Week 3. Numerical methods for PDEs

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Learning objectives

At the end of this week you will be able to:

Define and analyze numerical methods to solve systems governed by Partial Differential Equations. This entails:

1. Define the the system of PDEs that characterize the behavior of structures composed by rods and beams
2. Define numerical methods to solve a system of PDEs
3. Implement a solver for a system of PDEs
4. Analize and justify the results

3.1

Introduction to Finite Differences

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Finite Differences

Where does the term “Finite Differences” come from?

The goal is to go evaluate an infinitesimal quantity $\frac{\partial f(x,t)}{\partial x}$ using finite operation.

The definition of a (partial) derivative is:

$$\frac{\partial f(x,t)}{\partial x} = \lim_{b \to a} \frac{f(b,t) - f(a,t)}{b - a}$$

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Answer: No, in general this limit is not computable (division by zero).

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Answer: No, in general this limit is not computable (division by zero).

We cannot obtain the exact value of the derivative (in general), so we’ll have to approximate it. Here it comes the notion of finite difference. Instead of computing the limit, we just select a point *b* “close enough” to *a*:

$$\frac{\partial f(x,t)}{\partial x} \approx \frac{f(x + \Delta x, t) - f(x,t)}{\Delta x}$$

Finite Differences

Is there only one FD scheme to compute derivatives?

We have seen that we can approximate $\frac{\partial f(x,t)}{\partial x}$ using Finite Differences by:

$$\frac{\partial f(x,t)}{\partial x} \approx \frac{f(x+\Delta x,t) - f(x,t)}{\Delta x}$$

Consider the following two additional expressions:

$$\frac{\partial f(x,t)}{\partial x} \approx \frac{f(x,t) - f(x-\Delta x,t)}{\Delta x}$$

$$\frac{\partial f(x,t)}{\partial x} \approx \frac{f\left(x+\frac{\Delta x}{2},t\right) - f\left(x-\frac{\Delta x}{2},t\right)}{\Delta x}$$

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Question: Will they give the same result? If not which one is the best?

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Question: Will they give the same result? If not which one is the best?

Answer: Not the same result. Second option gives 2nd order accuracy, usually preferable.

Finite Differences

Let's analyse the error of the three different options using a TSE

Forward: $\frac{\partial f(x,t)}{\partial x} \approx \frac{f(x+\Delta x,t)-f(x,t)}{\Delta x}$, **Backward**: $\frac{\partial f(x,t)}{\partial x} \approx \frac{f(x,t)-f(x-\Delta x,t)}{\Delta x}$, **Central**: $\frac{\partial f(x,t)}{\partial x} \approx \frac{f\left(x+\frac{\Delta x}{2},t\right)-f\left(x-\frac{\Delta x}{2},t\right)}{\Delta x}$

Exercise: Using Taylor Series, determine the order of accuracy of these schemes

3.2

Solving PDEs with Finite Differences

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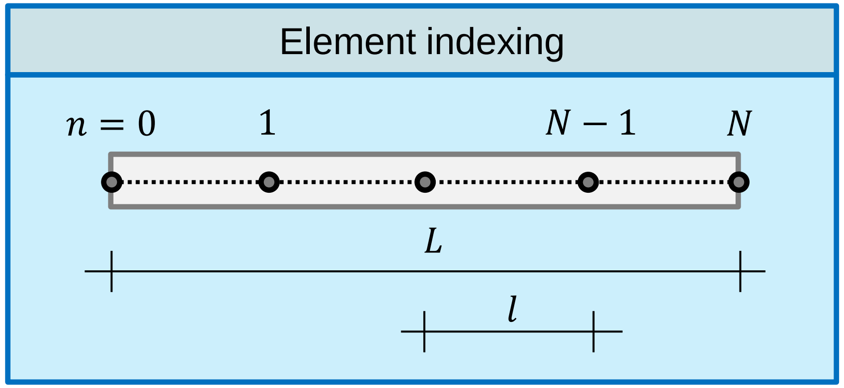
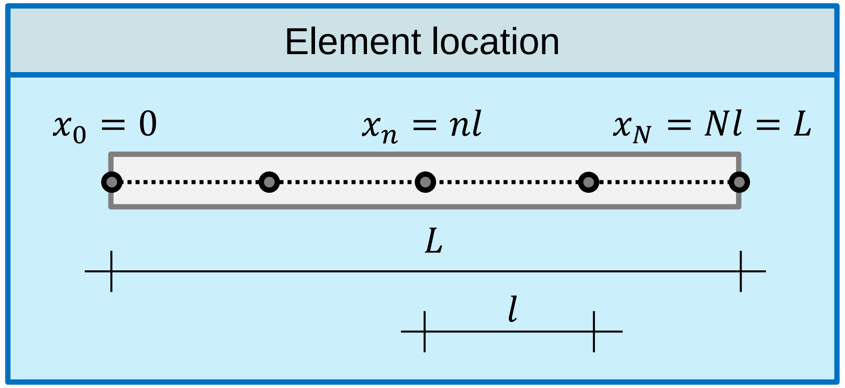
Solving the rod equation using Finite Differences

Consider the EOM of a rod (a PDE):

$$\rho A\ddot{u} + EAu'' = 0 \quad \text{where } u(x,t) \;\; \forall \;\; x \in (0,L)$$

- Let's use $N+1$ points, indexed with n, starting at $n=0$ and ending at $n=N$
- Element size $l = \frac{L}{N}$
- Point n is located at $x_n = nl$
- The displacement of the rod at x_n, $u(x_n)$ is simplified as u_n





Solving the rod equation using Finite Differences

Consider the EOM of a rod (a PDE):

$$\rho A\ddot{u} + EAu'' = 0 \qquad \text{where } u(x,t) \ \forall \ x \in (0, L)$$

The domain of the rod has been discretized. Each point n has to satisfy the EOM and so:

$$\rho A\ddot{u}_n - EAu_n'' = 0$$

Now we just need a way to compute the second derivative in space. Let's use central difference:

$$u_n'' = \frac{1}{l^2}\big(u(x_{n-1}) - 2u(x_n) + u(x_{n+1})\big) + O(\Delta_x^2)$$

$$u_n'' = \frac{1}{l^2}(u_{n-1} - 2u_n + u_{n+1}) + O(\Delta_x^2)$$

Substitute the FD relation into the EOM to obtain the EOM of the element n:

$$\rho A\ddot{u}_n - \frac{EA}{l^2}(u_{n-1} - 2u_n + u_{n+1}) = 0, \qquad \forall\, n = 0..N$$

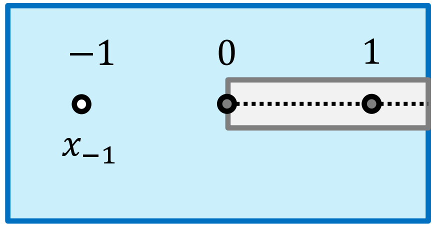
Note that this gives $N + 1$ equations!! Not 1.

Solving the rod equation using Finite Differences

What happens with the boundary nodes?

Consider the EOM at the first point $n = 0$:

$$\rho A\ddot{u}_0 - \frac{EA}{l^2}(u_{-1} - 2u_0 + u_1) = 0$$



u_{-1} falls outside the domain of the rod! This is called a ghost point.

Solving the rod equation using Finite Differences

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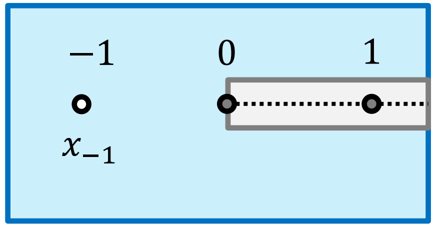

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Question: How do we handle this?

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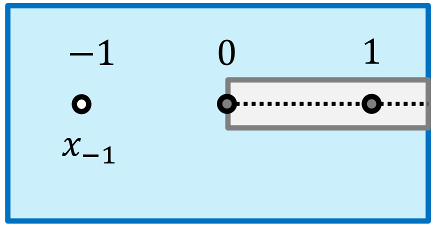

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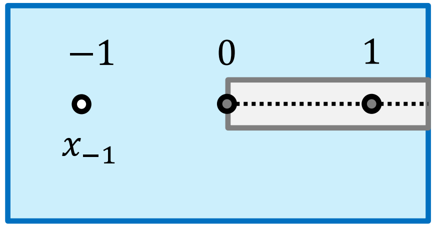

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Let's consider the two possible BCs: a prescribe force or a prescribed displacement

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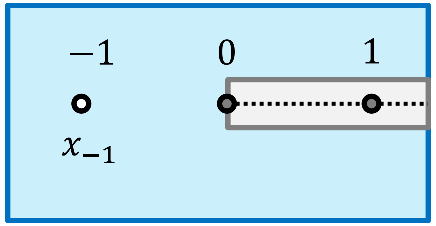
Solving the rod equation using Finite Differences



Prescribing the displacement

Consider a prescribed displacement at $x = 0$:

$$u(0,t) = u_0(t) = \delta(t) = U\cos(\Omega t)$$

Since this gives the displacement u_0 at any point in time, no longer have to solve the EOM at $n = 0$. So the ghost point disappears.

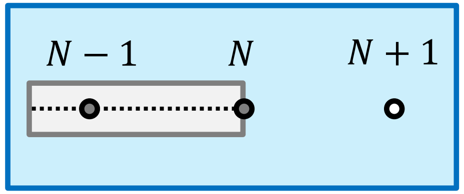
But u_0 is also part of the EOM of point $n = 1$:

$$\rho A\ddot{u}_1 - \frac{EA}{l^2}(u_0 - 2u_1 + u_2) = 0 \quad \rightarrow \quad \rho A\ddot{u}_1 - \frac{EA}{l^2}(\delta(t) - 2u_1 + u_2) = 0$$

Simply move it to the right hand side:

$$\rho A\ddot{u}_1 - \frac{EA}{l^2}(-2u_1 + u_2) = \frac{\delta(t)E}{l^2}$$

Solving the rod equation using Finite Differences



Prescribing the force

At the other edge of the beam there is another ghost point, $N+1$.

Consider a force $F(t) = f\cos(\Omega t)$ at $x = L$:

$$EAu'(L) = F(t)$$

Use central difference to approximate this derivative

$$u'(L) = \frac{u_{N+1} - u_{N-1}}{2l}$$

Substitute and isolate for u_{N+1}:

$$EA\frac{u_{N+1} - u_{N-1}}{2l} = F(t)$$

$$u_{N+1} = u_{N-1} + \frac{2lF(t)}{EA}$$

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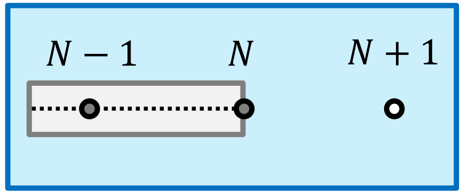
Solving the rod equation using Finite Differences

Prescribing the force



With this expression for u_{N+1} the ghost point can be removed for all equations it shows up in. In this case it is only the EOM of node N:

$$\rho A\ddot{u}_N - \frac{EA}{l^2}(u_{N-1} - 2u_N + u_{N+1}) = 0$$

Substitute the expression for u_{N+1} into the EOM of node N:

$$\rho A\ddot{u}_N - \frac{EA}{l^2}\left(u_{N-1} - 2u_N + u_{N-1} + \frac{2lF(t)}{EA}\right) = 0$$

$$\rho A\ddot{u}_N - \frac{EA}{l^2}(2u_{N-1} - 2u_N) = \frac{EA}{l^2}\frac{2lF(t)}{EA} = \frac{2F(t)}{l}$$

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Solving the rod equation using Finite Differences

How do we simplify the implementation?

We have the following set of equations:

- $u_0(t) = \delta(t) = U\cos(\Omega t)$
- $\rho A\ddot{u}_1 - \frac{EA}{l^2}(-2u_1 + u_2) = \frac{\delta(t)E}{l^2}$
- $\rho A\ddot{u}_n - \frac{EA}{l^2}(u_{n-1} - 2u_n + u_{n+1}) = 0, \qquad \forall\, n = 2, \ldots N-1$
- $\rho A\ddot{u}_N - \frac{EA}{l^2}(2u_{N-1} - 2u_N) = \frac{EA}{l^2}\frac{2lF(t)}{EA} = \frac{2F(t)}{l}$

Solving the rod equation using Finite Differences

How do we simplify the implementation?

Consider EOM for a fixed-forced rod, where $N+1=5$.

Without applying boundary conditions, we would have:

$$\rho A\ddot{u}_n - \frac{EA}{l^2}(u_{n-1} - 2u_n + u_{n+1}) = 0, \qquad \forall\, n = 0, \ldots N$$

In a matrix form:

$$\ddot{\boldsymbol{u}} = \frac{\mathrm{E}}{\rho l}\begin{bmatrix} 1 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 \end{bmatrix}_{N+1,N+3} \begin{bmatrix} u_{-1} \\ u_0 \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix}$$

Enforcing displacement:

$$\ddot{\boldsymbol{u}} = \frac{\mathrm{E}}{\rho l}\begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} + \begin{bmatrix} \frac{E\delta(t)}{\rho l^2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving the rod equation using Finite Differences

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Enforcing force:

$$\ddot{\boldsymbol{u}} = \frac{\mathrm{E}}{\rho l}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} \frac{E\delta(t)}{\rho l^2} \\ 0 \\ 0 \\ \frac{2F(t)}{\rho l A} \end{bmatrix}$$

Final matrix form:

$$\ddot{\boldsymbol{u}} = \boldsymbol{K}\boldsymbol{u} + \boldsymbol{F} \qquad \text{with } \boldsymbol{K} = \frac{\mathrm{E}}{\rho l}\begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 2 & -2 \end{bmatrix}, \; \boldsymbol{F} = \begin{bmatrix} \frac{E\delta(t)}{\rho l^2} \\ 0 \\ 0 \\ \frac{2F(t)}{\rho l A} \end{bmatrix}$$

Solving the rod equation using Finite Differences

Exercise: Go through tutorial 3.1 on solving the dynamic motion of a rod using Finite Differences

3.3

Finite Elements

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Finite Elements

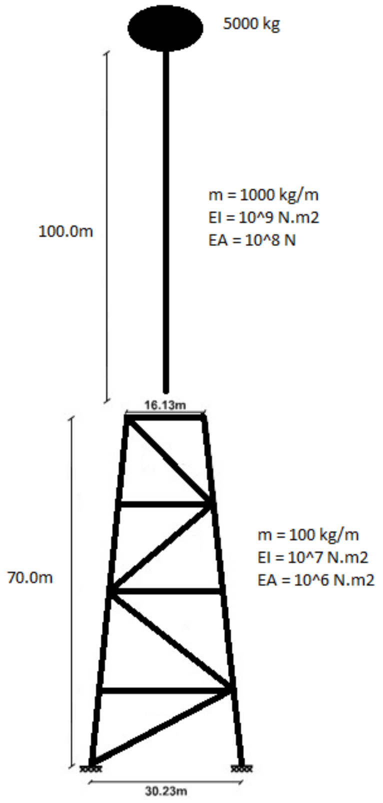
The basic concepts of Finite Element method:

Goal: Model a structure by dividing it into an equivalent system of many smaller structures or units (finite elements)

Allows to obtain a set of algebraic equations to solve for an unknown quantity

It can handle:

- Irregular Boundaries
- General Loads
- Different Materials
- Different Boundary Conditions
- Variable Element Size
- Easy Modification
- Dynamics
- Nonlinear Problems (Geometric or Material)



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Finite Elements

How do we define a FE method in practice?

A bit of abstract math… 😱

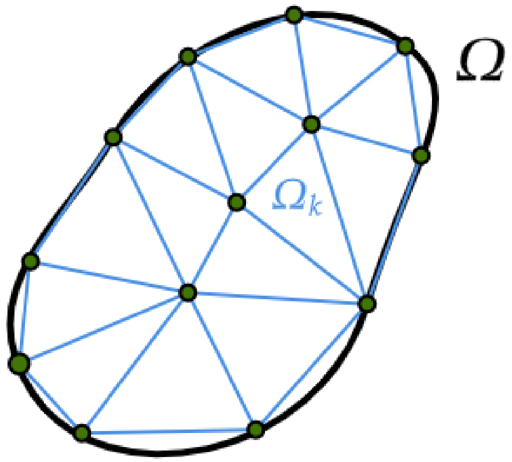
Let's consider we want to solve a PDE:

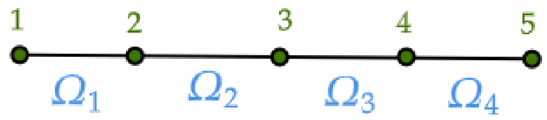
$$L(u) = f \ \ in \ \Omega$$

($L(u)$ can contain derivatives of arbitrary order in time and space)

1. Discretize the domain Ω:

- n_e elements Ω_{k} for $k = 1, \ldots, n_e$
- n_n nodes (in 1D $n_n = n_e + 1$)





Finite Elements

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Let's consider we want to solve a PDE:

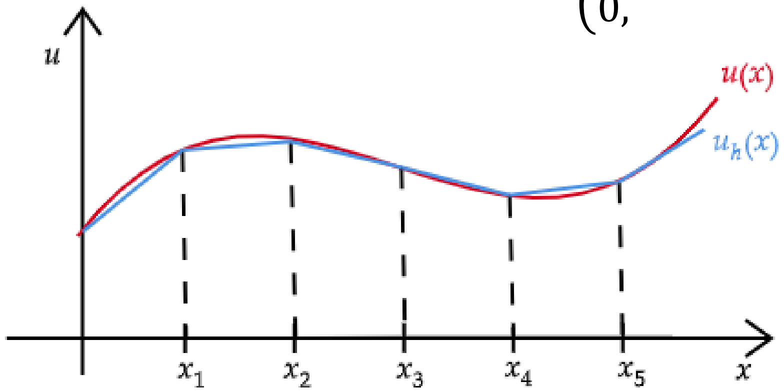
$$L(u) = f \;\; in\, \Omega$$

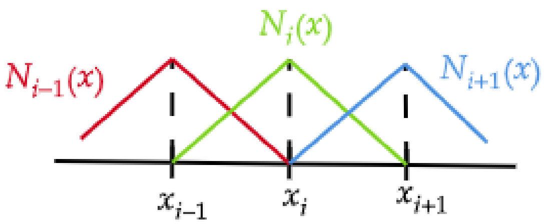
($L(u)$ can contain derivatives of arbitrary order in time and space)

2. Approximate the solution with element-wise shape functions (typically polynomials)

For each node (i) we define piece-wise interpolation functions defined on the set of elements S_i that are attached to the node i:

$$N_i(x) = \begin{cases} p(x), & x \in \Omega_k \;\forall\; k \in S_i \\ 1, & x = x_i \\ 0, & \text{otherwise} \end{cases}$$





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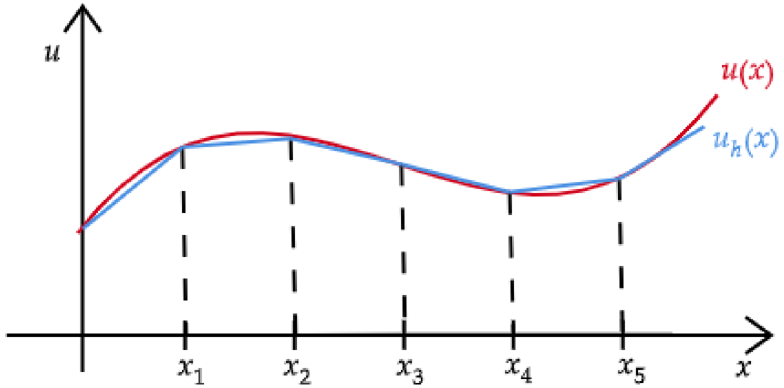
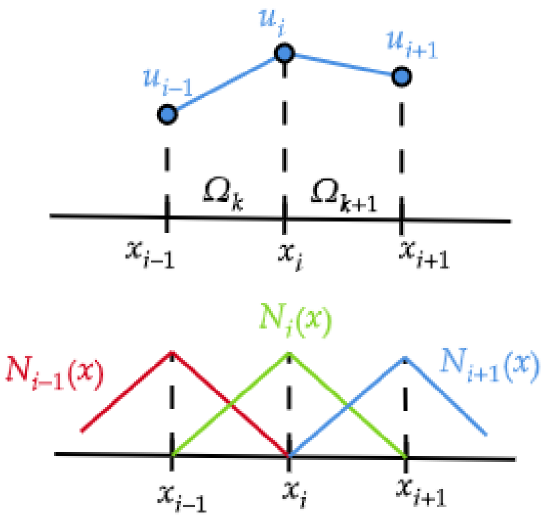
2. Approximate the solution with element-wise shape functions (typically polynomials)

With these shape functions and the value of the function at the node u_i, we can interpolate any solution as:

$$u(x) = \sum_{i=1}^{n_n} N_i(x)u$$





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Finite Elements

How do we define a FE method in practice?

A bit of abstract math… 😱

Let's consider we want to solve a PDE:

$$L(u) = f \ \ in\, \Omega$$

3. The problem is satisfied in a weak sense (virtual work principle)

Define the **weak form** at each element k:

1. starting from the strong form $L(u) = f$
2. Multiply by a virtual displacement (v)
3. Integrate over the element

$$a(u,v)_k \coloneqq \int_{\Omega_k} L(u) \cdot v \, d\Omega_k$$

$$b(v)_k \coloneqq \int_{\Omega_k} f \cdot v \, d\Omega_k$$

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Finite Elements

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Let's consider we want to solve a PDE:

$$L(u) = f \;\; in\, \Omega$$

3. The problem is satisfied in a weak sense (virtual work principle)

The **weak form** has to be satisfied for any function v in the space of solutions:

$$a(u,v) = \sum_{k=1}^{n_e} a(u,v)_k = \sum_{k=1}^{n_e} b(v)_k = b(v) \qquad \forall v$$

In particular, we can select v to be function from the space of shape functions $N_i(x)$ for all $i = 1, \dots, n_n$

Finite Elements

How do we define a FE method in practice?

A bit of abstract math… 😱

Let's consider we want to solve a PDE:

$$L(u) = f \ \ in\ \Omega$$

4. Construct the global algebraic system

- Take $v = N_i$ for all $i = 1, \ldots, n_n$
- Assume the solution is given by

$$u(x) = \sum_{j=1}^{n_n} N_j(x) u_j$$

$$a(u, N_i) = b(N_i) \qquad \forall i = 1, \ldots, n_n$$

$$\sum_{k=1}^{n_e} \left[\int_{\Omega_k} L\left(\sum_{j=1}^{n_n} N_j(x) u_j \right) N_i(x)\, d\Omega_k \right] = \sum_{k=1}^{n_e} \int_{\Omega_k} f N_i(x)\, d\Omega_k \quad \Rightarrow \quad \sum_{j=1}^{n_n} \left[\sum_{k=1}^{n_e} \int_{\Omega_k} L(N_j) N_i\, d\Omega_k \right] u_j = \sum_{k=1}^{n_e} \int_{\Omega_k} f N_i(x)\, d\Omega_k$$

Finite Elements

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$$\sum_{j=1}^{n_n}\left[\sum_{k=1}^{n_e}\int_{\Omega_k} L(N_j)N_i \, d\Omega_k\right]u_j = \sum_{k=1}^{n_e}\int_{\Omega_k} f N_i(x)\, d\Omega_k \quad \Rightarrow \quad \sum_{j=1}^{n_n}[A_{ij}]u_j = b_i \quad \forall i = 1, \dots, n_n \quad \Rightarrow \quad \boldsymbol{Au} = \boldsymbol{b}$$

Finite Elements

Finite Elements vs Finite Differences?

FEM pros and cons:

- Easy to consider different types of boundary conditions (equations do not change)
- Easy do consider heterogeneous domains (each element can have distinct properties)
- Easy to consider irregular domains (non-squared, irregular discretization)
- A direct relation between nodal displacements and nodal forces (internal forces) is obtained

FD pros and cons:

- Easy to understand (very intuitive)

3.4

Solving PDEs with Finite Elements

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Solving the rod equation using Finite Elements

Consider the EOM of a rod (a PDE):

$$\rho A\ddot{u} + EAu'' = 0 \quad \text{where} \;\; u(x,t) \;\; \forall \;\; x \in (0, L)$$

Remember the steps of a FEM:

1. Discretize the domain
2. Define shape functions
3. Define elemental weak form
4. Assemble the global system

Solving the rod equation using Finite Elements

1. **Discretize the domain**

Solving the rod equation using Finite Elements

2. **Define shape functions**

We consider that the axial displacement can be well approximated using piece-wise linear functions:

$$N_i^k(x) = a + bx$$

How do we define *a* and *b*?

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Solving the rod equation using Finite Elements

3. Define elemental weak form

Starting from the strong form (EOM):

$$m\,\ddot{u}(x) - EA\,u''(x) = q(x)$$

1. Multiply by a test function v and integrate over an element:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega - \int_{\Omega_k} EA\,u''(x)v(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega$$

Solving the rod equation using Finite Elements

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$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega - \int_{\Omega_k} EA\,u''(x)v(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega$$

2. Integrate by parts the terms with second-order derivative:

$$-\int_{\Omega_k} EA\,u''(x)v(x)d\Omega = \int_{\Omega_k} EA\,u'(x)v'(x)d\Omega \; - EAu'(x)v(x)\Big|_{x=x_a}^{x=x_b}$$

Solving the rod equation using Finite Elements

3. Define elemental weak form

Starting from the strong form (EOM):

$$m\,\ddot{u}(x) - EA\,u''(x) = q(x)$$

1. Multiply by a test function v and integrate over an element:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega - \int_{\Omega_k} EA\,u''(x)v(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega$$

2. Integrate by parts the terms with second-order derivative:

$$-\int_{\Omega_k} EA\,u''(x)v(x)d\Omega = \int_{\Omega_k} EA\,u'(x)v'(x)d\Omega \; - EAu'(x)v(x)\Big|_{x=x_a}^{x=x_b}$$

3. Final weak form:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega + \int_{\Omega_k} EA\,u'(x)v'(x)d\Omega \; - EAu'(x)v(x)\Big|_{x=x_a}^{x=x_b} = \int_{\Omega_k} q(x)v(x)d\Omega$$

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Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Final weak form:

$$\int_{\Omega_k} m\, \ddot{u}(x) v(x) d\Omega + \int_{\Omega_k} EA\, u'(x) v'(x) d\Omega = \int_{\Omega_k} q(x) v(x) d\Omega + T(x) v(x) \Big|_{x=x_a}^{x=x_b}$$

Choose $v(x) = N_i(x)$ for all $i = 1, \ldots, n_n$.

In a given element k between nodes a and b, the only non-zero shape functions will be $N_a(x)$ and $N_b(x)$.

Replace the solution by the approximated function:

$$u(x) = \sum_{j=1}^{n_n} N_j(x) u_j, \qquad u'(x) = \sum_{j=1}^{n_n} N_j'(x) u_j, \qquad \ddot{u}(x) = \sum_{j=1}^{n_n} N_j(x) \ddot{u}_j$$

$$\sum_{j=a,b} \left[\int_{\Omega_k} m\, N_j(x) N_i(x) d\Omega \right] \ddot{u}_j + \sum_{j=a,b}^{n_n} \left[\int_{\Omega_k} EA\, N_j'(x) N_i'(x) d\Omega \right] u_j = \int_{\Omega_k} q(x) N_i(x) d\Omega + T(x) N_i(x) \Big|_{x=x_a}^{x=x_b}$$

$$M_{ij}^k \ddot{u}_j + K_{ij}^k u_j = Q_i^k + S_i^k \qquad\qquad \boldsymbol{M}^k \ddot{\boldsymbol{u}} + \boldsymbol{K}^k \boldsymbol{u} = \boldsymbol{Q}^k + \boldsymbol{S}^k$$

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Let's compute the mass matrix

Using linear shape functions and noting $h = x_b - x_a$:

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$

$$M_{aa} = \int_{x_a}^{x_b} m\, N_a(x) N_a(x) d\Omega = ?$$

$$M_{ab} = M_{ba} = \int_{x_a}^{x_b} m\, N_a(x) N_b(x) d\Omega = ?$$

$$M_{bb} = \int_{x_a}^{x_b} m\, N_b(x) N_b(x) d\Omega = ?$$

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Let's compute the mass matrix

Using linear shape functions and noting $h = x_b - x_a$:

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$

$$M_{aa} = \int_{x_a}^{x_b} m\, N_a(x) N_a(x) d\Omega = \frac{m}{h^2} \frac{(x_b - x)^3}{3} \Big|_{x_a}^{x_b} = \boldsymbol{\frac{mh}{3}}$$

$$M_{ab} = M_{ba} = \int_{x_a}^{x_b} m\, N_a(x) N_b(x) d\Omega = \frac{m}{h^2}\left[-\frac{(x - x_a)^3}{3} + \frac{h(x - x_a)^2}{2}\right] \Big|_{x_a}^{x_b} = \boldsymbol{\frac{mh}{6}}$$

$$M_{bb} = \int_{x_a}^{x_b} m\, N_b(x) N_b(x) d\Omega = -\frac{m}{h^2} \frac{(x_b - x)^3}{3} \Big|_{x_a}^{x_b} = \boldsymbol{\frac{mh}{3}}$$

$$\boldsymbol{M}^k = \frac{mh}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

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Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Looking carefully at the internal forces:

$$S_i^k = T(x)N_i(x)\Big|_{x=x_a}^{x=x_b} = T(x_b)N_i(x_b) - T(x_a)N_i(x_a)$$

When $i = a$: $S_a^k = -T(x_a)$

When $i = b$: $S_b^k = T(x_b)$

Since $S_a^{k+1} = -S_b^k$, when adding the contributions of the internal forces to the global system they will cancel.

We only need to account for the internal forces at the boundary → Structure reactions.

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Looking carefully at the internal forces:

$$S_i^k = T(x)N_i(x)\Big|_{x=x_a}^{x=x_b} = T(x_b)N_i(x_b) - T(x_a)N_i(x_a)$$

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When $i = b$: $S_b^k = T(x_b)$

Since $S_a^{k+1} = -S_b^k$, when adding the contributions of the internal forces to the global system they will cancel.

We only need to account for the internal forces at the boundary → Structure reactions.

Solving the rod equation using Finite Elements

3. **Define elemental weak form**

Let's compute the stiffness matrix

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$

$$\boldsymbol{K}^k = ?$$

Solving the rod equation using Finite Elements

3. Define elemental weak form

Let's compute the stiffness matrix

$$N_a(x) = \frac{x_b - x}{h}, \quad N_a'(x) = -\frac{1}{h}, \quad N_b(x) = \frac{x - x_a}{h}, \quad N_b'(x) = \frac{1}{h}$$

$$K_{aa} = \int_{x_a}^{x_b} EA\, N_a'(x) N_a'(x) d\Omega = \frac{EA}{h^2} x \Big|_{x_a}^{x_b} = \boldsymbol{\frac{EA}{h}}$$

$$K_{ab} = K_{ba} = \int_{x_a}^{x_b} EA\, N_a'(x) N_b'(x) d\Omega = -\frac{EA}{h^2} x \Big|_{x_a}^{x_b} = -\boldsymbol{\frac{EA}{h}}$$

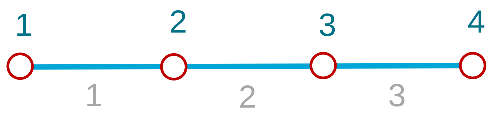
$$K_{bb} = \int_{x_a}^{x_b} EA\, N_b'(x) N_b'(x) d\Omega = \frac{EA}{h^2} x \Big|_{x_a}^{x_b} = \boldsymbol{\frac{EA}{h}}$$

$$\boldsymbol{K}^k = \frac{EA}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Solving the rod equation using Finite Elements

4. **Assemble the global system**

From the elemental weak forms we have: $M_{ij}^k \ddot{u}_j + K_{ij}^k u_j = Q_i^k$



$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}\ddot{u}_1\\ \ddot{u}_2\\ \ddot{u}_3\\ \ddot{u}_4\end{bmatrix} + \begin{bmatrix} & & & \\ & & & \\ & & & \\ & & & \end{bmatrix}\begin{bmatrix}u_1\\ u_2\\ u_3\\ u_4\end{bmatrix} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix} + \begin{bmatrix}S_1\\ 0\\ 0\\ S_4\end{bmatrix}$$

Solving the rod equation using Finite Elements

4. **Assemble the global system**

From the elemental weak forms we have: $M_{ij}^k \ddot{u}_j + K_{ij}^k u_j = Q_i^k$

$$\begin{bmatrix} M_{11}^1 & M_{12}^1 & 0 & 0 \\ M_{21}^1 & M_{22}^1 & & \\ 0 & & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 & & \\ 0 & & & \\ 0 & & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 \\ \\ \end{bmatrix} + \begin{bmatrix} S_1 \\ 0 \\ 0 \\ S_4 \end{bmatrix}$$

When $k = 1$: $M_{ij}^1 = K_{ij}^1 = Q_i^1 = 0$ for all $j, i \neq 1{,}2$

Solving the rod equation using Finite Elements

4. Assemble the global system

From the elemental weak forms we have: $M_{ij}^k \ddot{u}_j + K_{ij}^k u_j = Q_i^k$

$$\begin{bmatrix} M_{11}^1 & M_{12}^1 & 0 & 0 \\ M_{21}^1 & M_{22}^1 + M_{22}^2 & M_{23}^1 & 0 \\ 0 & M_{32}^2 & M_{33}^2 & \\ 0 & 0 & & \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 \\ 0 & K_{32}^2 & K_{33}^2 & \\ 0 & 0 & & \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 + Q_2^2 \\ Q_3^2 \\ \end{bmatrix} + \begin{bmatrix} S_1 \\ 0 \\ 0 \\ S_4 \end{bmatrix}$$

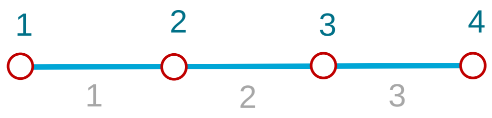
When $k = 1$: $M_{ij}^1 = K_{ij}^1 = Q_i^1 = 0$ for all $j, i \neq 1,2$

When $k = 2$: $M_{ij}^2 = K_{ij}^2 = Q_i^2 = 0$ for all $j, i \neq 2,3$

Solving the rod equation using Finite Elements

4. Assemble the global system

From the elemental weak forms we have: $M_{ij}^k \ddot{u}_j + K_{ij}^k u_j = Q_i^k$



$$\begin{bmatrix} M_{11}^1 & M_{12}^1 & 0 & 0 \\ M_{21}^1 & M_{22}^1 + M_{22}^2 & M_{23}^2 & 0 \\ 0 & M_{32}^2 & M_{33}^2 + M_{33}^3 & M_{34}^3 \\ 0 & 0 & M_{43}^3 & M_{44}^3 \end{bmatrix} \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 \\ 0 & K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^3 \\ 0 & 0 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^3 \end{bmatrix} + \begin{bmatrix} S_1 \\ 0 \\ 0 \\ S_4 \end{bmatrix}$$

When $k = 1$: $M_{ij}^1 = K_{ij}^1 = Q_i^1 = 0$ for all $j, i \neq 1,2$

When $k = 2$: $M_{ij}^2 = K_{ij}^2 = Q_i^2 = 0$ for all $j, i \neq 2,3$

When $k = 3$: $M_{ij}^3 = K_{ij}^3 = Q_i^3 = 0$ for all $j, i \neq 3,4$

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Solving the rod equation using Finite Elements

- **Bonus: apply boundary conditions**

Assume we have known displacement at node 1: $u_1 = U, \ddot{u}_1 = \ddot{U}$

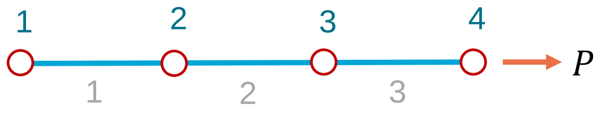
$$\begin{bmatrix} M_{11}^1 & M_{12}^1 & 0 & 0 \\ M_{21}^1 & M_{22}^1 + M_{22}^2 & M_{23}^1 & 0 \\ 0 & M_{32}^2 & M_{33}^2 + M_{33}^3 & M_{34}^3 \\ 0 & 0 & M_{43}^3 & M_{44}^3 \end{bmatrix} \begin{bmatrix} \ddot{U} \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{11}^1 & K_{12}^1 & 0 & 0 \\ K_{21}^1 & K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 \\ 0 & K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^3 \\ 0 & 0 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} U \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Q_1^1 \\ Q_2^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^3 \end{bmatrix} + \begin{bmatrix} S_1 \\ 0 \\ 0 \\ S_4 \end{bmatrix}$$

$$\begin{bmatrix} M_{22}^1 + M_{22}^2 & M_{23}^1 & 0 \\ M_{32}^2 & M_{33}^2 + M_{33}^3 & M_{34}^3 \\ 0 & M_{43}^3 & M_{44}^3 \end{bmatrix} \begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 \\ K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^3 \\ 0 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} Q_2^1 + Q_2^2 \\ Q_3^2 + Q_3^3 \\ Q_4^3 \end{bmatrix} + \begin{bmatrix} M_{21}^1 \ddot{U} + K_{21}^1 U \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ S_4 \end{bmatrix}$$

Solving the rod equation using Finite Elements

- **Bonus: apply boundary conditions**

Assume we have a force applied at node 4: $q(x) = P\delta(x - x_4), S_4 = 0$ (free node => no internal force)



$$Q_i^k = \int_{\Omega_k} q(x)N_i(x)d\Omega = \int_{\Omega_k} P\delta(x - x_4)N_i(x)d\Omega = PN_i(x_4)$$

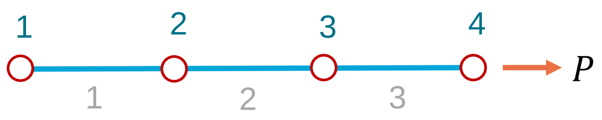
$$N_i(x_4) = \begin{cases} 0, & \text{for } i = 1,2,3 \\ 1, & \text{for } i = 4 \end{cases}$$

$$\begin{bmatrix} M_{22}^1 + M_{22}^2 & M_{23}^1 & 0 \\ M_{32}^2 & M_{33}^2 + M_{33}^3 & M_{34}^3 \\ 0 & M_{43}^3 & M_{44}^3 \end{bmatrix} \begin{bmatrix} \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \end{bmatrix} + \begin{bmatrix} K_{22}^1 + K_{22}^2 & K_{23}^2 & 0 \\ K_{32}^2 & K_{33}^2 + K_{33}^3 & K_{34}^3 \\ 0 & K_{43}^3 & K_{44}^3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ P \end{bmatrix} - \begin{bmatrix} M_{21}^1 \ddot{U} \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} K_{21}^1 U \\ 0 \\ 0 \end{bmatrix}$$

Solving the rod equation using Finite Elements

- **Bonus: apply boundary conditions**

The final expression:



$$\frac{mh}{6}\begin{bmatrix}4 & 1 & 0\\1 & 4 & 1\\0 & 1 & 2\end{bmatrix}\begin{bmatrix}\ddot{u}_2\\\ddot{u}_3\\\ddot{u}_4\end{bmatrix}+\frac{EA}{h}\begin{bmatrix}2 & -1 & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_2\\u_3\\u_4\end{bmatrix}=\begin{bmatrix}-\frac{mh}{6}\ddot{U}+\frac{EA}{h}U\\0\\P\end{bmatrix}$$

$$S_1 = M_{1j}\ddot{u}_j + K_{1j}u_j = \frac{mh}{6}\left(2\ddot{U}+\ddot{u}_2\right)+\frac{EA}{h}(U-u_2)$$

3.5

Finite Elements for beams

Solving the Euler-Bernoulli beam equation using Finite Elements

Consider the EOM of a rod (a PDE):

$$m\,\ddot{u}(x) + EI\,u''''(x) = q(x) \qquad \text{where } u(x,t) \;\; \forall \;\; x \in (0, L)$$

Remember the steps of a FEM:

1. Discretize the domain
2. Define shape functions
3. Define elemental weak form
4. Assemble the global system

Solving the Euler-Bernoulli beam equation using Finite Elements

1. **Discretize the domain**

Solving the Euler-Bernoulli beam equation using Finite Elements

2. Define the shape functions

We consider that the beam motion can be well approximated using piece-wise cubic functions:

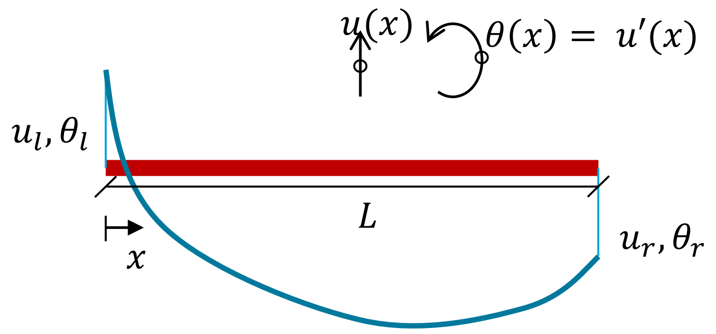
$$N_i^k(x) = a_i + b_i x + c_i x^2 + d_i x^3$$

We will use vertical displacements and rotations as unknowns. How do we define *a, b, c* and *d* for each DOF?

At each element we'll have 4 DOFs (2 at each node):

$$u(x) = \sum_{j=1}^{n_{dof}} N_j(x) w_j \qquad \text{and} \qquad \theta(x) = \sum_{j=1}^{n_{dof}} N_j'(x) w_j$$

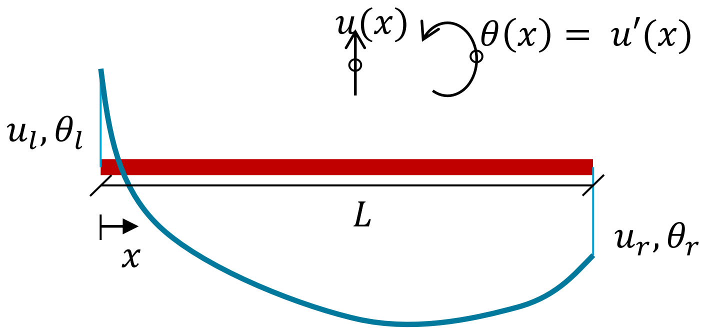
$$\boldsymbol{w} = [u_l \quad \theta_l \quad u_r \quad \theta_r]^T$$



Solving the Euler-Bernoulli beam equation using Finite Elements

2. Define the shape functions

We want each shape function associated to each DOF to be 1 at the DOF location and 0 at the other DOFs:



$$u_l = u(x_l) = \sum_{j=1}^{n_{dof}} N_j(x_l) w_j$$

$$\theta_l = \theta(x_l) = \sum_{j=1}^{n_{dof}} N_j'(x_l) w_j$$

$$u_r = u(x_r) = \sum_{j=1}^{n_{dof}} N_j(x_r) w_j$$

$$\theta_r = \theta(x_r) = \sum_{j=1}^{n_{dof}} N_j'(x_r) w_j$$

Solving the Euler-Bernoulli beam equation using Finite Elements

2. Define the shape functions

We want each shape function associated to each DOF to be 1 at the DOF location and 0 at the other DOFs:

$$\begin{aligned} u_l &= u(x_l) = \textstyle\sum_{j=1}^{n_{dof}} N_j(x_l) w_j \\ \theta_l &= \theta(x_l) = \textstyle\sum_{j=1}^{n_{dof}} N_j'(x_l) w_j \\ u_r &= u(x_r) = \textstyle\sum_{j=1}^{n_{dof}} N_j(x_r) w_j \\ \theta_r &= \theta(x_r) = \textstyle\sum_{j=1}^{n_{dof}} N_j'(x_r) w_j \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} N_1(x_l) & N_2(x_l) & N_3(x_l) & N_4(x_l) \\ N_1'(x_l) & N_2'(x_l) & N_3'(x_l) & N_4'(x_l) \\ N_1(x_r) & N_2(x_r) & N_3(x_r) & N_4(x_r) \\ N_1'(x_r) & N_2'(x_r) & N_3'(x_r) & N_4'(x_r) \end{bmatrix} \begin{bmatrix} u_l \\ \theta_l \\ u_r \\ \theta_r \end{bmatrix} = \begin{bmatrix} u_l \\ \theta_l \\ u_r \\ \theta_r \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} N_1(x_l) & N_2(x_l) & N_3(x_l) & N_4(x_l) \\ N_1'(x_l) & N_2'(x_l) & N_3'(x_l) & N_4'(x_l) \\ N_1(x_r) & N_2(x_r) & N_3(x_r) & N_4(x_r) \\ N_1'(x_r) & N_2'(x_r) & N_3'(x_r) & N_4'(x_r) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} N_1(x_l) \\ N_1'(x_l) \\ N_1(x_r) \\ N_1'(x_r) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & x_l & x_l^2 & x_l^3 \\ 0 & 1 & 2x_l & 3x_l^2 \\ 1 & x_r & x_r^2 & x_r^3 \\ 0 & 1 & 2x_r & 3x_r^2 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{cases} N_1(x) = 1 - \frac{3x^2}{h^2} + \frac{2x^3}{h^3} \rightarrow N_1'(x) = -\frac{6x}{h^2} + \frac{6x^2}{h^3} \\ N_2(x) = x - \frac{2x^2}{h} + \frac{x^3}{h^2} \rightarrow N_2'(x) = 1 - \frac{4x}{h} + \frac{3x^2}{h^2} \\ N_3(x) = \frac{3x^2}{h^2} - \frac{2x^3}{h^3} \rightarrow N_3'(x) = \frac{6x}{h^2} - \frac{6x^2}{h^3} \\ N_4(x) = -\frac{x^2}{h} + \frac{x^3}{h^2} \rightarrow N_4'(x) = -\frac{2x}{h} + \frac{3x^2}{h^2} \end{cases}$$

Solving the Euler-Bernoulli beam equation using Finite Elements

3. Elemental weak form

Starting from the strong form (EOM):

$$m\,\ddot{u}(x) + EI\,u''''(x) = q(x)$$

Multiply by a test function v and integrate over an element:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega + \int_{\Omega_k} EI\,u''''(x)v(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega$$

Integrate by parts the terms with fourth-order derivative (twice):

$$\int_{\Omega_k} EI\,u''''(x)v(x)d\Omega = -\int_{\Omega_k} EI\,u'''(x)v'(x)d\Omega + EIu'''(x)v(x)\Big|_{x=x_a}^{x=x_b}$$

$$= \int_{\Omega_k} EI\,u''(x)v''(x)d\Omega - \underbrace{EIu''(x)}_{M(x)}v'(x)\Big|_{x=x_a}^{x=x_b} + \underbrace{EIu'''(x)}_{V(x)}v(x)\Big|_{x=x_a}^{x=x_b}$$

Final weak form:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega + \int_{\Omega_k} EI\,u''(x)v''(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega + [M(x)v'(x) - V(x)v(x)]\Big|_{x=x_a}^{x=x_b}$$

Solving the Euler-Bernoulli beam equation using Finite Elements

3. Elemental weak form

Final weak form:

$$\int_{\Omega_k} m\,\ddot{u}(x)v(x)d\Omega + \int_{\Omega_k} EI\,u''(x)v''(x)d\Omega = \int_{\Omega_k} q(x)v(x)d\Omega + [M(x)v'(x) - V(x)v(x)]\Big|_{x=x_a}^{x=x_b}$$

Choose $v(x) = N_i(x)$ for all $i = 1, \dots, n_{dof}$. In a given element k between nodes a and b, the only non-zero shape functions will be $N_a(x)$ and $N_b(x)$.

Replace the solution by the approximated function:

$$u(x) = \sum_{j=1}^{n_n} N_j(x)w_j, \quad u''(x) = \sum_{j=1}^{n_n} N_j''(x)w_j, \quad \ddot{u}(x) = \sum_{j=1}^{n_n} N_j(x)\ddot{w}_j$$

$$\sum_{j=1}^{n_{dof}} \left[\int_{\Omega_k} m\,N_j(x)N_i(x)d\Omega\right]\ddot{w}_j + \sum_{j=1}^{n_{dof}} \left[\int_{\Omega_k} EI\,N_j''(x)N_i''(x)d\Omega\right]u_j = \int_{\Omega_k} q(x)N_i(x)d\Omega + [M(x)N_i'(x) - V(x)N_i(x)]\Big|_{x=x_a}^{x=x_b}$$

$$M_{ij}^k\ddot{u}_j + K_{ij}^k u_j = Q_i^k + S_i^k \qquad \boldsymbol{M}^k\ddot{\boldsymbol{u}} + \boldsymbol{K}^k\boldsymbol{u} = \boldsymbol{Q}^k + \boldsymbol{S}^k$$

Solving the Euler-Bernoulli beam equation using Finite Elements

3. Elemental weak form

Let’s compute the mass matrix

Using the shape functions that we found in the previous slide:

```
clc
clear
syms x l EI m real
N1 = 1-3*(x/l)^2+2*(x/l)^3;
N2 = x - 2*l*(x/l)^2 + l*(x/l)^3;
N3 = 3*(x/l)^2-2*(x/l)^3;
N4 = -1*(x/l)^2+1*(x/l)^3;

N = [N1 N2 N3 N4];

M = m*int(N.'*N,x,0,l);
M*420/m/l
```

```
[   156,    22*l,     54,  -13*l]
[  22*l,  4*l^2,   13*l, -3*l^2]
[    54,    13*l,    156,  -22*l]
[ -13*l, -3*l^2, -22*l,   4*l^2]
```

$$\boldsymbol{M}^k = \frac{mh}{420}\begin{bmatrix} 156 & 22h & 54 & -13h \\ 22h & 4h^2 & 13h & -3h^2 \\ 54 & 13h & 156 & -22h \\ -13h & -3h^2 & -22h & 4h^2 \end{bmatrix}$$

Solving the Euler-Bernoulli beam equation using Finite Elements

3. Elemental weak form

Looking carefully at the internal forces:

$$S_i = [M(x)N_i'(x) - V(x)N_i(x)]\Big|_{x=x_a}^{x=x_b} = [M(x_b)N_i'(x_b) - V(x_b)N_i(x_b)] - [M(x_a)N_i'(x_a) - V(x_a)N_i(x_a)]$$

When $i = 1$: $S_1 = V(x_a)$

When $i = 2$: $S_2 = -M(x_a)$

When $i = 3$: $S_3 = -V(x_b)$

When $i = 4$: $S_4 = M(x_b)$

Since $S_1 = -S_3$ and $S_2 = -S_4$, when adding the contributions of the internal forces to the global system they will cancel.

We only need to account for the internal forces at the boundary → Structure reactions.

3. Elemental weak form

Let's compute the stiffness matrix

Using the shape functions that we found in the previous slide:

```
clc
clear
syms x l EI m real
N1 = 1-3*(x/l)^2+2*(x/l)^3;
N2 = x - 2*l*(x/l)^2 + l*(x/l)^3;
N3 = 3*(x/l)^2-2*(x/l)^3;
N4 = -l*(x/l)^2+l*(x/l)^3;

N = [N1 N2 N3 N4];

d2N = diff(diff(N,x),x);

K = EI*int(d2N.'*d2N,x,0,l);

K/EI
```

$$\boldsymbol{K}^k = \frac{EI}{h}\begin{bmatrix} \frac{12}{h^2} & \frac{6}{h} & -\frac{12}{h^2} & \frac{6}{h} \\ \frac{6}{h} & 4 & -\frac{6}{h} & 2 \\ -\frac{12}{h^2} & -\frac{6}{h} & \frac{12}{h^2} & -\frac{6}{h} \\ \frac{6}{h} & 2 & -\frac{6}{h} & 4 \end{bmatrix}$$

```
[  12/l^3,  6/l^2, -12/l^3,  6/l^2]
[   6/l^2,    4/l,  -6/l^2,    2/l]
[ -12/l^3, -6/l^2,  12/l^3, -6/l^2]
[   6/l^2,    2/l,  -6/l^2,    4/l]
```

Solving the Euler-Bernoulli beam equation using Finite Elements

4. **Assemble the global system**

Same process as the axial rod…

Solving the Euler-Bernoulli beam equation using Finite Elements

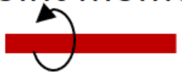
Bonus: more on loading options

Point force

$$p(x) = P_0\,\delta(x - x_0)$$

$$\int_0^L \mathbf{N}^{\mathrm{T}} p\,\mathrm{d}x = P_0 \begin{bmatrix} N_1(x_0) \\ N_2(x_0) \\ N_3(x_0) \\ N_4(x_0) \end{bmatrix}$$

Point moment



$$p(x) = M_0 \lim_{\Delta \to 0} \left(\frac{\delta\left(x - x_0 - \frac{\Delta}{2}\right) - \delta\left(x - x_0 + \frac{\Delta}{2}\right)}{\Delta} \right) = M_0 \dot{\delta}(x - x_0)$$

$$\int_0^L \mathbf{N}^{\mathrm{T}} p\,\mathrm{d}x = M_0 \begin{bmatrix} N_1'(x_0) \\ N_2'(x_0) \\ N_3'(x_0) \\ N_4'(x_0) \end{bmatrix}$$

Distributed force

$$p(x)$$

$$\int_0^L \mathbf{N}^{\mathrm{T}} p\,\mathrm{d}x = \int_0^L \begin{bmatrix} N_1(x) \\ N_2(x) \\ N_3(x) \\ N_4(x) \end{bmatrix} p(x)\,\mathrm{d}x$$

Distributed moment

$$m(x)$$

$$\int_0^L \mathbf{N}^{\mathrm{T}} p\,\mathrm{d}x = \int_0^L \begin{bmatrix} N_1'(x) \\ N_2'(x) \\ N_3'(x) \\ N_4'(x) \end{bmatrix} m(x)\,\mathrm{d}x$$

3.6

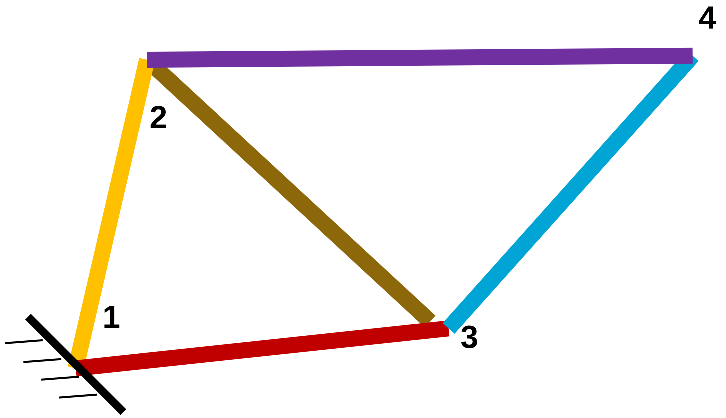
Finite Element Method for multiple elements

TU Delft

Finite Elements for a space frame structure

Let's see the FEM in practice for a structure with multiple elements:

- Remember the steps of a FEM:
  1. Discretize the domain
  2. Define shape functions
  3. Define elemental weak form
  4. Assemble the global system

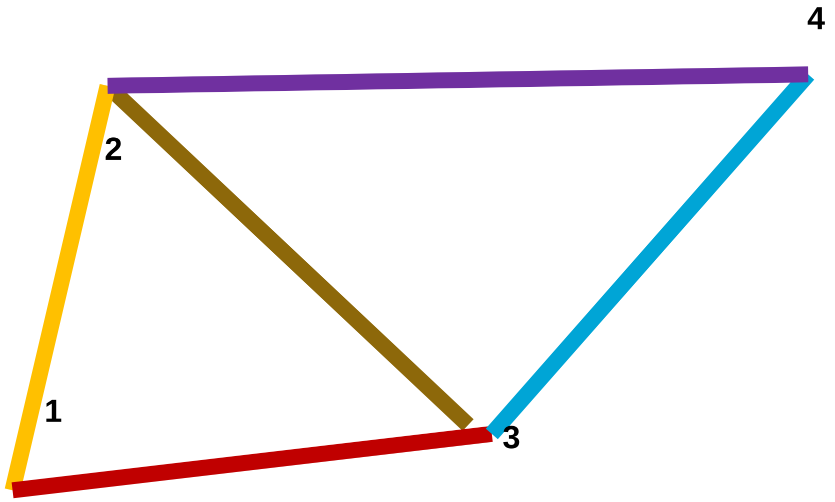


Consider a structure that is subject to axial displacement and bending. At each element the EOM has to be satisfied:

$$m\,\ddot{u}(x) - EA\,u''(x) = q_h(x)$$
$$m\,\ddot{v}(x) + EI\,v''''(x) = q_v(x)$$

Finite Elements for a space frame structure

1. **Discretize the domain**



Finite Elements for a space frame structure

2. Define the shape functions

We consider that the axial displacement is approximated with piece-wise linear functions and the vertical motion motion can be well approximated using piece-wise cubic functions:

$$N_{u_i}^k(x) = a_i + b_i x$$

$$N_{v_i}^k(x) = c_i + d_i x + e_i x^2 + f_i x^3$$

We will use horizontal and vertical displacements and rotations as unknowns.

Follow the same process to define *a, b, c, d, e* and *f.*

$$\begin{cases} N_1(x) = \dfrac{h-x}{h} & \rightarrow N_1'(x) = -\dfrac{1}{h} \\ N_2(x) = \dfrac{x}{h} & \rightarrow N_2'(x) = \dfrac{1}{h} \\ N_3(x) = 1 - \dfrac{3x^2}{h^2} + \dfrac{2x^3}{h^3} & \rightarrow N_3'(x) = -\dfrac{6x}{h^2} + \dfrac{6x^2}{h^3} \\ N_4(x) = x - \dfrac{2x^2}{h} + \dfrac{x^3}{h^2} & \rightarrow N_4'(x) = 1 - \dfrac{4x}{h} + \dfrac{3x^2}{h^2} \\ N_5(x) = \dfrac{3x^2}{h^2} - \dfrac{2x^3}{h^3} & \rightarrow N_5'(x) = \dfrac{6x}{h^2} - \dfrac{6x^2}{h^3} \\ N_6(x) = -\dfrac{x^2}{h} + \dfrac{x^3}{h^2} & \rightarrow N_6'(x) = -\dfrac{2x}{h} + \dfrac{3x^2}{h^2} \end{cases}$$

Finite Elements for a space frame structure

3. Elemental weak form

Starting from the strong form (EOM):

$$m\,\ddot{u}(x) - EA\,u''(x) = q_h(x)$$
$$m\,\ddot{v}(x) + EI\,v''''(x) = q_v(x)$$

1. Multiply by a test function v and integrate over an element.
2. Integrate by parts the terms with fourth-order derivative (twice).
3. Choose $v(x) = N_i(x)$ for all $i = 1, \dots, n_{dof}$.
4. Replace the solution by the approximated function.

Finite Elements for a space frame structure

3. Elemental weak form

Starting from the strong form (EOM):

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2. Integrate by parts the terms with fourth-order derivative (twice).
3. Choose $v(x) = N_i(x)$ for all $i = 1, \dots, n_{dof}$.
4. Replace the solution by the approximated function.

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{K}\mathbf{u} = \begin{bmatrix} Q_{l,\text{ext}} \\ P_{l,\text{ext}} \\ M_{l,\text{ext}} \\ Q_{r,\text{ext}} \\ P_{r,\text{ext}} \\ M_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -T_l \\ -V_l \\ -M_l \\ T_r \\ V_r \\ M_r \end{bmatrix}$$

$$\mathbf{M} = \frac{mL}{420}\begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22L & 0 & 54 & -13L \\ 0 & 22L & 4L^2 & 0 & 13L & -3L^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13L & 0 & 156 & -22L \\ 0 & -13L & -3L^2 & 0 & -22L & 4L^2 \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \frac{EA}{L} & 0 & 0 & \frac{-EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & \frac{6EI}{L^2} & 0 & \frac{-12EI}{L^3} & \frac{6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{4EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{2EI}{L} \\ \frac{-EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & \frac{-12EI}{L^3} & \frac{-6EI}{L^2} & 0 & \frac{12EI}{L^3} & \frac{-6EI}{L^2} \\ 0 & \frac{6EI}{L^2} & \frac{2EI}{L} & 0 & \frac{-6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}$$

Finite Elements for a space frame structure

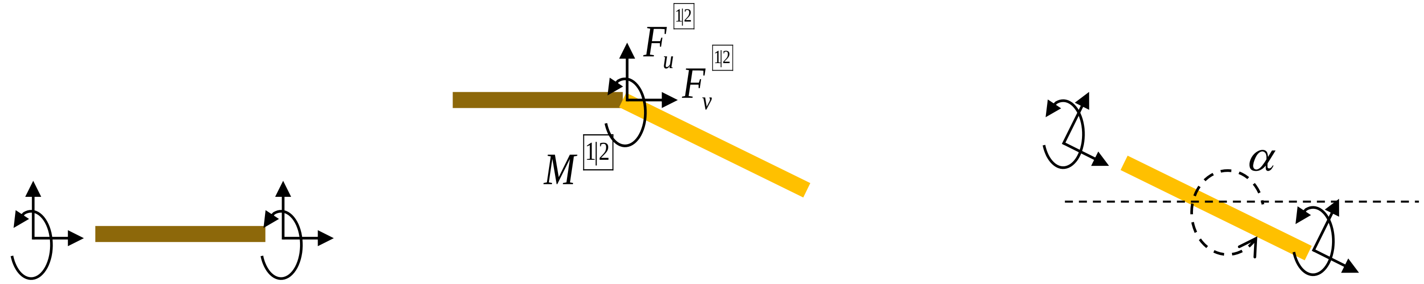
4. **Assemble the global system**

Same process as the axial rod and beam… But careful! The elements have different orientation!!!

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!



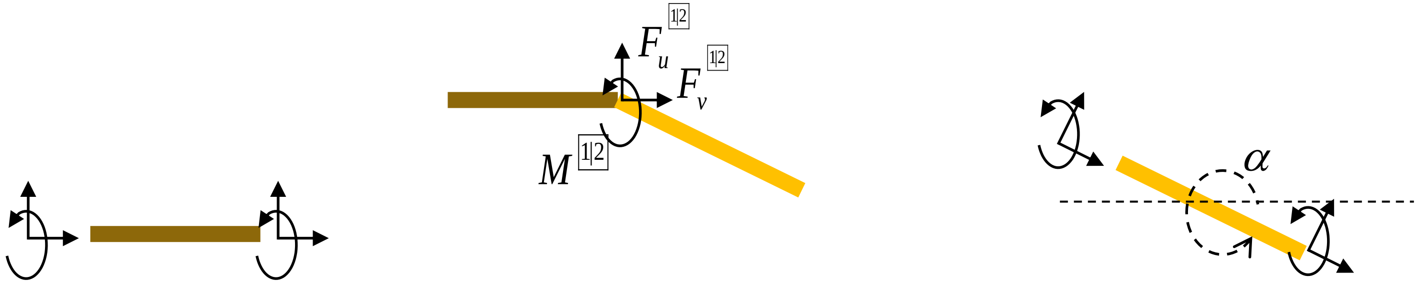
$$\begin{bmatrix} \mathbf{M}_{ll}^{[1]} & \mathbf{M}_{lr}^{[1]} \\ \mathbf{M}_{rl}^{[1]} & \mathbf{M}_{rr}^{[1]} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{[1]} \\ \ddot{\mathbf{u}}_{r}^{[1]} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{[1]} & \mathbf{K}_{lr}^{[1]} \\ \mathbf{K}_{rl}^{[1]} & \mathbf{K}_{rr}^{[1]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{[1]} \\ \mathbf{u}_{r}^{[1]} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{[1]} \\ \mathbf{P}_{r,\text{ext}}^{[1]} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{[1]} \\ \mathbf{S}_{r}^{[1]} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{[2]} & \mathbf{M}_{lr}^{[2]} \\ \mathbf{M}_{rl}^{[2]} & \mathbf{M}_{rr}^{[2]} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{[2]} \\ \ddot{\mathbf{u}}_{r}^{[2]} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{[2]} & \mathbf{K}_{lr}^{[2]} \\ \mathbf{K}_{rl}^{[2]} & \mathbf{K}_{rr}^{[2]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{[2]} \\ \mathbf{u}_{r}^{[2]} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{[2]} \\ \mathbf{P}_{r,\text{ext}}^{[2]} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{[2]} \\ \mathbf{S}_{r}^{[2]} \end{bmatrix}$$

Finite Elements for a space frame structure

4. Assemble the global system

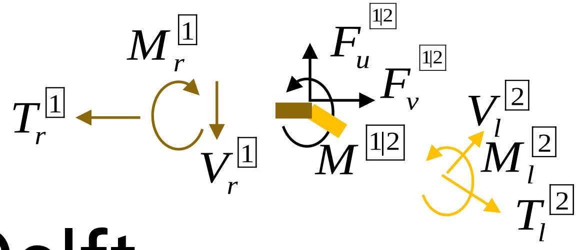
Same process as the axial rod and beam… But careful! The elements have different orientation!!!



$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{1}} & \mathbf{M}_{lr}^{\boxed{1}} \\ \mathbf{M}_{rl}^{\boxed{1}} & \mathbf{M}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{1}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{1}} & \mathbf{K}_{lr}^{\boxed{1}} \\ \mathbf{K}_{rl}^{\boxed{1}} & \mathbf{K}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{1}} \\ \mathbf{u}_{r}^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{\boxed{1}} \\ \mathbf{P}_{r,\text{ext}}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{1}} \\ \mathbf{S}_{r}^{\boxed{1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{2}} & \mathbf{M}_{lr}^{\boxed{2}} \\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{2}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{2}} & \mathbf{K}_{lr}^{\boxed{2}} \\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{2}} \\ \mathbf{u}_{r}^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{\boxed{2}} \\ \mathbf{P}_{r,\text{ext}}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{2}} \\ \mathbf{S}_{r}^{\boxed{2}} \end{bmatrix}$$

At the node:



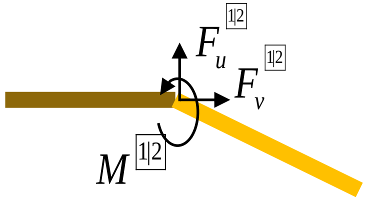
Equilibrium of forces

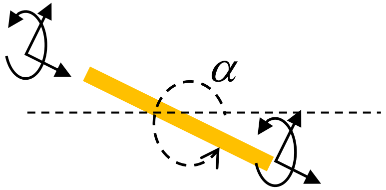
$$\begin{cases} F_v^{\boxed{1|2}} = T_r^{\boxed{1}} - T_l^{\boxed{2}} \\ F_u^{\boxed{1|2}} = \ldots - V_l^{\boxed{2}} \\ M^{\boxed{1|2}} = M_r^{\boxed{1}} \ldots M_l^{\boxed{2}} \end{cases}$$

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!

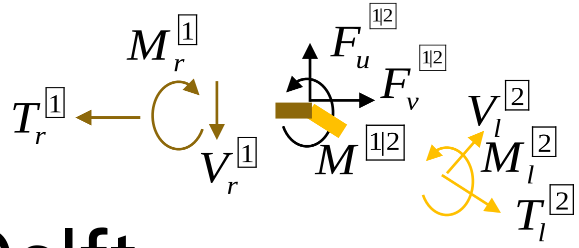




$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{1}} & \mathbf{M}_{lr}^{\boxed{1}} \\ \mathbf{M}_{rl}^{\boxed{1}} & \mathbf{M}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{1}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{1}} & \mathbf{K}_{lr}^{\boxed{1}} \\ \mathbf{K}_{rl}^{\boxed{1}} & \mathbf{K}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{1}} \\ \mathbf{u}_{r}^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{\boxed{1}} \\ \mathbf{P}_{r,\text{ext}}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{1}} \\ \mathbf{S}_{r}^{\boxed{1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{2}} & \mathbf{M}_{lr}^{\boxed{2}} \\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{2}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{2}} & \mathbf{K}_{lr}^{\boxed{2}} \\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{2}} \\ \mathbf{u}_{r}^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{\boxed{2}} \\ \mathbf{P}_{r,\text{ext}}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{2}} \\ \mathbf{S}_{r}^{\boxed{2}} \end{bmatrix}$$

At the node:



Equilibrium of forces

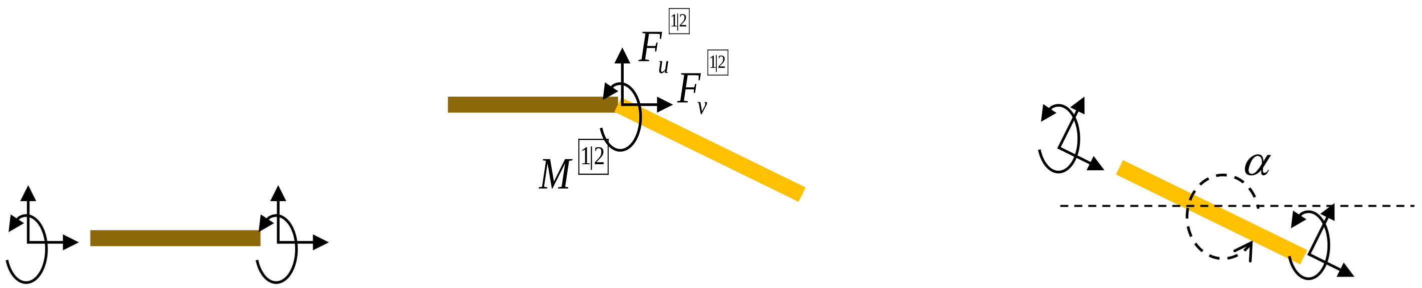
$$\begin{cases} F_v^{\boxed{1|2}} = T_r^{\boxed{1}} - T_l^{\boxed{2}}\cos(\alpha) + V_l^{\boxed{2}}\sin(\alpha) \\ F_u^{\boxed{1|2}} = V_r^{\boxed{1}} - T_l^{\boxed{2}}\sin(\alpha) - V_l^{\boxed{2}}\cos(\alpha) \\ M^{\boxed{1|2}} = M_r^{\boxed{1}} - M_l^{\boxed{2}} \end{cases}$$

$$\rightarrow \mathbf{F}^{\boxed{1|2}} = \mathbf{S}_r^{\boxed{1}} - \mathbf{T}^{\boxed{2}}\mathbf{S}_l^{\boxed{2}} \qquad \mathbf{T}^{\boxed{2}} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finite Elements for a space frame structure

4. Assemble the global system

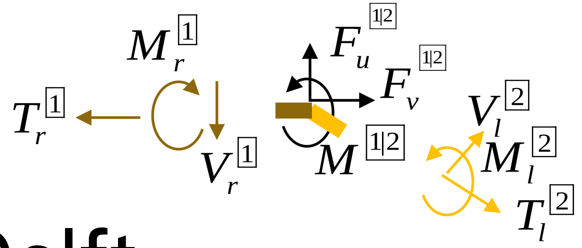
Same process as the axial rod and beam… But careful! The elements have different orientation!!!



$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{1}} & \mathbf{M}_{lr}^{\boxed{1}} \\ \mathbf{M}_{rl}^{\boxed{1}} & \mathbf{M}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{1}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{1}} & \mathbf{K}_{lr}^{\boxed{1}} \\ \mathbf{K}_{rl}^{\boxed{1}} & \mathbf{K}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{1}} \\ \mathbf{u}_{r}^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{1}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{1}} \\ \mathbf{S}_{r}^{\boxed{1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{2}} & \mathbf{M}_{lr}^{\boxed{2}} \\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{2}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{2}} & \mathbf{K}_{lr}^{\boxed{2}} \\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{2}} \\ \mathbf{u}_{r}^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{2}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{2}} \\ \mathbf{S}_{r}^{\boxed{2}} \end{bmatrix}$$

At the node:



Compatibility of displacements

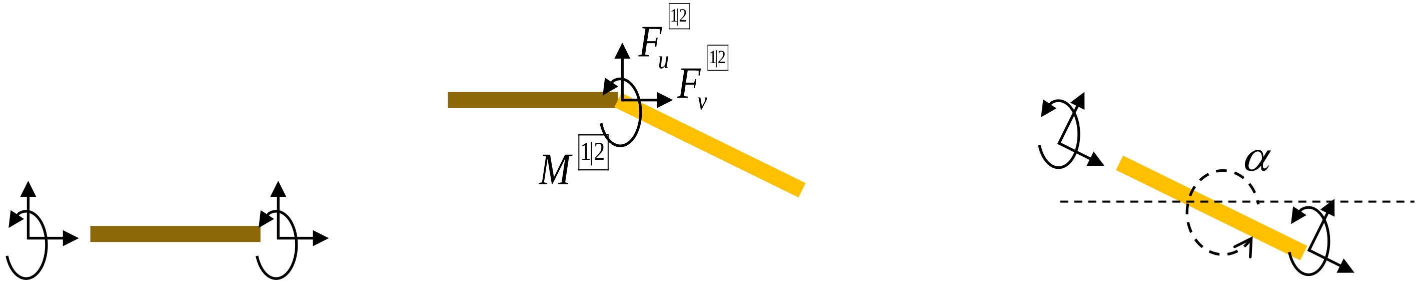
$$\begin{cases} v_r^{\boxed{1}} = v_l^{\boxed{2}} = v^{\boxed{1|2}} \\ u_r^{\boxed{1}} = u_l^{\boxed{2}} = u^{\boxed{1|2}} \\ \theta_r^{\boxed{1}} = \theta_l^{\boxed{2}} = \theta^{\boxed{1|2}} \end{cases} \qquad \begin{cases} v_r^{\boxed{1}} = v_l^{\boxed{2}} \cos(\alpha) - u_l^{\boxed{2}} \sin(\alpha) = v_r^{\boxed{1|2}} \\ u_r^{\boxed{1}} = v_l^{\boxed{2}} \sin(\alpha) + u_l^{\boxed{2}} \cos(\alpha) = u_r^{\boxed{1|2}} \\ \theta_r^{\boxed{1}} = \theta_l^{\boxed{2}} = \theta^{\boxed{1|2}} \end{cases}$$

$$\rightarrow \mathbf{u}^{\boxed{1|2}} = \mathbf{u}_r^{\boxed{1}} = \mathbf{T}^{\boxed{2}} \mathbf{u}_l^{\boxed{2}}$$

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!

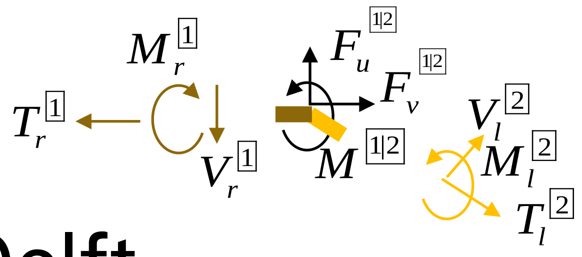


$$\begin{bmatrix} \mathbf{M}_{ll}^{[1]} & \mathbf{M}_{lr}^{[1]} \\ \mathbf{M}_{rl}^{[1]} & \mathbf{M}_{rr}^{[1]} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{[1]} \\ \ddot{\mathbf{u}}_{r}^{[1]} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{[1]} & \mathbf{K}_{lr}^{[1]} \\ \mathbf{K}_{rl}^{[1]} & \mathbf{K}_{rr}^{[1]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{[1]} \\ \mathbf{u}_{r}^{[1]} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{[1]} \\ \mathbf{P}_{r,\text{ext}}^{[1]} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{[1]} \\ \mathbf{S}_{r}^{[1]} \end{bmatrix}$$

Transformation matrix

$$\begin{bmatrix} \mathbf{M}_{ll}^{[2]} & \mathbf{M}_{lr}^{[2]} \\ \mathbf{M}_{rl}^{[2]} & \mathbf{M}_{rr}^{[2]} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{[2]} \\ \ddot{\mathbf{u}}_{r}^{[2]} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{[2]} & \mathbf{K}_{lr}^{[2]} \\ \mathbf{K}_{rl}^{[2]} & \mathbf{K}_{rr}^{[2]} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{[2]} \\ \mathbf{u}_{r}^{[2]} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{[2]} \\ \mathbf{P}_{r,\text{ext}}^{[2]} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{[2]} \\ \mathbf{S}_{r}^{[2]} \end{bmatrix}$$

At the node:



Equilibrium of forces

$$\mathbf{F}^{[1|2]} = \mathbf{S}_{r}^{[1]} - \mathbf{T}^{[2]}\mathbf{S}_{l}^{[2]}$$

Compatibility of displacements

$$\mathbf{u}^{[1|2]} = \mathbf{u}_{r}^{[1]} = \mathbf{T}^{[2]}\mathbf{u}_{l}^{[2]}$$

Transformation matrix T

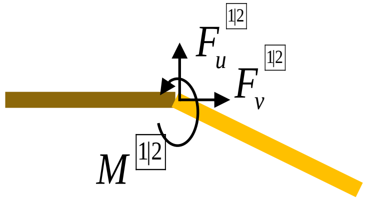
$$\mathbf{T}^{[2]} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

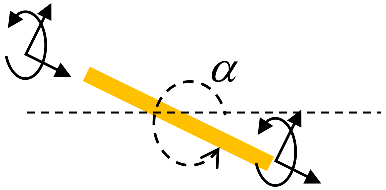
$$\mathbf{T}^{\mathrm{T}[2]}\mathbf{T}^{[2]} = \mathbf{I}$$

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!

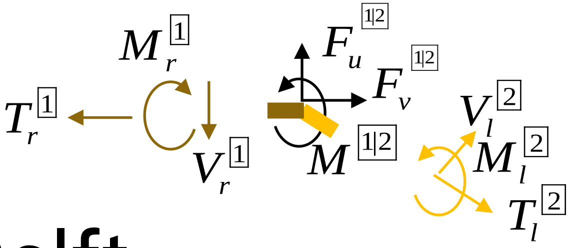




$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{1}} & \mathbf{M}_{lr}^{\boxed{1}} \\ \mathbf{M}_{rl}^{\boxed{1}} & \mathbf{M}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_l^{\boxed{1}} \\ \ddot{\mathbf{u}}_r^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{1}} & \mathbf{K}_{lr}^{\boxed{1}} \\ \mathbf{K}_{rl}^{\boxed{1}} & \mathbf{K}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_l^{\boxed{1}} \\ \mathbf{u}_r^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{1}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l^{\boxed{1}} \\ \mathbf{S}_r^{\boxed{1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{2}} & \mathbf{M}_{lr}^{\boxed{2}} \\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_l^{\boxed{2}} \\ \ddot{\mathbf{u}}_r^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{2}} & \mathbf{K}_{lr}^{\boxed{2}} \\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_l^{\boxed{2}} \\ \mathbf{u}_r^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{2}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l^{\boxed{2}} \\ \mathbf{S}_r^{\boxed{2}} \end{bmatrix}$$

At the node:



Equilibrium of forces

$$\mathbf{F}^{\boxed{1|2}} = \mathbf{S}_r^{\boxed{1}} - \mathbf{T}^{\boxed{2}}\mathbf{S}_l^{\boxed{2}}$$

Compatibility of displacements

$$\mathbf{u}^{\boxed{1|2}} = \mathbf{u}_r^{\boxed{1}} = \mathbf{T}^{\boxed{2}}\mathbf{u}_l^{\boxed{2}}$$

Transformation matrix T

$$\mathbf{T}^{\boxed{2}} = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) & 0 \\ \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

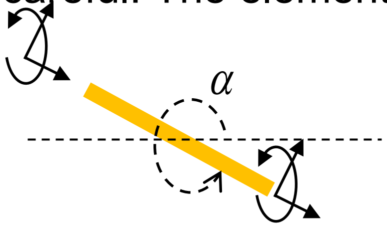
$$\mathbf{T}^{\mathrm{T}\boxed{2}}\mathbf{T}^{\boxed{2}} = \mathbf{I}$$

TU Delft

Finite Elements for a space frame structure

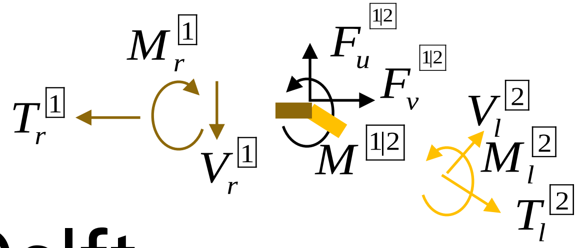
4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!



$$\begin{matrix}(xT^{2})\\(xT^{2})\end{matrix}\begin{bmatrix}\mathbf{M}_{ll}^{\boxed{2}} & \overset{(xT^{T2}T^{2})}{\mathbf{M}_{lr}^{\boxed{2}}}\\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_{l}^{\boxed{2}}\\ \ddot{\mathbf{u}}_{r}^{\boxed{2}}\end{bmatrix}+\begin{bmatrix}\mathbf{K}_{ll}^{\boxed{2}} & \overset{(xT^{T2}T^{2})}{\mathbf{K}_{lr}^{\boxed{2}}}\\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}}\end{bmatrix}\begin{bmatrix}\mathbf{u}_{l}^{\boxed{2}}\\ \mathbf{u}_{r}^{\boxed{2}}\end{bmatrix}=\begin{bmatrix}\mathbf{P}_{l,\text{ext}}^{\boxed{2}}\\ \mathbf{P}_{r,\text{ext}}^{\boxed{2}}\end{bmatrix}+\begin{bmatrix}-\mathbf{S}_{l}^{\boxed{2}}\\ \mathbf{S}_{r}^{\boxed{2}}\end{bmatrix}$$

At the node:



Equilibrium of forces

$$\mathbf{F}^{\boxed{1|2}}=\mathbf{S}_{r}^{\boxed{1}}-\mathbf{T}^{\boxed{2}}\mathbf{S}_{l}^{\boxed{2}}$$

Compatibility of displacements

$$\mathbf{u}^{\boxed{1|2}}=\mathbf{u}_{r}^{\boxed{1}}=\mathbf{T}^{\boxed{2}}\mathbf{u}_{l}^{\boxed{2}}$$

Transformation matrix T

$$\mathbf{T}^{\boxed{2}}=\begin{bmatrix}\cos(\alpha) & -\sin(\alpha) & 0\\ \sin(\alpha) & \cos(\alpha) & 0\\ 0 & 0 & 1\end{bmatrix}$$

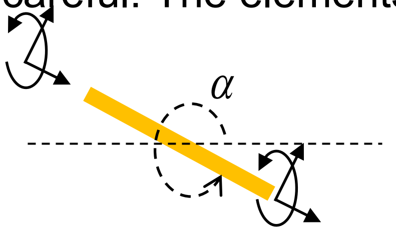
$$\mathbf{T}^{\mathrm{T}\boxed{2}}\mathbf{T}^{\boxed{2}}=\mathbf{I}$$

TU Delft

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!



$$\begin{bmatrix} \mathbf{T}^{\square}\mathbf{M}_{ll}^{\square}\mathbf{T}^{\mathrm{T}\square} & \mathbf{T}^{\square}\mathbf{M}_{lr}^{\square}\mathbf{T}^{\mathrm{T}\square} \\ \mathbf{T}^{\square}\mathbf{M}_{rl}^{\square}\mathbf{T}^{\mathrm{T}\square} & \mathbf{T}^{\square}\mathbf{M}_{rr}^{\square}\mathbf{T}^{\mathrm{T}\square} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\square}\ddot{\mathbf{u}}_{l}^{\square} \\ \mathbf{T}^{\square}\ddot{\mathbf{u}}_{r}^{\square} \end{bmatrix} + \begin{bmatrix} \mathbf{T}^{\square}\mathbf{K}_{ll}^{\square}\mathbf{T}^{\mathrm{T}\square} & \mathbf{T}^{\square}\mathbf{K}_{lr}^{\square}\mathbf{T}^{\mathrm{T}\square} \\ \mathbf{T}^{\square}\mathbf{K}_{rl}^{\square}\mathbf{T}^{\mathrm{T}\square} & \mathbf{T}^{\square}\mathbf{K}_{rr}^{\square}\mathbf{T}^{\mathrm{T}\square} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\square}\mathbf{u}_{l}^{\square} \\ \mathbf{T}^{\square}\mathbf{u}_{r}^{\square} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{\square}\mathbf{P}_{l,\mathrm{ext}}^{\square} \\ \mathbf{T}^{\square}\mathbf{P}_{r,\mathrm{ext}}^{\square} \end{bmatrix} + \begin{bmatrix} -\mathbf{T}^{\square}\mathbf{S}_{l}^{\square} \\ \mathbf{T}^{\square}\mathbf{S}_{r}^{\square} \end{bmatrix}$$

$$\begin{bmatrix} \tilde{\mathbf{M}}_{ll}^{\square} & \tilde{\mathbf{M}}_{lr}^{\square} \\ \tilde{\mathbf{M}}_{rl}^{\square} & \tilde{\mathbf{M}}_{rr}^{\square} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\square}\ddot{\mathbf{u}}_{l}^{\square} \\ \mathbf{T}^{\square}\ddot{\mathbf{u}}_{r}^{\square} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ll}^{\square} & \tilde{\mathbf{K}}_{lr}^{\square} \\ \tilde{\mathbf{K}}_{rl}^{\square} & \tilde{\mathbf{K}}_{rr}^{\square} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{\square}\mathbf{u}_{l}^{\square} \\ \mathbf{T}^{\square}\mathbf{u}_{r}^{\square} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{\square}\mathbf{P}_{l,\mathrm{ext}}^{\square} \\ \mathbf{T}^{\square}\mathbf{P}_{r,\mathrm{ext}}^{\square} \end{bmatrix} + \begin{bmatrix} -\mathbf{T}^{\square}\mathbf{S}_{l}^{\square} \\ \mathbf{T}^{\square}\mathbf{S}_{r}^{\square} \end{bmatrix}$$

Element matrices in Global coordinates

$$\tilde{\mathbf{M}}_{\alpha\beta}^{\square} = \mathbf{T}^{\square}\mathbf{M}_{\alpha\beta}^{\square}\mathbf{T}^{\mathrm{T}\square}$$

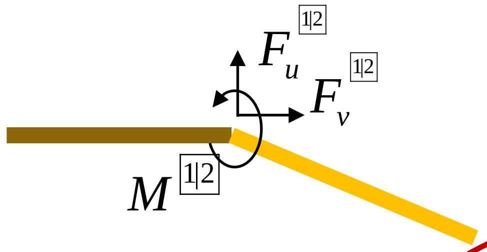
$$\tilde{\mathbf{K}}_{\alpha\beta}^{\square} = \mathbf{T}^{\square}\mathbf{K}_{\alpha\beta}^{\square}\mathbf{T}^{\mathrm{T}\square}$$

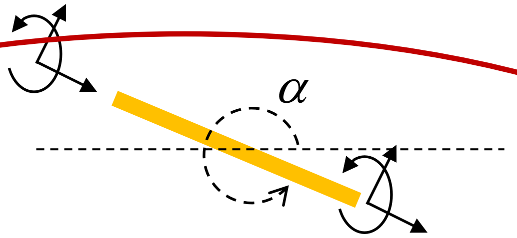
Element matrices in Local coordinates

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!





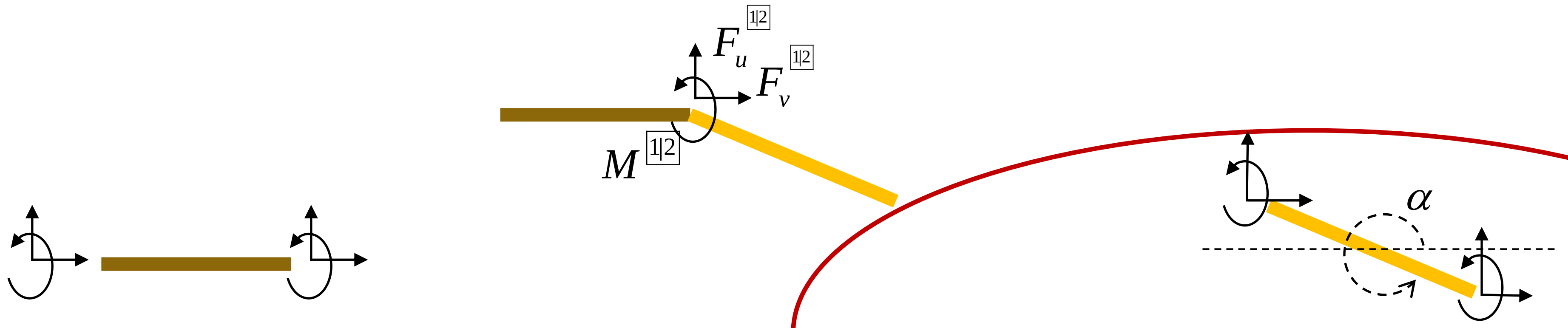
$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{1}} & \mathbf{M}_{lr}^{\boxed{1}} \\ \mathbf{M}_{rl}^{\boxed{1}} & \mathbf{M}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{1}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{1}} & \mathbf{K}_{lr}^{\boxed{1}} \\ \mathbf{K}_{rl}^{\boxed{1}} & \mathbf{K}_{rr}^{\boxed{1}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{1}} \\ \mathbf{u}_{r}^{\boxed{1}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{1}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{1}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{1}} \\ \mathbf{S}_{r}^{\boxed{1}} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{M}_{ll}^{\boxed{2}} & \mathbf{M}_{lr}^{\boxed{2}} \\ \mathbf{M}_{rl}^{\boxed{2}} & \mathbf{M}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{\boxed{2}} \\ \ddot{\mathbf{u}}_{r}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{\boxed{2}} & \mathbf{K}_{lr}^{\boxed{2}} \\ \mathbf{K}_{rl}^{\boxed{2}} & \mathbf{K}_{rr}^{\boxed{2}} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{\boxed{2}} \\ \mathbf{u}_{r}^{\boxed{2}} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\mathrm{ext}}^{\boxed{2}} \\ \mathbf{P}_{r,\mathrm{ext}}^{\boxed{2}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{\boxed{2}} \\ \mathbf{S}_{r}^{\boxed{2}} \end{bmatrix}$$

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!



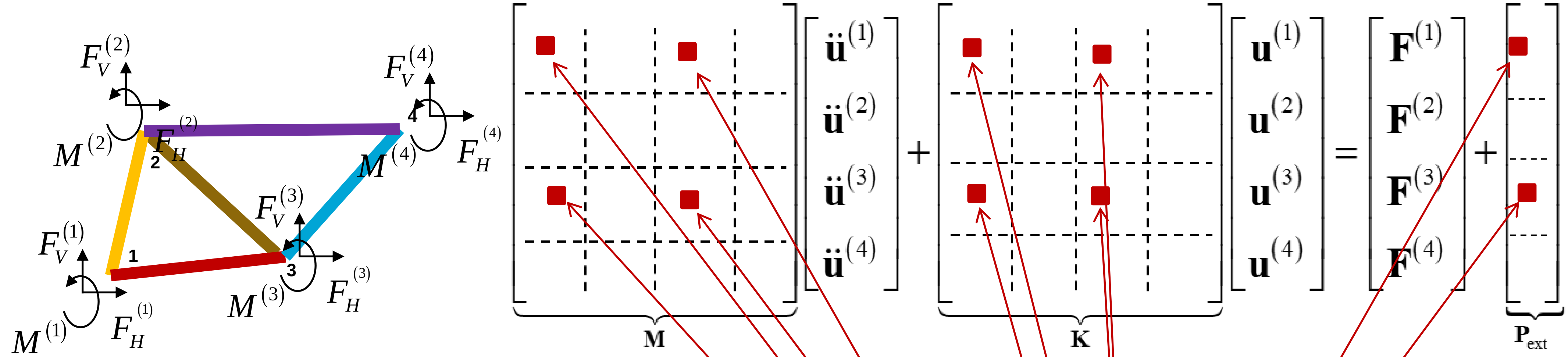
$$\begin{bmatrix} \mathbf{M}_{ll}^{(1)} & \mathbf{M}_{lr}^{(1)} \\ \mathbf{M}_{rl}^{(1)} & \mathbf{M}_{rr}^{(1)} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_{l}^{(1)} \\ \ddot{\mathbf{u}}_{r}^{(1)} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll}^{(1)} & \mathbf{K}_{lr}^{(1)} \\ \mathbf{K}_{rl}^{(1)} & \mathbf{K}_{rr}^{(1)} \end{bmatrix} \begin{bmatrix} \mathbf{u}_{l}^{(1)} \\ \mathbf{u}_{r}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}}^{(1)} \\ \mathbf{P}_{r,\text{ext}}^{(1)} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_{l}^{(1)} \\ \mathbf{S}_{r}^{(1)} \end{bmatrix}$$

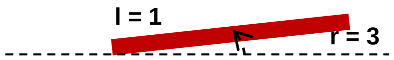
$$\begin{bmatrix} \tilde{\mathbf{M}}_{ll}^{(2)} & \tilde{\mathbf{M}}_{lr}^{(2)} \\ \tilde{\mathbf{M}}_{rl}^{(2)} & \tilde{\mathbf{M}}_{rr}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{(2)}\ddot{\mathbf{u}}_{l}^{(2)} \\ \mathbf{T}^{(2)}\ddot{\mathbf{u}}_{r}^{(2)} \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ll}^{(2)} & \tilde{\mathbf{K}}_{lr}^{(2)} \\ \tilde{\mathbf{K}}_{rl}^{(2)} & \tilde{\mathbf{K}}_{rr}^{(2)} \end{bmatrix} \begin{bmatrix} \mathbf{T}^{(2)}\mathbf{u}_{l}^{(2)} \\ \mathbf{T}^{(2)}\mathbf{u}_{r}^{(2)} \end{bmatrix} = \begin{bmatrix} \mathbf{T}^{(2)}\mathbf{P}_{l,\text{ext}}^{(2)} \\ \mathbf{T}^{(2)}\mathbf{P}_{r,\text{ext}}^{(2)} \end{bmatrix} + \begin{bmatrix} -\mathbf{T}^{(2)}\mathbf{S}_{l}^{(2)} \\ \mathbf{T}^{(2)}\mathbf{S}_{r}^{(2)} \end{bmatrix}$$

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!





$$\begin{bmatrix}\mathbf{M}_{ll} & \mathbf{M}_{lr}\\ \mathbf{M}_{rl} & \tilde{\mathbf{M}}_{rr}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_l\\ \ddot{\mathbf{u}}_r\end{bmatrix}+\begin{bmatrix}\mathbf{K}_{ll} & \mathbf{K}_{lr}\\ \mathbf{K}_{rl} & \mathbf{K}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{bmatrix}=\begin{bmatrix}\mathbf{P}_{l,\text{ext}}\\ \mathbf{P}_{r,\text{ext}}\end{bmatrix}+\begin{bmatrix}-\mathbf{S}_l\\ \mathbf{S}_r\end{bmatrix}\xrightarrow{\mathbf{T}}\begin{bmatrix}\tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr}\\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_l\\ \ddot{\mathbf{u}}_r\end{bmatrix}+\begin{bmatrix}\tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr}\\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{bmatrix}=\begin{bmatrix}\tilde{\mathbf{P}}_{l,\text{ext}}\\ \tilde{\mathbf{P}}_{r,\text{ext}}\end{bmatrix}+\begin{bmatrix}-\mathbf{S}_l\\ \mathbf{S}_r\end{bmatrix}$$

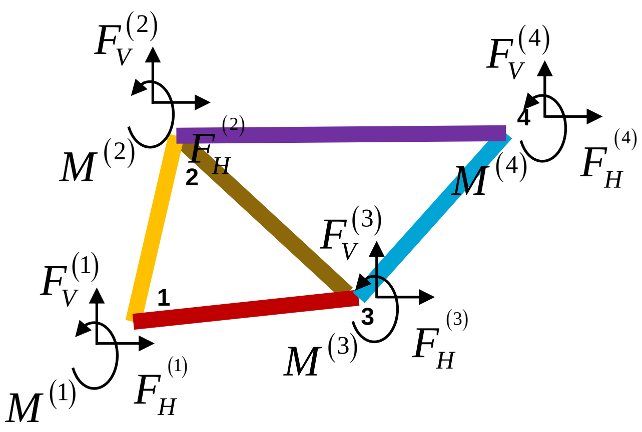
System in local coordinates

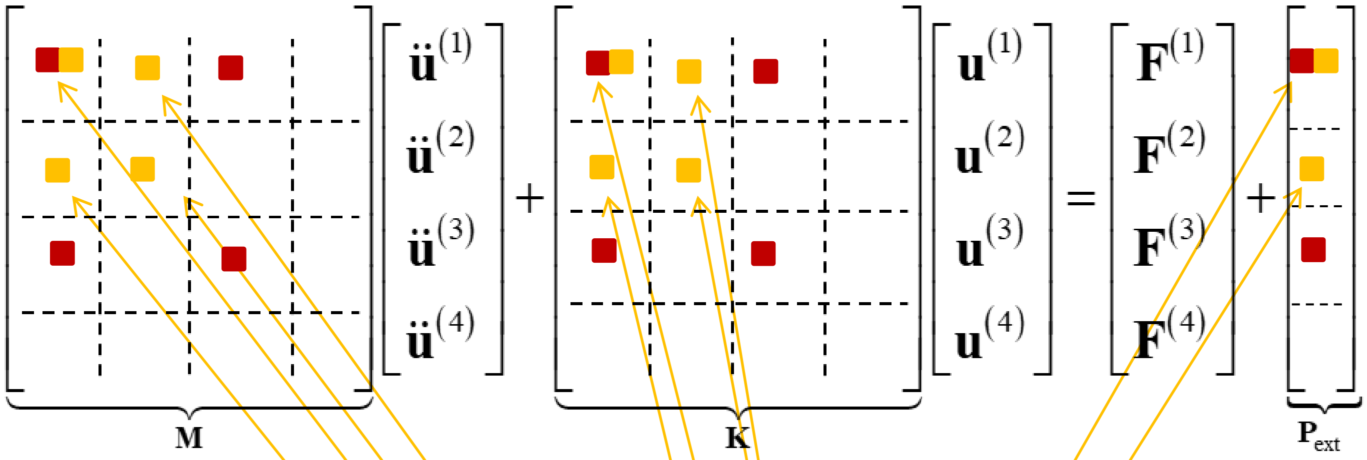
System in global coordinates

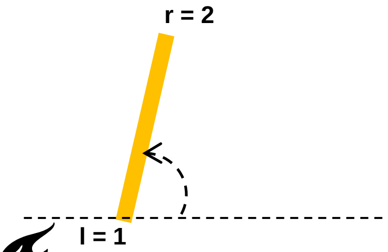
Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!







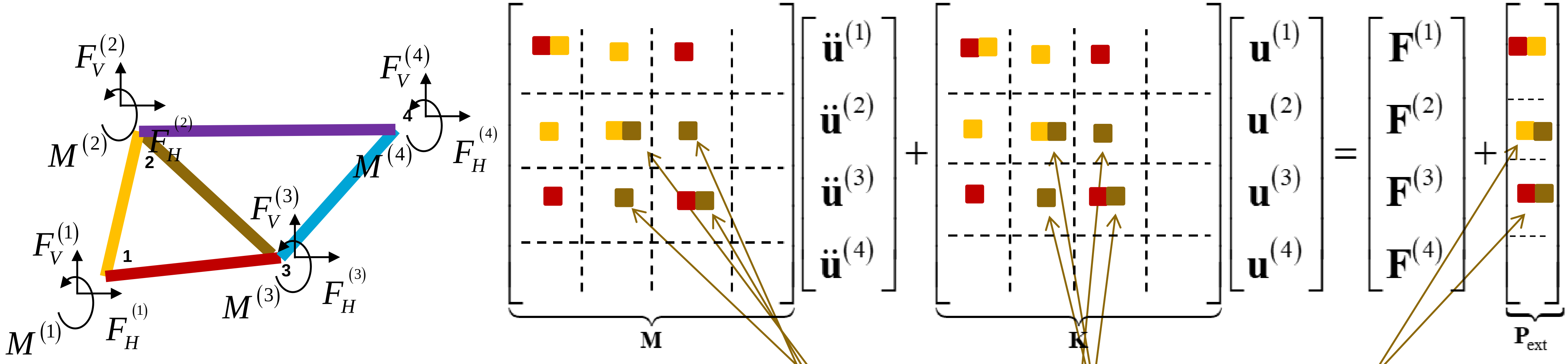
$$\begin{bmatrix} \mathbf{M}_{ll} & \mathbf{M}_{lr} \\ \mathbf{M}_{rl} & \tilde{\mathbf{M}}_{rr} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lr} \\ \mathbf{K}_{rl} & \mathbf{K}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}} \\ \mathbf{P}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix} \xrightarrow{\mathbf{T}} \begin{bmatrix} \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr} \\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr} \\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr} \end{bmatrix} \begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix}$$

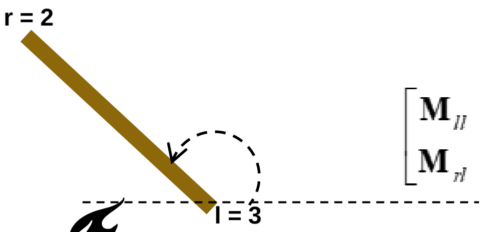
System in local coordinates → System in global coordinates

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!





$$\begin{bmatrix}\mathbf{M}_{ll} & \mathbf{M}_{lr}\\ \mathbf{M}_{rl} & \tilde{\mathbf{M}}_{rr}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_l\\ \ddot{\mathbf{u}}_r\end{bmatrix}+\begin{bmatrix}\mathbf{K}_{ll} & \mathbf{K}_{lr}\\ \mathbf{K}_{rl} & \mathbf{K}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{bmatrix}=\begin{bmatrix}\mathbf{P}_{l,\mathrm{ext}}\\ \mathbf{P}_{r,\mathrm{ext}}\end{bmatrix}+\begin{bmatrix}-\mathbf{S}_l\\ \mathbf{S}_r\end{bmatrix} \xrightarrow{\mathbf{T}} \begin{bmatrix}\tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr}\\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr}\end{bmatrix}\begin{bmatrix}\ddot{\mathbf{u}}_l\\ \ddot{\mathbf{u}}_r\end{bmatrix}+\begin{bmatrix}\tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr}\\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr}\end{bmatrix}\begin{bmatrix}\mathbf{u}_l\\ \mathbf{u}_r\end{bmatrix}=\begin{bmatrix}\tilde{\mathbf{P}}_{l,\mathrm{ext}}\\ \tilde{\mathbf{P}}_{r,\mathrm{ext}}\end{bmatrix}+\begin{bmatrix}-\mathbf{S}_l\\ \mathbf{S}_r\end{bmatrix}$$

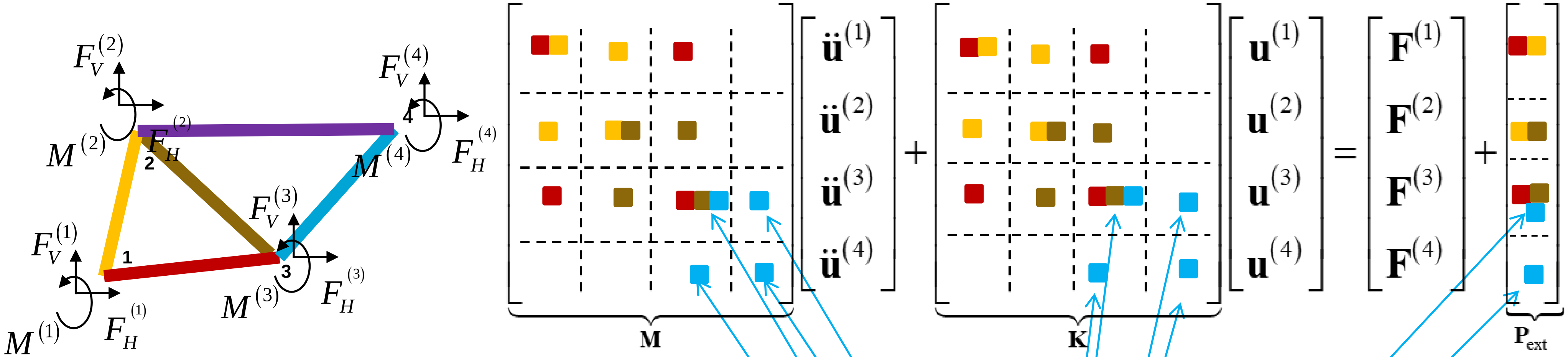
System in local coordinates

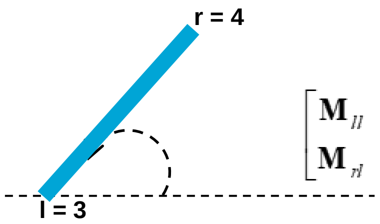
System in global coordinates

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!





$$\begin{bmatrix} \mathbf{M}_{ll} & \mathbf{M}_{lr} \\ \mathbf{M}_{rl} & \mathbf{M}_{rr} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lr} \\ \mathbf{K}_{rl} & \mathbf{K}_{rr} \end{bmatrix}\begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}} \\ \mathbf{P}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix} \xrightarrow{\mathbf{T}} \begin{bmatrix} \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr} \\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr} \\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr} \end{bmatrix}\begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix}$$

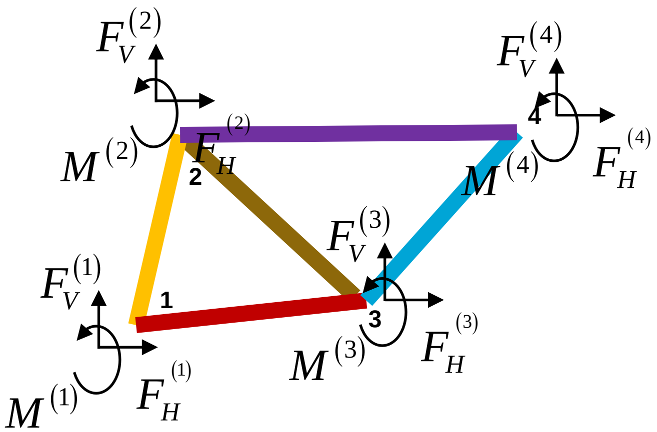
System in local coordinates

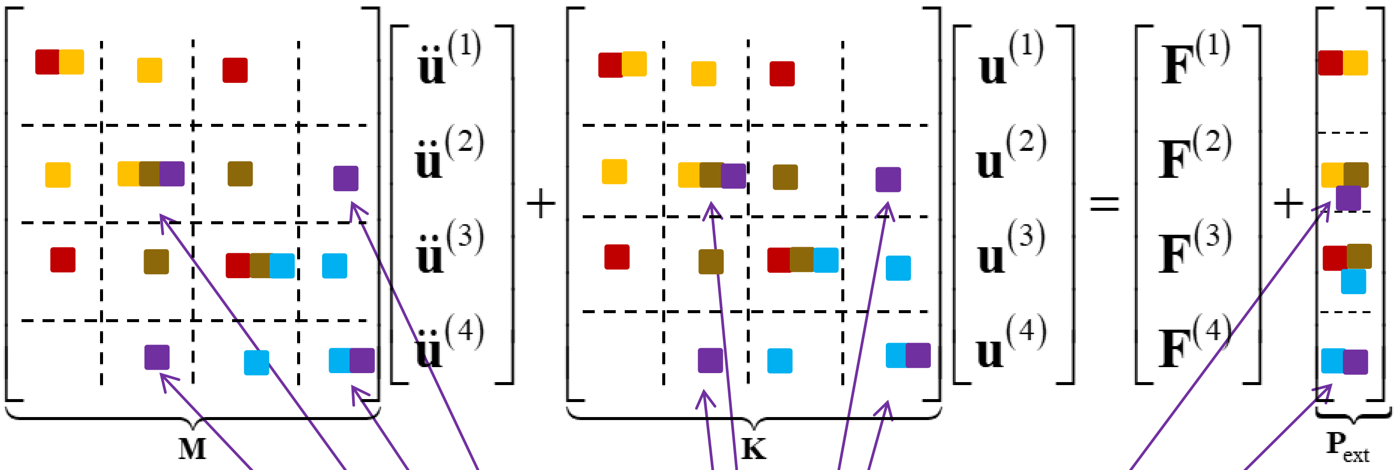
System in global coordinates

Finite Elements for a space frame structure

4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!





$$\begin{bmatrix} \mathbf{M}_{ll} & \mathbf{M}_{lr} \\ \mathbf{M}_{rl} & \tilde{\mathbf{M}}_{rr} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{ll} & \mathbf{K}_{lr} \\ \mathbf{K}_{rl} & \mathbf{K}_{rr} \end{bmatrix}\begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{l,\text{ext}} \\ \mathbf{P}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix} \xrightarrow{\mathbf{T}} \begin{bmatrix} \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr} \\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{u}}_l \\ \ddot{\mathbf{u}}_r \end{bmatrix} + \begin{bmatrix} \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr} \\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr} \end{bmatrix}\begin{bmatrix} \mathbf{u}_l \\ \mathbf{u}_r \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} \end{bmatrix} + \begin{bmatrix} -\mathbf{S}_l \\ \mathbf{S}_r \end{bmatrix}$$

System in local coordinates

System in global coordinates

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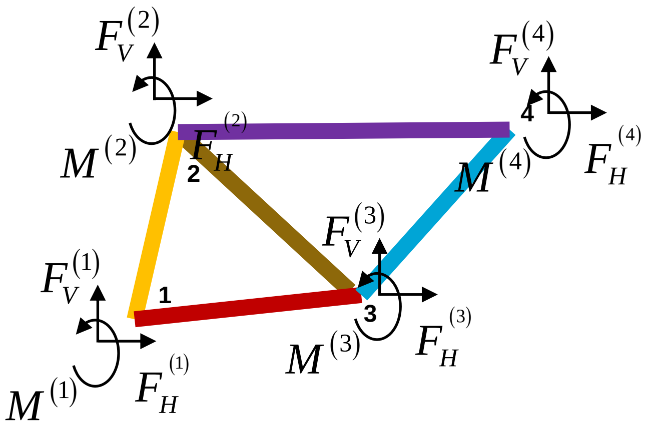
4. Assemble the global system

Same process as the axial rod and beam… But careful! The elements have different orientation!!!



$$\mathbf{M} = \begin{bmatrix} \tilde{\mathbf{M}}_{ll} + \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr} & \tilde{\mathbf{M}}_{lr} & \\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr} + \tilde{\mathbf{M}}_{rr} + \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{lr} \\ \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{lr} & \tilde{\mathbf{M}}_{rr} + \tilde{\mathbf{M}}_{ll} + \tilde{\mathbf{M}}_{ll} & \tilde{\mathbf{M}}_{lr} \\ & \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rl} & \tilde{\mathbf{M}}_{rr} + \tilde{\mathbf{M}}_{rr} \end{bmatrix}$$

$$\mathbf{K} = \begin{bmatrix} \tilde{\mathbf{K}}_{ll} + \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr} & \tilde{\mathbf{K}}_{lr} & \\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr} + \tilde{\mathbf{K}}_{rr} + \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{lr} \\ \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{lr} & \tilde{\mathbf{K}}_{rr} + \tilde{\mathbf{K}}_{ll} + \tilde{\mathbf{K}}_{ll} & \tilde{\mathbf{K}}_{lr} \\ & \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rl} & \tilde{\mathbf{K}}_{rr} + \tilde{\mathbf{K}}_{rr} \end{bmatrix}$$

$$\mathbf{P}_{\text{ext}} = \begin{bmatrix} \tilde{\mathbf{P}}_{l,\text{ext}} + \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} + \tilde{\mathbf{P}}_{r,\text{ext}} + \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} + \tilde{\mathbf{P}}_{l,\text{ext}} + \tilde{\mathbf{P}}_{l,\text{ext}} \\ \tilde{\mathbf{P}}_{r,\text{ext}} + \tilde{\mathbf{P}}_{r,\text{ext}} \end{bmatrix}$$

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The final recipe

1. Number all the nodes of your structure
2. Loop on the elements
   a. Define left and right node
   b. Calculate local matrices **K**, **M** and load vector **P**ext
   c. Calculate orientation angle (α)
   d. Calculate rotation matrix **T** and calculate global **K*** and **M*** matrices and global **P*** ext vector

$$\mathbf{M}^* = \mathbf{T}\,\mathbf{M}\,\mathbf{T}^T$$

$$\mathbf{K}^* = \mathbf{T}\,\mathbf{K}\,\mathbf{T}^T$$

$$\mathbf{P}^*_{ext} = \mathbf{T}\,\mathbf{P}_{ext}$$

   e. Assemble global matrices **M***, **K*** and vector **P***ext in “big matrices”
   f. The internal forces of each element will cancel with external applied forces, neighbouring elements and eventual reactions from supports, and thus are not needed for the final system of equations
3. Loop on nodes
   a. assemble applied external point forces on “big” F vector
   b. (attention: if forces applied at directly at the nodes have been already considered as forces applied at the elements, then skip this part)

Thank you for your attention

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