Week 7. Geometrically nonlinear structures

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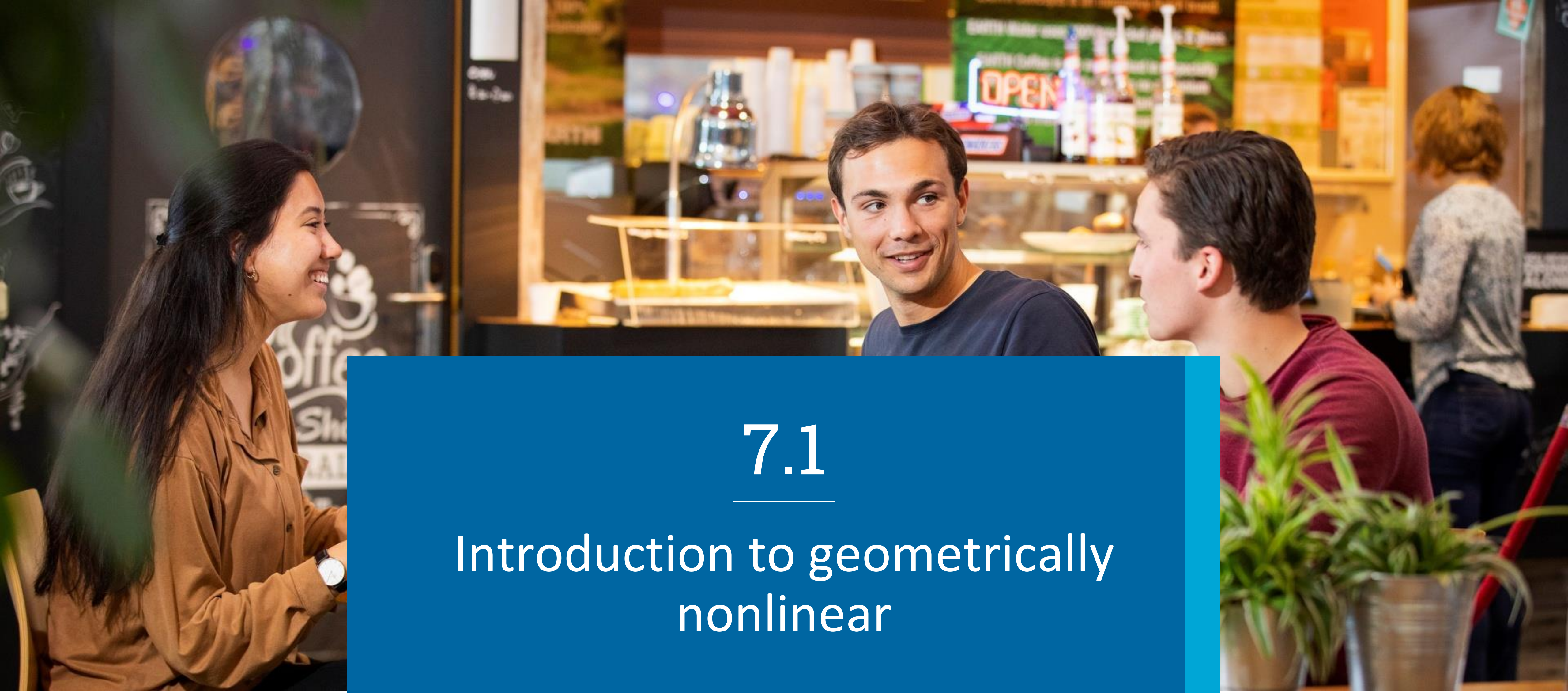
Learning objectives

At the end of this week you will be able to:

Define numerical methods to analyse systems governed by nonlinear Partial Differential Equations. This entails:

1. Define the formulation that characterizes the dynamics of structures subject to large deformations
2. Define numerical methods to solve nonlinear systems of PDEs
3. Analize and justify the results



7.1

Introduction to geometrically nonlinear

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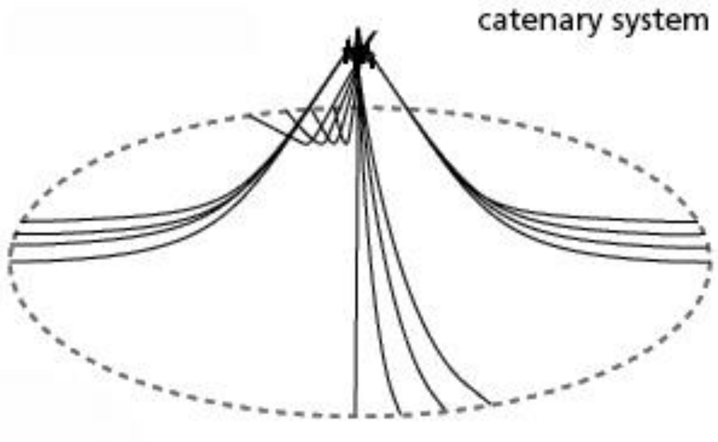
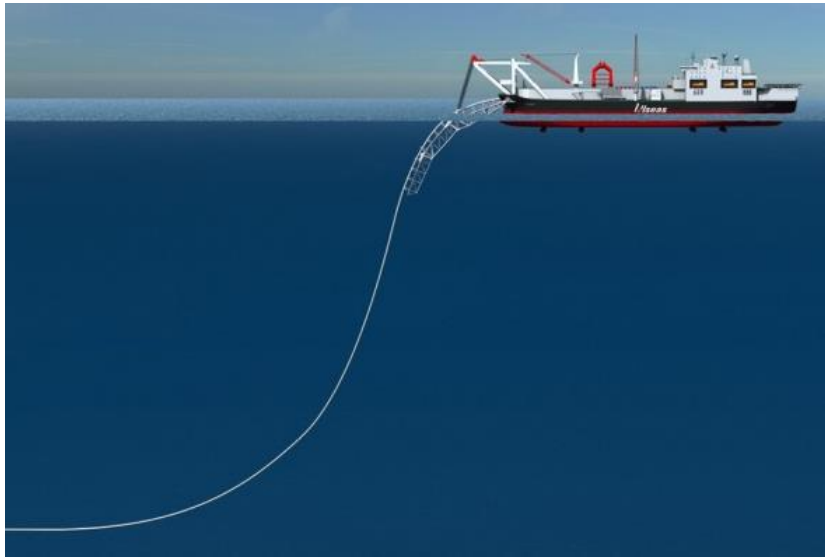
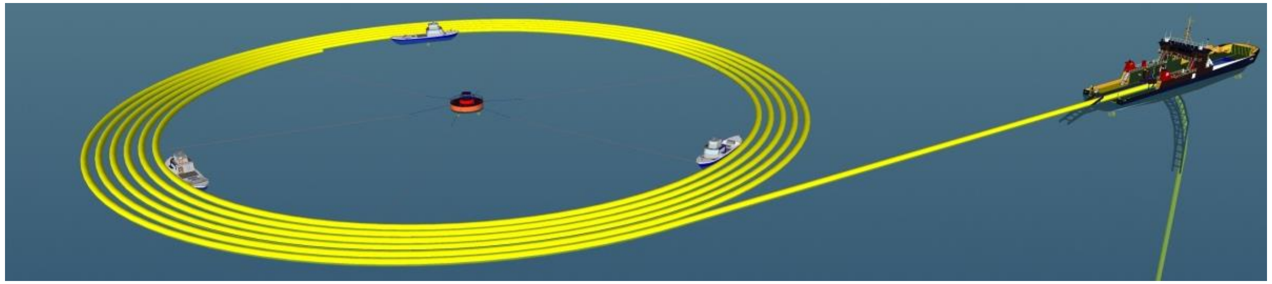
What is a geometrically nonlinear structure?

A geometrically nonlinear (GNL) structure is a structure whose deformation cannot be assumed to be small.

Examples:







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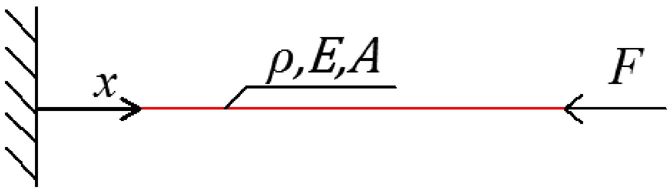
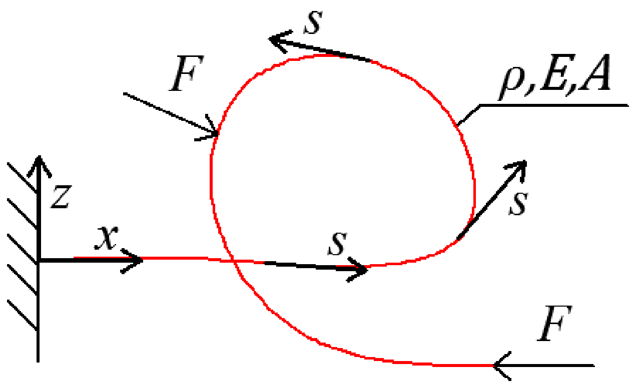
How do we model a geometrically nonlinear structure?

- When deriving the EOMs of a “normal” Euler-Bernoulli beam, rod or other linear 1D objects, **small displacements** are assumed. Why? So that all trigonometric functions can be approximated by a first order TSE (linearize).
- In that case the solution (displacement) is found with respect to the original (neutral) position → **Local Coordinates System (LCS)**
- For GNL elements we need **large displacements**, so taking the solution w.r.t. the neutral position is not useful. Instead we will use the coordinates with respect to a reference point → **Global Coordinates System (GLS)**





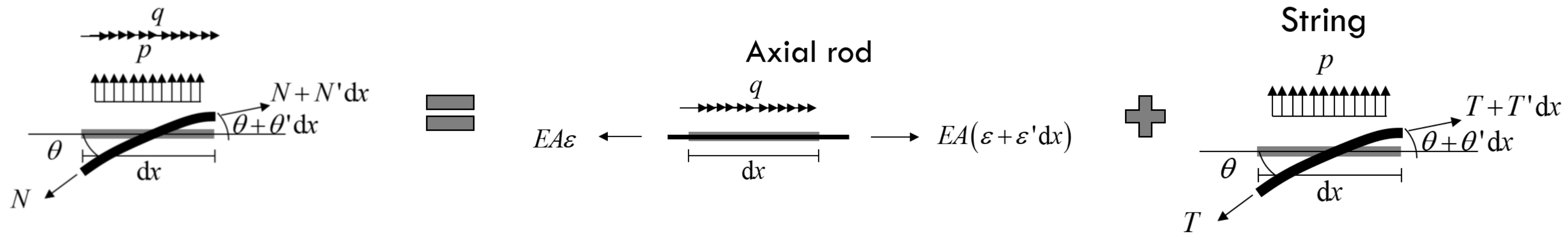
How do we model a geometrically nonlinear structure?

What are the differences between Local and global coordinates systems?

- Coordinate systems
  - local coordinate system (LCS): x_i and x_j
  - global coordinate system (GCS): $\hat{x}_i$ and $\hat{x}_j$.
- LCS (geometrically linear systems only)
  - $\boldsymbol{q}_n$ contains only the deformations with respect to a reference location
  - A transformation is required from LCS to GCS
  - The entire structure can move as long as it remains straight, i.e. linear.
- GCS (both geometrically linear and nonlinear system)
  - $\boldsymbol{q}_n$ contains the location of each node
  - Already in GCS
  - No restrictions on displacement, i.e. it does not have to remain straight, can be a circle.

Coupling a rod and a string?

A GNL element can be simplified into an uncoupled rod and string:



If we had small deformations:

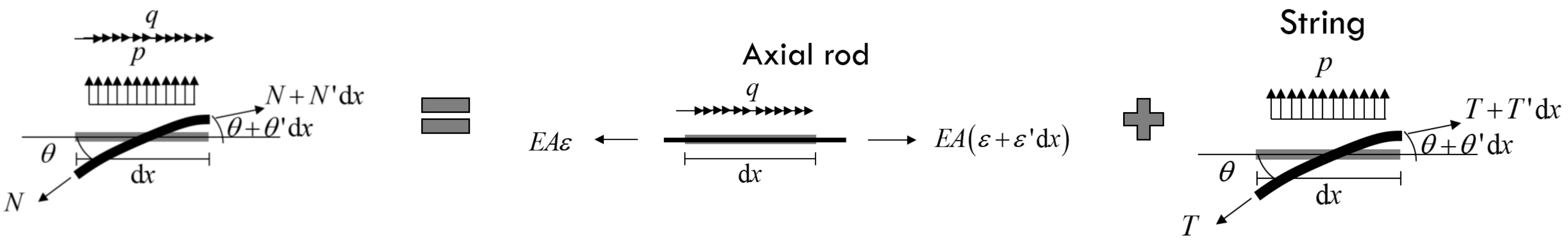
Rod EOM: $m\ddot{u} - EAu'' = q$

String EOM: $m dx \ddot{v} = p\ dx + (T + T'dx)\underbrace{\sin(\theta + \theta' dx)}_{\theta+\theta' dx} - T\underbrace{\sin(\theta)}_{\theta}$

$$m\ddot{v} - Tv'' - \cancel{T'v'} = p$$

Coupling a rod and a string?

A GNL element can be simplified into an uncoupled rod and string:



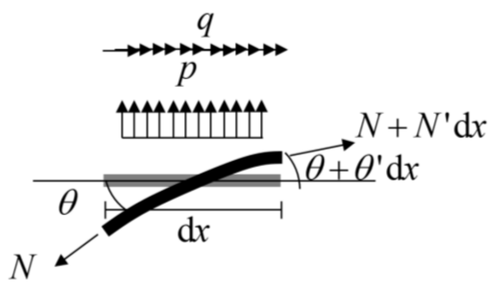
If we had small deformations:

(uncoupled) Discrete system (after applying FEM): $\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \mathbf{p} + \mathbf{q}$

$$\mathbf{K} = \frac{1}{L}\begin{bmatrix} EA & 0 & -EA & 0 \\ 0 & T & 0 & -T \\ -EA & 0 & EA & 0 \\ 0 & -T & 0 & T \end{bmatrix} \qquad \mathbf{M} = \frac{mL}{6}\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} u_l \\ v_l \\ u_r \\ v_r \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 0 \\ \frac{pL}{2} \\ 0 \\ \frac{pL}{2} \end{bmatrix} \qquad \mathbf{q} = \begin{bmatrix} \frac{qL}{2} \\ 0 \\ \frac{qL}{2} \\ 0 \end{bmatrix}$$

Coupling a rod and a string?

When taking large deformations: axially deformed string



Second Newton's law

$$\begin{cases} m\,\mathrm{d}x\,\ddot{u} = q\mathrm{d}x + (N + N'\mathrm{d}x)\cos(\theta + \theta'\mathrm{d}x) - N\cos(\theta) \\ m\,\mathrm{d}x\,\ddot{v} = p\mathrm{d}x + (N + N'\mathrm{d}x)\sin(\theta + \theta'\mathrm{d}x) - N\sin(\theta) \end{cases}$$

$$\begin{cases} m\ddot{u} = q - N\theta'\sin(\theta) + N'\cos(\theta) \\ m\ddot{v} = p + N\theta'\cos(\theta) + N'\sin(\theta) \end{cases}$$

Constitutive law (material)

$$N = EA\varepsilon$$

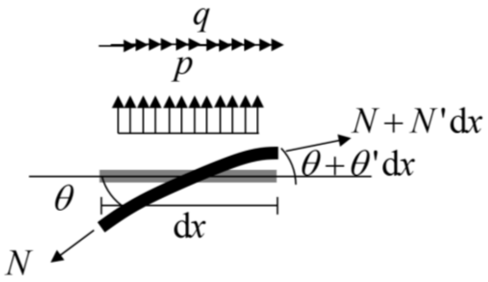
Geometry law

$$\theta = \tan^{-1}\left(\frac{v'}{1+u'}\right) \qquad \theta' = \frac{v''(1+u') - v'u''}{v'^2 + (1+u')^2}$$

$$\varepsilon = \sqrt{(1+u')^2 + v'^2} - 1 \qquad \varepsilon' = \frac{u''(1+u') + v''v'}{\sqrt{(1+u')^2 + v'^2}}$$

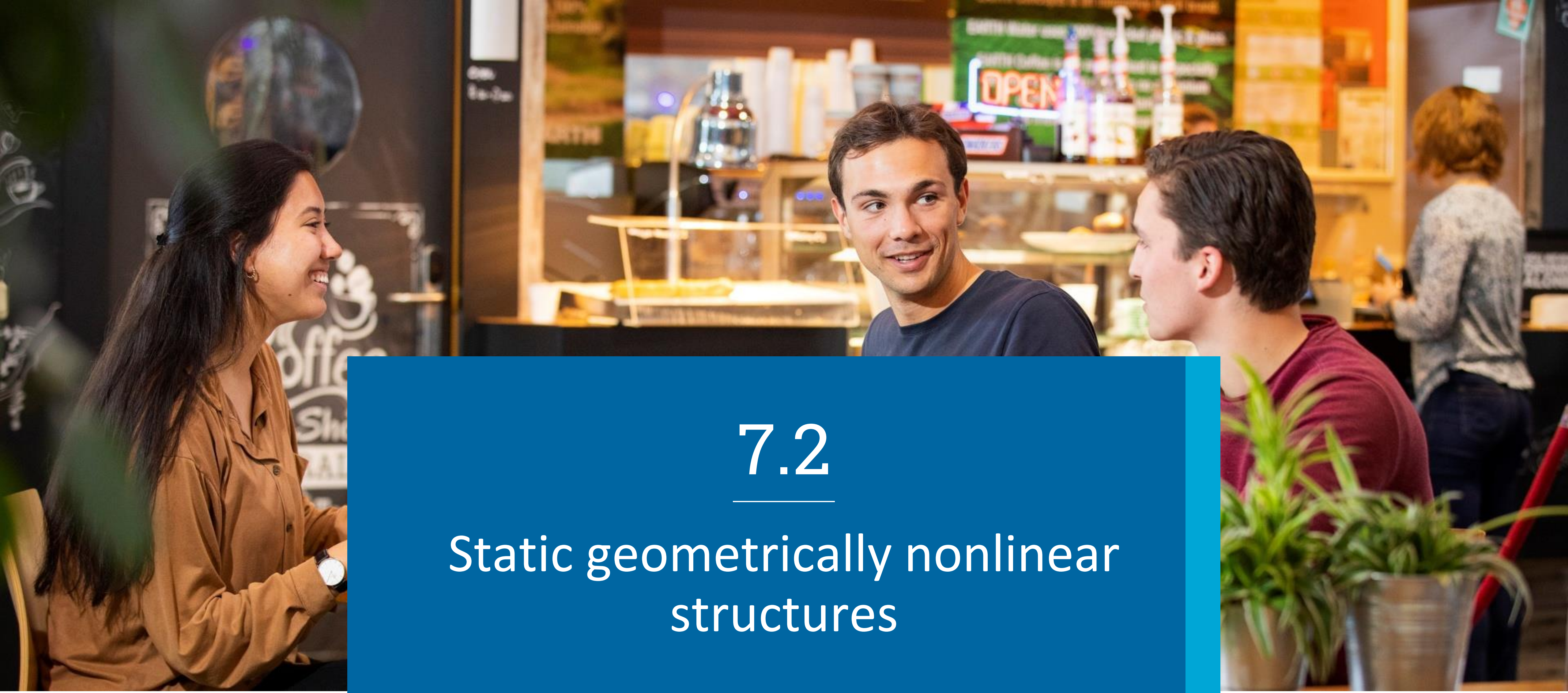
Coupling a rod and a string?

When taking large deformations: axially deformed string



$$
\begin{cases}
m\ddot{u} - EAv'\left(u''v' - v''(u'+1)\right)\dfrac{\sqrt{(u'+1)^2 + v'^2} - 1}{\left((u'+1)^2 + v'^2\right)^{\frac{3}{2}}} - EA(u'+1)\dfrac{v''v' + u''(u'+1)}{(u'+1)^2 + v'^2} = q \\
m\ddot{v} - EA\dfrac{v'\left(v''v' + u''(u'+1)\right)}{(u'+1)^2 + v'^2} + EA\dfrac{\sqrt{(u'+1)^2 + v'^2} - 1}{\left((u'+1)^2 + v'^2\right)^{\frac{3}{2}}}\left(u''v' - v''(u'+1)\right)(u'+1) = p
\end{cases}
$$

We are engineers: let us simplify!!!!!



7.2

Static geometrically nonlinear structures

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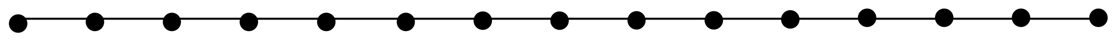
Solving a static GNL structure

How can we simplify the problem?

Using FEM! ☺

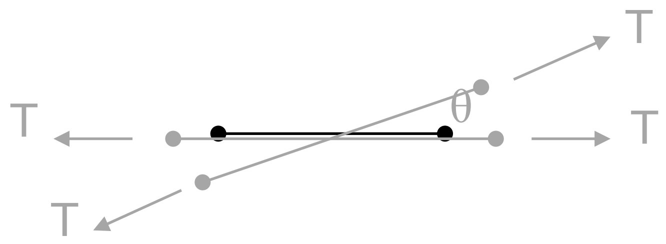
Steps to follow:

1) Divide the string into smaller strings (elements)



For each small string, we know its axial stiffness, its unit mass, and its tensionless length.

Let us assume that for each string, the deformation happens only axially: it can rotate and extend, no shear nor bending deformations



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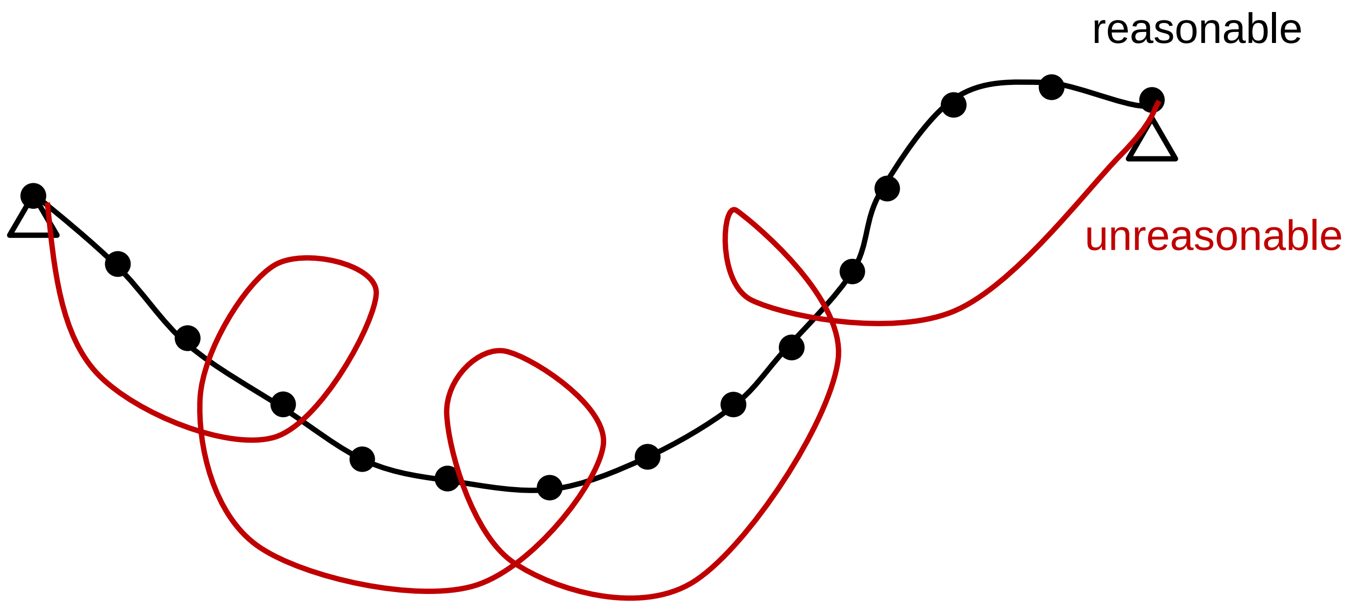
Steps to follow:

2) Assume a deformed shape that respects the boundary conditions and that is “reasonable”



Solving a static GNL structure

How can we simplify the problem?

Using FEM! ☺

Steps to follow:

3) Based on the deformed shape (displaced position of the nodes), we can calculate for each spring the extension, the tension, the orientation angle (and its forces at the nodes)

$$\varepsilon = \frac{\sqrt{\left(x_{0,r} + u_{x,r} - x_{0,l} - u_{x,l}\right)^2 + \left(y_{0,r} + u_{y,r} - y_{0,l} - u_{y,l}\right)^2} - L}{L}$$

$$\theta = \tan^{-1}\left(\frac{y_{0,r} + u_{y,r} - y_{0,l} - u_{y,l}}{x_{0,r} + u_{x,r} - x_{0,l} - u_{x,l}}\right)$$

$$T = EA\varepsilon$$

$$F_{x,l} = -T\cos\theta \qquad F_{x,r} = T\cos\theta$$

$$F_{y,l} = -T\sin\theta \qquad F_{y,l} = T\sin\theta$$

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Solving a static GNL structure

How can we simplify the problem?

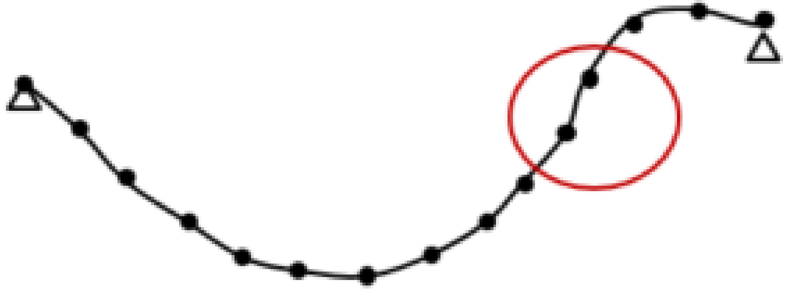
Using FEM! ☺

Steps to follow:

4) Apply the same process as seen in Module 3… with some differences:

- Now the weak form is nonlinear, it depends not only on the string properties (EA, L), but also on the position of its nodes (from which we calculate tension and orientation)

- After integration the contribution of the internal forces and external forces needs to balance at the nodes



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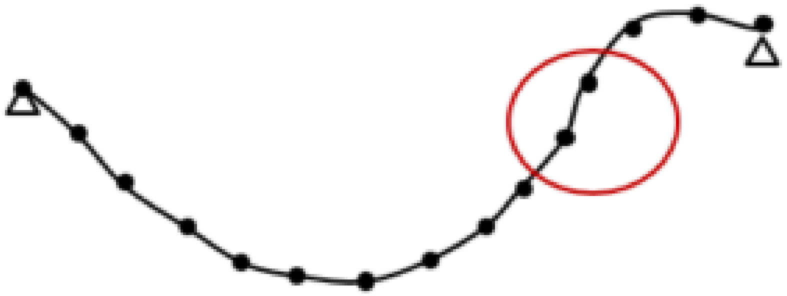

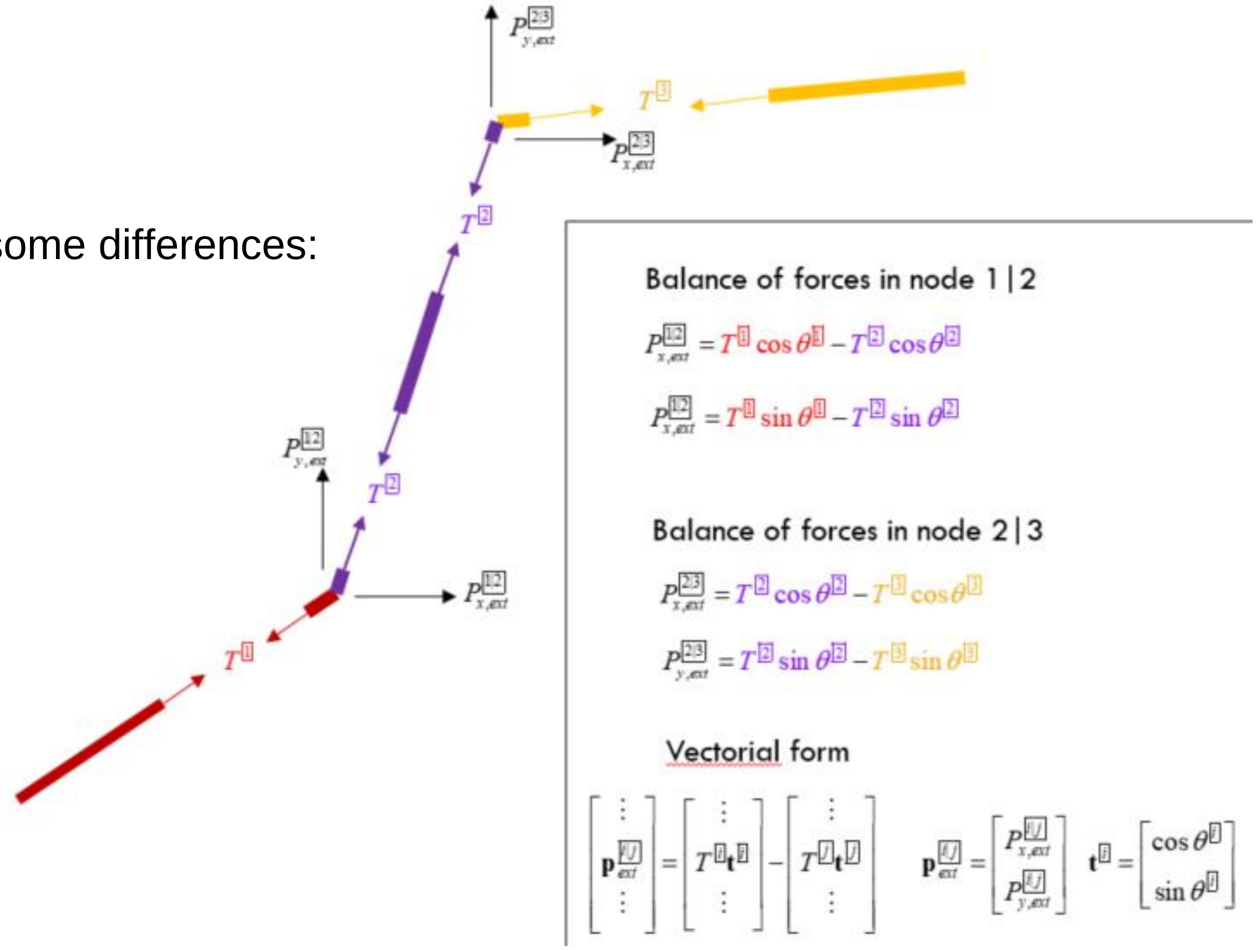

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Solving a static GNL structure

How can we simplify the problem?

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Steps to follow:

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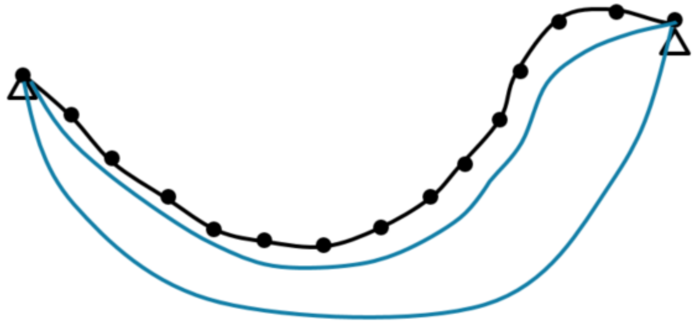


$$\begin{bmatrix} \vdots \\ \mathbf{p}_{ext}^{\boxed{i|j}} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ T^{\boxed{i}}\mathbf{t}^{\boxed{i}} \\ \vdots \end{bmatrix} - \begin{bmatrix} \vdots \\ T^{\boxed{j}}\mathbf{t}^{\boxed{j}} \\ \vdots \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} \vdots \\ \mathbf{p}_{ext}^{\boxed{i|j}} \\ \vdots \end{bmatrix} - \begin{bmatrix} \vdots \\ T^{\boxed{i}}\mathbf{t}^{\boxed{i}} \\ \vdots \end{bmatrix} + \begin{bmatrix} \vdots \\ T^{\boxed{j}}\mathbf{t}^{\boxed{j}} \\ \vdots \end{bmatrix} = 0 \quad \rightarrow \quad \mathbf{P}_{ext} - \mathbf{f}_{\text{int}}\left(\mathbf{x}_0 + \mathbf{u}\right) = 0$$

Wait! Not that easy!! Now the entries on the internal force depend on $\boldsymbol{u}$! We have a nonlinear system!!

Solving a static GNL structure

How can we simplify the problem?

Using FEM! ☺

Steps to follow:

5) Solve the nonlinear system:

Let's write the residual of the nonlinear system: $\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u})$

Apply TSE…

$$\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{R}(\boldsymbol{u}^*) + \frac{\partial \boldsymbol{R}(\boldsymbol{u}^*)}{\partial \boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}^*) = \underbrace{\boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}^*)}_{\boldsymbol{R}(\boldsymbol{u}^*)} - \boldsymbol{K}(\boldsymbol{u}^*)\delta\boldsymbol{u}$$

$\boldsymbol{K}(\boldsymbol{u}^*)$ is the derivative of $\boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u})$ with respect to the FEM coefficients $\boldsymbol{u}$

$\boldsymbol{K}(\boldsymbol{u}^*)$ is the linearized stiffness matrix. Remember that we assumed that each string can only extend and rotate (no shear nor bending), and thus the stiffness matrix of each string corresponds to the coupled matrix of ROD+STRING

$$\mathbf{K} = \frac{1}{L}\begin{bmatrix} EA & 0 & -EA & 0 \\ 0 & T & 0 & -T \\ -EA & 0 & EA & 0 \\ 0 & -T & 0 & T \end{bmatrix} \qquad T = EA\varepsilon$$

Solving a static GNL structure

How can we simplify the problem?

Using FEM! ☺

Steps to follow:

5) Solve the nonlinear system:

Let's write the residual of the nonlinear system: $\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u})$

Apply TSE…

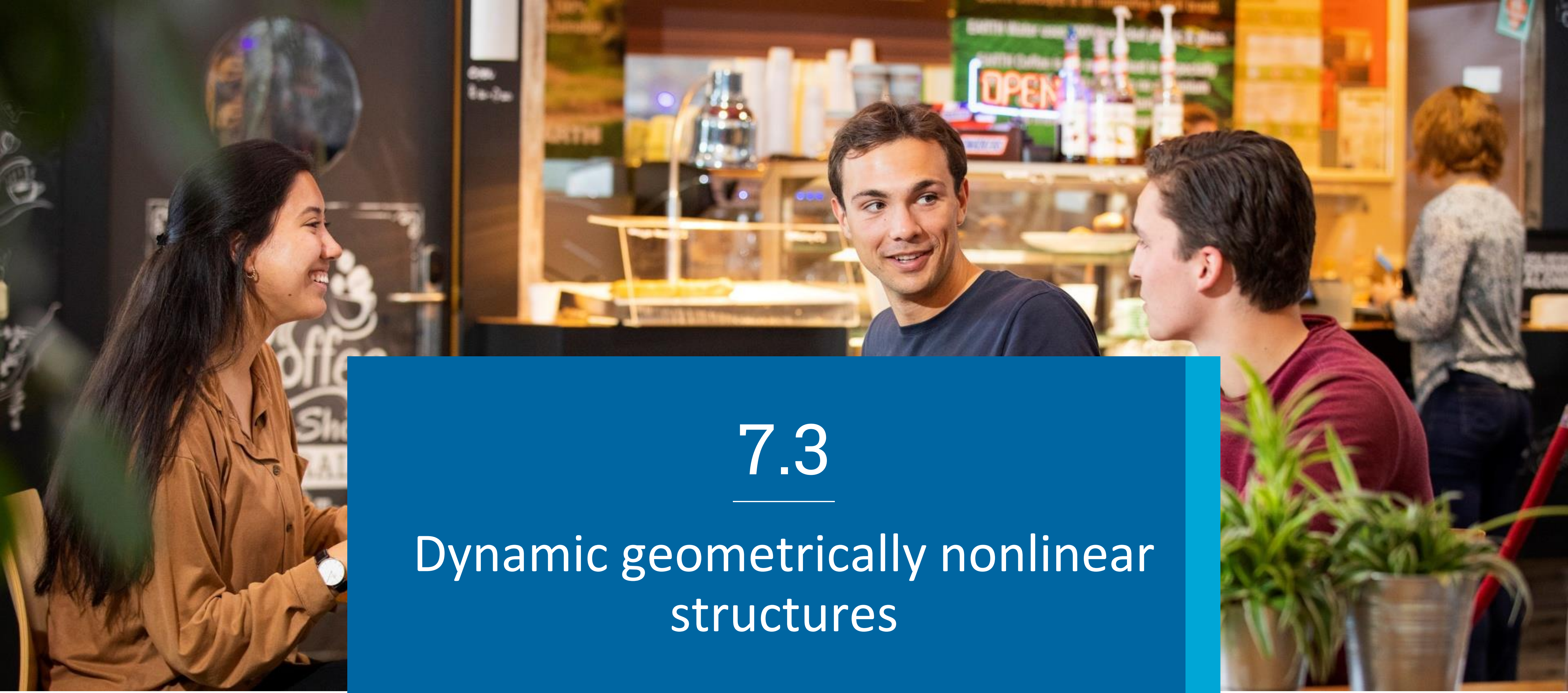
$$\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{R}(\boldsymbol{u}^*) + \frac{\partial \boldsymbol{R}(\boldsymbol{u}^*)}{\partial \boldsymbol{u}}(\boldsymbol{u} - \boldsymbol{u}^*) = \underbrace{\boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}^*)}_{\boldsymbol{R}(\boldsymbol{u}^*)} - \boldsymbol{K}(\boldsymbol{u}^*)\boldsymbol{\delta u}$$

We want $\boldsymbol{R}(\boldsymbol{u}) = 0$ → Newton-Raphson method:

1. Take initial guess $\boldsymbol{u}^{\boldsymbol{0}}$
2. Iterate while $\boldsymbol{R}(\boldsymbol{u}^{i}) > \boldsymbol{tol}$:
   - Compute increment: $\boldsymbol{\delta u}^{\boldsymbol{i}} = \boldsymbol{K}(\boldsymbol{u}^{\boldsymbol{i-1}})^{-1}\boldsymbol{R}(\boldsymbol{u}^{\boldsymbol{i-1}})$
   - Update nodal position: $\boldsymbol{u}^{\boldsymbol{i}} = \boldsymbol{u}^{\boldsymbol{i-1}} + \boldsymbol{\partial u}^{\boldsymbol{i}}$
   - Compute new residual: $\boldsymbol{R}(\boldsymbol{u}^{\boldsymbol{i}}) = \boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}^{i})$

Summary

- We have seen how to solve a GNL structure using FEM
- All the same steps for the linear FEM + Newton-Raphson algorithm.



7.3

Dynamic geometrically nonlinear structures

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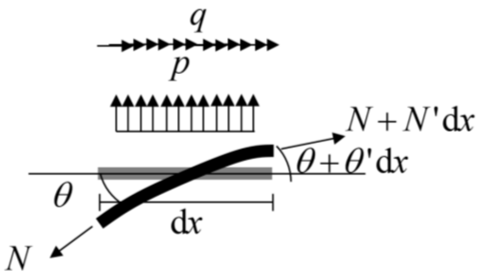
Solving a dynamic GNL structure

Let's go back to the axially deformed string formulation



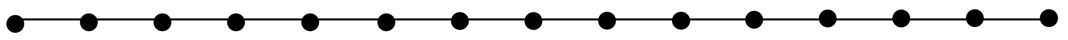
$$
\begin{cases}
m\ddot{u} - EAv'\left(u''v' - v''(u'+1)\right)\dfrac{\sqrt{(u'+1)^2 + v'^2} - 1}{\left((u'+1)^2 + v'^2\right)^{\frac{3}{2}}} - EA(u'+1)\dfrac{v''v' + u''(u'+1)}{(u'+1)^2 + v'^2} = q \\
m\ddot{v} - EA\dfrac{v'\left(v''v' + u''(u'+1)\right)}{(u'+1)^2 + v'^2} + EA\dfrac{\sqrt{(u'+1)^2 + v'^2} - 1}{\left((u'+1)^2 + v'^2\right)^{\frac{3}{2}}}\left(u''v' - v''(u'+1)\right)(u'+1) = p
\end{cases}
$$

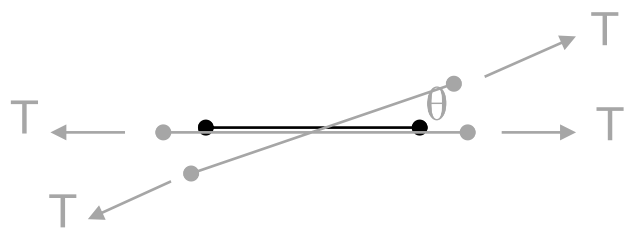
The mass term is linear 👍. But why? Shouldn't it depend on the element configuration?

Solving a dynamic GNL structure

Let's take the same steps as in the static case:

1) Divide the string into smaller strings (elements)





2) Assume a deformed shape that respects the boundary conditions and that is "reasonable"

3) Based on the deformed shape (displaced position of the nodes), we can calculate for each spring the extension, the tension, the orientation angle

$$\varepsilon = \frac{\sqrt{\left(x_{0,r} + u_{x,r} - x_{0,l} - u_{x,l}\right)^2 + \left(y_{0,r} + u_{y,r} - y_{0,l} - u_{y,l}\right)^2} - L}{L}$$

$$\theta = \tan^{-1}\left(\frac{y_{0,r} + u_{y,r} - y_{0,l} - u_{y,l}}{x_{0,r} + u_{x,r} - x_{0,l} - u_{x,l}}\right)$$

$$T = EA\varepsilon$$

$$F_{x,l} = -T\cos\theta \qquad F_{x,r} = T\cos\theta$$

$$F_{y,l} = -T\sin\theta \qquad F_{y,l} = T\sin\theta$$

Solving a dynamic GNL structure

Let’s take the same steps as in the static case:

4) Apply the same process as seen in Module 4…

Static: $\boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}) = 0$

Dynamic: $\boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}) = \boldsymbol{M}\ddot{\boldsymbol{u}}$

- Global assembly:

1. Calculate position of nodes
2. Calculate extension, tension, and orientation
3. Calculate local stiffness matrix **M** and rotation matrix **T**
4. Rotate matrix **M**: $\widetilde{\boldsymbol{M}} = \boldsymbol{T}\boldsymbol{M}\boldsymbol{T}^{\boldsymbol{T}}$
5. Assemble $\widetilde{\boldsymbol{M}}$ at the slots corresponding to the string’s nodes

Solving a dynamic GNL structure

Let’s take the same steps as in the static case:

4) Build elemental matrices and assemble into the global system

But $\boldsymbol{T}$ depends on the solution $\boldsymbol{u}$. Shouldn’t $\widetilde{\boldsymbol{M}}$ depend on $\boldsymbol{u}$?

Solving a dynamic GNL structure

Let’s take the same steps as in the static case:

5) Solve the system:

Let’s write the residual of the nonlinear system:

$$\boldsymbol{R}(\boldsymbol{u}) = \boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}) - \boldsymbol{M}\ddot{\boldsymbol{u}}$$

If we use an explicit time integration, we can re-write it as:

$$\boldsymbol{M}\ddot{\boldsymbol{u}}_{n+1} = \boldsymbol{F}_{ext} - \boldsymbol{F}_{int}(\boldsymbol{x}_0 + \boldsymbol{u}_n)$$

$\boldsymbol{M}$ is linear, so no Newton-Raphson required! Just use the usual ODE solver.

Note that:

- Since $\boldsymbol{M}$ is linear, we only need to compute it once (at the beginning of the simulation)
- Since $\boldsymbol{F}_{int}$ depends on $\boldsymbol{u}_n$, we’ll have to re-compute it at every time step.

Summary

- We have seen how to solve a dynamic problem in a GNL structure using FEM
- All the same steps for the linear FEM.
- We have seen that we don’t need to use a Newton-Raphson algorithm if we use an explicit time integration.

Thank you for your attention

Oriol Colomés

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