Week 2. Dynamics of rigid bodies

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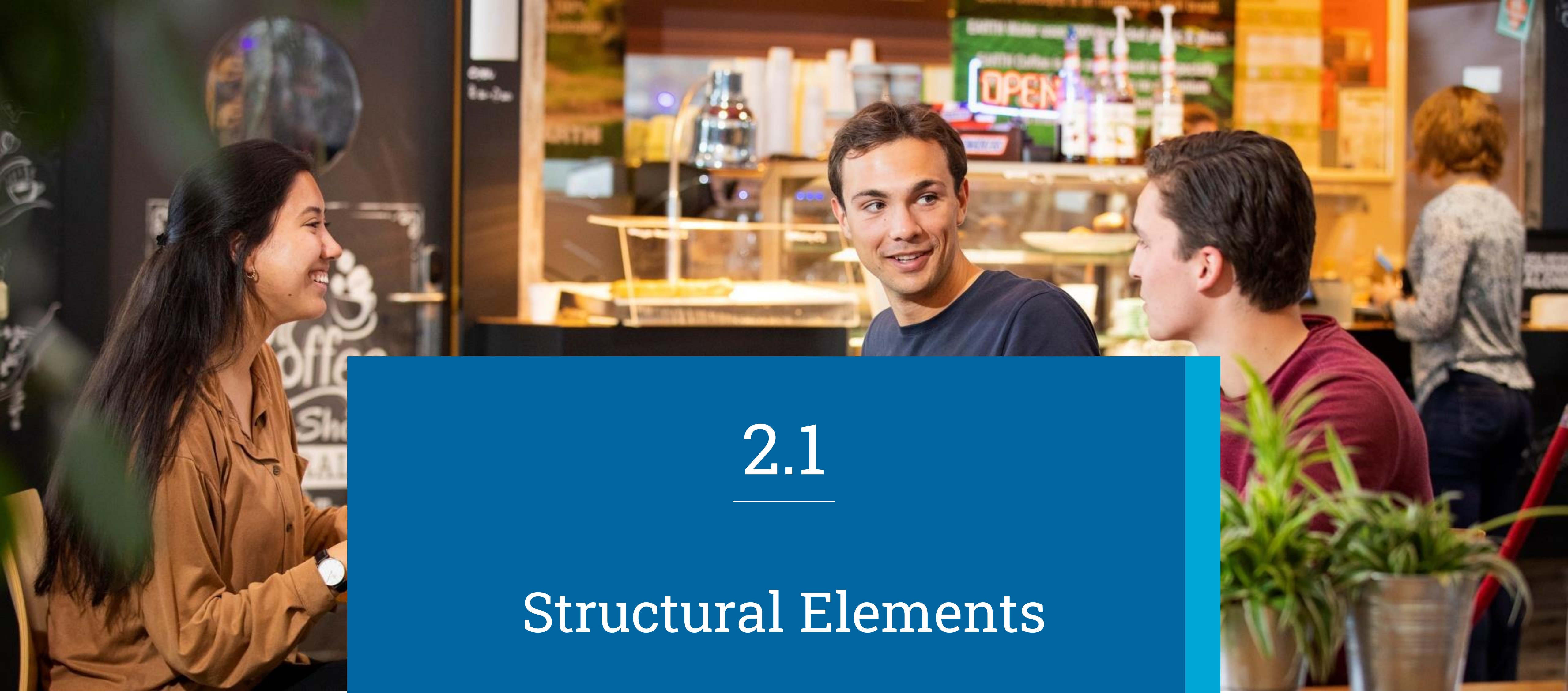
TU Delft

Learning objectives

At the end of this week you will be able to:

Define and analyze numerical methods to solve the dynamic motion of rigid body systems. This entails:

1. Characterize a structure as a set of point masses, rigid bodies, rods and beams interacting between each other
2. Define the Equations of Motion of a system through a Hamiltonian approach
3. Define the linearized Equation of Motion of a nonlinear system
4. Define numerical methods to solve a system of ODEs
5. Implement a solver for a system of ODEs
6. Analize and justify the results



2.1

Structural Elements

Types of structural elements

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- Point mass
- Rigid body
- Rods and bars
- Beams

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▪ Beams (Euler-Bernoulli)	Density (ρ), Young's modulus (E), cross section area (A) and inertia (I)

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Types of structural elements

In this unit we will deal with the following **structural connections**:

- Rigid connection
- (Elastic) hinge
- Springs (k)
- Dampers (c)

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Equations of motion

For a **point mass** we will typically consider the mass-damper-spring system:

$$m\ddot{x} + c\dot{x} + kx = F$$

For a **rigid body** we will include rotation, in 2D we will usually have 3 Degrees of freedom (DOFs). Assuming no coupling between DOFs, the Equation of motion (EOM) of a rigid body could be defined as:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} c_x & 0 & 0 \\ 0 & c_y & 0 \\ 0 & 0 & c_\theta \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} + \begin{bmatrix} k_x & 0 & 0 \\ 0 & k_y & 0 \\ 0 & 0 & k_\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ F_\theta \end{bmatrix}$$

Equations of motion

For a **rod**, the equation of motion is defined as (u is the rod elongation):

$$\rho A(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}\right) = p(x)$$

For an **Euler-Bernoulli beam**, the equation of motion is defined as (v is the beam deflection):

$$\rho A(x)\frac{\partial^2 v}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 v}{\partial x^2}\right) = q(x)$$

2.2

Lagrangian mechanics

Introduction

We want to obtain the **EOMs of an nDOF system** whose individual bodies have to satisfy certain constraints. For example, two rigid bodies that are connected (same displacement) at 1 point.

We want a general way of deriving EOM that could be used for all systems

Introduction

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Question: how?

Introduction

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We want a general way of deriving EOM that could be used for all systems

Question: how?

Answer: Using the Hamilton's principle and the Lagrange formalism (Euler-Lagrange equation).

Hamilton's principle

*"the dynamics of a physical system are determined by a variational problem for a functional based on a single function, **the Lagrangian**, which may contain all physical information concerning the system and the forces acting on it.*

*The variational problem is equivalent to and allows for the derivation of the differential **equations of motion of the physical system**."*

Let's start with some definitions:

$\boldsymbol{q}(t)$: Vector of degrees of freedom of a given system

$S(\boldsymbol{q})$: Action functional (takes a function and returns a scalar)

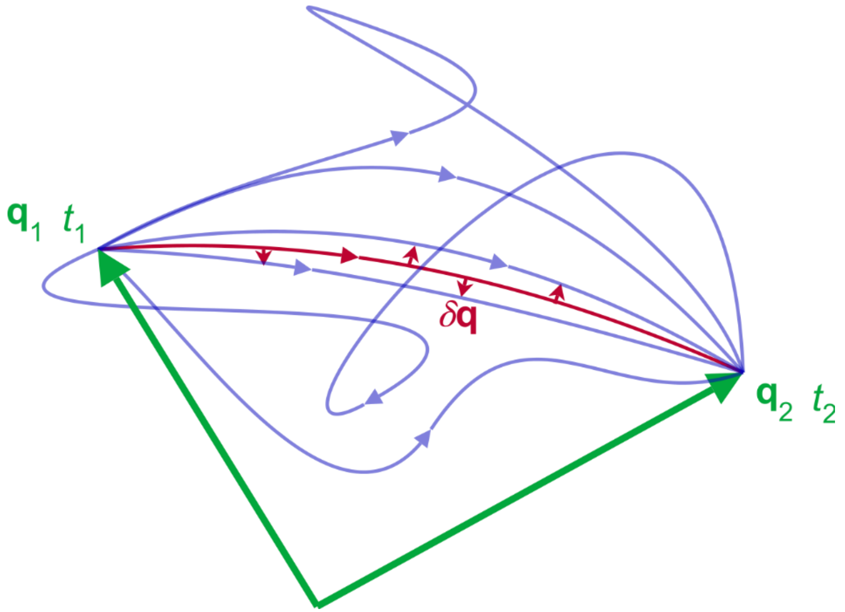
$L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)\big)$: Lagrangian function

$$S(\boldsymbol{q}) = \int_{t_1}^{t_2} L\big(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)\big)\, dt$$

Hamilton's principle

Hamilton's principle:

The true evolution $\boldsymbol{q}(t)$ of a system described by N generalized coordinates $\boldsymbol{q} = (q_1, q_2, \dots, q_N)$ between two specified states $\boldsymbol{q}_1 = \boldsymbol{q}(t_1)$ and $\boldsymbol{q}_2 = \boldsymbol{q}(t_2)$ at two specified times t_1 and t_2 is a stationary point (a point where the variation is zero) of the action functional S.



$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = 0$$

Euler-Lagrange equations

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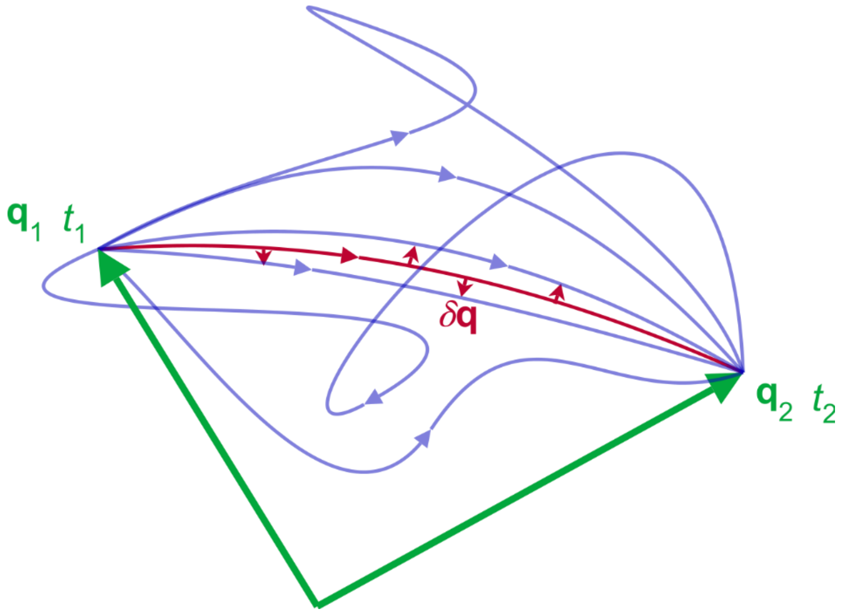

$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = 0 \Rightarrow \frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0$$

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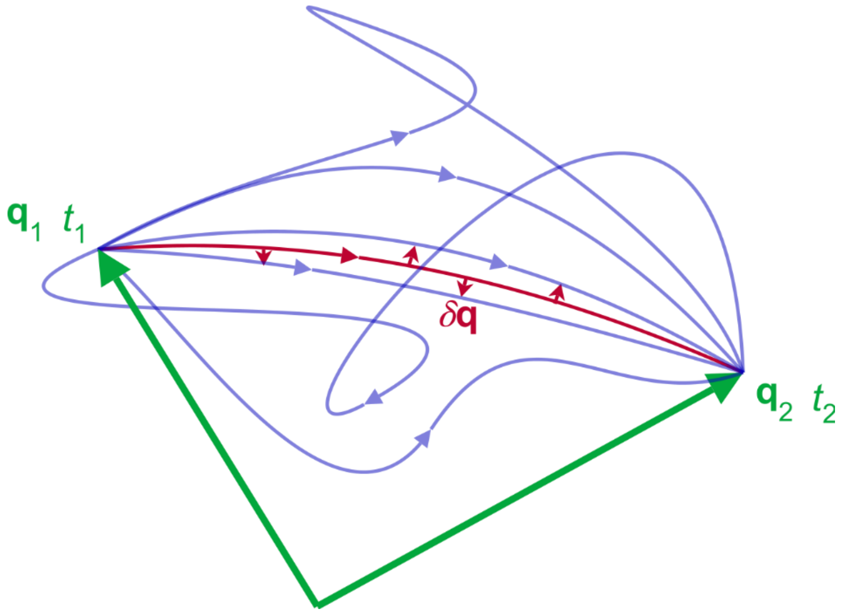


$$\frac{\delta S}{\delta \boldsymbol{q}(t)} = 0 \Rightarrow \underbrace{\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0}$$

Euler-Lagrange equations

(derivation in structural dynamics)

Euler-Lagrange equations

What is a Lagrangian?

The Lagrangian of a system, $\mathcal{L}$, is given by:

$$\mathcal{L} = T - V - W$$

Where T is the total kinetic energy of the system, V the total potential energy of the system and W the external work.

Euler-Lagrange equations

What is a Lagrangian?

The Lagrangian of a system, $\mathcal{L}$, is given by:

$$\mathcal{L} = T - V - W$$

Where T is the total kinetic energy of the system, V the total potential energy of the system and W the external work.

General strategy for constructing $\mathcal{L}$

1. **Kinematic equations**: describe the location (GCS) of each relevant point using the generalized coordinates of the system (DOFs).

   Relevant are: points that have mass, connection points, etc.

   If multiple bodies are connected to each other, express the location of consecutive points relative to each other.

2. **Energy equations**: write down the kinetic and potential energy of all elements.
3. Substitute the kinematic relations into the energy equations.
4. Done!

Euler-Lagrange equations

Deriving equations of motion from the Lagrangian:

The EOMs of a system can be obtained Euler-Lagrange equations:

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0$$

So, to derive the EOM we need two steps:

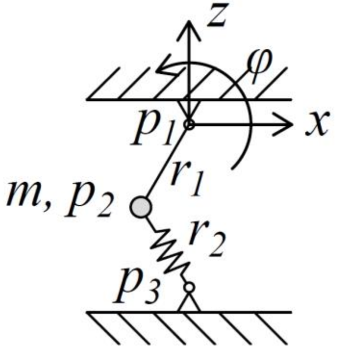
1. Construct L
2. Take derivatives.

For large systems, we will do the second step with Python/Maple (to simplify the process)

Example

Derive the EOM of the following system:

1. Construct the Lagrangian



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Example

Derive the EOM of the following system:

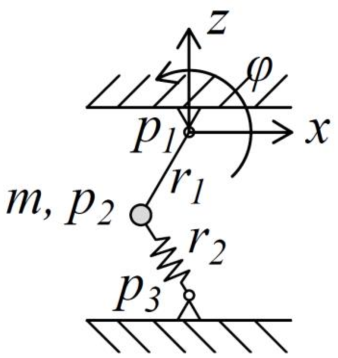
1. Construct the Lagrangian
   1. Kinematic realations:

$$x_2 = x_1 + r_1\cos(\varphi)$$
$$z_2 = z_1 + r_1\sin(\varphi)$$

$$\dot{x}_2 = \dot{x}_1 - r_1\sin(\varphi)\,\dot{\varphi}$$
$$\dot{z}_2 = \dot{z}_1 + r_1\cos(\varphi)\,\dot{\varphi}$$

   2. Energy equations:

$$T = \frac{1}{2}m|v|^2 = \frac{1}{2}m\left(\sqrt{\dot{x}_2^2 + \dot{z}_2^2}\right)^2 = \frac{1}{2}m(\dot{x}_2^2 + \dot{z}_2^2) = \frac{1}{2}mr_1^2\dot{\varphi}^2$$
$$V = \frac{1}{2}k\Delta l^2 = \frac{1}{2}k\left(\sqrt{(x_3 - x_2)^2 + (z_3 - z_2)^2} - r_2\right)^2$$



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Example

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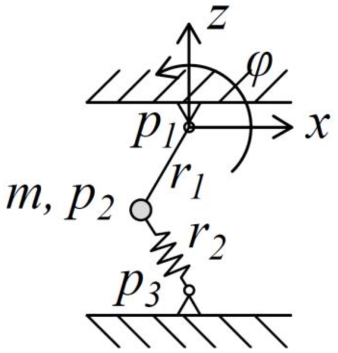

$$x_2 = x_1 + r_1\cos(\varphi)$$
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$$V = \frac{1}{2}k\Delta l^2 = \frac{1}{2}k\left(\sqrt{(x_3 - x_2)^2 + (z_3 - z_2)^2} - r_2\right)^2$$

   3. Lagrangian:

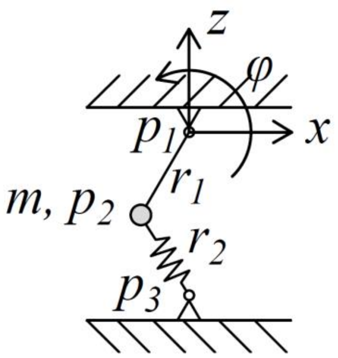
$$L = \frac{1}{2}mr_1^2\dot{\varphi}^2 - \frac{1}{2}k\left(\sqrt{(x_3 - x_2)^2 + (z_3 - z_2)^2} - r_2\right)^2$$

Example

Derive the EOM of the following system:

2. Take derivatives

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0$$



What Degrees of freedom do we have in this problem?

2.3

Linearization

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Linearization

In some (most) cases, the equations of motion of a system are nonlinear.

Question: Why would we be interested in linearizing a nonlinear system?

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In some (most) cases, the equations of motion of a system are nonlinear.

Question: Why would we be interested in linearizing a nonlinear system?

- Faster to compute responses
- Get natural frequencies
- Use a frequency domain approach
- …

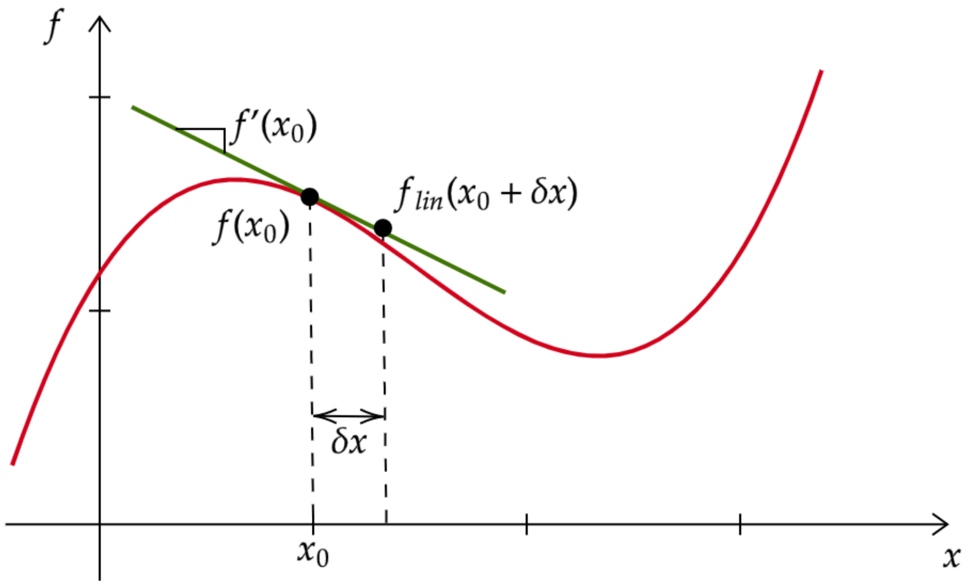
Linearization

Question: How do we linearize?

Answer: Taylor series, truncated at 1st order term

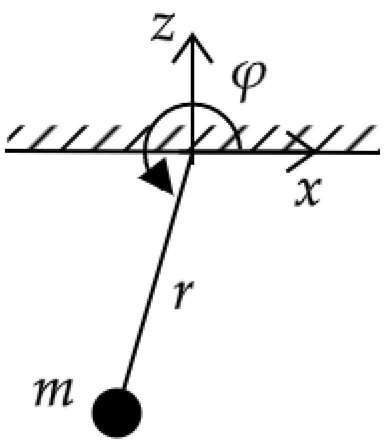
$$f(x) \approx f_{lin}(x) = f(x_0) + (x - x_0)f'(x_0)$$

$$f_{lin}(x_0 + \delta x) = f(x_0) + f'(x_0)\delta x$$



Example

Exercice: Find the equation of motion of a pendulum and its linearized version



Kinematic relations

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Energy equations:

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$$V = mgr\sin(\varphi)$$

Lagrangian:

$$L = \frac{1}{2}mr_1^2\dot{\varphi}^2 - mgr\sin(\varphi)$$

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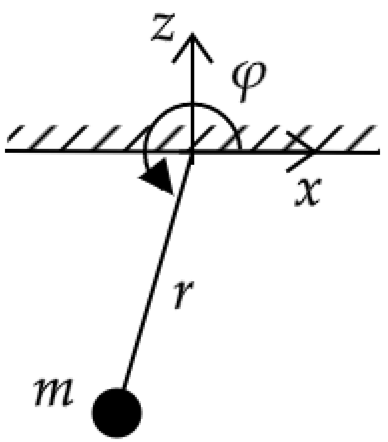

Euler-Lagrange equations

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0$$

$$\frac{\partial L}{\partial \varphi} = \frac{\partial\left(\frac{1}{2}mr^2\dot{\varphi}^2 - mgr\sin(\varphi)\right)}{\partial \varphi} = -\mathrm{mgr}\cos(\varphi)$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{\partial\left(\frac{1}{2}mr^2\dot{\varphi}^2 - mgr\sin(\varphi)\right)}{\partial \dot{\varphi}} = mr^2\dot{\varphi}$$

$$\ddot{\boldsymbol{\varphi}} = -\frac{\boldsymbol{g}}{\boldsymbol{r}}\,\mathbf{cos}(\boldsymbol{\varphi})$$

Example

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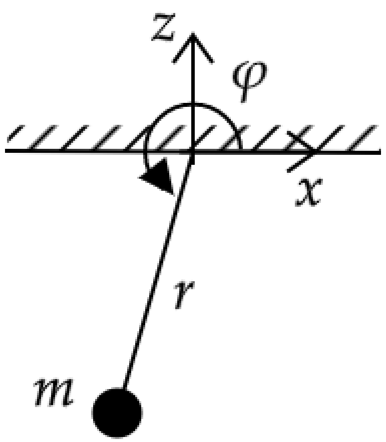

Now we want to linearize the EOM:

$$\ddot{\varphi} = -\frac{g}{r}\cos(\varphi) = \mathrm{f}(\varphi)$$

$$f(\varphi) \approx \tilde{f}(\varphi) = f(\varphi_0) + (\varphi - \varphi_0) f'(\varphi_0)$$

$$f'(\varphi_0) = \frac{g}{r}\sin(\varphi_0)$$

$$\tilde{f}(\varphi_0 + \delta\varphi) = -\frac{g}{r}\cos(\varphi_0) + \frac{g}{r}\sin(\varphi_0)\,\delta\varphi$$

If $\varphi_0 = \frac{3\pi}{2}$:

$$\ddot{\varphi} = f(\varphi) \approx -\frac{g}{r}\delta\varphi$$

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Thank you for your attention

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