Counting of States

Hirosi Ooguri

Verlinde Symposium

14 – 15 July 2022

In the fall of 1986, Tohru Eguchi and I were working on conformal field theories on Riemann surfaces.

Eguchi was spending his sabbatical in Paris, and we communicated by snail mails and telex.

1986, 10, 12

江口様

前略

以下の 2つのことがわかりましたので、ご報告します。

(A) primary field ϕ の descendant $\chi = \sum_{\{m\}} a_{\{m\}} L_{-m_1} \cdots L_{-m_N} \phi$ の

$L_m \chi = 0 \ (m \geq 1)$ なるものがあると、$\chi(z) = 0$ とおいて得

微分方程式は Riemann 面上の局所座標系の変換 z↦

の下で不変である。

(B) $\psi(z)$: (complex) free spinor , $J(z)$: $U(1)$ current.

$\langle J(z)\psi(w_1)\psi(w_2)\rangle$ に Ward 恒等式を適用して得られる式に

Fay の公式 $\vartheta(\gamma + \int_a^c \omega)\vartheta(\gamma + \int_b^d \omega) E(c,b) E(a,d)$

$+ \vartheta(\gamma + \int_b^c \omega)\vartheta(\gamma + \int_a^d \omega) E(c,a) E(d,b)$

$= \vartheta(\gamma + \int_{a+b}^{c+d} \omega)\vartheta(\gamma) E(c,d) E(a,b)$

の特別な場合になっている。

12 October

```
KDD23
112+
21 1332
KDD PTASC 03
J23472

PROF. T. EGUCHI
LABORATOIRE DE PHYSIQUE THEORIQUE
ECOLE NORMALE SUPERIEURE
24 RUE LHOMOND
75231 PARIS
CEDEX 5, FRANCE

FAY'S FORMULA AND GENERALIZATIONS MEAN 2N POINT FERMION AMP IS
VACUUM AMP WITH BG GAUGE FIELD (N SOURCES AND N SINKS).
BOSONIZATION RULE IS DERIVED FROM IT.  W-ID IS ITS COROLLARY.
BITNET WILL BE AVAILABLE IN A WEEK.
     H. OOGURI

NNNN
KDD PTASC 03
```

10/15 江口研

15 October

71

$\langle \exp(\oint_{z=\infty} dz\, j(z) f(z)) \rangle \times \vartheta[^a_b](0|\Omega)$

$= \exp(\sum_{n,m\geq 1} Q_{n,m} x_n x_m)\, \vartheta[^a_b](-\sum_{n\geq 1} \vec{A}_n x_n | \Omega)$ (7)

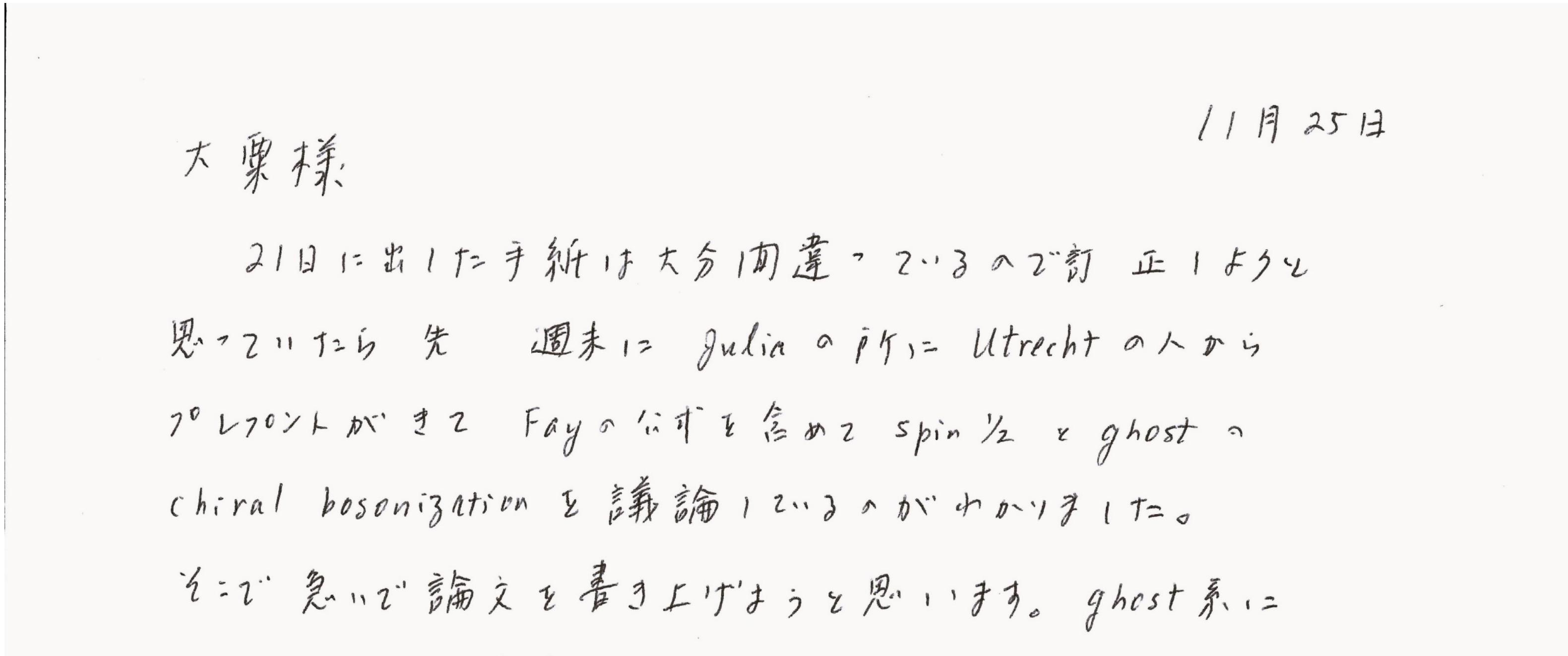
となります。これで current についての generating function が求まりました。注目すべきことに、この式の右辺を $x = (x_1, x_2, \cdots, x_n, \cdots)$ の函数とみて $\tau(x)$ と書くことにすると、テータ函数の加法定理（Fayの公式）を使えば、$\tau(x)$ が KP方程式系の広田の双一次方程式を満たしていることが示されます。逆に KP方程式系の準周期解は、縮退したRiemann面まで考えれば、式(7)の右辺のようなもので尽されることが知られているので、

current代数の generating function $\langle \exp(\oint_{z=0} dz\, j(z) f(z)) \rangle$	⟺	KP方程式系の準周期解 $\tau(x) = e^{Qx^2}\vartheta[^a_b](-\vec{A}x|\Omega)$

という対応が成り立ちます。

17 November

Then, … I receive this letter from Eguchi dated on 25 November:



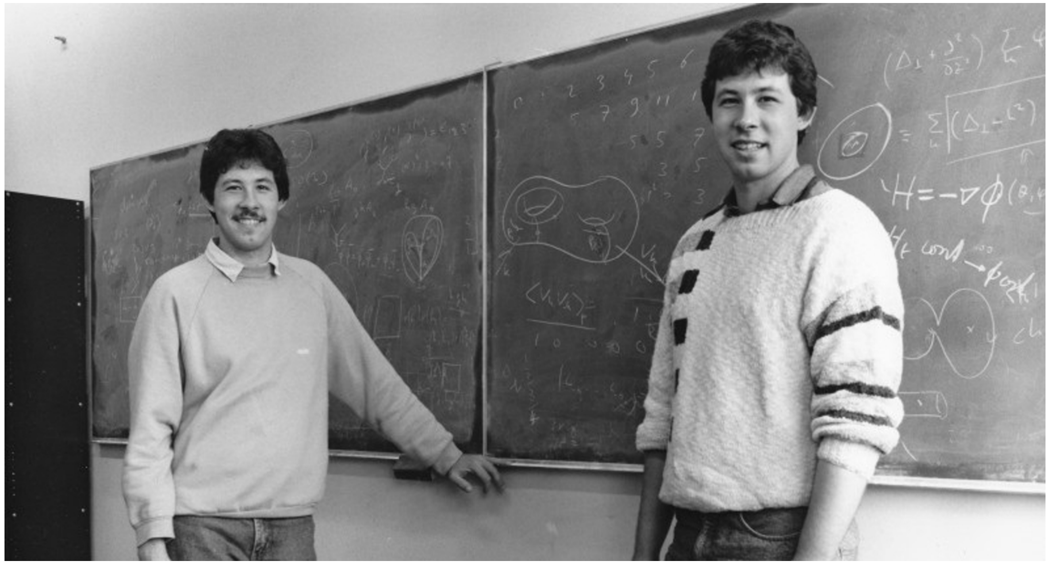
11月25日

大栗様

21日に出した手紙は大分間違っているので訂正しようと思っていたら先週末にJuliaの所にUtrechtの人からプレプリントがきてFayの公式を含めてspin ½とghostのchiral bosonizationを議論しているのがわかりました。そこで急いで論文を書き上げようと思います。ghost系に

"…, last weekend, [Bernard] Julia received a preprint from Utrecht, where chiral bosonization of spin ½ and ghost fields are discussed using Fay's identity. I think we should finish writing our paper soon. …"

We were scooped.



"…, last weekend, [Bernard] Julia received a preprint from Utrecht, where chiral bosonization of spin ½ and ghost fields are discussed using Fay's identity. I think we should finish writing our paper soon. …"

Chiral Bosonization

Nuclear Physics B288 (1987) 357–396
North-Holland, Amsterdam

CHIRAL BOSONIZATION, DETERMINANTS AND THE STRING PARTITION FUNCTION

Erik VERLINDE and Herman VERLINDE

Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, 3508 TA Utrecht, The Netherlands

Received 17 November 1986

Volume 187, number 1,2 PHYSICS LETTERS B 19 March 1987

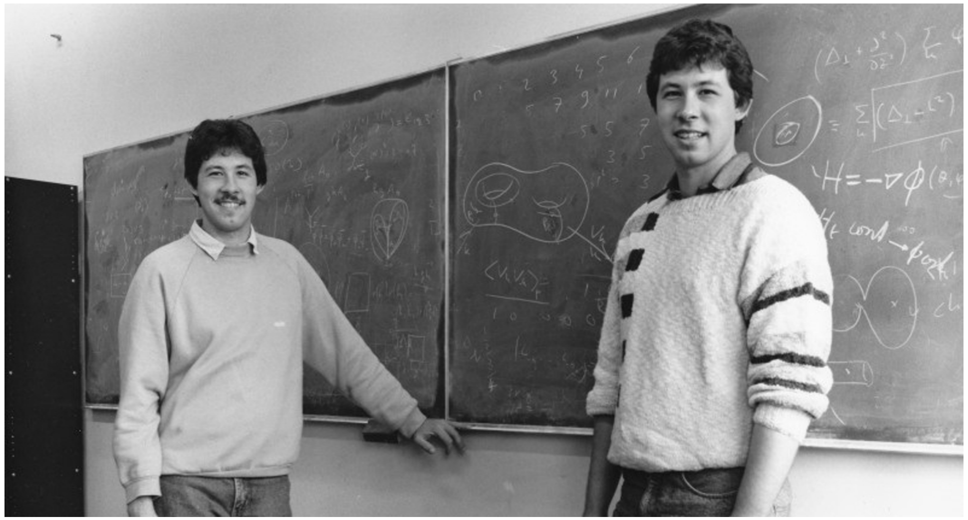
CHIRAL BOSONIZATION ON A RIEMANN SURFACE

Tohru EGUCHI [1]
Laboratoire de Physique Théorique de l'Ecole Normale Supérieure [2], 24 rue Lhomond, F-75231 Paris Cedex 05, France

and

Hirosi OOGURI
Department of Physics, University of Tokyo, Tokyo, Japan

Received 23 December 1986



Chiral Bosonization

Nuclear Physics B288 (1987) 357–396
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Hirosi OOGURI
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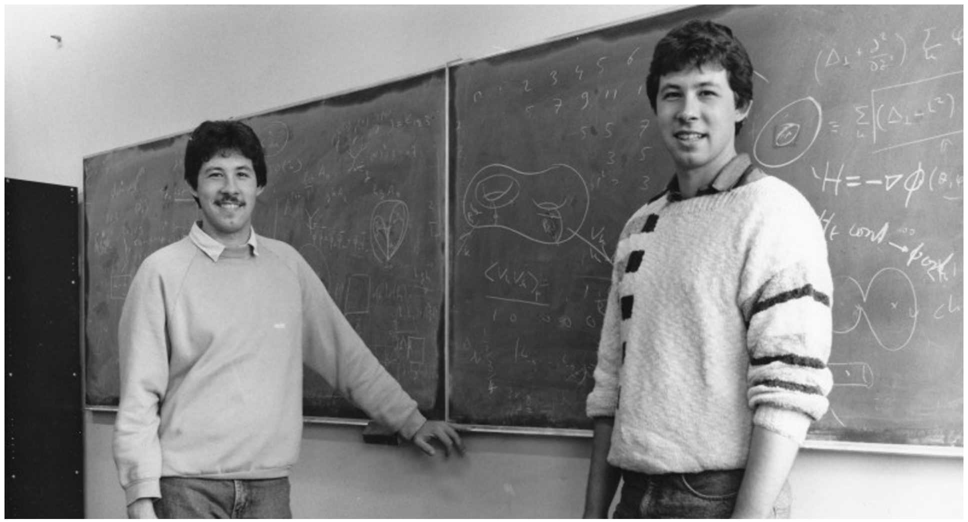
Received 23 December 1986

402 citations

111 citations



Chiral Bosonization

Nuclear Physics B288 (1987) 357–396
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CHIRAL BOSONIZATION, DETERMINANTS AND THE STRING PARTITION FUNCTION

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Department of Physics, University of Tokyo, Tokyo, Japan

Received 23 December 1986

Volume 180, number 3 PHYSICS LETTERS B

ANALYTIC FIELDS ON RIEMANNIAN SURFACES

V.G. KNIZHNIK

L.D. Landau Institute for Theoretical Physics

Received 27 June 1986

Commun. Math. Phys. 112, 567–590 (1987)

Communications in Mathematical Physics
© Springer-Verlag 1987

Analytic Fields on Riemann Surfaces. II

V.G. Knizhnik

Landau Institute for Theoretical Physics, Moscow, USSR

Received March 1, 1987

Born in 1962

21 January:	Erik and Herman Verlinde
20 February:	Vadim Knizhnik
13 March:	H.O.
29 March:	Igor Klebanov

Volume 192, number 1,2 PHYSICS LETTERS B 25 June 1987

MULTILOOP CALCULATIONS IN COVARIANT SUPERSTRING THEORY

Erik VERLINDE and Herman VERLINDE
Institute for Theoretical Physics, Princetonplein 5, PO Box 80.006, 3508 TA Utrecht, The Netherlands

Received 9 February 1987; revised manuscript received 3 April 1987

We give an explicit construction of the superstring multiloop amplitudes in terms of theta functions. We analyse the correlation functions of the space–time supersymmetry current and find that these contain unphysical poles. Using BRST-invariance we show that these poles have no physical effect for on-shell amplitudes, and that the partition function is given by a total derivative on moduli space.

115, 649–690 (1988)

Communications in Mathematical Physics

$C=1$ Conformal Field Theories on Riemann Surfaces

Robbert Dijkgraaf, Erik Verlinde, and Herman Verlinde
Institute for Theoretical Physics, Princetonplein 5, P.O. Box 80.006, NL-3508 TA Utrecht, The Netherlands

Nuclear Physics B300 [FS22] (1988) 360–376
North-Holland, Amsterdam

FUSION RULES AND MODULAR TRANSFORMATIONS IN 2D CONFORMAL FIELD THEORY

Erik VERLINDE

Institute for Theoretical Physics, University of Utrecht, P. O. Box 80.006, 3508 TA Utrecht, The Netherlands

Received 15 March 1988

Nieuw Amsterdam
1988 - 1989

Boston, March 1993

Aspen, July 1989

Paper with Cumrun and Erik

Letters in Mathematical Physics (2005) 74:311–342
DOI 10.1007/s11005-005-0022-x

Hartle–Hawking Wave-Function for Flux Compactifications: the Entropic Principle

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[1] *California Institute of Technology 452-48, Pasadena, CA 91125, USA.*
e-mail: ooguri@theory.caltech.edu
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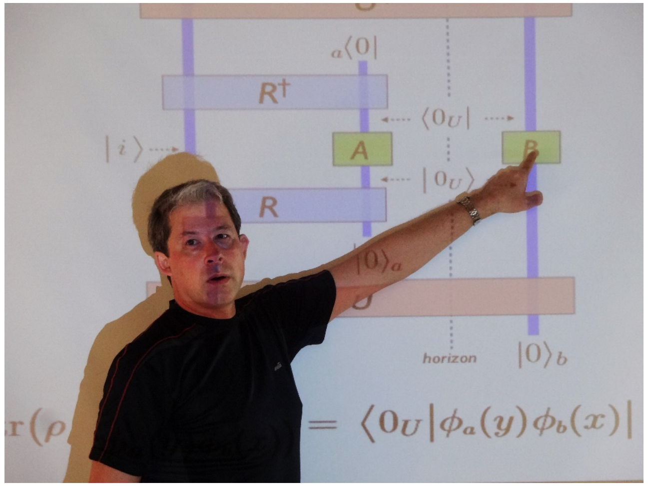
(Received: 21 March 2005)

Abstract. We argue that the topological string partition function, which has been known to correspond to a wave-function, can be interpreted as an exact "wave-function of the universe" in the mini-superspace sector of physical superstring theory. This realizes the idea



[hep-th] 29 Nov 2012

Black Hole Entanglement and Quantum Error Correction

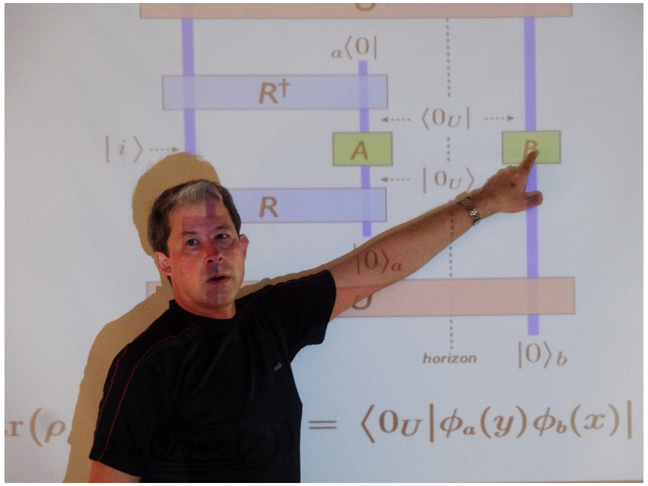
Erik Verlinde[1*] and Herman Verlinde[2,3†]

[1]Institute for Theoretical Physics, University of Amsterdam, Amsterdam, The Netherlands

[2]Department of Physics, Princeton University, Princeton, NJ 08544, USA

[2]Princeton Center for Theoretical Science, Princeton, NJ 08544, USA

Simons Symposium on Quantum Entanglement
February 2012



[hep-th] 29 Nov 2012

Black Hole Entanglement and Quantum Error Correction

Erik Verlinde[1*] and Herman Verlinde[2,3†]

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[2]Department of Physics, Princeton University, Princeton, NJ 08544, USA
[2]Princeton Center for Theoretical Science, Princeton, NJ 08544, USA

"… one can reconstruct local effective field theory observables that probe the black hole interior, ... The reconstruction makes use of the formalism of quantum error correcting codes, …"

Precursor to:

- Papadodimas, Raju: 1310.6335
- Almheiri, Dong, Harlow: 1411.7041
- Harlow: 1607.03901
- Akers, Engelhardt, Harlow, Penington, Vardhan: to appear

…

Counting of States

1996 – 1997

Elliptic Genera of Symmetric Products and Second Quantized Strings

Robbert Dijkgraaf[1], Gregory Moore[2],
Erik Verlinde[3] and Herman Verlinde[4]

[1] *Mathematics Dept, Univ. of Amsterdam, 1018 TV Amsterdam*

[2] *Physics Dept, Yale University, New Haven, CT 06520*

[3] *Theory Division, CERN, CH-1211 Geneva 23, and*
Inst. for Theor. Physics, University of Utrecht, 3508 TA Utrecht

[4] *Inst. for Theor. Physics, Univ. of Amsterdam, 1018 XE Amsterdam*

BPS Spectrum of the Five-Brane and Black Hole Entropy

Robbert Dijkgraaf

Department of Mathematics
University of Amsterdam, 1018 TE Amsterdam

Erik Verlinde

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Institute for Theoretical Physics
Universtity of Utrecht, 3508 TA Utrecht

and

Herman Verlinde

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University of Amsterdam, 1018 XE Amsterdam
and
Joseph Henry Laboratories
Princeton University, Princeton, NJ 08544

Counting Dyons in $N = 4$ String Theory

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Princeton University, Princeton, NJ 08544

Matrix String Theory

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Department of Mathematics
University of Amsterdam, 1018 TV Amsterdam

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Universtity of Utrecht, 3508 TA Utrecht

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Institute for Theoretical Physics
University of Amsterdam, 1018 XE Amsterdam

Asymptotic Density of States in High Energy

Nuclear Physics B270 [FS16] (1986) 186–204
North-Holland, Amsterdam

OPERATOR CONTENT OF TWO-DIMENSIONAL CONFORMALLY INVARIANT THEORIES

John L. CARDY

Department of Physics, University of California, Santa Barbara CA 93106, USA

Received 22 November 1985
(Revised 3 January 1986)

August 2000

On the Holographic Principle in
a Radiation Dominated Universe

ERIK VERLINDE

Joseph Henry Laboratories Princeton University
Princeton, New Jersey 08544

$d = 2$

$d \geq 2$
and holographic

Asymptotic Density of States in High Energy **with Global Symmetry**

The probability P_R of a randomly chosen state of CFT_d with global symmetry G at high temperature T to be in an irreducible unitary representation R is:

- $P_R = \dfrac{(\dim R)^2}{|G|}$, when G is a finite group.

Harlow + H.O.: 2111.04725

- $$P_R = (\dim R)^2 \left(\frac{4\pi}{b\,T^{d-1}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b\,T^{d-1}}\right] ,$$

when G is a compact Lie group.

Kang, Lee + H.O.: 2111.04725

CFT_d at High Temperature

$$\mathrm{Tr}\left[\, e^{-\beta H} \,\right] \approx \exp(a\, T^{d-1} + \cdots) \qquad \left(a = \frac{\pi^2}{6}(c_L + c_R) \text{ for } d = 2 \right)$$

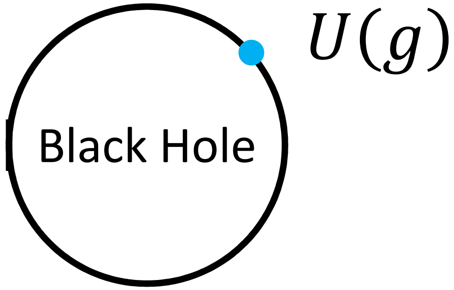
If CFT_d has global symmetry G:

$$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right] \approx \delta(g, 1) \times (\, T\text{-dependent factor} \,)$$

Harlow and I showed this when CFT is **holographic** and G is a **finite group**. First, I am going to review the derivation.

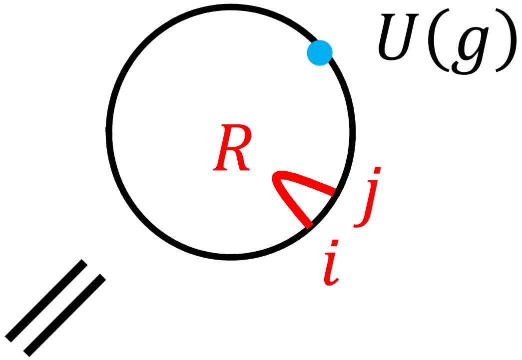
I will then present a general argument which works for **any unitary CFT** and for **any compact groups** including Lie groups.

If CFT is holographic,

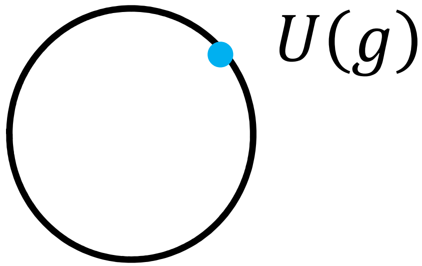
$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right] =$ 

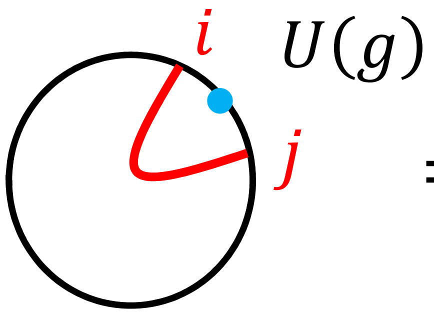
at high temperature.

For any representation R:

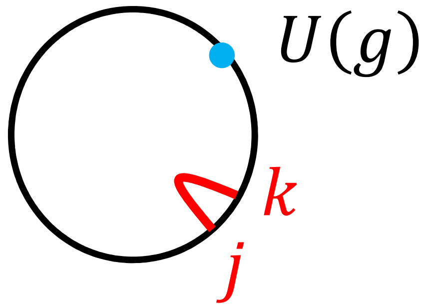


$\rightarrow \delta_i^j$

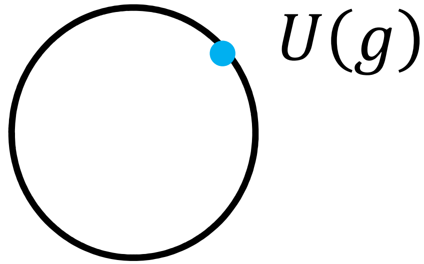




$= D_R(g)_i^k$

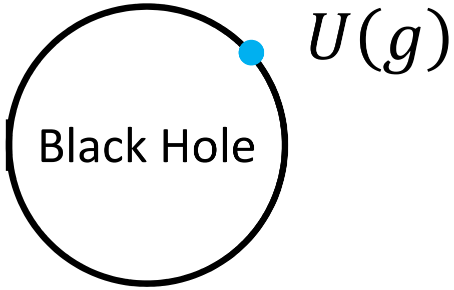


$\rightarrow D_R(g)_i^j$



If CFT is holographic,

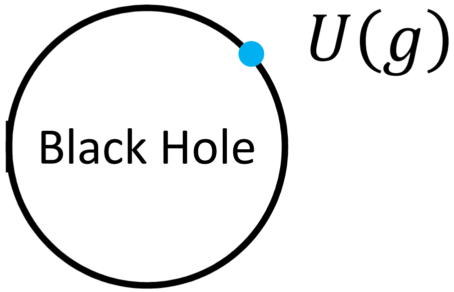
$$\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] =$$



at high temperature.

For any representation R:

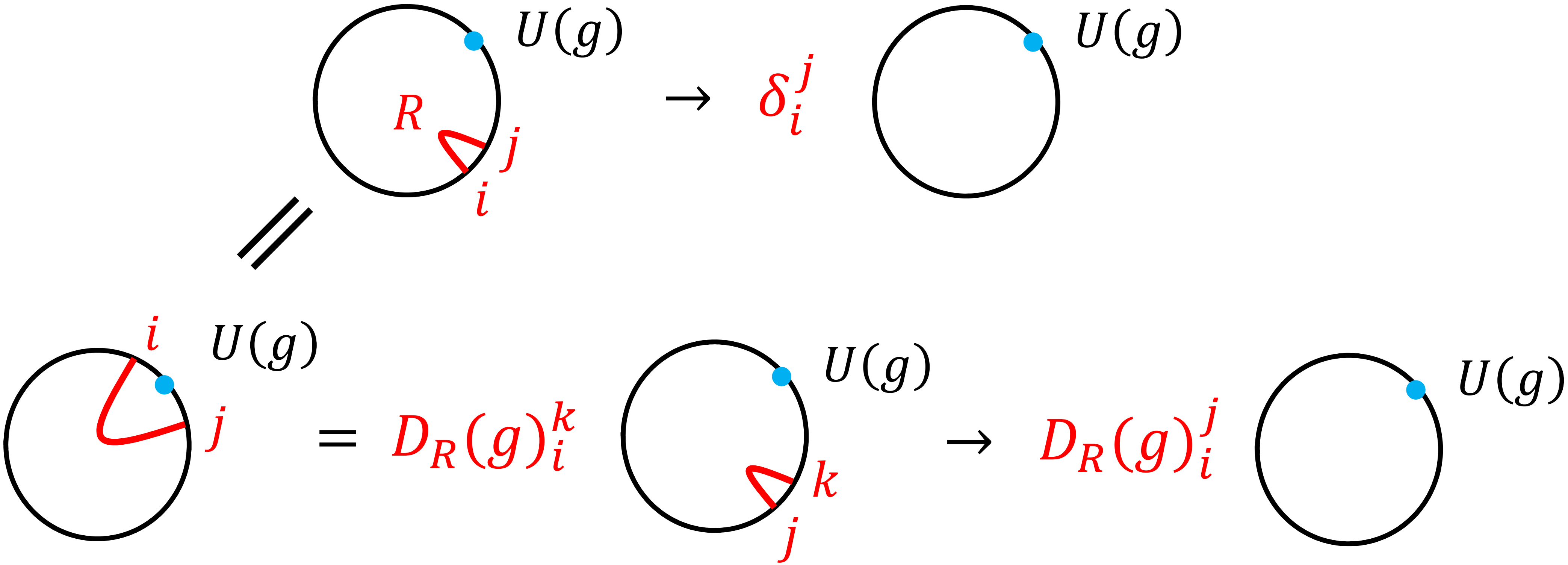
$$\left(\delta_i^j - D_R(g)_i^j\right)$$



$$\approx 0$$

Therefore,

$$\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] \approx \delta(g,1) \times (\, T\text{-dependent factor}\,)$$



- This argument does not apply when G is a Lie group since the gauge flux may be non-zero.
- It would also be nice to generalize this to non-holographic theories.

High-Temperature Effective Field Theory

* Thanks to David Simmons-Duffin for discussion

Consider a background gauge field coupled to G on $S^1_\beta \times \Sigma_{d-1}$.
At high temperature, we can write down a low energy effective action by dimensionally reducing on S^1_β,

$$S_{\rm eff} = \int_{\Sigma_{d-1}} \sqrt{g} dx^{d-1} \left(T^{d-1} V\left(e^{i\phi}\right) + \cdots \right) \ ,$$

where $e^{i\phi} \in G$ is the holonomy around S^1_β.
The kinetic terms such as $(D\phi)^2$ and F^2 are suppressed by $1/T$.

By diffeomorphism invariance, the potential $V\left(e^{i\phi}\right)$ at high temperature is independent of the geometry of Σ_{d-1}.

High-Temperature Effective Field Theory

$$S_{\mathrm{eff}} = \int_{\Sigma_{d-1}} \sqrt{g}\, dx^{d-1} \left(T^{d-1} V\left(e^{i\phi}\right) + \cdots \right)$$

By setting ϕ to be constant,

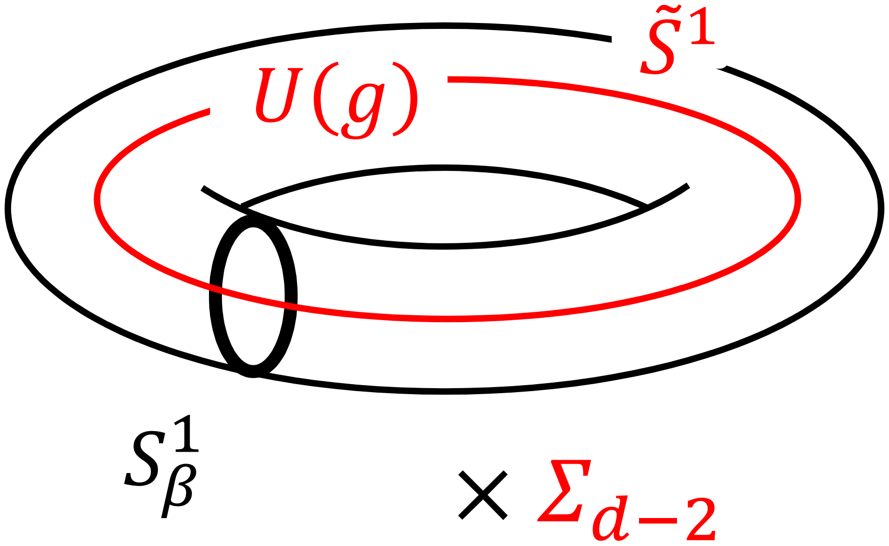
$$\mathrm{Tr}\left[\, U\left(g = e^{i\phi}\right) e^{-\beta H} \,\right] \approx e^{-T^{d-1}\,\mathrm{vol}(\Sigma_{d-1}) V\left(e^{i\phi}\right)} \times \mathrm{Tr}\left[\, e^{-\beta H} \,\right]$$

By choosing $\Sigma_{d-1} = \tilde{S}^1 \times \Sigma_{d-2}$, exchanging S^1_β and $\tilde{S}^1$, and rescaling the whole spacetime $S^1_\beta \times \tilde{S}^1 \times \Sigma_{d-2}$ by T, we can identify:

$V(g)$ = **tension of the domain wall** implementing the twist by g.

High-Temperature Effective Field Theory

$$\mathrm{Tr}\left[\, U\left(g = e^{i\phi}\right) e^{-\beta H} \,\right] \approx e^{-T^{d-1}\,\mathrm{vol}(\Sigma_{d-1}) V\left(e^{i\phi}\right)} \times \mathrm{Tr}\left[\, e^{-\beta H} \,\right]$$



After the rescaling, we can interpret:

$$T \times \left(T^{d-2}\,\mathrm{vol}(\Sigma_{d-2})\right) \times V(g)$$

- T ↑ Circumference of rescaled $\tilde{S}^1$
- $T^{d-2}\,\mathrm{vol}(\Sigma_{d-2})$ ↑ Volume of rescaled Σ_{d-2}
- $V(g)$ ↑ Domain wall tension

Since it costs energy to twist,
the tension $V(g)$ has the global minimum at $g = 1$.

High-Temperature Effective Field Theory

Since $V(g)$ has the global minimum at $g = 1$,

$$\exp\left(-\,T^{d-1}\,\mathrm{vol}(\Sigma_{d-1})V(g)\right) \propto \delta(g,1) \text{ at high temperature.}$$

Therefore,

$$\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right] \approx \delta(g,1) \times (\,T\text{-dependent factor}\,)\,.$$

If G is a finite group, $\delta(g,1) = \sum_R \frac{\dim R}{|G|}\,\chi_R(g)$.

The density of high energy states $\rho_R(E)$ transforming in R is then,

$$\rho_R(E) = \frac{(\dim R)^2}{|G|}\,\rho(E)\,.$$

High-Temperature Effective Field Theory

If G is a compact Lie group,

$\mathrm{vol}(\Sigma_{d-1})V\left(e^{i\phi}\right) = a - \frac{b}{4}\langle\phi,\phi\rangle + \cdots$, for some $b \geq 0$, and

$\mathrm{Tr}\left[\,U\left(g = e^{i\phi}\right)e^{-\beta H}\,\right] \approx \exp\left(a\,T^{d-1} - \frac{b}{4}T^{d-1}\langle\phi,\phi\rangle + \cdots\right).$

Since $T^{(\mathrm{d}-1)\dim G/2}e^{-\frac{b}{4}T^{d-1}\langle\phi,\phi\rangle}$ is a solution to the heat equation on the group manifold G with $\tau = 1/T^{d-1}$ as the time variable,

$$\frac{\mathrm{Tr}\left[\,U(g)\,e^{-\beta H}\,\right]}{\mathrm{Tr}\left[\,e^{-\beta H}\,\right]} \approx \left(\frac{4\pi}{b\,T^{d-1}}\right)^{\dim G/2} \sum_R \dim R \cdot \chi_R(g)\exp\left(-\frac{c_2(R)}{b\,T^{d-1}}\right)$$

High-Temperature Effective Field Theory

If G is a compact Lie group,

$$\frac{\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]}{\mathrm{Tr}[\, e^{-\beta H} \,]} \approx \left(\frac{4\pi}{b\, T^{d-1}}\right)^{\dim G/2} \sum_{R} \dim R \cdot \chi_R(g) \exp\left(-\frac{c_2(R)}{b\, T^{d-1}}\right)$$

The probability P_R of a randomly chosen state of at high temperature T to be in a representation R is:

$$P_R = \ (\dim R)^2 \ \left(\frac{4\pi}{b\, T^{d-1}}\right)^{\dim G/2} \exp\left(-\frac{c_2(R)}{b\, T^{d-1}}\right)$$

Examples

We have verified

$$\mathrm{Tr}\left[\, U\left(g = e^{i\phi}\right) e^{-\beta H} \,\right] \approx \exp\left(a\, T^{d-1} - \frac{b}{4} T^{d-1}\langle \phi, \phi \rangle + \cdots \right).$$

for free field theories and holographic CFTs, and computed the coefficients a and b.

- For free massless scalars with $G = U(1)$,
  - $a = 2\zeta(d)$ and $b = 4\zeta(d-2)$ for $d \geq 4$.
- For free Weyl spinors with $G = U(1)$,
  - $a = 3\zeta(3), b = 16 \log 2$ for $d = 3$
  - $a = \zeta(2) = \pi^2/6, b = 1$ for $d = 2$

Examples

In holographic CFTs with non-abelian G,
there are **two types of bulk geometries** relevant to
$\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]$ above the Hawking-Page transition:

1. The Reissner–Nordström solution with a commutative gauge field in the non-abelian theory by $U(1)^{\mathrm{rank}\,G} \in G$
2. The genuinely non-abelian solution (*i.e.*, with non-abelian hair)

Their relative stability has been an outstanding question.

Reviews: Volkov, Galt'sov: 9810070
Winstanley: 0801.0527

Examples

In holographic CFTs with non-abelian G,
there are **two types of bulk geometries** relevant to $\mathrm{Tr}\left[\, U(g)\, e^{-\beta H} \,\right]$ above the Hawking-Page transition:

- Our task is simplified since the two solutions converge at high temperature.
- We found that the **solution with non-abelian hair is more stable** if we take $1/T$ effects into account.

$$a = \left(\frac{4\pi}{d}\right)^{d-1} \frac{w_{d-1}\ell^{d-1}}{4dG_N}\,, \qquad b = \left(\frac{4\pi}{d}\right)^{d-2} \frac{4(d-2)w_{d-1}\ell^{d-1}}{e^2}$$

G_N: Newton constant, e: gauge coupling, ℓ: AdS radius, w_{d-1}: area of the unit sphere

Asymptotic Density of States in High Energy **with Global Symmetry**

The probability P_R of a randomly chosen state of at high temperature T to be in a representation R is:

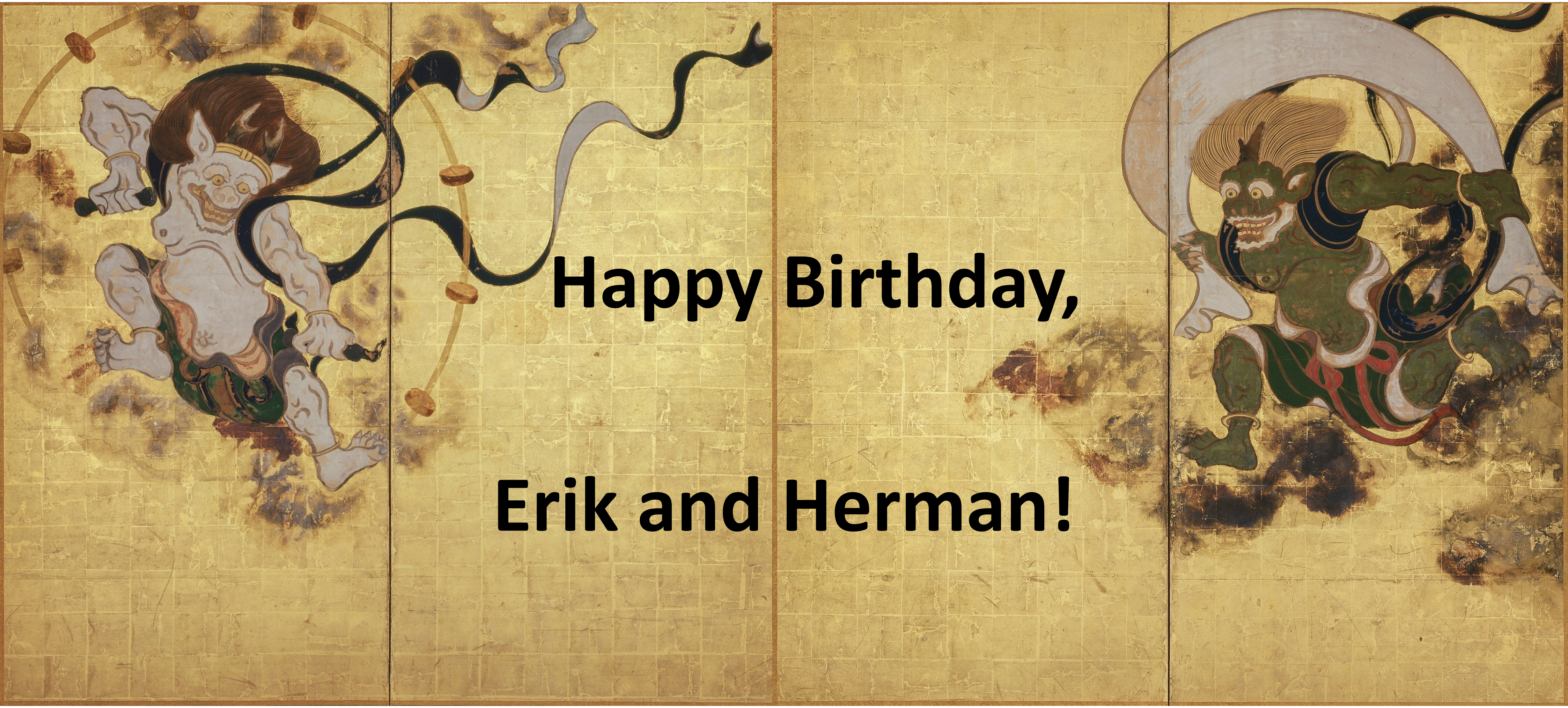
- $$P_R = \frac{(\dim R)^2}{|G|}\,,$$ when G is a finite group.

Harlow + H.O.: 2111.04725

- $$P_R = (\dim R)^2 \left(\frac{4\pi}{b\,T^{d-1}}\right)^{\dim G/2} \exp\left[-\frac{c_2(R)}{b\,T^{d-1}}\right]\,,$$

when G is a compact Lie group.

Kang, Lee + H.O.: 2111.04725



I wish you many more adventures and discoveries.