Problem 1. Let S be a non-empty polyhedral cone, i.e., $S := \{x \in \mathbb{R}^n : Ax \leq 0\}$, for some $A \in \mathbb{R}^{m \times n}$. Recall that S° denotes the polar cone of S.

1. Consider another non-empty polyhedral cone $\bar{S}$. Show that $(S \cap \bar{S})^\circ = S^\circ + \bar{S}^\circ$.
2. Consider two systems of linear equalities and inequalities:

$$x : \sum_{i=1}^{n} x_i > 0, \; x \in S \quad \text{and}$$

$$y : \; A^\top y = e, \; y \in \mathbb{R}_+^m, \text{ where } e \text{ is the vector of all ones.}$$

   Prove that these systems cannot be feasible simultaneously. Briefly explain how this question could be related to polar cones.

3. Assume that one of the above two systems is definitely infeasible. Does the other one have to be feasible? Motivate your answer using the results discussed in class.

Problem 2. Consider $S, \bar{S} \subseteq \mathbb{R}^n$, and let $H := \{x : a^\top x = b\}$ be a hyperplane with a normal vector a and a constant term b that separates S and $\bar{S}$. Recall that H *strictly* separates S and $\bar{S}$ if $a^\top x < b < a^\top y$ for all $x \in S$, $y \in \bar{S}$.

1. Let both S and $\bar{S}$ be open, not necessarily convex. Prove that H strictly separates S and $\bar{S}$.
2. Provide an example of *disjoint* S and $\bar{S}$ such that both of them are non-empty, closed and convex, and they cannot be strictly separated by a hyperplane. Explain why your example satisfies the required conditions.