Week 4. Numerical methods for PDEs in 2D

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TU Delft

Learning objectives

At the end of this week you will be able to:

Define and analyze numerical methods to solve systems governed by Partial Differential Equations in 2-dimensional domains. This entails:

1. Define the the system of PDEs that characterize the behavior of linear elastic structures in 2D
2. Define numerical methods to solve a system of PDEs
3. Implement a solver for a system of PDEs
4. Analize and justify the results

4.1

The isoparametric map

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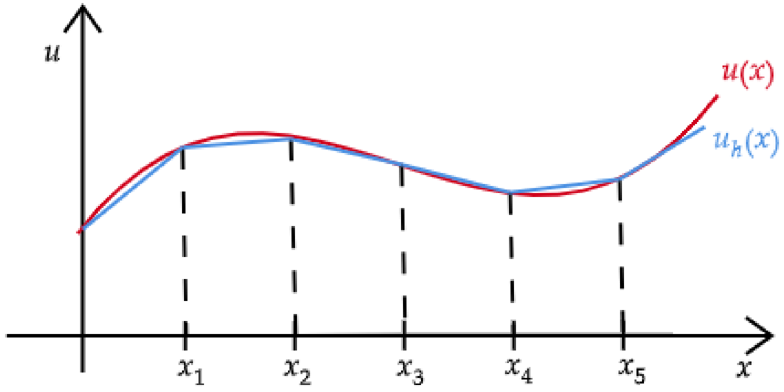
The isoparametric map

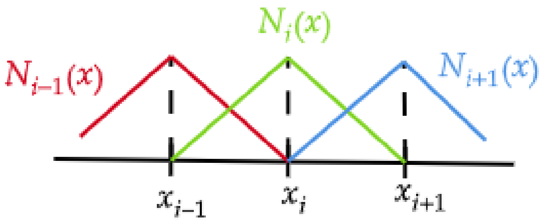
In the previous week we learned how to define Finite Element functions through the use of shape functions:

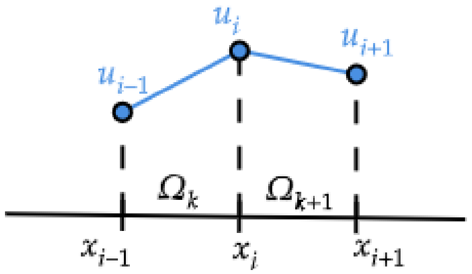
$$u(x) = \sum_{i=1}^{n_n} N_i(x) u_i$$

For each node (i) we define piece-wise interpolation functions defined on the set of elements S_i that are attached to the node i:

$$N_i(x) = \begin{cases} p(x), & x \in \Omega_k \ \forall \ k \in S_i \\ 1, & x = x_i \\ 0, & \text{otherwise} \end{cases}$$







The isoparametric map

Problem: we need to define the shape functions for each element!!

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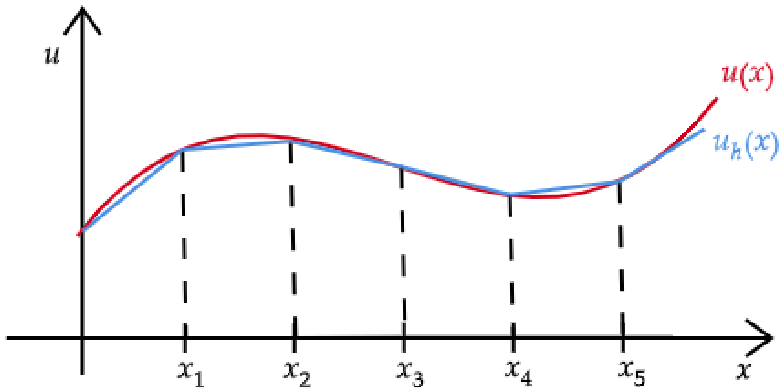
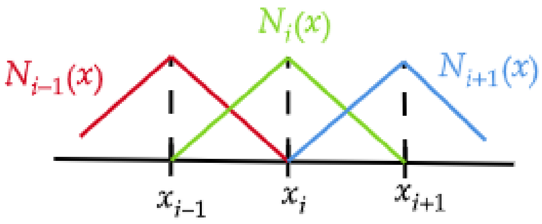
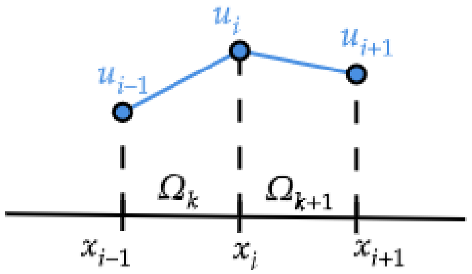




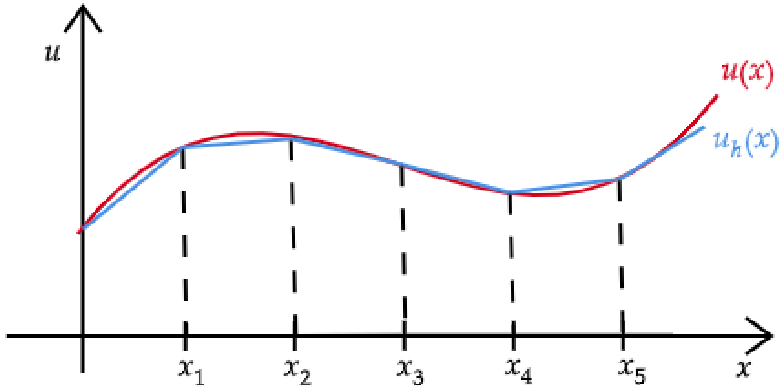


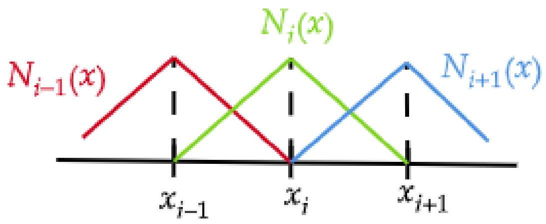
The isoparametric map

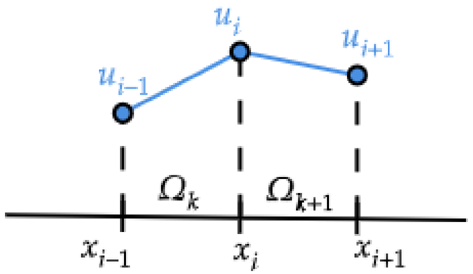
Problem: we need to define the shape functions for each element!!

For piece-wise linear shape functions at element k between nodes a and b we have: $h = x_b - x_a$:

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$







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The isoparametric map

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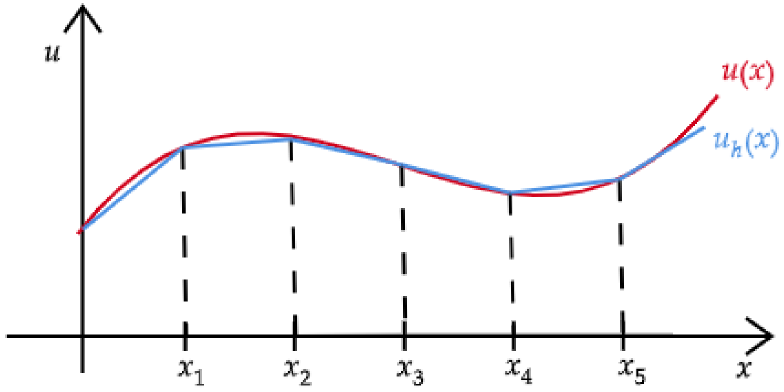
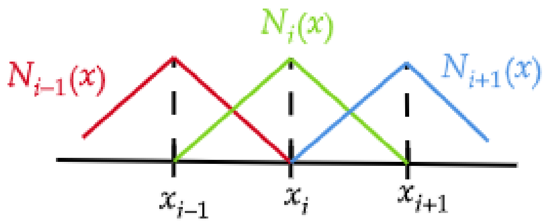
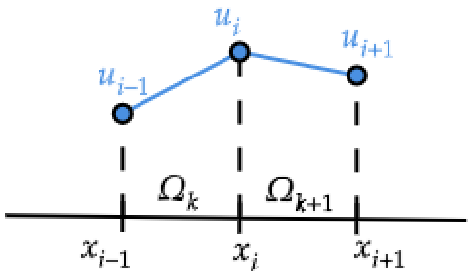
This is not practical and defeats the purpose of generality of the Finite Element method!

Solution?



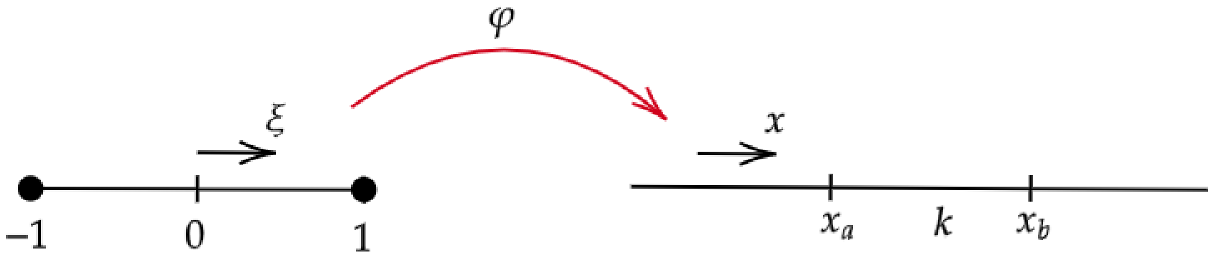




The isoparametric map

Solution: The isoparametric map

We express the coordinates of an element in terms of a fixed reference element



$$x = \varphi_k(\xi)$$

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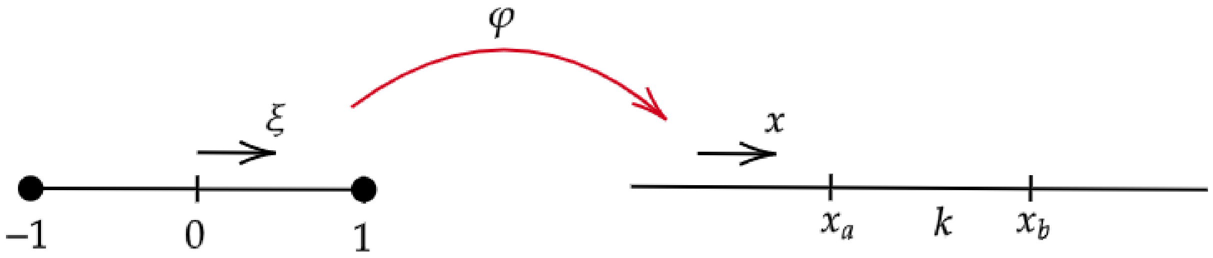
The isoparametric map

Solution: The isoparametric map

We express the coordinates of an element in terms of a fixed reference element



$$x = \varphi_k(\xi)$$

In particular, we will use a linear map at each element:

$$x(\xi)\Big|_k = x_a \frac{(1-\xi)}{2} + x_b \frac{1+\xi}{2} = x_a N_a(\xi) + x_b N_b(\xi)$$

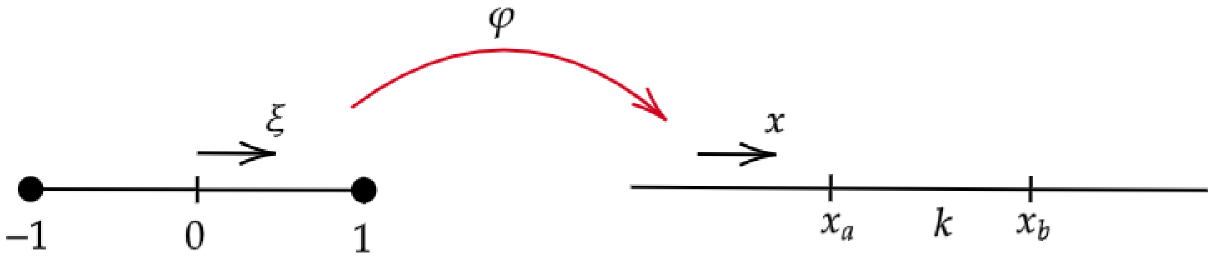
With

$$N_a(\xi) = \frac{(1-\xi)}{2} \qquad N_b(\xi) = \frac{1+\xi}{2}$$

The isoparametric map

Solution: The isoparametric map

We express the coordinates of an element in terms of a fixed reference element



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With

$$N_a(\xi) = \frac{(1-\xi)}{2} \qquad N_b(\xi) = \frac{1+\xi}{2}$$

The shape functions don't depend on x!

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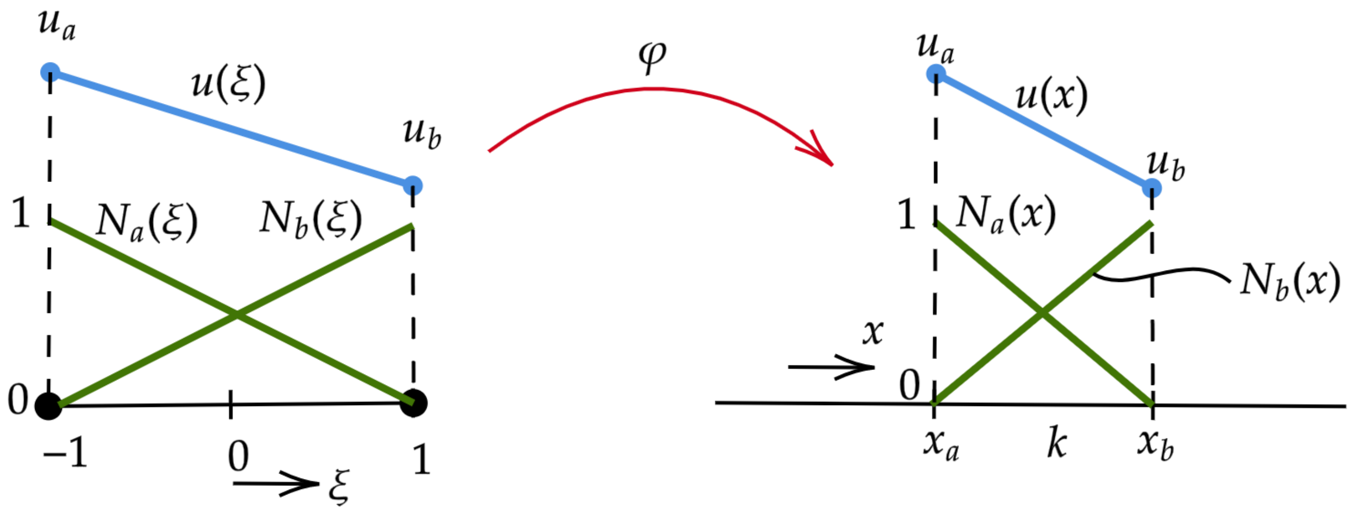
The isoparametric map

Now for a given element k we can also express the solution in terms of the Reference Element coordinates:



$$u(\xi)\Big|_k = u_a N_a(\xi) + u_b N_b(\xi)$$

The isoparametric map

Let's see how this applies to the elemental weak form…

We had that for the rod equation in a given element k between nodes a and b, the only non-zero shape functions will be $N_a(x)$ and $N_b(x)$.

$$u(x) = \sum_{j=1}^{n_n} N_j(x)u_j, \quad u'(x) = \sum_{j=1}^{n_n} N_j{}'(x)u_j, \quad \ddot{u}(x) = \sum_{j=1}^{n_n} N_j(x)\ddot{u}_j$$

$$\sum_{j=a,b}\left[\int_{\Omega_k} m\, N_j(x)N_i(x)d\Omega\right]\ddot{u}_j + \sum_{j=a,b}^{n_n}\left[\int_{\Omega_k} EA\, N_j'(x)N_i'(x)d\Omega\right]u_j = \int_{\Omega_k} q(x)N_i(x)d\Omega + T(x)N_i(x)\Big|_{x=x_a}^{x=x_b}$$

The isoparametric map

Let's see how this applies to the elemental weak form…

Using the isoparametric map we have that:

$$u(x)\Big|_k = u\big(x(\xi)\big)\Big|_k = \sum_{j=a,b} N_j(\xi)u_j$$

$$u'(x)\Big|_k = \frac{\partial u(x)}{\partial x}\Big|_k = \frac{\partial u\big(x(\xi)\big)}{\partial x}\Big|_k = \frac{\partial u\big(x(\xi)\big)}{\partial \xi}\frac{\partial \xi}{\partial x}\Big|_k = \sum_{j=a,b} \frac{\partial N_j(\xi)}{\partial \xi}\frac{\partial \xi}{\partial x}u_j$$

$$\ddot{u}(x)\Big|_k = \sum_{j=a,b} N_j(\xi)\ddot{u}_j$$

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The isoparametric map

Let's see how this applies to the elemental weak form…

We know that $x = \varphi_k(\xi)$, with a linear map at each element:

$$x(\xi)\Big|_k = x_a \frac{(1-\xi)}{2} + x_b \frac{1+\xi}{2} = x_a N_a(\xi) + x_b N_b(\xi)$$

With

$$N_a(\xi) = \frac{(1-\xi)}{2} \qquad N_b(\xi) = \frac{1+\xi}{2}$$

Then, we can evaluate $\frac{\partial N_j(\xi)}{\partial \xi}$ and $\frac{\partial \xi}{\partial x}$:

$$\frac{\partial N_a(\xi)}{\partial \xi} = -\frac{1}{2} \qquad \frac{\partial N_b(\xi)}{\partial \xi} = \frac{1}{2}$$

$$\frac{\partial \xi}{\partial x} = \left(\frac{\partial x}{\partial \xi}\right)^{-1} = \left(\frac{x_b - x_a}{2}\right)^{-1} = \left(\frac{h_k}{2}\right)^{-1} = \frac{2}{h_k}$$

The term $\frac{\partial x}{\partial \xi}$ is usually known as the *Jacobian* of the map and denoted as J_k. **Note that this is the only quantity that depends on the element!**

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The isoparametric map

Let’s see how this applies to the elemental weak form.

Now we know how to define all the mapped shape functions and derivatives

$$u(x)\Big|_k = u\big(x(\xi)\big)\Big|_k = \sum_{j=a,b} \boldsymbol{N_j}(\boldsymbol{\xi})u_j$$

$$u'(x)\Big|_k = \frac{\partial u(x)}{\partial x}\Big|_k = \frac{\partial u\big(x(\xi)\big)}{\partial x}\Big|_k = \frac{\partial u\big(x(\xi)\big)}{\partial \xi}\frac{\partial \xi}{\partial x}\Big|_k = \sum_{j=a,b} \frac{\boldsymbol{\partial N_j}(\boldsymbol{\xi})}{\boldsymbol{\partial \xi}}\boldsymbol{J_k^{-1}}u_j$$

$$\ddot{u}(x)\Big|_k = \sum_{j=a,b} \boldsymbol{N_j}(\boldsymbol{\xi})\ddot{u}_j$$

The isoparametric map

Let's see how this applies to the elemental weak form.

Going back to the elemental matrices, at an element k between nodes a and b we will have:

$$u(x)|_k \sum_{j=a,b} N_j(\xi)u_j, \quad u'(x)|_k = \sum_{j=a,b} \frac{\partial N_j(\xi)}{\partial \xi} J_k^{-1} u_j, \quad \ddot{u}(x)|_k = \sum_{j=a,b} N_j(\xi)\ddot{u}_j$$

$$\sum_{j=a,b} \left[\int_{\Omega_k} m\, N_j(\xi) N_i(\xi) d\Omega \right] \ddot{u}_j + \sum_{j=a,b}^{n_n} \left[\int_{\Omega_k} EA \left(N_j'(\xi) J_k^{-1} \right) \left(N_i'(\xi) J_k^{-1} \right) d\Omega \right] u_j = \int_{\Omega_k} q(x(\xi)) N_i(\xi) d\Omega + T(x(\xi)) N_i(\xi) \Big|_{\xi=-1}^{\xi=1}$$

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The isoparametric map

Let's see how this applies to the elemental weak form.

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$$u(x)|_k \sum_{j=a,b} N_j(\xi)u_j, \quad u'(x)|_k = \sum_{j=a,b} \frac{\partial N_j(\xi)}{\partial \xi} J_k^{-1} u_j, \quad \ddot{u}(x)|_k = \sum_{j=a,b} N_j(\xi)\ddot{u}_j$$

$$\sum_{j=a,b} \left[\int_{\Omega_k} m\, N_j(\xi) N_i(\xi) d\Omega \right] \ddot{u}_j + \sum_{j=a,b}^{n_n} \left[\int_{\Omega_k} EA \left(N_j'(\xi) J_k^{-1}\right)\left(N_i'(\xi) J_k^{-1}\right) d\Omega \right] u_j = \int_{\Omega_k} q(x(\xi)) N_i(\xi) d\Omega + T(x(\xi)) N_i(\xi) \Big|_{\xi=-1}^{\xi=1}$$

Since we want to evaluate the functions at the reference element, **we also need to map the integral bounds!**

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The isoparametric map

Let's see how this applies to the elemental weak form…

Mapping the integral:

$$\int_{\Omega_k} f(x)\,dx = \int_{\Omega_{ref}} f(\xi)\frac{\partial x}{\partial \xi}d\xi = \int_{\Omega_{ref}} f(\xi)J_k d\xi = \int_{-1}^{1} f(\xi)J_k d\xi$$

Leads to the final elemental weak form:

$$\sum_{j=a,b}\left[\int_{-1}^{1} m\, N_j(\xi)N_i(\xi)J_k d\xi\right]\ddot{u}_j + \sum_{j=a,b}\left[\int_{-1}^{1} EA\left(N_j'(\xi)J_k^{-1}\right)\left(N_i'(\xi)J_k^{-1}\right)J_k d\xi\right]u_j = \int_{-1}^{1} q(x(\xi))N_i(\xi)J_k d\xi + T(x(\xi))N_i(\xi)\Big|_{\xi=-1}^{\xi=1}$$

$$M_{ij}^k\ddot{u}_j + K_{ij}^k u_j = Q_i^k + S_i^k \qquad \boldsymbol{M}^k\ddot{\boldsymbol{u}} + \boldsymbol{K}^k\boldsymbol{u} = \boldsymbol{Q}^k + \boldsymbol{S}^k$$

The isoparametric map

We can check that we recover the same expression as when evaluating the shape functions in the physical space x

Using linear shape functions and noting $h = x_b - x_a$:

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a{}'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$

$$M_{aa} = \int_{x_a}^{x_b} m\, N_a(x) N_a(x) d\Omega = -\frac{m}{h^2} \frac{(x_b - x)^3}{3} \Big|_{x_a}^{x_b} = \frac{\boldsymbol{mh}}{\mathbf{3}}$$

$$M_{ab} = M_{ba} = \int_{x_a}^{x_b} m\, N_a(x) N_b(x) d\Omega = \frac{m}{h^2}\left[\frac{x_b x^2}{2} + \frac{x_a x^2}{2} - x_a x_b x - \frac{x^3}{3}\right]\Big|_{x_a}^{x_b} = \frac{\boldsymbol{mh}}{\mathbf{6}}$$

$$M_{bb} = \int_{x_a}^{x_b} m\, N_b(x) N_b(x) d\Omega = \frac{m}{h^2} \frac{(x - x_a)^3}{3} \Big|_{x_a}^{x_b} = \frac{\boldsymbol{mh}}{\mathbf{3}}$$

$$\boldsymbol{M}^k = \frac{mh}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The isoparametric map

We can check that we recover the same expression as when evaluating the shape functions in the physical space x

Using linear shape functions in the reference element and noting $J_k = \frac{h}{2}$:

$$N_a(\xi) = \frac{1-\xi}{2}, \quad N_a'(\xi) = -\frac{1}{2}, \quad N_b(x) = \frac{\xi+1}{2}, \quad N_b'(x) = \frac{1}{2}$$

$$M_{aa} = \int_{-1}^{1} m\, N_a(\xi) N_a(\xi) J_k d\xi = -\frac{mh}{8}\frac{(1-\xi)^3}{3}\Big|_{-1}^{1} = \boldsymbol{\frac{mh}{3}}$$

$$M_{ab} = M_{ba} = \int_{-1}^{1} m\, N_a(\xi) N_b(\xi) J_k d\xi = \frac{mh}{8}\left[\xi - \frac{\xi^3}{3}\right]\Big|_{-1}^{1} = \boldsymbol{\frac{mh}{6}}$$

$$M_{bb} = \int_{-1}^{1} m\, N_b(\xi) N_b(\xi) J_k d\xi = \frac{mh}{8}\frac{(\xi+1)^3}{3}\Big|_{x_a}^{x_b} = \boldsymbol{\frac{mh}{3}}$$

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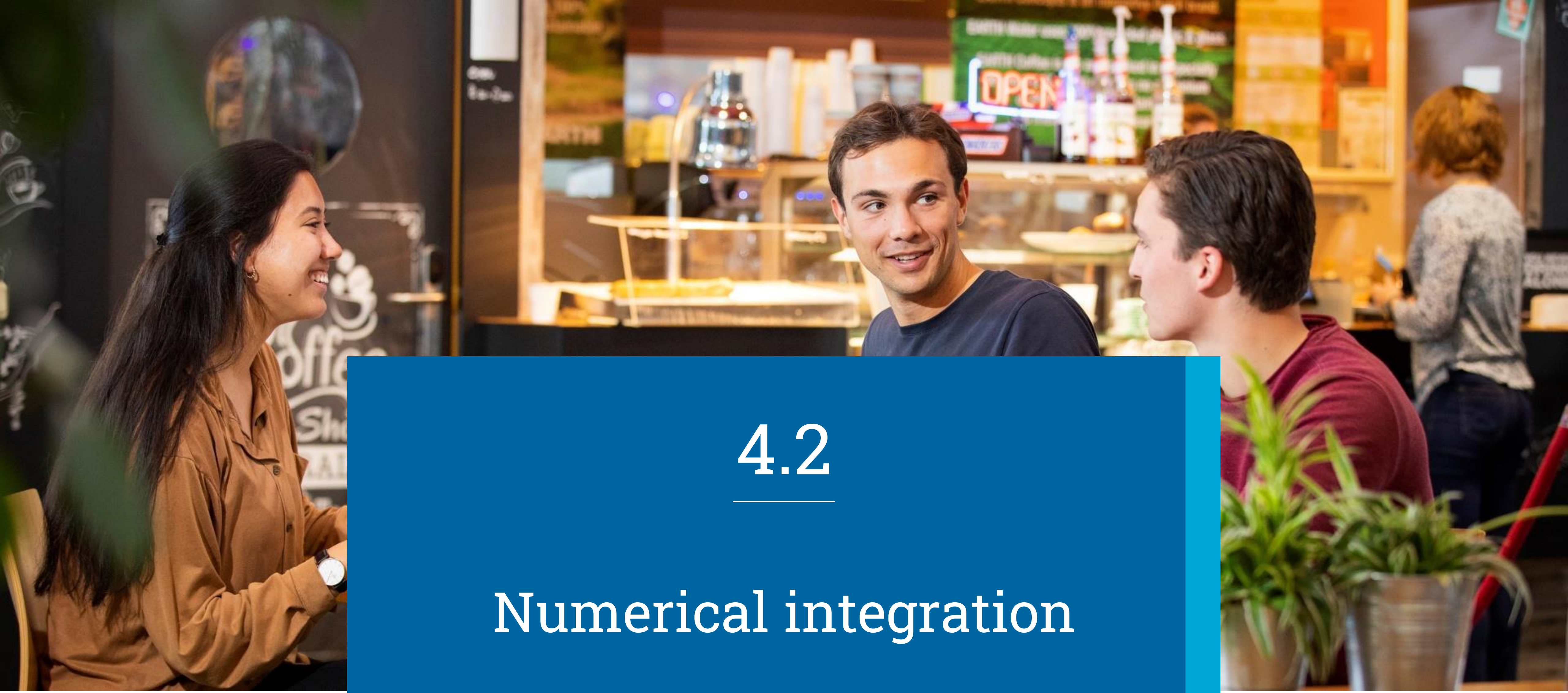
The isoparametric map

We can check that we recover the same expression as when evaluating the shape functions in the physical space x

The same applies to the **stiffness** matrix!



4.2

Numerical integration

Numerical integration

Now that we can define quantities at the reference Finite Element, numerical integration it's much easier

We approximate the integral over an element by a sum of weighted function evaluations:

$$\int f(x)dx \approx \sum_{i=1}^{n_{qp}} f(x_i)w_i$$

If f is a polynomial of degree r, the numerical integral is exact if we have $n_{qp} \geq \frac{r+1}{2}$

Polynomial degree of f	Number of integration points n	Integration points x_i	Associated weight factors w_i
1st degree or lower	1	$x_1 = 0$	$w_1 = 2$
3rd degree or lower	2	$x_1 = 1/\sqrt{3}$ $x_2 = -1/\sqrt{3}$	$w_1 = 1$ $w_2 = 1$
5th degree or lower	3	$x_1 = \sqrt{0.6}$ $x_2 = 0$ $x_3 = -\sqrt{0.6}$	$w_1 = 5/9$ $w_2 = 8/9$ $w_3 = 5/9$

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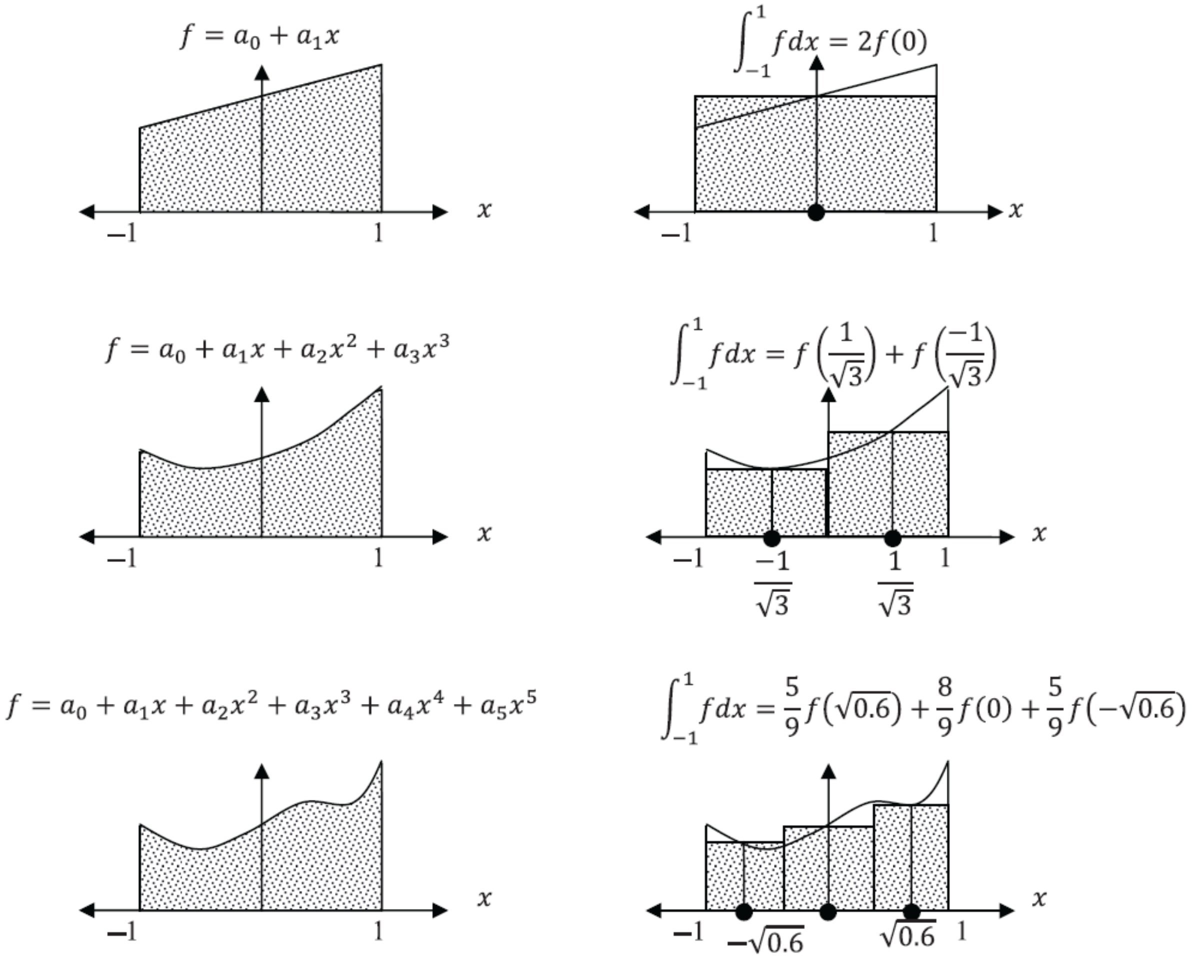
Numerical integration



Numerical integration

Since the reference element does not change, we can evaluate the shape functions and derivatives at the quadrature points and re-use the values for all elements!

Let's see how this looks for the mass matrix…

Computing symbolic integrals:

$$N_a(\xi) = \frac{1-\xi}{2}, \quad N_a'(\xi) = -\frac{1}{2}, \quad N_b(x) = \frac{\xi+1}{2}, \quad N_b'(x) = \frac{1}{2}$$

$$M_{aa} = \int_{-1}^{1} m\, N_a(\xi) N_a(\xi) J_k d\xi = -\frac{mh}{8} \frac{(1-\xi)^3}{3}\Big|_{-1}^{1} = \boldsymbol{\frac{mh}{3}}$$

$$M_{ab} = M_{ba} = \int_{-1}^{1} m\, N_a(\xi) N_b(\xi) J_k d\xi = \frac{mh}{8}\left[\xi - \frac{\xi^3}{3}\right]\Big|_{-1}^{1} = \boldsymbol{\frac{mh}{6}}$$

$$M_{bb} = \int_{-1}^{1} m\, N_b(\xi) N_b(\xi) J_k d\xi = \frac{mh}{8} \frac{(\xi+1)^3}{3}\Big|_{x_a}^{x_b} = \boldsymbol{\frac{mh}{3}}$$

$$\boldsymbol{M}^k = \frac{mh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Numerical integration

Let's see how this looks for the mass matrix…

Computing numerical integrals (we need 2 quadrature points to evaluate quadratic polynomials):

$$\xi_1 = -\frac{1}{\sqrt{3}}, \qquad \xi_2 = \frac{1}{\sqrt{3}}$$

$$N_{a,1}(\xi_1) = \frac{1-\xi_1}{2} = \frac{\sqrt{3}+1}{2\sqrt{3}}, \quad N_{a,2}(\xi_2) = \frac{1-\xi_2}{2} = \frac{\sqrt{3}-1}{2\sqrt{3}}, \quad N_{b,1}(\xi_1) = \frac{\xi_1+1}{2} = \frac{\sqrt{3}-1}{2\sqrt{3}}, \quad N_{b,2}(\xi_2) = \frac{\xi_2+1}{2} = \frac{\sqrt{3}+1}{2\sqrt{3}}$$

$$M_{aa} = \int_{-1}^{1} m\, N_a(\xi) N_a(\xi) J_k d\xi = \frac{mh}{2}\left[\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)^2 \cdot 1 + \left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)^2 \cdot 1\right] = \boldsymbol{\frac{mh}{3}}$$

$$M_{ab} = M_{ba} = \int_{-1}^{1} m\, N_a(\xi) N_b(\xi) J_k d\xi = \frac{mh}{2}\left[\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right) \cdot 1 + \left(\frac{\sqrt{3}-1}{2\sqrt{3}}\right)\left(\frac{\sqrt{3}+1}{2\sqrt{3}}\right) \cdot 1\right]\Bigg|_{-1}^{1} = \boldsymbol{\frac{mh}{6}}$$

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$$\boldsymbol{M}^k = \frac{mh}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Numerical integration

Exercise: Implement the isoparametric map and numerical integration on the reference element for Workshop 6. Assess the computing times between:

- Symbolic integration at each element
- Numerical integration at each element
- Numerical integration at the reference element using the isoparametric map.

4.3

Linear Elasticity

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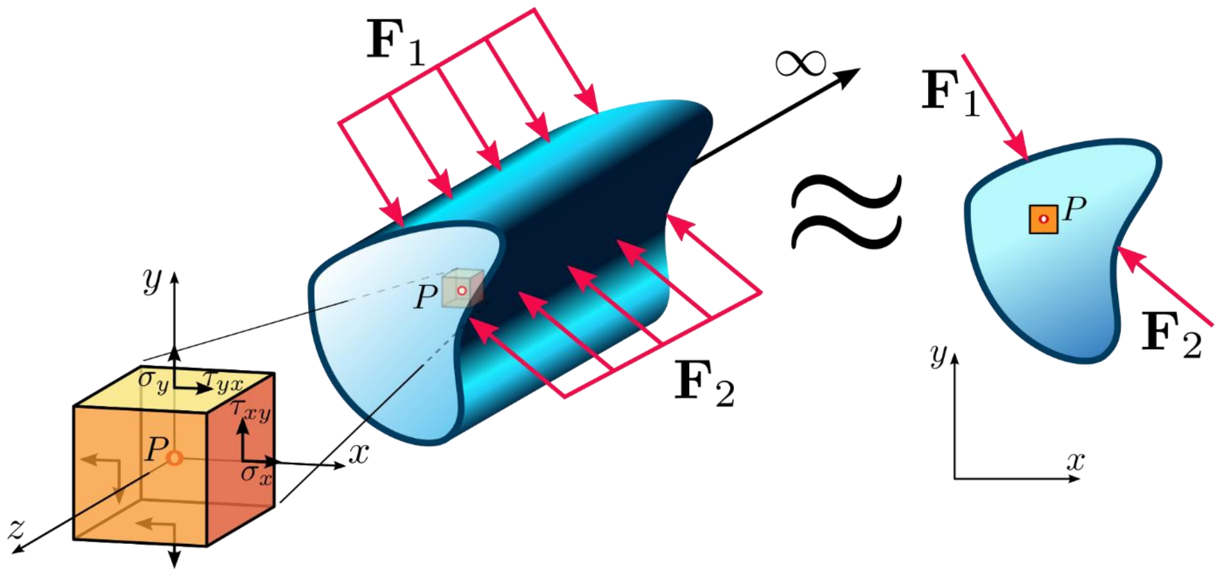
Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

The cross section of the structure is only allowed to deform in the plane



Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

At every point of the cross section we will have a tensor field denoting the stress (σ_{ij}) and a tensor field denoting the strain (ε_{ij})

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}$$

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \varepsilon_{11} & \varepsilon_{12} & 0 \\ \varepsilon_{21} & \varepsilon_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Remember that we can link the strain with the displacement field:

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \qquad \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

We know the relation between strain and stress for linear elastic problems:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

We know the relation between strain and stress for linear elastic problems:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

With C_{ijkl} a fourth order tensor that depends on material properties.

Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

We know the relation between strain and stress for linear elastic problems:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

With C_{ijkl} a fourth order tensor that depends on material properties.

To simplify the analysis, in this course we will consider isotropic linear elastic materials, which leads to the following constitutive relation:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

With μ and λ the Lamé's constants, defined as:

$$\mu = \frac{E}{2(1+\nu)}, \qquad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}$$

Constitutive law for linear elastic problems

In this course we will consider 2-dimensional elastic problems in **plane strain:**

We know the relation between strain and stress for linear elastic problems:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$$

With C_{ijkl} a fourth order tensor that depends on material properties.

To simplify the analysis, in this course we will consider isotropic linear elastic materials, which leads to the following constitutive relation:

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$$

Since we are in plane strain, we will only work with $i, j = 1,2$, knowing that $\sigma_{33} = \nu(\sigma_{11} + \sigma_{22})$

Equation of motion for linear elastic problems

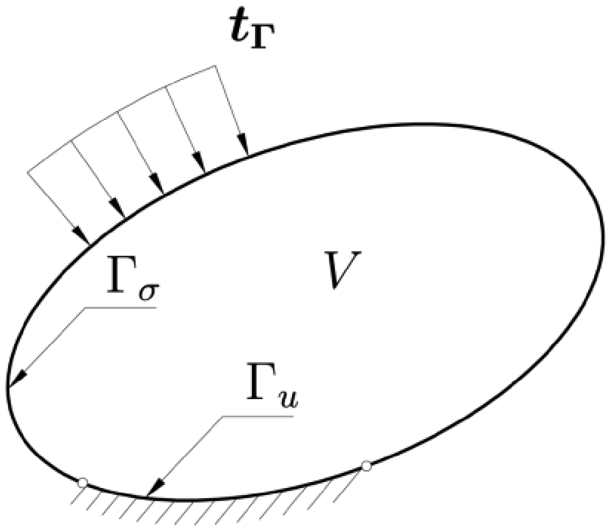
Once we know the relation between stress and strain (and displacements), we can define the **strong form** of the problem.

From conservation of momentum, we have:

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad in\ \Omega$$

With appropriate boundary conditions:

$$u = u_D \quad on\ \Gamma_u$$
$$\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t}_{\boldsymbol{\Gamma}} \quad on\ \Gamma_N$$



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4.4

Finite Elements for linear Elasticity

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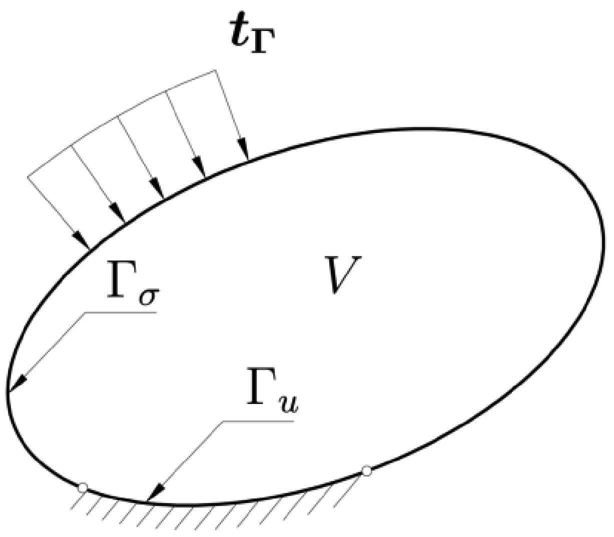
Solving linear elasticity using Finite Elements

Starting from the **strong form** of the problem, we can apply the Finite Element recipe to solve for the displacements in a cross section.

$$-\nabla \cdot \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f} \quad in\ \Omega$$

$$u = u_D \quad on\ \Gamma_u$$
$$\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t}_{\boldsymbol{\Gamma}} \quad on\ \Gamma_N$$



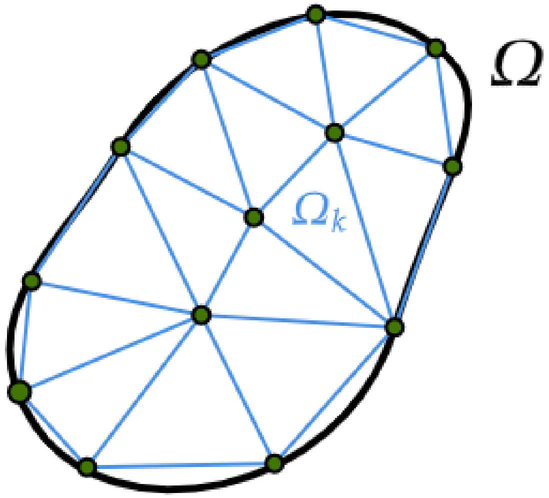
Remember the steps of a FEM:

1. Discretize the domain
2. Define shape functions
3. Define elemental weak form
4. Assemble the global system

Solving linear elasticity using Finite Elements

1. **Discretize the domain**

We use external libraries or structured grids in regular shapes (assume we can get a list of elements with node connectivities)

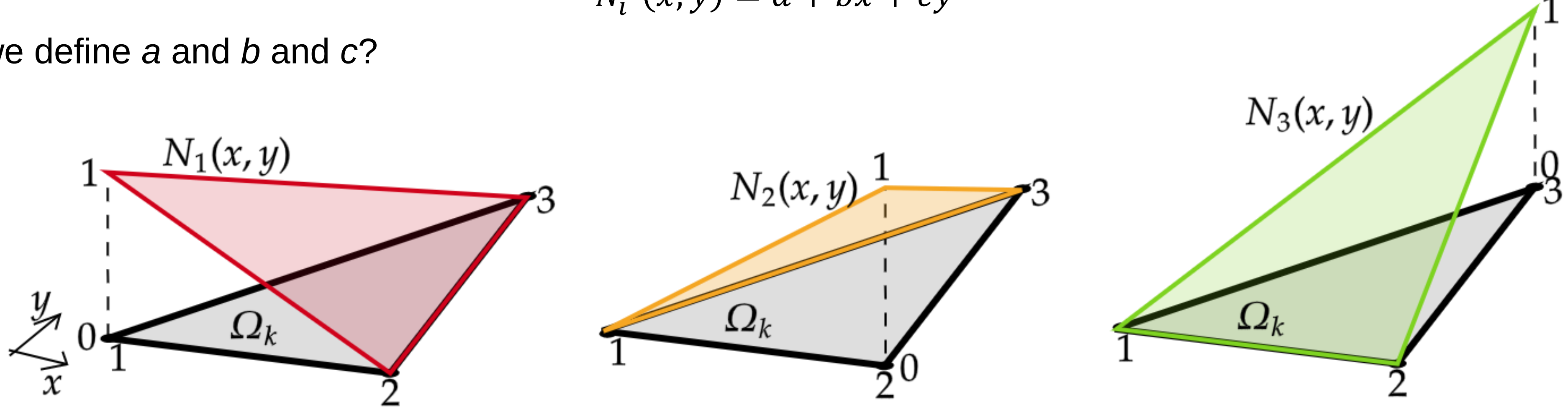


2. Define shape functions

We consider that the axial displacement can be well approximated using piece-wise linear functions:

$$N_i^k(x, y) = a + bx + cy$$

How do we define *a* and *b* and *c*?



Solving linear elasticity using Finite Elements

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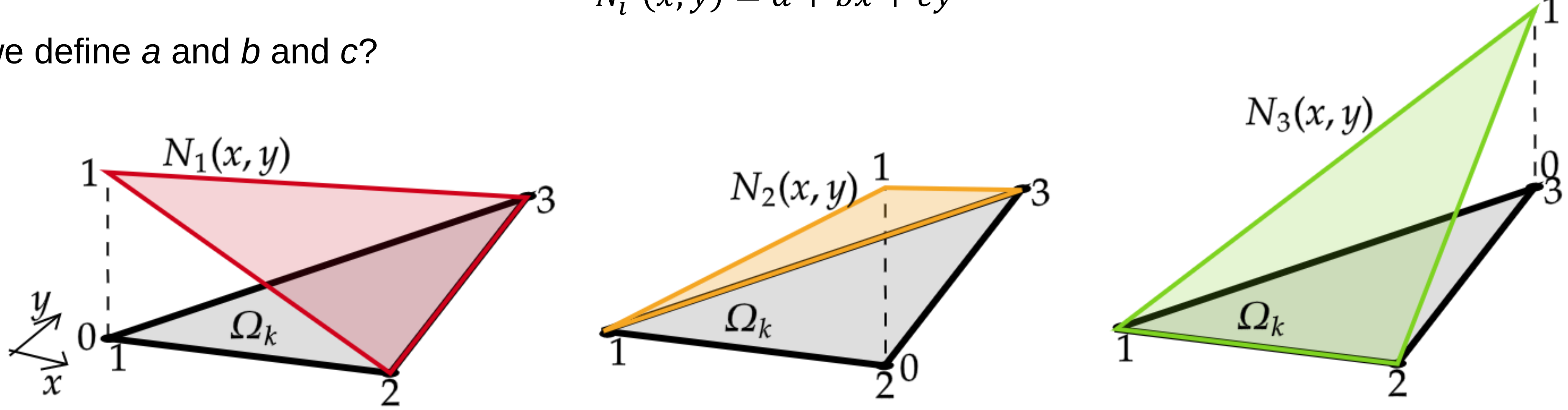


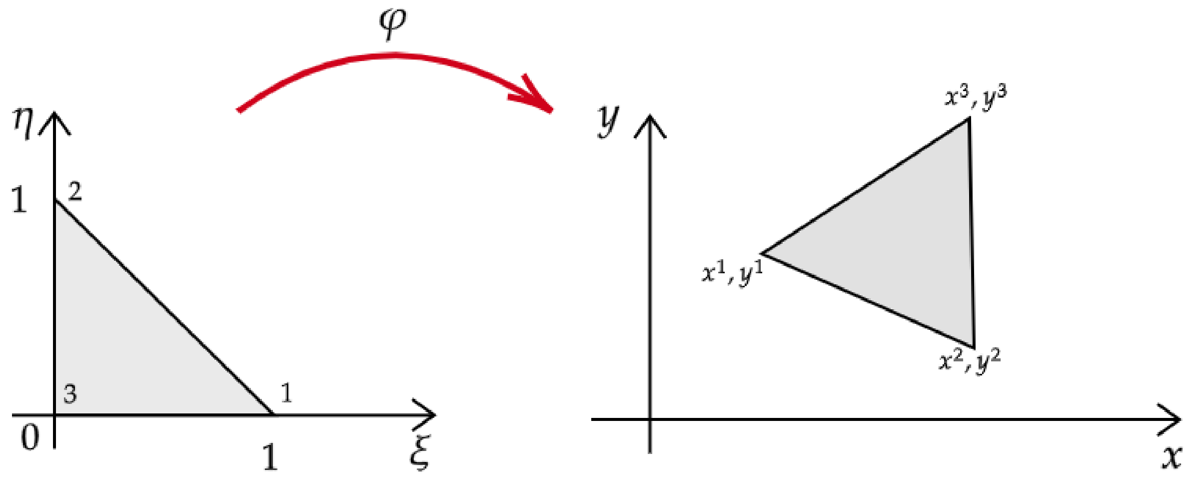
Do we need to define these functions for all elements?

Solving linear elasticity using Finite Elements

2. **Define shape functions**

Isoparametric element to the rescue!

$$N_1(\xi,\eta) = \xi$$
$$N_2(\xi,\eta) = \eta$$
$$N_3(\xi,\eta) = 1 - \eta - \xi$$



Solving linear elasticity using Finite Elements

2. **Define shape functions**

Isoparametric element to the rescue!

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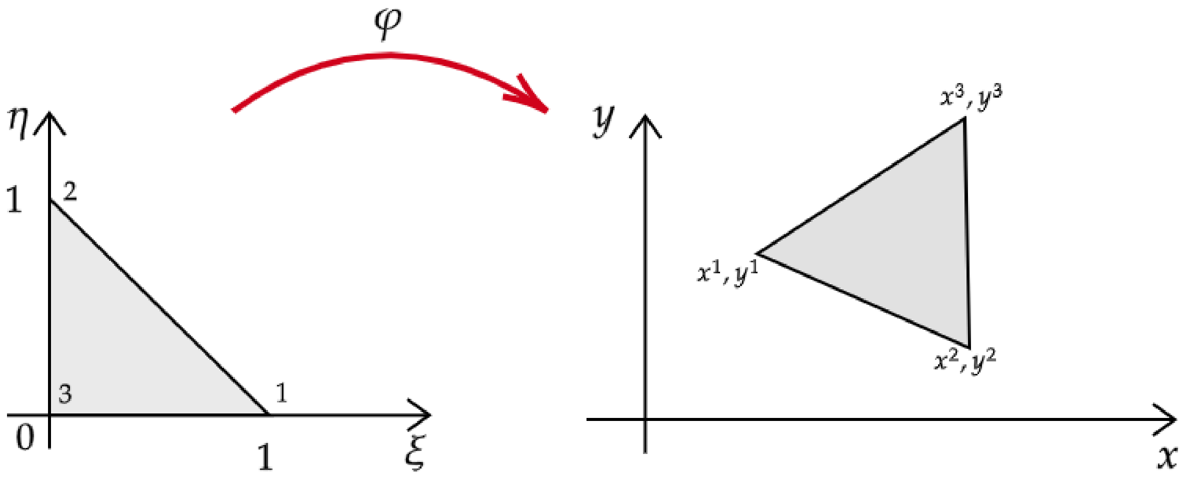
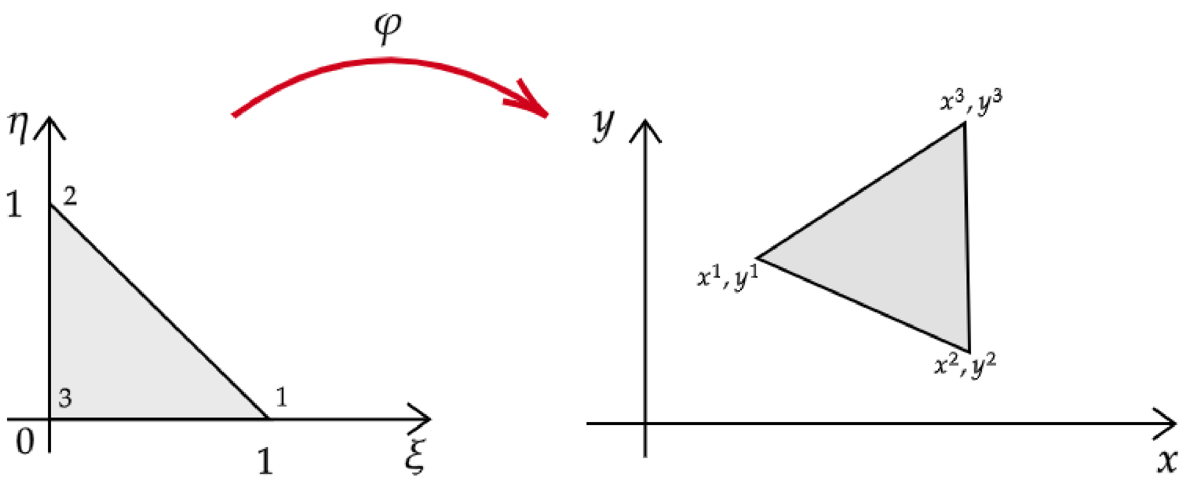


Now we have derivatives in two directions:

$$\frac{\partial N_1}{\partial \xi} = 1, \qquad \frac{\partial N_1}{\partial \eta} = 0$$
$$\frac{\partial N_2}{\partial \xi} = 0, \qquad \frac{\partial N_2}{\partial \eta} = 1$$
$$\frac{\partial N_3}{\partial \xi} = -1, \qquad \frac{\partial N_3}{\partial \eta} = -1$$

2. Define shape functions



We will also need the jacobian $\boldsymbol{J}_k$ (not anymore a scalar…)

$$x(\xi,\eta)|_k = x^1 N_1(\xi,\eta) + x^2 N_2(\xi,\eta) + x^3 N_3(\xi,\eta)$$
$$y(\xi,\eta)|_k = y^1 N_1(\xi,\eta) + y^2 N_2(\xi,\eta) + y^3 N_3(\xi,\eta)$$

Then, we can evaluate $\boldsymbol{J}_k$:

$$\boldsymbol{J}_k = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{pmatrix}, \qquad J_{k_{ij}} = \frac{\partial x_i}{\partial \xi_j} \text{ (abuse of notation } x_1 = x, x_2 = y, \xi_1 = \xi, \xi_2 = \eta)$$

$$\boldsymbol{J}_k = \begin{pmatrix} x^1 - x^3 & x^2 - x^3 \\ y^1 - y^3 & y^2 - y^3 \end{pmatrix}$$

Solving linear elasticity using Finite Elements

3. Define elemental weak form

Starting from the strong form (using Einstein notation for brevity):

$$-\frac{\partial \sigma_{ki}}{\partial x_k} = f_i \quad in\ \Omega$$

1. Multiply by a test function v_i and integrate over an element (vectorial test function since the unknown is also vectorial):

$$-\int_{\Omega_k} \frac{\partial \sigma_{ki}}{\partial x_k} v_i d\Omega = \int_{\Omega_k} f_i\, v_i\, d\Omega$$

Solving the rod equation using Finite Elements

3. Define elemental weak form

Starting from the strong form (using Einstein notation for brevity):

$$-\frac{\partial \sigma_{ki}}{\partial x_k} = f_i \quad in\ \Omega$$

1. Multiply by a test function v and integrate over an element:

$$-\int_{\Omega_k} \frac{\partial \sigma_{kl}}{\partial x_k} v_l d\Omega = \int_{\Omega_k} f_l\, v_l\, d\Omega$$

2. Integrate by parts :

$$-\int_{\Omega_k} \frac{\partial \sigma_{kl}}{\partial x_k} v_l d\Omega = \int_{\Omega_k} \sigma_{kl} \frac{\partial v_l}{\partial x_k} d\Omega - \int_{\partial\Omega_k} n_k \sigma_{kl} v_l d\Gamma$$

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

(trick) Since the stress tensor is symmetric:

$$\sigma_{kl}\frac{\partial v_l}{\partial x_k} = \frac{1}{2}\left(\sigma_{kl}\frac{\partial v_l}{\partial x_k} + \sigma_{lk}\frac{\partial v_l}{\partial x_k}\right) = \frac{1}{2}\left(\sigma_{kl}\frac{\partial v_l}{\partial x_k} + \sigma_{kl}\frac{\partial v_k}{\partial x_l}\right) = \sigma_{kl}\varepsilon^*_{kl}$$

With $\varepsilon^*_{kl} = \frac{1}{2}\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right)$ (virtual strain)

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

(trick) Since the stress tensor is symmetric:

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With $\varepsilon_{kl}^* = \frac{1}{2}\left(\frac{\partial v_l}{\partial x_k} + \frac{\partial v_k}{\partial x_l}\right)$ (virtual strain)

Introducing the constitutive relation for linear elasticity: $\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij}$

$$\int_{\Omega_k} \sigma_{kl}\frac{\partial v_l}{\partial x_k} d\Omega - \int_{\partial\Omega_k} n_k\sigma_{kl}v_l d\Gamma = \int_{\Omega_k} 2\mu\varepsilon_{ij}\varepsilon_{ij}^* + \lambda\varepsilon_{kk}\varepsilon_{ll}^* d\Omega - \int_{\partial\Omega_k} n_k\sigma_{kl}v_l d\Gamma$$

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

What about the integrals on the boundary?

- The contributions on the internal edges will cancel
- The contributions on the traction boundary can be replaced by the traction: $\boldsymbol{\sigma}(\boldsymbol{u}) \cdot \boldsymbol{n} = \boldsymbol{t}_{\Gamma}$
- The contributions from the fixed degrees of freedom are removed from the system

Final weak form:

$$\int_{\Omega_k} 2\mu\varepsilon_{ij}\varepsilon_{ij}^{*} + \lambda\varepsilon_{kk}\varepsilon_{ll}^{*} d\Omega = \int_{\Omega_k} f_l v_l \, d\Omega + \int_{\Gamma_N} t_{\Gamma_l} v_l d\Gamma$$

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Final weak form:

$$\int_{\Omega_k} 2\mu\varepsilon_{ij}\varepsilon_{ij}^* + \lambda\varepsilon_{kk}\varepsilon_{ll}^* d\Omega = \int_{\Omega_k} f_l v_l \, d\Omega + \int_{\Gamma_N} t_{\Gamma_l} v_l d\Gamma$$

Introducing the Finite Element approximation $u^k(x,y) = \sum_j N_j(x,y)u_j^{\boldsymbol{k}}$ and selecting $v^i(x,y) = N_i(x,y)$, we can re-write the elemental stiffness matrix as:

$$K_{aibk} = \int_{\Omega_k} C_{ijkl} \frac{\partial N_a}{\partial x_j} \frac{\partial N_b}{\partial x_l} d\Omega$$

With

$$C_{ijkl} = \frac{E}{2(1+\nu)}\left(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right) + \frac{E\nu}{1-\nu^2}\delta_{ij}\delta_{kl}$$

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Solving the rod equation using Finite Elements

4. **Assemble the global system**

Local to global matrix assembly like in the 1D case

Thank you for your attention

Oriol Colomés

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