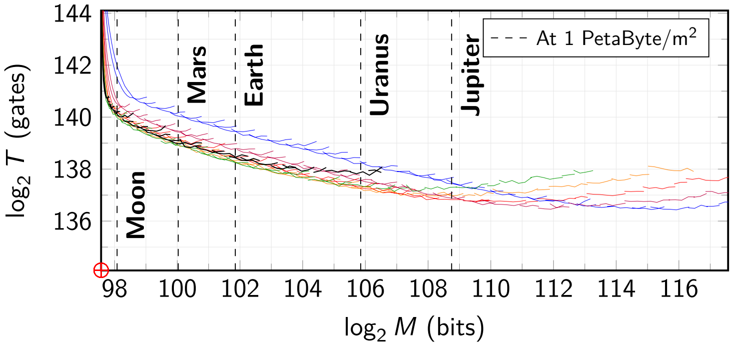

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Hard to systematize

- Getting those curve is costly 40 core-days for dim 384
- Not feasible in an estimator

? An analytic estimate would be much preferable !

Alternative instantiation of BDGL16

? Can we do better than Random Product Codes ?