EAST 24 Sept 16-20 2024.

Algebraic K-Theory & Trace Methods

K-Theory Why? How? ✓ Examples ✓

Cyclic Homology Why? How? ✓ Examples ✓

Initially taken for granted
- do you want/need anything about this?

Premise:

K-theory is interesting
&
we badly want to calculate it.

Problem:

Very hard to calculate - essentially only finite fields ☹

when trace methods enter

Idea:

Approximate K-theory
by a functor
- you can calculate
- which allows you to extend the calculations you do have to fat enough neighborhoods to capture interesting stuff.

$$
\begin{tikzcd}
 & \mathbb{S}[\Omega X] \arrow[dr] & & \\
\mathbb{S} \arrow[ur] \arrow[dr] \arrow[d] & & \mathbb{S}[\tau,X] \arrow[dd] & \\
MU \arrow[r] & \mathbb{Z} \arrow[ur] \arrow[d] & \subseteq \mathcal{O}_F \arrow[r] & F \\
 & \boxed{\mathbb{F}_p} \arrow[r] & \mathbb{F}_p[\tau,X] &
\end{tikzcd}
$$

Plan

$$\begin{tikzcd}
\mathrm{Cat}_\infty^{perf} \arrow[r, "K"] \arrow[d] & \mathrm{Spt} \arrow[d] \\
\mathrm{Cat}_\infty^{poly} \arrow[r] & \mathrm{Spc}
\end{tikzcd}$$

Barwick h.al.

ANALOGY (Lie groups)

$K_0 A = $ iso'd f.g. proj A-mod $/ \sim$ — are projectives free? ideal class gp. finiteness obstruction

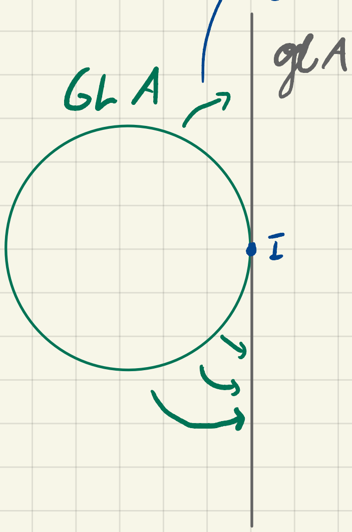
$K_1 A \cong GL\,A / EA = H_1\, BGL\,A$. — can we do Gauss?

$$H_i\, KA = H_i\, \underbrace{\boxed{GL\,A}}_{\text{Think of this as a Lie group.}} \qquad i > 0$$

Think of this as a Lie group.

Trace methods indeed start from "Chern classes / GLA".

log - locally an isomorphism



Known in many cases

coalg str on H_* of spaces

$$K_i A \otimes \mathbb{Q} \overset{i>0}{\cong} \mathrm{Prim}\, H_i(\mathbb{K}A; \mathbb{Q}) = \mathrm{Prim}\, H_i(GL\,A; \mathbb{Q})$$

$$\mathrm{Prim}\, H_i(\mathfrak{gl}\,A; \mathbb{Q}) = ?$$

Tsygan Loday Quillen (early 80's)

$$\mathrm{Prim}\, H_i(\mathfrak{gl}\, A; \mathbb{Q}) \cong \text{"}HC_{i-1}\, A \otimes \mathbb{Q}\text{"}$$

important features:

there's an invariant theory story

$$C_n \subseteq \Sigma_n \curvearrowright \mathfrak{gl}_n A$$

A "free" action is peeled away leaving only cyclic actions (more precisely?)

- cyclic homology / Connes
- Morita invariance - removes the matrices

orbits of the cyclic action

Cf. differential forms / Chern classes

$$\Omega^*_A = \text{Kähler differentials}$$

$$= \Lambda^*_A \Omega^1_A \qquad \Omega^1_A = A\{da\}_{a \in A} / d(ab) = a\,db + (da)\cdot b$$

$$\begin{array}{ccc} K_1 A & \longrightarrow & \Omega^1_A \\ \uparrow & & \cup \\ GL_1 A \ni g & \longmapsto & d\log g = \frac{dg}{g} \end{array}$$

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In the smooth case

$$\begin{tikzcd} \Omega^*_A \arrow[r, phantom, "\cong"] \arrow[d, "d"] & HH_* A \arrow[d] \\ \Omega^{*+1}_A \arrow[r, phantom, "\cong"] & HH_{*+1} A \end{tikzcd}$$

differential ($\leftrightarrow$ curvature)

Connes' B-operator ($\leftrightarrow$ cyclic structure)

Homology = de Rham Homology = cyclic homology

LQT

$$\mathrm{Prim}\, H_*(gl\, A; \mathbb{Q}) \cong HC_{*-1} A \otimes \mathbb{Q}.$$

GOAL:

Realize this idea: get a theory T like

Ω^*_A – the tangent space or even better

H^*_{dR} – knows also about curvature (why?: so that it isn't only a linear approximation)

$K \to T$ which is an equivalence locally

What does K-theory measure?

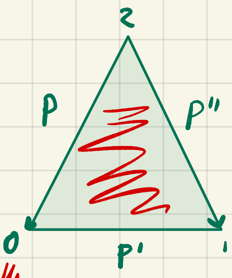
A - ring $\qquad \mathcal{P}_A$ fin gen proj

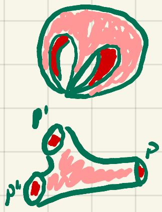
$\qquad\qquad\qquad\qquad \Downarrow$

$$0 \to P' \hookrightarrow P \twoheadrightarrow P'' \to 0 \text{ surj}$$

choice

$$P \cong P' \oplus P''$$





Idea:

K-Theory is the universal device "forgetting" (functorially) this choice.

in the sense that

$$K\left(\begin{array}{c}\text{exact sequences}\\ 0 \to P' \to P \to P'' \to 0\end{array}\right) \xrightarrow{\sim} K(\text{source}) \times K(\text{target})$$

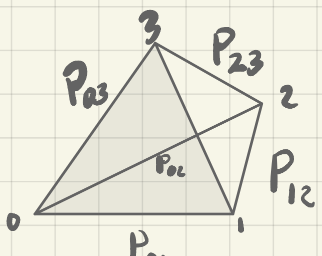
equivalence

NOTE: even
[if SES's do not split,
K-theory should still
split them]

(as driven home by Waldhausen's S-construction:

q-simplices are flags of higher coherencies

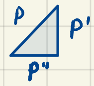
$$[q] \longmapsto \left\{ \begin{array}{ccccccc|c}
P_{01} & \rightarrowtail & P_{02} & \rightarrowtail & P_{03} \rightarrowtail & \cdots & \rightarrowtail P_{0q} & \quad i<j<k \\
 & & \downarrow & & \downarrow & & \downarrow & \text{s.t. } P_{ij} \rightarrowtail P_{ik} \\
 & & P_{12} & \rightarrowtail & P_{13} \rightarrowtail & \cdots & \rightarrowtail P_{1q} & \text{SES.} \qquad \downarrow \\
 & & & & \downarrow & & \downarrow & \qquad\qquad P_{jk} \\
 & & & & P_{23} \rightarrowtail & & \rightarrowtail P_{2q} & \\
 & & & & & & \downarrow & \\
 & & & & & \vdots & P &
\end{array} \right\}$$



Break till here.

RIGHT K_0

$$K_0 \mathcal{C} := \pi_0 \Omega S\mathcal{C} = \pi_1 S\mathcal{C}$$



$P' \to P \to P''$ SES

$$= \text{free gp}(\mathrm{ob}\,\mathcal{C})/\sim$$

$\longleftarrow$ because $P \cong Q \to 0$ SES.

$$= \text{free gp}(\mathrm{iso\ cl}\,\mathcal{C})/\sim$$

$\longleftarrow$ because $P' \oplus P'' \cong P'' \oplus P'$

$$= \mathbb{Z}\{\mathrm{iso\ cl}\,\mathcal{C}\} / [P'] + [P''] \sim [P]$$

ADDITIVITY

$$K_0 S_2 \mathcal{C} \rightleftarrows K_0 \mathcal{C} \oplus K_0 \mathcal{C} \qquad \text{isomorphism}$$

$$[P' \to P \to P''] \longmapsto ([P'], [P''])$$

$$[P' \to P' \oplus P'' \to P''] \longleftarrow\!\shortmid \quad \text{— " —}$$

Pf

$\circlearrowright$ id ✓

$\circlearrowleft$ is id because of the SES in $S_2 \mathcal{C}$

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & P' \arrow[r, equal] \arrow[d, equal] & P' \arrow[r] \arrow[d] & 0 \arrow[r, equal] \arrow[d] & 0 \\
0 \arrow[r] & P' \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & P'' \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & P'' \arrow[r, equal] \arrow[d] & P'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

: $[P' \to P \to P''] = [P' = P' \to 0] + [0 \to P'' \to P'']$

only depends on $[P']$ & $[P'']$.

$\square$

Thm (Waldh[ausen], McCarthy)

$k \longmapsto S^{(k)}\mathcal{C} \circlearrowleft \Sigma_k$ permute simpl[icial] direction is a "positive symmetric spectrum"

$$\mathcal{C} \longrightarrow \Omega S\mathcal{C} \xrightarrow{\sim} \Omega^2 S^2\mathcal{C} \xrightarrow{\sim} \qquad \xrightarrow{\sim} \Omega^\infty S^\bullet \mathcal{C}$$

$S^{(2)}\mathcal{C} = \operatorname{diag}\{[s],[t] \mapsto S_s \boxed{S_t \mathcal{C}}\}$

as category, $S_t\mathcal{C}$ has all maps $\nabla \to \nabla$

(cof's satisfy a Reedy-Type condition)

$$S S_2 \mathcal{C} \xrightarrow{\sim} S\mathcal{C} \times S\mathcal{C}$$

induction

$$S S_q \mathcal{C} \xrightarrow{\sim} S\mathcal{C}^{\times q}$$

"bar construction"

$$\mathcal{C} \to \Omega S\mathcal{C} \xrightarrow{\quad\sim\quad} \Omega^2 SS\mathcal{C} \xrightarrow{\sim} \qquad \xrightarrow{\sim} \Omega^k S^{(k)}\mathcal{C} \xrightarrow{\sim}$$

$$\begin{array}{ccc} S^1 \wedge \mathcal{C} & \longrightarrow & S\mathcal{C} \\ \| & & \downarrow \\ \{q \mapsto S^1_q \wedge \mathcal{C}\} & & \mathcal{C}^{\times q} \\ \| & & \\ \mathcal{C}^{\vee q} & & \end{array}$$

$\underbrace{k \longmapsto S^{(k)}\mathcal{C}}_{K\mathcal{C}}$ is a "positively fibrant" symmetric spectrum.

Since $K\mathcal{C}$ is connective &

$$\Omega^\infty K\mathcal{C} \simeq \Omega S\mathcal{C}$$

we can choose whether we want to work stably or unstably according to what's convenient.

On uniqueness I

unstable &
- exact category version

(1992)
[McCarthy's insight: "mix F & S"].
cf Tabuada et al ~ 13

K-theory is initial / htt among "additive functors under the core"

In the sense that given

1) $\Omega i S \overset{x}{\Rightarrow} F : \mathcal{E}x \to \mathcal{S}pc$

2)
$$\begin{tikzcd} \Omega i S\mathcal{C} \arrow[r, "x_\mathcal{C}"] \arrow[d] \arrow[dr, phantom, "\circ"] & F\mathcal{C} \arrow[d, "s"] \\ \Omega^2 i SS\mathcal{C} \arrow[r, "\Omega x_{S\mathcal{C}}"'] & \Omega F S\mathcal{C} \end{tikzcd}$$
"Additive"

3)
$$\begin{tikzcd} & i\mathcal{C} \arrow[dl] \arrow[dr, "\eta_\mathcal{C}"] & \\ \Omega i S\mathcal{C} \arrow[rr, "x_\mathcal{C}"'] & & F\mathcal{C} \end{tikzcd}$$
"under the core"

$\Rightarrow$
$$\begin{tikzcd} & \Omega i S\mathcal{C} \arrow[dl, "x_\mathcal{C}"'] \arrow[dr, "\Omega\eta_{S\mathcal{C}}"] \arrow[d, phantom, "\circ"] & \\ F\mathcal{C} \arrow[rr, "\sim"] & & \Omega F S\mathcal{C} \end{tikzcd}$$

$\Uparrow$

$$\begin{tikzcd} \Omega i S\mathcal{C} \arrow[d, "s"] \arrow[dr, bend left, "\Omega\eta_{S\mathcal{C}}"] \arrow[dd, bend right=60, "x_\mathcal{C}"'] & \\ \Omega^2 i SS\mathcal{C} \arrow[r, "\Omega x_{S\mathcal{C}}"] & \Omega F S\mathcal{C} \\ F\mathcal{C} \arrow[ur, bend right, "s"'] & \end{tikzcd}$$
commutes.

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2

With spectral enrichment & equivalences
(as opposed to isomorphisms)
one simply localises the above.

Weak equivalences

Waldhausen sets up the theory for
"categories w/ cofibrations & weak equivalences"

~~Axioms?~~ done ["Waldhausen cats"]

Note When $\mathcal{C}$ has a notion of weak eq.

$w\mathcal{C}$ (as opposed to isos only)

then we localize along these.

One way to do this is to replace $\mathcal{C}$

by the simpl cat $N^w\mathcal{C} := \{ [q] \mapsto \mathrm{Fun}^{\mathrm{full}}([q], w\mathcal{C}) \overset{\mathrm{full}}{\subseteq} \mathrm{Fun}([q], \mathcal{C}) \}$

(inheriting whatever enrichment $\mathcal{C}$ may have)

& setting "reversal of priorities"

$$K\mathcal{C} := K N^w\mathcal{C} = k \longmapsto S^{(k)} N^w \mathcal{C} = \text{"} wS^{(k)}\mathcal{C} \text{"}$$

It is also possible to localize so that the weak equivalences are actually replaced by isomorphisms - an advantage when theory applied to more "fragile" functors.
cf. DGM §5.

Lemma. [3.1.4.1].

K-theory of split radical extensions

we only do $\mathbb{Z}$-algs now

a "radical extension": is a map $B \xrightarrow{f} A$ of ( )-algebras

s.t. $\forall/\exists\, n$ s.t.

$$\begin{tikzcd} GL_n B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & GL_n A \arrow[d] \\ M_n B \arrow[r] & M_n A \end{tikzcd}$$

is cartesian

[For rings this is eq. to the usual def & $n=1$] includes Hensel, nilpotent...

Nakayama $M \in$ f.g B-mod & $MI = M \Rightarrow M = 0$

[& for connective $\mathbb{S}$-alg it is eq. to $\pi_0 B \to \pi_0 A$ being radical]

Assume f is split radical ext of rings $B = A \oplus I \twoheadrightarrow A$

[$\ker\{GL_n(A\oplus I) \to GL_n A\} = (1 + M_n I)^\times \cong M_n A \otimes_A I \cong \mathrm{Hom}_A(A^n, A^n \otimes_A I)$]

$$K B = \{ n \mapsto S^{(n)} P_B \}$$

"move along isomorphisms"

$$\simeq \{ n \mapsto \coprod_{m \in S^{(n)} P_A} \underbrace{B\, \mathrm{Hom}(m, m \otimes_A I)}_{\text{a connected space!}} \}$$

$$\& \; KB/KA \simeq \{ n \mapsto \bigvee_{m \in S^{(n)} P_A} B\, \mathrm{Hom}(m, m \otimes_A I) \}$$

(note: $\mathrm{Hom}'(m, m \otimes_A I)$ is a gp w. composition

$f, g : m \to m \otimes_A I$

$$f \cdot g = (1+f)(1+g) - 1 = f + g + f \circ g$$

$$f \circ g : m \xrightarrow{g} m \otimes_A I \xrightarrow{f \otimes 1} m \otimes_A I \otimes_A I \xrightarrow{\mu} m \otimes_A I$$

When $I^2 = 0$ we get

$$f \cdot g = f + g \;\;\&$$

$$BA\mathrm{mod}(m, m \otimes_A I) \cong A\text{-}\mathrm{mod}(m, m \otimes_A BI) \;!$$

K-theory of split square zero extensions

When $I^2=0$ we get

$$f\cdot g = f+g \quad \&$$

$$BA\text{mod}(m, m\otimes_A I) \cong A\text{-mod}(m, m\otimes_A BI)$$

!

Lemma For I an A-bimodule, let

$$A\oplus I \to A$$

be the sq. zero extension

Then

$$\frac{K(A\oplus I)}{KA} \simeq \left\{ n \longmapsto \bigvee_{m\in S^{(n-1)}P_A} \mathrm{Hom}(m, m\otimes_A BI) \right\}_n$$

Say someth abt "Endomorphisms"?!

$$\frac{K\,A[\varepsilon]/\varepsilon^2}{KA} \simeq \Sigma\frac{K\,\mathrm{End}\,A}{KA}$$

zero endos

each of these are just spaces!

Recap & start constructing the trace

$$K_0 \quad \overset{\text{exact}}{\underline{}} \quad \overset{\text{iso}}{\underline{}} \qquad \text{ab}$$

$$K : \text{cat w. cofs \& we.} \longrightarrow \text{Spt}$$

initial among functors s.t.

$$\text{Free } \overset{\mathbb{Z}}{\text{ab gp}} \backslash \overset{\text{under}}{} \text{iso } \mathcal{C} \longrightarrow K_0 \mathcal{C}$$

$$\text{Free } \underset{\Sigma^\infty}{\text{Spt}} \backslash w\mathcal{C} \longrightarrow K\mathcal{C}$$

$$K_0 S_2 \mathcal{C} \xrightarrow{\simeq} K_0 \mathcal{C} \times K_0 \mathcal{C}$$

$$K S_2 \mathcal{C} \xrightarrow{\simeq} K\mathcal{C} \times K\mathcal{C}$$

\+ Morita invariance & preservation of filtered colimits.

—— a more precise statement is given by Blumberg, Tabuada & Gepner (2013)

So to find invariants for K-theory we should look for functors F satisfying

1) $\Sigma^\infty w\mathcal{C} \longrightarrow F\mathcal{C}$

2) $F S_2 \mathcal{C} \xrightarrow{\sim} F\mathcal{C} \times F\mathcal{C}$

\+ Morita invariance & preservation of filtered colimits.

– Develop:

2) essentially says $F S\mathcal{C} \simeq \Sigma F\mathcal{C}$.

"F is stabilized by S"

THE TRACE – OUTLINE

$$HH(A) = HH(A|\mathbb{S}) \quad (= THH\,A)$$

[but $HH_i A \to HH_i(A|\mathbb{Z})$ is iso for $i \leq 1$ because $\mathbb{S} \to \mathbb{Z}$ is 1-connected]

$$\begin{array}{ccc} x & \longmapsto & (x, \mathrm{id}_x) \in \bigoplus_c \mathrm{End}(c) \\ \mathrm{core}\,\mathcal{C} & \longrightarrow & HH\,\mathcal{C} \end{array}$$

$HH\, S_2 P_A \simeq HH\, P_A \times HH\, P_A$ $\quad(\Rightarrow)$ Additivity

$HH\, P_A \simeq HH\, F_A$ $\quad$ Cofinality

$HH\, F_A \simeq HH\, A.$ $\quad$ Morita invariance

$$\boxed{KA = \mathrm{core}\, S^\bullet P_A \longrightarrow HH\, S^\bullet P_A \xleftarrow[\text{Additivity}]{\sim} HH\, P_A \xleftarrow[\text{Morita + Cof.}]{\sim} HH\, A.}$$

$\uparrow$ bispectrum

For $F = HH_0$ these conditions are satisfied

Additivity

$$HH_0\mathcal{C} = \bigoplus_{c\in ob\,\mathcal{C}} \mathrm{End}\,c \underset{f_1 f_0}{\overset{f_0 f_1}{\leftleftarrows}} \bigoplus_{c_0,c_1} \mathcal{C}(c_1,c_0)\otimes\mathcal{C}(c_0,c_1) \quad f_0\otimes f_1$$

First do $\boxed{HH_0 S^2 P_A \cong HH_0 P_A \otimes HH_0 P_A}$

All SES' in P_A split, so

$$\left(HH_0\mathcal{C} \leftarrow \bigoplus_{c\in\mathcal{C}} \mathrm{End}\,c \leftleftarrows \bigoplus_{c_0,c_1} \mathcal{C}(c_1,c_0)\otimes\mathcal{C}(c_0,c_1) \right)$$

$$\mathrm{End}\left(\begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix}\right) \hookleftarrow \left\{ \begin{tikzcd} p' \arrow[r] \arrow[d] & p' \arrow[d] \\ p \arrow[d] & p \arrow[d] \\ p'' \arrow[r] \arrow[uur] & p'' \end{tikzcd} \right\}$$

Note $d_0 \sim d_1$: $\nearrow$, 0 $\longleftarrow\!|$ $\left(\nearrow\right)\otimes\left(\xrightarrow{=}\right) \in \mathrm{Hom}\left(\begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix}, \begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix}\right) \otimes \mathrm{Hom}\left(\begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix}, \begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix}\right)$

So "diagonals" die & $S_2 P_A \longrightarrow P_A \times P_A$ $\quad \begin{matrix} p' \\ \downarrow \\ p \\ \downarrow \\ p'' \end{matrix} \mapsto (p', p'')$

induce iso on $HH_\bullet$

for $\bullet > 0$ extend w. explicit simpl. homotopy

$$\begin{tikzcd} p' \arrow[r, equal] \arrow[d] & p' \arrow[r, "f'"] \arrow[d] & p' \arrow[r, equal] \arrow[d] & p' \arrow[d] \\ p'\oplus p'' \arrow[r, "\cong"] \arrow[d] & p \arrow[r, "f"] \arrow[d] & p \arrow[r, "\cong"] \arrow[d] & p'\oplus p'' \arrow[d] \\ p'' \arrow[r, equal] & p'' \arrow[r, "f''"] & p'' \arrow[r, equal] & p'' \end{tikzcd}$$

$$\alpha^{-1} f \alpha = \begin{bmatrix} f' & F \\ 0 & f'' \end{bmatrix} \left(\cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} f' & 0 \\ 0 & 0 \end{bmatrix} \right.$$

$$-\shortparallel- \cdot \quad \begin{bmatrix} f' & F \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & F \\ & f'' \end{bmatrix} \left(\cdot \begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix} \right. = \begin{bmatrix} 0 & F \\ & f'' \end{bmatrix}$$

$$-\shortparallel- \quad \begin{bmatrix} 0 & 0 \\ 0 & f'' \end{bmatrix}$$

Morita Invariance

Ex $\quad F = HH_0 \qquad \overset{\text{f.g free } A\text{-mod}}{\mathcal{F}_A} \longleftrightarrow \overset{\text{full subcat rk } 1}{\{A\}} = A$

$$HH_0 \mathcal{F}_A \xleftarrow{\sim} HH_0 A$$ induced by $\tilde{\mathcal{F}}_A(A^m, A^n) \cong M_{m\times n} A$

cofinal $\quad$ filtered colim. $\quad$ rk $\leq k$

$$\mathcal{P}_A \supseteq \mathcal{F}_A \supseteq \dots \supseteq \mathcal{F}_A^k \supseteq \quad \supseteq \mathcal{F}_A^2 \supseteq \mathcal{F}_A^1 = A$$

$$HH_0 \mathcal{F}_A^2 \simeq HH_0 A$$

Endomorphisms only depend on their trace

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[dr, "b"] & A \\ A \arrow[ur, "c"] \arrow[r, "d"'] & A \end{tikzcd} \;\sim\; \begin{tikzcd} A \arrow[r, "\boxed{a+d}"] & A \\ A & A \end{tikzcd}$$

trace!

↘ & ↗ ~ 0 as before

$$\xrightarrow{d} \;\sim\; \xrightarrow{d} \;\longleftarrow\!\!| \;\; \overset{d}{\nearrow} \otimes \overset{1}{\searrow}$$

endos only depend on their support.

$$\begin{tikzcd} A \arrow[r, "a"] & A \\ A & A \end{tikzcd} \;\sim\; A \xrightarrow{a} A \qquad \begin{tikzcd} A \arrow[r, "a"] & A \arrow[r, "1"] & A \\ A & & A \end{tikzcd}$$

Cofinality

$\mathcal{F} \subseteq \mathcal{P}$

$\forall \; P \in \mathcal{P} \quad \exists \quad F(P) \;\;\&$

$$\begin{tikzcd} P \arrow[r, "i_P"] \arrow[dr, equal] & F(P) \arrow[d, "\pi_P"] & \in \mathcal{P} \\ & P & \end{tikzcd}$$

e.g $\mathcal{F}_A \subseteq \mathcal{P}_A$.

$$\boxed{HH\,\mathcal{F} \xrightarrow{\sim} HH\,\mathcal{P}}$$

On HH_0:

$$P \xrightarrow{f} P \;\sim\; F(P) \xrightarrow{i_P f \pi_P} F(P)$$

$$\begin{tikzcd} & P \arrow[dr, "i_P"] & & \\ P \arrow[ur] & & F(P) \arrow[r, "\pi_P"] & P \end{tikzcd}$$

BGT 13

bare the red spurs

Even better in terms of

1.12

Thm: $E : \mathrm{Cat}^{ex}_\infty \longrightarrow \mathrm{Spt}$ additive (∞-cat formulation)

Cat^{ex}_∞: small stable ∞-cats. & exact functors (preserve fin. lims & colims) — cf BGT13 §2

Def $F : \mathcal{A} \to \mathcal{B} \in \mathrm{Cat}^{ex}_\infty$ is a 'Morita eq.' if $\mathrm{Mod}\,\mathcal{A} \to \mathrm{Mod}\,\mathcal{B}$ eq.

E is 'additive' if it inverts Morita equivalences, preserves filtered colimits, split exact seq.

$$\begin{tikzcd} \mathcal{A} \arrow[r, "ff"] \arrow[d] & \mathcal{B} \arrow[d] \arrow[dr] & \\ * \arrow[r] & \mathcal{E} & \sim \mathcal{B}/\mathcal{A} \end{tikzcd}$$

in $\mathrm{Cat}^{perf}_\infty \leftarrow$ id. compl. Exact seq. split

in Cat^{ex}_∞ (split) ex. as id.compl. is

Then

$$\boxed{\mathrm{Map}(K, E) \simeq \underbrace{E(\mathrm{Spt}^{\omega})}_{\text{comp.}} \in \mathrm{Spt}.}$$

In part.

$$\{\text{nat'l transfos } K \to HH\}/_{\text{ht}} \cong HH_0\,\mathbb{S} = \mathbb{Z}.$$

additive

$$\text{Dennis trace} \longleftrightarrow 1$$

TC is not an additive invariant, but it is approximated by such — doesn't preserve filt colims (but TC/p does CMM!) — BGT 10.11

$$TC \simeq \operatorname{holim} \{ \to TC^n \to TC^{n-1} \to \cdots \to TC^1 \}, \quad TC^1 = THH.$$

additive

So a $K \to TC$ consists of a system of compatible

$K \to TC^n$'s $\leftarrow$ class. by $TC^n(\mathbb{S})$

& so by $TC\,\mathbb{S} = \lim TC^n\,\mathbb{S} \sim_p \mathbb{S} \vee \Sigma \mathbb{CP}^\infty_{-1}$

& hence after completion at p $\{K \to_p TC\}/_{\text{ht}} \cong \pi_0\,TC\,\mathbb{S} = \mathbb{Z}_p$.

cytr $\leftrightarrow 1 \in \mathbb{Z}_p$

TC doesn't preserve filt colims because it involves bad limits, ~~like~~ $\lim_{\mathbb{N}}$ & $\lim_{S^1}$ (fixed pts) — badly present in NS!

The extraordinary thing is that

- finite fixpts of THH does (cyclo)
- mod p it does. CMM

$$\boxed{D_1 K(A)(P) \simeq THH(A, \Sigma P)}$$

" Stable K-theory is THH "

$$D_1 K(A)(P) := \lim_{\substack{|\Sigma^n| \to 0 \\ (n \to \infty)}} \frac{K(A \oplus \Sigma^n P) - K(A)}{\Sigma^n}$$

sq zero.

$\Omega^n K(A \oplus \Sigma^n P)/KA$

'derivative at A in the direction of P '

$$\simeq \Omega^\infty \left(n \longmapsto \bigoplus_{c \in S^\bullet P_A} Hom(c, c \otimes_A \Sigma^{n+1} P) \right) \quad \left(\cong \bigoplus_{c \in S^\bullet P_A.} Hom(c, c \otimes BP) \right)$$

bi sp.

(ab gp valued / spt)

$$\wr\wr$$

$$\Sigma^{n+1} \bigoplus_c Hom(c, c \otimes P)$$

Claim :

$$\bigoplus_{c \in S^\bullet P_A} Hom(c, c \otimes P) \simeq THH(S^\bullet P_A, Hom(-, - \otimes_A P))$$

$$\simeq THH(A, P)$$

So that " $K^S = THH$ " :

$$D_1 K(A)(P) \simeq \Sigma THH(A, P)$$

$(K^S(A, P) := \Sigma^{-1} D_1 K(A)(P).)$

Theorem [Stable K-theory is THH]

$S^\bullet$, Hom $\in$ Spt.

Everything stabilized, so $\oplus = \vee$, $\otimes = \wedge$

so $HH = THH$

$\boxed{\text{Must prove}}$

$$D_1 K(A)(P) \simeq HH(A, BP)$$

Know $\longrightarrow$ $\wr\wr$ $\qquad\qquad$ $\wr\wr$ $\longleftarrow \sim$ (Since HH is additive, Morita invariant, cof. etc)

$$\bigoplus_{c \in S^\bullet P_A} \mathrm{Hom}(c, c \otimes BP) \to \Big\{[q] \mapsto \underbrace{\bigoplus_{\substack{c_0, \dots, c_q \\ \in S^\bullet P_A}} \mathrm{Hom}(c_0, c_q \otimes BP) \otimes \bigotimes_{i=1}^{q} \mathrm{Hom}(c_i, c_{i-1})}_{F_q}\Big\} = HH(S^\bullet P_A, BP)$$

stabilized!

$$F_0 \underset{\text{face}}{\overset{\text{deg}}{\rightleftarrows}} F_q$$

$$\boxed{\text{face deg} = \mathrm{id}}$$

$\deg(\alpha) = \alpha \otimes \mathrm{id} \otimes \dots \otimes \mathrm{id}$

$\mathrm{face}(\alpha_0 \otimes \alpha_1 \otimes \ \otimes \alpha_q) = \alpha_0 \cdots \alpha_q : c_q \to c_q \otimes BP$

The original pf was performed in an environment where it made sense to chase elements & so we'll present it as such

See next page.

Must prove

That $\boxed{\deg \mathrm{face} \sim \mathrm{id}}$ is seen as follows: $\qquad I = BP$

define $A, B : F_q \longrightarrow F_q S_2$

$\alpha_0 \otimes \dots \otimes \alpha_q \overset{A}{\longmapsto}$

$$\begin{tikzcd}
0 \otimes I \arrow[d] & c_q \arrow[l] \arrow[d, dash] \arrow[r, equal] & c_q \arrow[d, dash] \arrow[r, equal] & \cdots \arrow[r, equal] & c_q \arrow[d, dash] & 0 \arrow[l] \arrow[d, dash] \\
c_q \otimes I \arrow[d, equal] & c_0 + c_q \arrow[l, "\alpha_0 + 0"'] \arrow[d, dash] & c_1 + c_q \arrow[l, "\alpha_1 + 1"'] \arrow[d, dash] & \cdots \arrow[l] & c_{q-1} + c_q \arrow[l] \arrow[d, dash] & c_q \arrow[l, "(\alpha_q, 1)"'] \arrow[d, equal] \\
c_q \otimes I & c_0 \arrow[l, "\alpha_0"'] & c_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l] & c_{q-1} \arrow[l] & c_q \arrow[l, "\alpha_q"']
\end{tikzcd}$$

$\overset{B}{\longmapsto}$

$$\begin{tikzcd}
c_q \otimes I \arrow[d, equal] & c_q \arrow[l, "\alpha_0 \cdots \alpha_q"'] \arrow[d, "\alpha_1 \cdots \alpha_q"'] \arrow[r, equal] & \cdots \arrow[r, equal] & c_q \arrow[d, "\alpha_q"'] \arrow[r, equal] & c_q \arrow[d, equal] \\
c_q \otimes I \arrow[d] & c_0 + c_q \arrow[l, "\alpha_0 + 0"'] \arrow[d, "-1", "\alpha_1 \cdots \alpha_q"'] & \cdots \arrow[l, "\alpha_1 + 1"'] & c_{q-1} + c_q \arrow[l] \arrow[d, "-1", "\alpha_q"'] & c_q \arrow[l, "(\alpha_q, 1)"'] \arrow[d] \\
0 \otimes I & c_0 \arrow[l] & \cdots \arrow[l, "\alpha_1"] & c_{q-1} \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

Note

$$\boxed{d_2 A = d_0 B = 0 \qquad d_1 A = d_1 B}$$

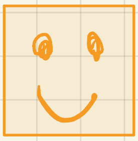
$$\boxed{F_q S_2 \xrightarrow{d_1 \sim d_0 + d_2} F_q}$$

& so

$$\mathrm{id} = d_0 A = d_0 A + \underset{=0}{d_2 A} \sim d_1 A$$

$$\|$$

$$\deg\mathrm{face} = d_2 B = \underset{=0}{d_0 B} + d_2 B \sim d_1 B$$



$$t_{\mathrm{id}}(x) = \left\{ \begin{tikzcd}
0 \arrow[d] & c_p \arrow[l] \arrow[d, "i_1"] \arrow[r, equal] & c_p \arrow[d, "i_1"] \arrow[r, equal] & \cdots \arrow[d, "i_1"] \arrow[r, equal] & c_p \arrow[d] & 0 \arrow[l] \\
c_p \arrow[d, equal] & c_p \oplus c_0 \arrow[l, "\alpha_0 \pi_2"'] \arrow[d, "\pi_2"] & c_p \oplus c_1 \arrow[l, "\mathrm{id} \oplus \alpha_1"'] \arrow[d, "\pi_2"] & \cdots \arrow[l, "\mathrm{id} \oplus \alpha_2"'] \arrow[d, "\pi_2"] & c_p \oplus c_{p-1} \arrow[l, "\mathrm{id} \oplus \alpha_{p-1}"'] \arrow[d, equal] & c_p \arrow[l, "(\mathrm{id} \oplus \alpha_p)\Delta"'] \\
c_p & c_0 \arrow[l, "\alpha_0"'] & c_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l, "\alpha_2"'] & c_{p-1} \arrow[l, "\alpha_{p-1}"'] & c_p \arrow[l, "\alpha_p"']
\end{tikzcd} \right.$$

and

$$t_\beta(x) = \left\{ \begin{tikzcd}
c_p \arrow[d, equal] & c_p \arrow[l, "\beta"'] \arrow[d, "(1 \oplus \beta_1)\Delta"] \arrow[r, equal] & c_p \arrow[d, "(1 \oplus \beta_2)\Delta"] \arrow[r, equal] & \cdots \arrow[r, equal] & c_p \arrow[d, "(1 \oplus \beta_p)\Delta"] \arrow[r, equal] & c_p \arrow[d, equal] \\
c_p \arrow[d] & c_p \oplus c_0 \arrow[l, "\alpha_0 \pi_2"'] \arrow[d, "\beta_1 - 1"] & c_p \oplus c_1 \arrow[l, "\mathrm{id} \oplus \alpha_1"'] \arrow[d, "\beta_2 - 1"] & \cdots \arrow[l, "\mathrm{id} \oplus \alpha_2"'] & c_p \oplus c_{p-1} \arrow[l, "\mathrm{id} \oplus \alpha_{p-1}"'] \arrow[d, "\beta_p - 1"] & c_p \arrow[l, "(\mathrm{id} \oplus \alpha_p)\Delta"'] \arrow[d] \\
0 & c_0 \arrow[l] & c_1 \arrow[l, "\alpha_1"'] & \cdots \arrow[l, "\alpha_2"'] & c_{p-1} \arrow[l, "\alpha_{p-1}"'] & 0 \arrow[l]
\end{tikzcd} \right.$$

where i_j (resp. π_j) is the j^{th} inclusion (resp. projection), Δ is the diagonal and $\beta_k = \prod_{k \le i \le q} \alpha_i$. Note the identities

$$d_0 T_{\mathrm{id}} = \mathrm{id}, \qquad d_2 T_\beta = \sigma \circ \delta, \qquad d_2 T_{\mathrm{id}} = d_0 T_\beta = 0 \qquad \text{and} \qquad d_1 T_{\mathrm{id}} = d_1 T_\beta.$$

Hence

$$\mathrm{id} \;=\; d_0 T_{\mathrm{id}} \simeq d_1 T_{\mathrm{id}} = d_1 T_\beta \simeq d_2 T_\beta = \sigma \circ \delta. \qquad \square$$

Huge cheat: A & B are not well defined as written

I've used the model - for an Ab-cat w fin coproducts
(so that EM-spt defined internally)

$$HH(\mathcal{C}, M) \simeq \Big\{ [q] \mapsto \bigoplus_{\boxed{c_0 \leftarrow \cdots \leftarrow c_q \in \mathcal{C}}} M(c_0, c_q) \Big\} = H(\mathbb{Z}\mathcal{C}, M)$$

- in set

as opposed to

$$HH(\mathcal{C}, M) \simeq \Big\{ [q] \mapsto \bigoplus_{c_0\, c_q} HM(c_0, c_q) \otimes \bigotimes_{i=1}^{q} H\mathcal{C}(c_i, c_{i+1}) \Big\}$$

That these are equivalences follows by $\boxed{BM}$ since

$\rightarrow$ $\underset{a\text{-conn}}{X} \wedge \underset{b\text{-conn}}{K(G, b+1)} \longrightarrow \mathbb{Z}[X] \otimes K(G, b+1)$

is $2a+b+2$ - connected

<u>and</u> a purely <u>homological argument</u> (both theories are H_*-theories wrt the same projectives & have same H_0)

Details in ch 3 in DGM

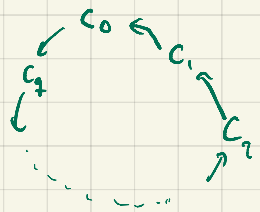
The cyclic structure

can be replaced by a bimodule — assume all flat

$$HH\,\mathcal{C} = \left\{[q] \mapsto \bigoplus_{c_0 \dots c_q} \mathcal{C}(c_0, c_q) \otimes \mathcal{C}(c_1, c_0) \otimes \quad \otimes\, \mathcal{C}(c_q, c_{q-1})\right\}$$

face	d_i	compose
degeneracies	s_i	insert id
cyclic structure	t	rotate

cyclic object



$\boxed{\text{EVIL}}$?

In some world with underlying sets.

fact: $\Lambda^o \xrightarrow{X} \mathcal{S}et$ cyclic set $\Rightarrow$ $|X_j| \circlearrowleft S^1$

$j \cup \!|$

Δ^o & categorical fixed pts

$$|X|^{S^1} = \{ z \in X_0 \mid s_0 x = t s_0 x \}$$

$\boxed{\text{Ex}}$ if $(c, f : c \to c) \in HH_0\,\mathcal{C}$, then $s_0 z = (c, f, id_c)$

(z = zero simplex; c = ob $\mathcal{C}$)

$$t s_0 z = (c, id_c, f)$$

So $|HH\,\mathcal{C}|^{S^1} = \{(c, id_c)\} = \operatorname{core} \mathcal{C}.$

In part

$$|HH\, S\mathcal{C}|^{S^1} = \operatorname{core} S\mathcal{C} = K\mathcal{C}.$$

model dependent / not homotopy invariant

As this is "evil", the question is: how much can we recover with homotopical means.

Aside on

$\boxed{\text{Connes' cyclic cat } \Lambda}$

NS: App B

$$\Lambda_\infty = \{ \tfrac{1}{n}\mathbb{Z} \mid \text{monotone maps} \}, \quad n \geq 1$$

free action $\mathbb{Z} \ni r$

$f \longmapsto f + r$

on maps

$$\begin{array}{c} \Lambda_\infty \\ \downarrow \\ \Lambda_r = \Lambda_\infty / r\mathbb{Z} \\ \downarrow \\ \Lambda = \Lambda_1 \end{array} \qquad r \in \mathbb{Z}$$

$$\Lambda_\infty \cong \Lambda_\infty^{op} \qquad f : S \to T \iff f^* : T \to S$$

$$f^*(t) = \min\{ s \mid f(s) \geq t \}$$

$$\begin{tikzcd} \Delta \arrow[r, "j_\infty"] \arrow[rr, bend left, "j"] & \Lambda_\infty \arrow[r] & \Lambda \end{tikzcd}$$

$$[n] \longmapsto \tfrac{1}{n+1}\mathbb{Z}$$

$[n] = [0, 1, \ldots, n]$

$[n] \xrightarrow{\varphi} [m]$

0	0	0	0
1	$\frac{1}{n+1}$	1	$\frac{1}{m+1}$
2	$\frac{2}{n+1}$	2	$\frac{2}{m+1}$
⋮		⋮	
n	$\frac{n}{n+1}$	m	$\frac{m}{m+1}$

$j(\varphi)$: $\frac{i}{n+1} \longmapsto \frac{\varphi i}{m+1}$

Fact

$|\Lambda_\infty| \sim *$ & $|\Lambda| \sim */B\mathbb{Z} \sim K(\mathbb{Z}, 2) = BS^1$

$\Lambda^{op} \cong \Lambda$ A cyclic object is a functor from Λ^o

so a cyclic space $\Lambda^o \to Top$ realizes to an S^1-space

$BS^1 \to Top$.

Fact $ob\, \Delta \overset{j}{\cong} ob\, \Lambda$ & Λ is generated by Δ &

$$t : \tfrac{1}{n}\mathbb{Z} \xrightarrow{x \mapsto x + \frac{1}{n}} \tfrac{1}{n}\mathbb{Z}$$

Key input $\boxed{\text{("cyclotomy")}}$

"genuine equivariant"

BHM magic: fiber sequence (in their point set model)

$$\begin{tikzcd}
HH\mathcal{C}_{hC_{p^n}} \arrow[r] \arrow[d, equal] & HH\mathcal{C}^{C_{p^n}} \arrow[r, "\text{"}R\text{"}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & HH\mathcal{C}^{C_{p^{n-1}}} \arrow[d] \\
HH\mathcal{C}_{hC_{p^n}} \arrow[r, "N"] & HH\mathcal{C}^{hC_{p^n}} \arrow[r] & HH\mathcal{C}^{tC_{p^n}}
\end{tikzcd}$$

"geometric fixed points"

"Cyclotomy" $\quad HH\mathcal{C} \simeq \Phi^{C_p} HH\mathcal{C}. \sim \mathrm{cof}\{HH\mathcal{C}_{hC_p} \to HH\mathcal{C}^{C_p}\}.$

Why? Follows <u>formally</u> from (so, not so deep...?)

Spt as a <u>functor cat</u>, e.g orth spt

$\mathrm{Spt} = \{(\mathcal{I}, \oplus, 0) \to (\mathcal{S}, \wedge, *)\}$

orthogonal spt

both cats have $\boxed{\text{good diagonals}}$ (technically: are "highly selvant")

$$\begin{tikzcd}
X \arrow[r, "\simeq"] \arrow[dr, "\Delta"'] & (X \otimes X)^{C_2} \arrow[d] \\
& X \otimes X
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{I}^{BG} \arrow[r, "X"] \arrow[d, "\mathrm{Fix}^G"'] & \mathcal{S}^{BG} \arrow[r, "\mathrm{Fix}^G"] & \mathcal{S} \\
\mathcal{I} \arrow[urr, "\Phi^G X"'] & &
\end{tikzcd}$$

$\Phi^G X \leftarrow$ left Kan.

& cofs generated by representables.

(& the fact that HH is given by $\otimes$'s in Spt.)

NS uses this to avoid point set models,

e.g. define $\to$

$$\begin{tikzcd} \boxed{HH\mathcal{C}^{C_p}} \arrow[r, "\text{``}R\text{''}"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & HH\mathcal{C} \arrow[d, "\varphi"'] \\ HH\mathcal{C}^{hC_p} \arrow[r] & HH\mathcal{C}^{tC_p} \end{tikzcd}$$

— cyclotomic

in terms of objects only using the Borel structure

$$\begin{tikzcd} & HH \arrow[d] \\ HH^{hC_p} \arrow[r] \arrow[d] & HH^{tC_p} \\ HH^{C_{p^2}} \arrow[r] & (HH^{tC_p})^{hC_p} \end{tikzcd}$$

+ **Tate orbit lemma:** NS

$X \circlearrowleft C_{p^2}$ bounded below $\Rightarrow$

$$[X_{hC_p}]^{tC_{p^2}/C_p} \sim 0.$$

HH always/$\mathbb{S}$: "THH"

Leading to NS'

$$\boxed{TC \longrightarrow HH^{hS^1} \underset{\varphi^{hS^1}}{\overset{can}{\rightrightarrows}} \prod_p [HH^{tC_p}]^{hS^1} \quad \left(\overset{NS}{\sim} HH^{tS^1}{}^{\wedge}\right)}$$

prod compl.

$$can: HH^{hS^1} \simeq [HH^{hC_p}]^{hS^1/C_p} \simeq [HH^{hC_p}]^{hS^1} \longrightarrow [HH^{tC_p}]^{hS^1}$$

Why?

$$\boxed{D_1 TC(A)(P) \simeq HH(A, \Sigma P)} \simeq D_1 K(A,P)$$

Lie = $\boxed{\text{Sketch}}$ $HH(A \oplus P) \simeq \bigoplus_i HH^{[i]}(A,P)$

distributivity

summands with "$\# P_s = i$."

ex. $q=1$

$$(A\oplus P)\otimes(A\oplus P) = \underbrace{A\otimes A}_{HH^{[0]}(A,P)} \oplus \underbrace{[A\otimes P \oplus P\otimes A]}_{HH^{[1]}} \oplus \underbrace{P\otimes P}_{HH^{[2]}}$$

as connectivity of P increases

$$HH(A\oplus P) \overset{\text{highly conn}}{\rightsquigarrow} HH A \oplus HH^{[1]}(A,P)$$

$HH^{[1]}(A,P)$ P can be placed freely so.

free $/S^1$ so

[the remaining summands are "free enough"]

Tate vanishes

S^1 fix pts $= \Sigma(S^1\text{-orbits})$

$$D_1 TC(A)(P) \simeq \Sigma\, HH(A,P)$$

Why do we care?

K & TC are "analytic functors" & having the "same differential" $\Rightarrow$ they are "equal up to a constant" (within the "radius of convergence")

i.e. their "difference" $= \mathrm{fib}(K \to TC) =: K^{inv}$ is locally constant

If $B \to A$ is a nilpotent extension of connective $\mathbb{S}$-algebras, then

$$K^{inv} B \to K^{inv} A$$

is an equivalence

Goodwillie's conjecture

There are issues w/ limits & colimits commuting

We saw a model for K of radical extensions, but a priori it is hard to parse

K_* will generally have huge infinitely divisible pieces not well understood (e.g. $K_2\, k[[t]]$)

but after completion things calm a bit down ($K\, k[[t]] \xrightarrow[p]{\sim} \mathrm{holim}\, K\, k[t]/t^n$)

Not known to be the case generally, but in the commutative case this is ok.

CMM.

Hensel extensions are fine: $\boxed{B \to A \text{ hensel} \Rightarrow K^{inv} B \xrightarrow[\sim p]{} K^{inv} A}$

[uses that $\mathbb{S}/p^n\, TC$ preserves filtered colimits].

A Noeth [conn] "F-finite", A complete in the ideal I

$\Rightarrow$ CMM $KA \xrightarrow[\sim p]{} \mathrm{holim}\, K(A/I^n)$

Popescu approximation

Calculations on the TC - side

1) $THH\,\mathbb{F}_p \simeq \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_{\mathbb{S}} \mathbb{F}_p} \mathbb{F}_p$ — the "dual Steenrod algebra"

cf. $/\mathbb{Z}$:

$$HH(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p} \mathbb{F}_p.$$

remember: everything is derived

$$\pi_*(\mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p) = H_*(\mathbb{F}_p \otimes_{\mathbb{Z}} (\mathbb{Z} \xrightarrow{p} \mathbb{Z}))$$

$$= H_*(\mathbb{F}_p \xrightarrow{0} \mathbb{F}_p) = \mathbb{F}_p[t]/t^2 = E(t) \qquad |t| = 1 \qquad \text{"}t = \sigma p\text{"}$$

So

$$HH(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p \otimes_{\mathbb{F}_p[t]/t^2} \mathbb{F}_p = B(\mathbb{F}_p, \mathbb{F}_p[t]/t^2, \mathbb{F}_p)$$

$$= \{[q] \mapsto \mathbb{F}_p \otimes \mathbb{F}_p[t]/t^{2\,\otimes q} \otimes \mathbb{F}_p\}$$

nondegenerate elements $[\underbrace{t|t \dots |t}_{q}] = 1 \otimes \underbrace{t \otimes \dots \otimes t}_{q} \otimes 1$

$\gamma_q \sigma t$

$HH_q(\mathbb{F}_p/\mathbb{Z}) = \mathbb{F}_p$ (generated by $[t| \dots |t]$)

Shuffle product gives the algebra structure

$$\boxed{HH_*(\mathbb{F}_p/\mathbb{Z}) \cong \Gamma_{\mathbb{F}_p}([t])}$$

$$[\overbrace{t| \dots |t}^{a}][\overbrace{t| \dots |t}^{b}] = \binom{a+b}{b}[\overbrace{t| \dots |t}^{a+b}]$$

$$\gamma_a \cdot \gamma_b = \binom{a+b}{b}\gamma_{a+b}$$

Alternatively via the Greenlees SS

Given

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d] \\ \pi_0 A \arrow[r] & Q = \pi_0 A \otimes_A B \end{tikzcd}$$

connective $\mathbb{S}$-algebras

flatness assumption.

$$\Rightarrow \quad \pi_* A \otimes_{\pi_0 A} \pi_* Q \Rightarrow \pi_* B$$

cf Serre SS:

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ * \arrow[r] & B \end{tikzcd} \quad \text{(conn.)}$$

$$H_*(B,k) \otimes_k H_*(F,k) \Rightarrow H_*(E,k).$$

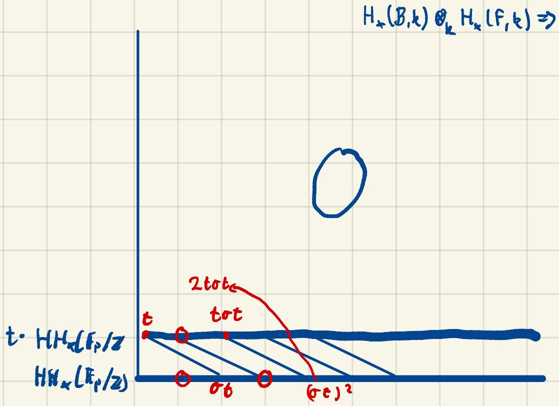
e.g.

$$\begin{tikzcd} \mathbb{F}_p \otimes_{\mathbb{Z}} \mathbb{F}_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathbb{F}_p \arrow[d] \\ \mathbb{F}_p \arrow[r] & HH(\mathbb{F}_p/\mathbb{Z}) \end{tikzcd}$$

$$E(t) \otimes HH_*(\mathbb{F}_p/\mathbb{Z}) \Rightarrow \mathbb{F}_p$$

Do it live

$HH(\mathbb{F}_p/\mathbb{Z}) = \Gamma_{\mathbb{F}_p}(\sigma t)$



Over $\mathbb{S}$: $\boxed{HH_*\mathbb{F}_p = \mathbb{F}_p[\mu]}$ $|\mu| = 2$ Bökstedt '8

$$HH\,\mathbb{F}_p = \mathbb{F}_p \otimes_{\mathbb{F}_p \otimes \mathbb{F}_p} \mathbb{F}_p$$

$|\tau_i| = |2p^i - 1|$, $|\xi_i| = |2p^i - 2|$

$$\pi_* \mathbb{F}_p \otimes_{\mathbb{F}_p} \mathbb{F}_p = \text{dual Steenrod} = \mathbb{F}_p[\tau_0, \tau_1, \dots \xi_1, \xi_2 \dots] \Big/ \begin{matrix} \tau_i^2 = \xi_{i+1} & p=2 \\ \tau_i^2 = 0 & p \text{ odd} \end{matrix}$$

(Easy) case $p=2$ $\quad \cong \mathbb{F}_p[\tau_0, \dots]$, so that

Tor SS $\quad E(\sigma\tau_0, \sigma\tau_1, \dots) \Rightarrow HH_*\mathbb{F}$ $\quad$ no possible diff's or additive extensions

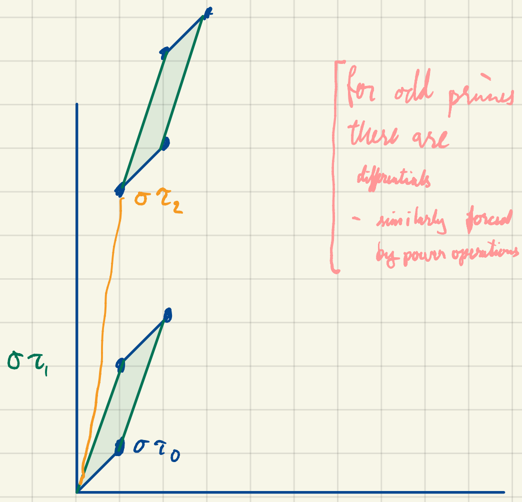
$$\Rightarrow HH_*\mathbb{F}_2 = \begin{cases} \mathbb{F}_2 & * \text{ even} \\ \mathbb{F}_2 & * \text{ odd} \end{cases}$$

(just as for $HH_*(\mathbb{F}_2/\mathbb{Z}) = \Gamma_{\mathbb{F}_2}(\sigma t) = \bigotimes_{i=1}^{\infty} E\,\gamma_i \sigma t$)

However, power operations imply

$$(\sigma\tau_i)^2 = Q^2 \sigma \tau_i$$
$$= \sigma Q^2 \tau_i = \sigma \tau_{i+1}$$

so $\quad HH_*\mathbb{F} = \mathbb{F}[\underset{\text{"}\mu\text{"}}{\sigma\tau_0}]$



[for odd primes there are differentials - similarly forced by power operations]

$$\begin{array}{cccccc}
 & 0 & 2 & 4 & 6 & \\
HH_*\mathbb{F}_p & 1 & \mu & \mu^2 & \mu^3 & \mu^4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & & & 0 & 0 & 0 \\
FlH_*(\mathbb{F}_p/\mathbb{Z}) & 1 & 0 & \gamma_2 & \gamma_3 = \gamma_3 & \gamma_4
\end{array}$$

That there are isos is forced by $\mathbb{S} \to \mathbb{Z}$ being 1-connected

but the rest of the map is zero!

On the (super) non algebraic nature of $HH\mathbb{F}_p$.

$$S^1: \quad HH\mathbb{F}_p = \mathbb{F}_p \otimes_{\underbrace{\mathbb{F}_p \otimes \mathbb{F}_p}_{A}} \mathbb{F}_p$$

- does not support divided power structure

  ⇒ not H(commutative simpl ring)

- has nontriv power operations

  ⇒ not H(cdga)

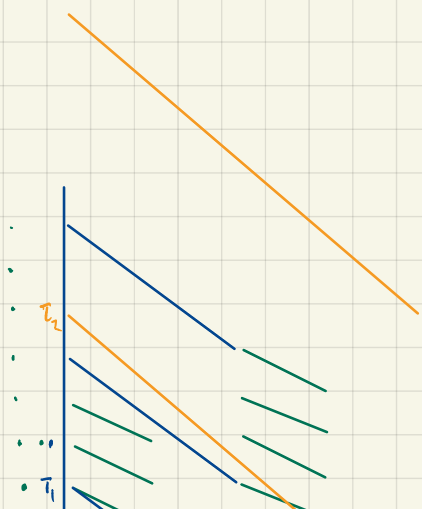
Old results by Mahowald et al ⇒ (simil / Bö's) $\mathbb{F}_p \otimes \mathbb{F}_p$ free E_2 $\mathbb{F}_p$-algebra

⇒ $HH\mathbb{F}_p$ free E_1 $\mathbb{F}_p$-algebra

Blumberg, Cohen, Schlichtkrull
Revisited recently.
Antieau, Krause, Nikolaus ?

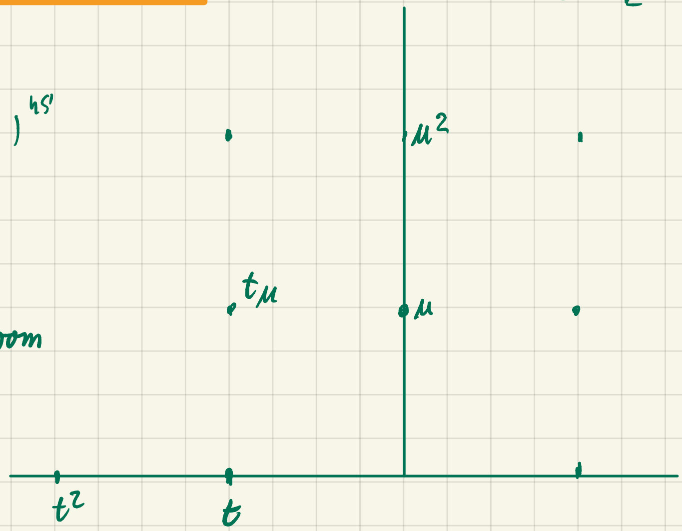
However: fully algebraic "one step higher."

$$S^2: \quad \mathbb{F}_p \otimes_{HH\mathbb{F}_p} \mathbb{F}_p = H(\mathbb{F}_p[\varepsilon]/\varepsilon^2) \qquad |\varepsilon| = 3.$$



Homotopy fix & Tate of $HH\mathbb{F}_p$

Tate $E^2 = E^\infty$



$$E^2 = H^{-s}(BS^1, HH_t\,\mathbb{F}_p) \Rightarrow \pi_{s+t}(HH\,\mathbb{F}_p)^{hS^1}$$
$$\overset{\shortparallel}{\mathbb{F}_p[t]\otimes\mathbb{F}_p[\mu]}$$

concentrated in even degs – no room for differentials

However there are extensions so that

$$t\mu = p$$

&

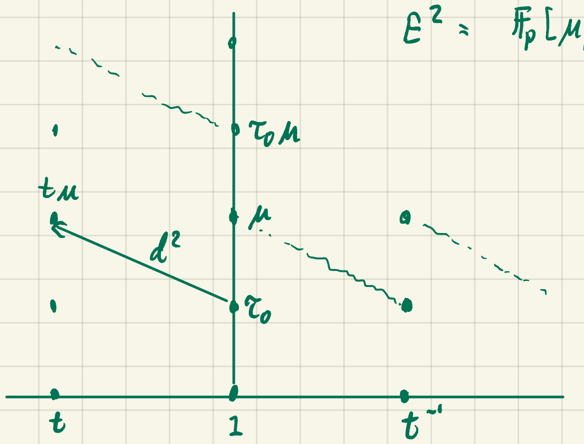
$$\boxed{\pi_* HH\mathbb{F}_p^{hS^1} = \mathbb{Z}_p[t,\mu]/t\mu=p \overset{\text{"can"}}{\hookrightarrow} \pi_* HH\mathbb{F}^{tS^1} = \mathbb{Z}_p[t^{\pm1},\mu]/t\mu=p}$$
$$\overset{\shortparallel\wr}{\mathbb{Z}_p[t^{\pm1}]}$$

One way to see $t\mu = p$ is as follows

$$\begin{array}{ccc} \overset{\tau_0}{S^1_+\wedge \mathbb{S}/p\wedge \mathbb{F}_p} & & \\ \downarrow & & \\ S^1_+\wedge \mathbb{S}/p\wedge HH\mathbb{F}_p & \longrightarrow & \mathbb{S}/p\wedge HH\mathbb{F}_p \\ \tau_0 & \longmapsto & \mu = \sigma\tau_0 \end{array}$$

$\Rightarrow$ in the Tate SS for $\mathbb{S}/p\wedge HH\mathbb{F}_p^{tS^1}$

$$E^2 = \mathbb{F}_p[\mu, t^{\pm1}]\otimes E\tau_0$$



Shows $p \mid t\mu$

so

$$\mathbb{S}/p_* HH\mathbb{F}_p^{tS^1} \cong \mathbb{F}_p[t^{\pm1}]$$

Now we know "can" & must find

$\boxed{\text{The Frobenius}}$

Consider

$$\begin{tikzcd}
 & & HH\mathbb{F}_p \arrow[d, "\varphi"] \\
HH\mathbb{F}_{p\,hC_p} \arrow[r] & HH\mathbb{F}_p^{hC_p} \arrow[r] & \boxed{HH\mathbb{F}_p^{tC_p}} \\
 & HH\mathbb{F}_p^{hS^1} \arrow[u] \arrow[r] & HH\mathbb{F}_p^{tS^1} \arrow[u]
\end{tikzcd}$$

cyclo

so $\mathbb{F}_p$-algebra

$HH\mathbb{F}_p^{tC_p}$ $\mathbb{F}_p$-algebra $\Rightarrow$

$p \in \pi_* HH\mathbb{F}_p^{tS^1}$ maps to zero

So the infinite cycle $t\mu$ in

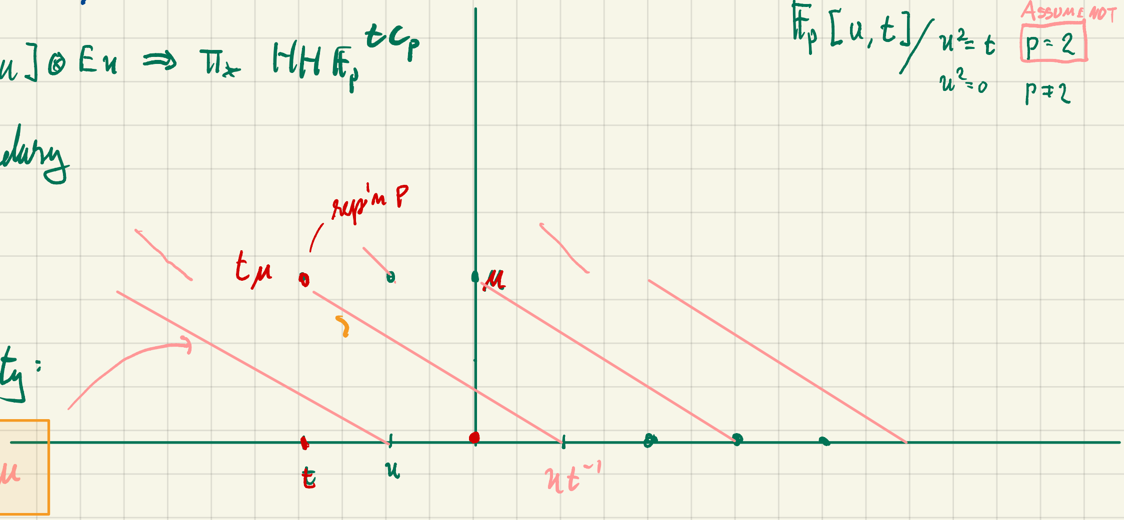
$\hat{E}^2 = \mathbb{F}_p[t^{\pm 1}, \mu] \otimes E u \Rightarrow \pi_* HH\mathbb{F}_p^{tC_p}$

must be a boundary

$H^*(BC_p; \mathbb{F}_p) =$

$\mathbb{F}_p[u,t]/u^2 = t$ $\boxed{p=2}$ ASSUME NOT

$\phantom{\mathbb{F}_p[u,t]/}u^2=0$ $p \neq 2$



Only possibility:

$\boxed{d^3 u = t^2 \mu}$

So $\boxed{\pi_* HH\mathbb{F}_p^{tC_p} = \mathbb{F}_p[t^{\pm 1}]}$

in hC_p ut^{-1}'s not there, so $t\mu^n$ survives

(& $\pi_* HH\mathbb{F}_p^{hC_p} = \mathbb{Z}/p^2[t, \mu_1] / t\mu_1 = p,\ pt = 0$)

Further analysis shows $\varphi : HH\mathbb{F}_p \xrightarrow{\text{iso in deg} \geq 0} HH\mathbb{F}_p^{tC_p}$ $\quad \varphi\mu = t^{-1}$ (arg. not included)

&

$$\pi_* HH\mathbb{F}_p^{hS^1} \xrightarrow{\text{can} - \varphi} \pi_* HH\mathbb{F}_p^{tS^1}$$

$$\mathbb{Z}_p[t,\mu]/t\mu = p \qquad\qquad \mathbb{Z}_p[t^{\pm 1},\mu]/t\mu = p$$

0 for $* = 0$

iso for $* \neq 0$

$$0 \to TC_0\,\mathbb{F}_p \to \underset{\mathbb{Z}_p}{\pi_0 HH\mathbb{F}_p^{hS^1}} \xrightarrow{1-1} \underset{\mathbb{Z}_p}{\pi_0 HH\mathbb{F}_p^{tS^1}} \to TC_{-1}\,\mathbb{F}_p \to 0$$

So.

$$\boxed{TC\,\mathbb{F}_p = \mathbb{F}_p[\partial]/\partial^2 \qquad |\partial| = -1}$$

(more generally

$$TC_*\,\mathbb{F}_q = \begin{cases} 0 & * \neq 0,1 \\ \mathbb{Z}_p & * = 0 \\ \operatorname{coker}(1-F): W\mathbb{F}_q \to W\mathbb{F}_q & * = -1 \end{cases}$$

ASIDE

$\boxed{\text{"Old" point of view by finite fixpts}}$

$$\begin{tikzcd} \boxed{HH\mathbb{F}_p^{C_p}} \arrow[r, "R"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & HH\mathbb{F}_p \arrow[d, "\varphi"] \\ HH\mathbb{F}_p^{hC_p} \arrow[r] & HH\mathbb{F}_p^{tC_p} \end{tikzcd}$$

$$\begin{tikzcd} \mu \in \mathbb{F}_p[\mu] \arrow[d, mapsto] \\ t^{-1} \in \mathbb{F}_p[t^{\pm 1}] \end{tikzcd}$$

$\mathbb{Z}/p^2[t,\mu]/t\mu = p$
$pt = 0$

$t \longmapsto t$
$\mu \longmapsto pt^{-1} = 0$

further analysis of orbits.

connective ☺

$\pi_* HH\mathbb{F}_p^{C_p} = \mathbb{Z}/p^2[\mu_1]$

$\pi_* HH\mathbb{F}_p^{C_{p^m}} = \mathbb{Z}/p^{m+1}[\mu_m]$

$\pi_* HH\mathbb{F}_p^{C_{p^{m-1}}} = \mathbb{Z}/p^m[\mu_{m-1}]$

$$\begin{tikzcd} \mu_m \arrow[d, "R"'] & \mu_m \arrow[d, "F"] \\ p\mu_{m-1} & \mu_{m-1} \end{tikzcd}$$

incl of fixed pts

The "TR-tower"

- just like Witt vectors

in fact, if $\pi_0 A$ commutative, then

HM:

$\pi_0 \operatorname{holim}_m HHA^{C_{p^m}} = W\pi_0 A$

$$\begin{tikzcd} TF_*\mathbb{F}_p \arrow[r, "1-R"] \arrow[d, phantom, "\cong"] & TF_*\mathbb{F}_p \arrow[d, phantom, "\cong"] \\ \mathbb{Z}_p[\mu] \arrow[r] & \mathbb{Z}_p[\mu] \end{tikzcd}$$

$\mu \mapsto (1-p)\mu$
inv

$\Downarrow$

$\boxed{TC\,\mathbb{F}_p = \mathbb{Z}[\partial]/\partial^2}$

These "Witt-like" extensions
is the genesis of the trace methods
evidence for Rognes' (& Madsen, Ausoni....)
Red shift conjecture

Resolved for E_∞-rings by Burklund, Schlank, Yuan

building on

Clausen Mathew Naumann Noel 20 — height $KA \leq$ height $A+1$
Land Mathew Meyer Tamme 20
Hahn - Wilson 20
Yuan ≤ 2 for Lubin Tate $E_n(L)$ 21

Note:

$|\partial| = -1$ A $\mathbb{F}_p$-algebra

$TC\mathbb{F}_p \underset{p}{\sim} \mathbb{Z}_p[\partial]/\partial^2 \longrightarrow TCA_p$

$\Rightarrow$ TCA_p algebraic

TC $\mathbb{Z}_p$ more complicated (there are diff's in the SSs)

but it starts w. the nice (Bö, FLS).

$$HH_*(\mathbb{Z}, \mathbb{F}_p) = E(\lambda_1) \otimes \mathbb{F}_p[\mu_1] \qquad |\lambda_1| = 2p-1, \quad |\mu_1| = 2p$$

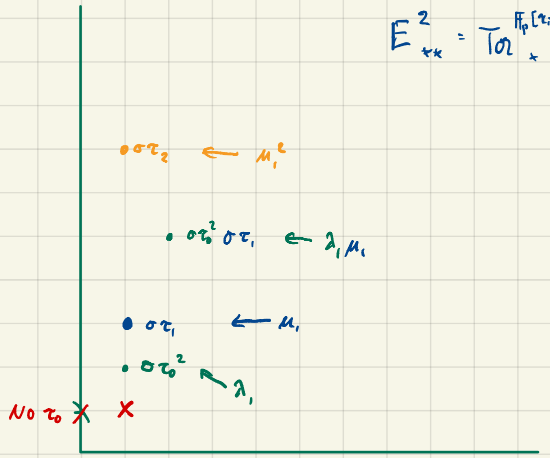
coefficients!

mult on
[$HH_*(\mathbb{Z})$ is a mess - sq zero]

$$\mathbb{F}_p \otimes \mathbb{Z} \xrightarrow{0 = p} \mathbb{F}_p \otimes \mathbb{Z} \to \mathbb{F}_p \otimes \mathbb{F}_p, \text{ so}$$

$\mathbb{F}_p \otimes \mathbb{Z}$ just as $\mathbb{F}_p \otimes \mathbb{F}_p$ but no τ_0 (but τ_0^2 is there when $p = 2$)

$$E^2_{**} = \mathrm{Tor}_*^{\mathbb{F}_p[\tau_i, \xi_i]_{i \geq 0}}(\mathbb{F}_p, \mathbb{F}_p) \Rightarrow HH_*(\mathbb{Z}, \mathbb{F}_p)$$



TC $\mathbb{Z}_p$ First calculated by Bö, Ma, Ro early 90's

Why is this the start of a successful theory?

For readability: move $\tau^{\text{p-compl}}_{\geq 0}$ outside to a blanket assumption.

1 $\quad K\mathbb{F}_p \simeq_p H\mathbb{Z}_p$ $\quad$ (Adams operations $\overset{\text{Quillen}}{\Rightarrow}$ [$A \xrightarrow{x \mapsto x^p} A$ iso $\Rightarrow$ $K_i A$ p div for $i>0$])

$$\begin{array}{ccc} K\mathbb{F}_p & \simeq_p & H\mathbb{Z}_p \\ & & \big\downarrow \wr \\ \tau_{\geq 0} TC\,\mathbb{F}_p & \simeq_p & H\mathbb{Z}_p \end{array}$$

must preserve 1 !

2

$$\begin{array}{c} K\,\mathbb{Z}/p^n \\ \big\downarrow \wr_p \\ \tau_{\geq 0} TC\,\mathbb{Z}/p^n \end{array}$$

since $\mathbb{Z}/p^n \to \mathbb{F}_p$ nilp ext

only very recently calculated $\quad$ Antieau Krause Nikolaus 24 $\quad$ prismatic coho

3

$$\begin{array}{c} K\,\mathbb{Z}_p \\ \big\downarrow \wr_p \\ \tau_{\geq 0} TC\,\mathbb{Z}_p \end{array}$$

because of Hesselholt Madsen TC / Panin · Suslin $K\,\mathbb{Z}_p \simeq_p \lim_n K\,\mathbb{Z}/p^n$ TC

or now directly from CMM

The $K\mathbb{Z}_p$ calculation by HM opens an entire pathway into other calculations in K-theory verifying the Lichtenbaum · Quillen conjecture for these cases

Theorem

$p > 0$ prime

k perfect field of char p

A connective E_∞-alg, $\pi_0 A$ a Wk-algebra f.g as a module

Then

$$K^{\text{inv}} A \simeq_p \Sigma^{-2} \operatorname{coker}\left(\pi_0 A \xrightarrow{1-\text{Frob}} \pi_0 A\right)$$

$\tau\; TC_{-1}\, \pi_0 A_p$

$K^{\text{inv}} = \text{fiber } K \to TC$

K-theory of local fields [HM Annals]

LQC ✓

$$A := \mathcal{O}_F \subseteq F \quad \leftarrow \text{CDV field, char } 0$$
$$\downarrow$$
$$k \quad \leftarrow \text{perf. char } = p > 2$$

HM 02 (Gal coho.)

$$K_{2s-1}(F, \mathbb{Z}/p^n) \cong H^1(F, \mu_{p^n}^{\otimes s})$$

$$K_{2s}(F, \mathbb{Z}/p^n) \cong H^0(F, \mu_{p^n}^{\otimes s}) \oplus H^2(F, \mu_{p^n}^{\otimes s+1})$$

Localization

$$Kk \longrightarrow K\mathcal{O}_F \longrightarrow KF$$

Problem: cyclotomy idea for goes "bad" for cats over in $H\mathbb{Q}$-modules essentially for the same reason that the Witt-ring construction splits for char zero fields fails to give anything interesting

$$HH\,\mathbb{Q}_p^{C_{p^n}} = \mathbb{Q}_p^{\times(n+1)}$$
$$TR(\mathbb{Q}_p, p) = \prod_{\mathbb{N}} \mathbb{Q}_p$$
$$TC(\mathbb{Q}_p, p) = \mathbb{Q}_p \quad (\text{fix of Frob})$$

Why? $K\mathcal{C} \to TC\,\mathcal{C}$ as such is fine, but the identification used in the K-theory localization doesn't work.

$$\begin{array}{ccccccccc} 0 \to & K_i\mathbb{Z}_p & \to & K_i\mathbb{Q}_p & \to & K_{i-1}\mathbb{F}_p & \to 0 \\ & \downarrow \mathbb{Z}_p & & \downarrow & & \downarrow \mathbb{Z}_p & \\ & TC_i\mathbb{Z}_p & & TC_i\mathbb{Q}_p & & TC_{i-1}\mathbb{F}_p & \end{array}$$

$$Kk \to K\mathcal{O}_F \to KF$$

K theory of $\mathcal{C} :=$ bounded cx / $P_{\mathcal{O}_F}$

$$\underset{KF}{\boxed{\mathcal{C}\ \ \text{torsion } H_*, \text{ q.i}}} \subseteq \underset{KA}{\boxed{\mathcal{C},\ \text{qi}}} \longrightarrow \boxed{\mathcal{C},\ \mathbb{Q}\text{ qi}}$$

$K(\ ) \simeq K(\text{bounded cx}/P_F) \simeq KF$

$$TC\,k \longrightarrow TC\,A \longrightarrow TC(\ )$$

Doesn't work for TC.

$$\begin{tikzcd}
K\,k \arrow[r] \arrow[d] \arrow[r, phantom, shift right=4ex, "\xleftarrow{\text{eq after compl \& truncation}}\rightarrow"] & K\,\mathcal{O}_F \arrow[r] \arrow[d] & K\,F \arrow[d] \\
TC\,k \arrow[r] & TC\,\mathcal{O}_F \arrow[r] & TC\left(\begin{smallmatrix}\text{bounded cx}/p_A\,k \\ \text{rat'l } q.i\end{smallmatrix}\right)
\end{tikzcd}$$

methods for calculating

$\boxed{\text{how to attack?}}$

inverting p in enrichment <u>not</u> an option.

log poles oh, k

still the tricks simplifying from cats of modules to (log) rings works.

Log ring (A, M, α) — ring, monoid — $M \xrightarrow{\alpha} (A, \cdot)$ monoid map

<u>Derivations</u> into an A-mod E

$A \xrightarrow{D} E$ derivation

$M \xrightarrow{D\log} E$ map of monoids

$\forall\, x \in M \quad \alpha(x) \cdot D\log x = D\alpha(x) \qquad \left(\text{"}D\log x = \frac{Dx}{x}\text{"}\right)$ if $M = GL_1 A$

Like Kähler diffs Ω^1_A univ. / deriv.

$\Omega^1_A \xrightarrow{\cong} HH_1 A \qquad da \mapsto a \otimes 1 + 1 \otimes a$

Log differential $\qquad \omega^1_{(A, GL_1 A)} \xrightarrow{\cong} HH_1(A \mid M)$

where

$\omega^1_{(A,M)} = \left[\Omega^1_A \oplus (R \otimes M^{gp})\right] / \, d\alpha_x = \alpha x \otimes x$

Analyzing the finite equivariant structure $TC(A \mid K)$ is reconstructed from a de Rham Witt-interpretation.

Denseness

Idea: Consider

$$\begin{tikzcd} X \arrow[r, phantom, "\subseteq"] \arrow[d] & Y \arrow[d, "f"] \\ \{z\} \arrow[r, phantom, "\subseteq"] & Z \end{tikzcd}$$

$\leftarrow$ cont $\in$ Spaces

$X \subseteq Y$ dense $\Rightarrow$ $f \equiv z$.

[for each y choose a seq x_n in X converging to y. Then $f(y) = \lim f(x_n) = z$]

$A \xrightarrow{f} B$ 1-conn (ex $\mathbb{S} \to \mathbb{Z}$, $\mathbb{S} \searrow MU \nearrow \mathbb{Z}$)

$\Rightarrow$ $A\text{-mod}^\omega \underset{f^*}{\overset{-\otimes_A B}{\rightleftarrows}} B\text{-mod}^\omega$ "dense" as demonstrated by Adams / Amitsur complex

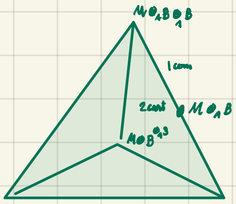
$$\begin{tikzcd} M \arrow[rr] \arrow[dr] \arrow[dd] & & M \otimes_A B \arrow[dd, "\leftarrow 1 \text{ conn}"] \\ & N_1 \arrow[ur] \arrow[dl] \arrow[dr, phantom, "\lrcorner"] & \\ M \otimes_A B \arrow[rr] & & M \otimes_A B \otimes_A B \end{tikzcd}$$

$N_0 = M \otimes_A B$

$M \xrightarrow{\sim} \operatorname{holim}_n N_n \xrightarrow{k\text{-connected}} N_k \to N_{k-1} \to \cdots \to N_0$

$N_k = \operatorname{holim}_{\text{cube}} \underbrace{M \otimes_A B \otimes_A \cdots \otimes_A B}_{\text{on vertices}}$

each d-dim'l subcube is d-cart.



Connectivity grows so quickly it is "easy" for a functor F (even to unstable cats) to be continuous so that

$$\begin{tikzcd} A\text{-mod}^\omega \arrow[d, "F"] & B\text{-mod}^\omega \arrow[l, "f^*"'] \arrow[d] \arrow[dl, phantom, "\sim"] \\ Sp & * \arrow[l] \end{tikzcd}$$

$F \simeq *$ $\Leftarrow$

this uses heavily the BM technology of Maxime. The cubes lie in a sweet spot where they are uniformly d-cartesian = $2d$-1-cocartesian even unstably so e.g. $BGL A^+$ constr can be used on the nose.

ex. Functors continuous in this sense

K, HH, ... TC, K^{inv}

ex If S is an cube, then

$$(T \subseteq S) \mapsto K(A \otimes \mathbb{Z}^{\otimes |T|}) \quad \text{is } |S|+1\text{-cartesian}$$

Pf.

For each $T \subseteq S$

$(U \subseteq T) \longmapsto A \otimes \mathbb{Z}^{\otimes |U|}$ is $|U|$-cartesian because $\mathbb{S} \to \mathbb{Z}$ is 1-connected

$\Rightarrow (U \subseteq T) \longmapsto \Omega^\infty (A \otimes \mathbb{Z}^{\otimes |U|})$ is — " —

By Blakers-Massey as explained in the prep talk of Maxime we're at the sweet spot:

$\boxed{\text{A cube of spaces has the prop. that all } T\text{-subcubes are } |T|\text{-cartesian} \iff \text{— " — } 2|T|-1\text{-cocartesian}}$

So, with such an incoming cube, a functor preserving <u>either</u> condition will have as output a cube of this sort

Hence

$S \supseteq T \longmapsto \Omega^\infty M_n(A \otimes \mathbb{Z}^{\otimes |T|})$ is $|T|$-cart (Ω^∞ cart ✓)

$S \supseteq T \longmapsto GL_n(A \otimes \mathbb{Z}^{\otimes |T|})$ — " — ($\pi_0(A) \xrightarrow{\simeq} \pi_0(A \otimes \mathbb{Z}^{\otimes |T|})$)

$S \supseteq T \longmapsto BGL_n(A \otimes \mathbb{Z}^{\otimes |T|})$ is $|T|+1$-cart. (B cart ✓)

$S \supseteq T \longmapsto BGL_n(A \otimes \mathbb{Z}^{\otimes |T|})^+$ — " — ($+$ cocart ok)

$S \supseteq T \longmapsto BGL_n(A \otimes \mathbb{Z}^{\otimes |T|})^+ \times K(\pi_0(A \otimes \mathbb{Z}^{\otimes |T|}))$

$\cong \Omega^\infty K(A \otimes \mathbb{Z}^{\otimes |T|})$ [here the vertices are not connected, so we lose uniformity at the lowest dim'l but we still get that]

The S-cube $T \mapsto \Omega^\infty K(A \otimes \mathbb{Z}^{\otimes T})$ is $|S|+1$-cartesian get

The S-cube $T \mapsto \cancel{\Omega^\infty} K(A \otimes \mathbb{Z}^{\otimes T})$ is $|S|+1$ - cartesian

Letting $|S|$ go to infinity we get

$$K A \longrightarrow \operatorname*{holim}_{\emptyset \neq T} K(A \otimes \mathbb{Z}^{\otimes |T|})$$

is an equivalence

Presumably, this could've been proven directly in spt - but I think it is a neat illustration of how you can get even unstable results

$A \xrightarrow{f} B$ 1-conn. map of $\mathbb{S}$-algebras

So, proving theorems abt these functors on A — K, TC, ...

often reduces to showing it for B

e.g. Goodwillie's conj : McCarthy

K^{inv} is locally constant on simpl rings + com $\mathbb{S}$-alg $C \otimes H\mathbb{Z} \simeq HC'$ (simpl'l ring) + Dundas ($H\mathbb{Z}$-alg $\downarrow f^*$ $\mathbb{S}$-alg) $\Rightarrow$ K^{inv} is locally constant on connective $\mathbb{S}$-alg.

Note 1) "$C \otimes H\mathbb{Z} \simeq HC'$" works in the associative case but fails in the commutative setting

2) Each $TC(C \otimes H\mathbb{Z}^{\otimes s})$ does not seem very accessible, so it is not as if $TC(C)$ can be calculated by these

However, replacing $\mathbb{S} \to \mathbb{Z}$ by $\mathbb{S} \to MU$, we get a better hold because

$$MU^{\otimes s+1} \simeq BU_+^{\times s} \wedge MU$$

BCS — HH of Thom spaces.

$$HH\,MU \simeq MU \otimes SU_+ \qquad HH_* MU \cong \mathbb{Z}[x_i] \otimes E(e_{2j+1})$$

Lazard $MU_* \cong \mathbb{Z}[x_i]$

$$HH_*(MU^{\otimes s}) \cong \mathbb{Z}[x_i] \otimes (E(j))^{\otimes s+1} \otimes (\mathbb{Z}[b_k])^{\otimes s}$$

$H_* MU$

[& descent SS for $HH_*\mathbb{S}$ = ANSS for $\pi_*\mathbb{S}$].

q: has anyone done a serious investigation on

$$\varphi : HH_* T^{\times S^1} \longrightarrow HH_* T^{\otimes S^1} \ ?$$

Algebraic K-theory of spaces

X connected

$K\,\mathbb{S}[\Omega X] = A(X)$

"algebraic K-theory of X"

Tells a lot about homeom / diffeo / PL-iso of high dim'l mflds.

especially $A* = K\mathbb{S}$ Disk Diffeo

$$\begin{tikzcd}
K\,\mathbb{S}[\Omega X] \arrow[r] \arrow[d] & \boxed{TC\,\mathbb{S}[\Omega X]} \arrow[d] \\
K(\mathbb{Z}[\pi_1 X]) \arrow[r] & TC(\mathbb{Z}[\pi_1 X])
\end{tikzcd}$$

so what is this?

In part

$$\begin{tikzcd}
A(*) = K\mathbb{S} \arrow[r] \arrow[d] & TC\,\mathbb{S} \arrow[d] \\
K\mathbb{Z} \arrow[r] & TC\mathbb{Z}
\end{tikzcd}$$

theory

motivic

(or if $\mathbb{S}_p$ is enough: trace as above)
$\downarrow$
$\mathbb{Z}_p$

just calculated

[note $TC\mathbb{Z} \simeq_p (TC\mathbb{Z}_p)$
but $K\mathbb{Z} \to K\mathbb{Z}_p$ widely different]

- $HH\mathbb{S} \simeq \mathbb{S}$

$HH_*\,\mathbb{S}[G] \simeq \Sigma^\infty \mathrm{Map}(S^1, BG)_+$ ← free loop space ΛBG

splitting comes from the diagonal $G^{\times p} \leftarrow G$

$$\begin{tikzcd}
HH\,\mathbb{S}[G]_{hC_{p^n}} \arrow[r] \arrow[d, equal] & HH\,\mathbb{S}[G]^{C_{p^n}} \arrow[r, "R"'] \arrow[d] & HH\,\mathbb{S}[G]^{C_{p^{n-1}}} \arrow[l, bend right, "S_n"'] \arrow[d] \\
\text{—"—} \arrow[r] & HH\,\mathbb{S}[G]^{hC_{p^n}} \arrow[r] & HH\,\mathbb{S}[G]^{tC_{p^n}}
\end{tikzcd}$$

Going to the limit gives cart diag after p-completion

$$\begin{tikzcd}
TC\,\mathbb{S}[G]_p \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner_p"] & \operatorname{holim}_F HH\,\mathbb{S}[G]_{hC_{p^n}} \arrow[d] \\
HH\,\mathbb{S}[G] \arrow[r, "1-FS_1"] \arrow[d, phantom, "\cong"] & HH\,\mathbb{S}[G] \arrow[d, phantom, "\cong"] \\
\Sigma\Lambda BG_+ \arrow[r, "1-\text{pth power}"] & \Sigma\Lambda BG_+
\end{tikzcd}$$

$\simeq_p \Sigma\, HH\,\mathbb{S}[G]_{hS^1} \simeq (\Sigma^{\infty+1}\Lambda BG_+)_{hS^1}$

[$BC_{p^\infty} \simeq_p BS^1$]

F = incl of fixed pts.

$G = *$

$$\boxed{TC\,\mathbb{S} \simeq_p \mathbb{S} \vee \Sigma\mathbb{CP}^\infty_{-1}}$$

← stunted proj space (with a cell in deg −2)

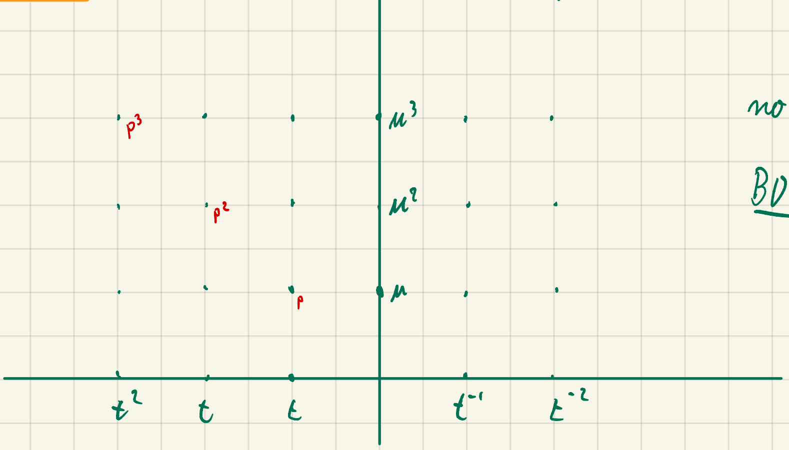
Problem:
$TC\mathbb{S} \to TC\mathbb{Z}$
not well understood.

$\boxed{TC\,\mathbb{F}_p}$

KAST

RYDD!

$$(HH\,\mathbb{F}_p)^{hS^1} \longrightarrow (HH\,\mathbb{F}_p)^{tS^1}$$



no diff'ls

BDT extension ((explain))

NEXT PAGE.

$|t| = (-2,0)$

$|\mu| = (0,2)$

$$E^2 = H^{-s}(BS^1, HH_t\,\mathbb{F}_p) \Rightarrow \pi_{s+t}(HH\,\mathbb{F}_p)^{hS^1}$$
$$\overset{\shortparallel}{\mathbb{F}_p[t] \otimes \mathbb{F}_p[\mu]}$$

$$\begin{array}{rcl} \pi_*(HH\mathbb{F}_p)^{hS^1} & = & \mathbb{Z}_p[t,\mu]/t\mu = p \\ \text{incl = can} \downarrow & \downarrow \varphi & \\ \pi_*(HH\mathbb{F}_p)^{tS^1} & = & \mathbb{Z}_p[t^{\pm 1},\mu]/t\mu = p \\ & \cong & \mathbb{Z}_p[t^{\pm 1}] \quad (\mu = p \cdot t^{-1}) \end{array}$$

Come back to the picture

$$\begin{array}{ccccccc} TR\,\mathbb{F}_p & = & HW\mathbb{F}_p & = & H\mathbb{Z}_p \\ 1\downarrow\ \downarrow F & & \downarrow 0 & & \downarrow 0 \\ TR\,\mathbb{F}_p & = & HW\mathbb{F}_p & = & H\mathbb{Z}_p \end{array}$$

$$\boxed{TC\,\mathbb{F}_p{}^{\wedge}_{p} = H\mathbb{Z}_p \vee \Sigma^{-1} H\mathbb{Z}_p}$$

see below (ref?)

can $\mu = \mu = pt^{-1}$ can $t = t$

$\varphi\,\mu = t^{-1}$ $\varphi\,t = p\,t$

$$(\text{can} - \varphi) \left\{ \begin{array}{l} 1 \mapsto 0 \\ \mu \mapsto (p-1)t^{-1} \\ t \mapsto (1-p)t \end{array} \right\} \text{ iso in } \forall \text{ degs} \neq 0$$

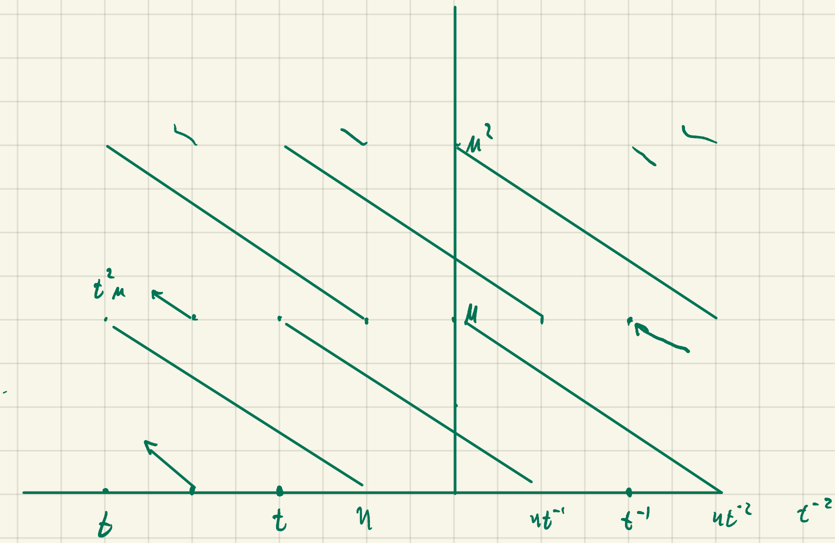
$$\boxed{0 \to TC_0\,\mathbb{F}_p{}^{\wedge}_{p} \to \mathbb{Z}_p \xrightarrow{0} \mathbb{Z}_p \to TC_{-1}\,\mathbb{F}_p{}^{\wedge}_{p} \to 0}$$

On the Frobenius.

$$(HH\,\mathbb{F}_p)^{hC_p} \longrightarrow (HH\,\mathbb{F}_p)^{tC_p}$$

$\pi_* \Uparrow$

$$H^{-s}(C_p, HH_t\mathbb{F}_p)$$



$$\pi_* HH\mathbb{F}_p = \mathbb{F}_p[\mu]$$
$$\varphi \downarrow$$
$$\pi_* HH\mathbb{F}_p{}^{tC_p} = \mathbb{F}_p[t^{\pm 1}].$$

$\varphi(\mu) = t^{-1}$ ← reference!

On S^1-homotopy fixed & Tate

KAST

$$HH\,A = A \otimes_{A^{o}\otimes A} A \qquad\qquad \sigma\tau_0 = \mu$$

$$\begin{tikzcd} HH\,A & \sigma\tau_0=\mu \\ S^1_+\wedge(A^{op}\otimes A) \arrow[u, "\sigma"] & \tau_0 \arrow[u, mapsto] \end{tikzcd} \quad \text{for } A=\mathbb{F}_p$$

$$\begin{tikzcd} A^{op}\otimes A \arrow[r] \arrow[d] & A \arrow[d] \\ A \arrow[r] & HH\,A \end{tikzcd}$$

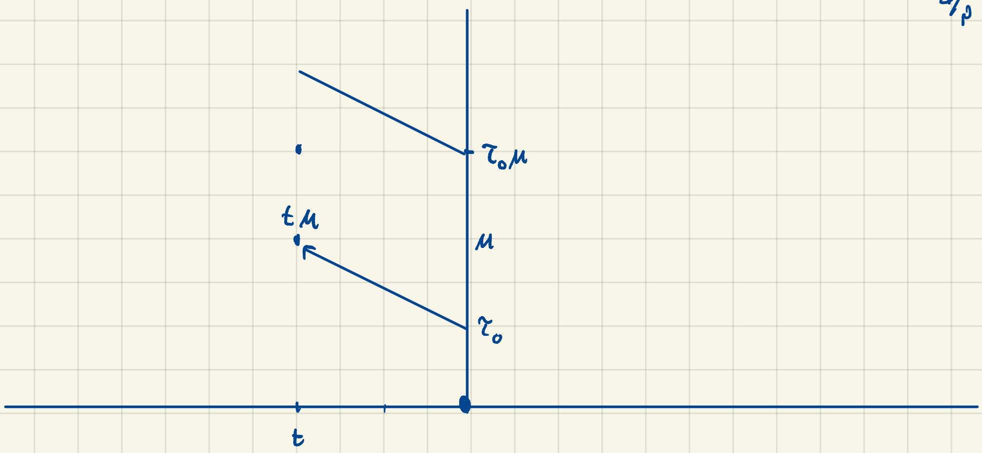
$\Rightarrow$ Under the S^1-action

$$\begin{tikzcd} S^1_+\wedge\mathbb{F}_p\wedge\mathbb{F}_p \arrow[r, "\text{unit}"] & S^1_+\wedge\mathbb{F}_p\wedge HH\mathbb{F}_p \arrow[r] & \mathbb{F}_p\wedge HH\mathbb{F}_p \\ & {[1]}\wedge\tau_0 \arrow[r, mapsto] & \mu \\ S^1_+\wedge\mathbb{S}/p\wedge\mathbb{F}_p \arrow[uu] \arrow[r] & S^1_+\wedge\mathbb{S}/p\wedge HH\mathbb{F}_p \arrow[r] & \mathbb{S}/p\wedge HH\mathbb{F}_p \end{tikzcd}$$

$$(\mathbb{S}/p_*\, HH\mathbb{F}_p = E\tau_0\otimes\mathbb{F}_p[\mu])$$

The d^2-differential in the S^1-Tate SS is induced by the S^1-action

$$\mathbb{F}_p[t^{\pm1}]\otimes E\tau_0\otimes\mathbb{F}_p[\mu] \Rightarrow \pi_*(\mathbb{S}/p\wedge HH\mathbb{F}_p)^{tS^1}$$
$$\cong \;\leftarrow \mathbb{S}/p \text{ finite}$$
$$\mathbb{S}/p_*(HH\mathbb{F}_p)^{tS^1}$$



This means that the class rep'n by $t\mu$ in $HH\mathbb{F}_p^{hS^1}$ & $HH\mathbb{F}_p^{tS^1}$ must be divisible by p yielding maximal extension so that

$$\hat{E}^\infty = \hat{E}^2 = \mathbb{F}_p[t^{\pm1},\mu] \Rightarrow \mathbb{Z}_p[t^{\pm1},\mu]/t\mu\doteq p = \pi_*\, HH\mathbb{F}_p^{tS^1}$$
$$\cup$$
$$E^\infty = E^2 = \mathbb{F}_p[t,\mu] \Rightarrow \mathbb{Z}_p[t,\mu]/t\mu\doteq p = \pi_*\, HH\mathbb{F}_p^{tS^1}$$

KAST

On finite fixed points

Consider

$$\begin{tikzcd} & & HH\mathbb{F}_p \arrow[d, "\varphi"] \\ HH\mathbb{F}_{p\,hC_p} \arrow[r] & HH\mathbb{F}_p^{hC_p} \arrow[r] & \boxed{HH\mathbb{F}_p^{tC_p}} \\ & HH\mathbb{F}_p^{hS^1} \arrow[u] \arrow[r] & HH\mathbb{F}_p^{tS^1} \arrow[u] \end{tikzcd}$$

φ — cyclo

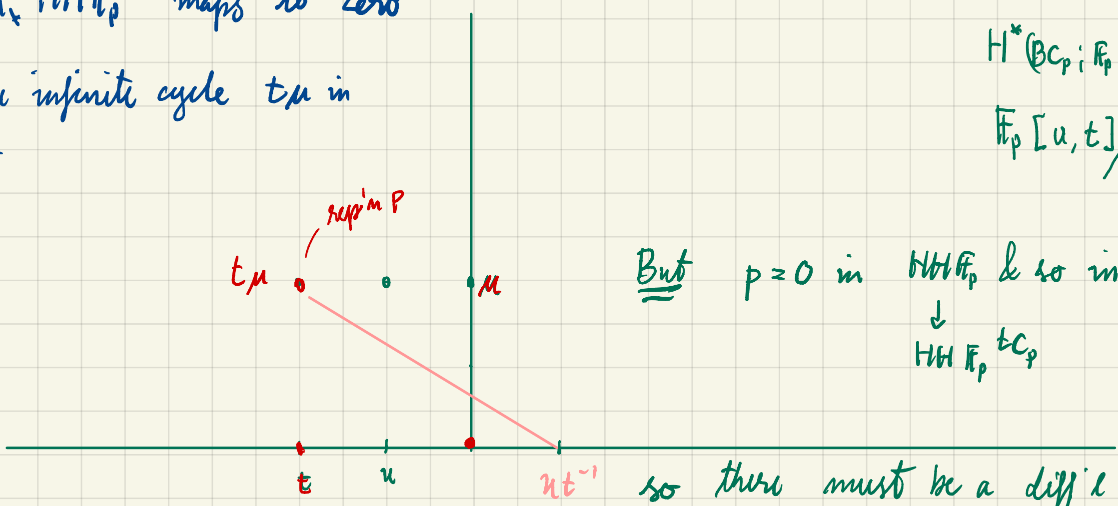
$HH\mathbb{F}_p^{tC_p}$ — so $\mathbb{F}_p$-algebra

$H^*(BC_p;\mathbb{F}_p) = \mathbb{F}_p[u,t]/\begin{matrix} u^2=t & p=2 \\ u^2=0 & p\neq 2\end{matrix}$

$HH\mathbb{F}_p^{tC_p}$ $\mathbb{F}_p$-algebra $\Rightarrow$

$p \in \pi_* HH\mathbb{F}_p^{tS^1}$ maps to zero

So the infinite cycle $t\mu$ in $\hat{E}^2$



<u>But</u> $p=0$ in $HH\mathbb{F}_p$ & so in
$\downarrow$
$HH\mathbb{F}_p^{tC_p}$

so there must be a diff'l killing it ($t\mu$) in SS $\Rightarrow HH\mathbb{F}_p^{tC_p}$

Only possibility:

$$\boxed{d^3 u = t^2\mu}$$

So: $\hat{E}^3 = \mathbb{F}_p[t^{\pm}] = \hat{E}^\infty$

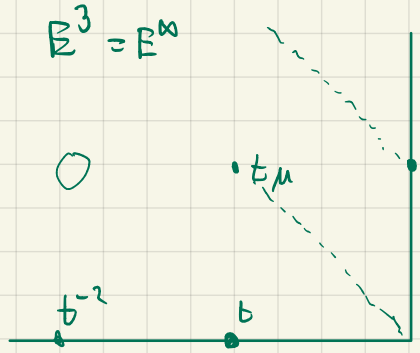
$= \pi_* HH\mathbb{F}_p^{tC_p}$

Fact $HH\mathbb{F}_p \xrightarrow{\varphi} HH\mathbb{F}_p^{tC_p}$

$\mathbb{F}_p[\mu] \longrightarrow \mathbb{F}_p[t^{\pm 1}] \qquad \varphi\mu \doteq t^{-1}$

In $HH\mathbb{F}_p^{hC_p}$ ut^{-1} is not there, so $t\mu$ is not a boundary (but all $t^{i+1}\mu$ are)

$E^3 = E^\infty$



So $\pi_* HH\mathbb{F}_p^{hC_p}$

$= \mathbb{Z}/p^2[t,\mu]/\begin{matrix} t\mu = p \\ pt = 0\end{matrix}$