Convex Analysis for Optimization - Exercise set 3

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Important note: you can certainly find the solutions to some of the below exercises. However, if you do so right at the start, you will not learn. So, try as much as you can on your own before searching for an existing solution.

Problem 1. Prove that the dual cone of any set $\emptyset \neq S \subseteq \mathbb{R}^n$ is indeed a cone and is convex, closed, and pointed. To prove the last fact easier, recall that d is a direction of recession of a non-empty closed convex set if and only if there is *one* point in the set from which we can move infinitely in the direction of d. Do convexity, closedness and pointedness hold for the polar cone and why?

Problem 2. Show that the cone

$$C = \left\{ x \in \mathbb{R}^{n+1} : \ x_{n+1} \geq \sqrt{\sum_{i=1}^{n} x_i^2} \right\}$$

is self-dual, i.e. that it is its own dual cone.

Problem 3. Prove that a closed convex set is an intersection of all closed half-spaces containing it.

Problem 4. Prove the following strong separation theorem. Let $S, \bar{S} \subseteq \mathbb{R}^n$ be nonempty and convex. There is a hyperplane strongly separating S and $\bar{S}$ if and only if $0 \notin \mathrm{cl}(S - \bar{S})$.

Problem 5. Show that if $f, g : \mathbb{R}^n \to \mathbb{R}$ are convex functions of one variable and g is nondecreasing, then $h(x) = g(f(x))$ is convex.

Problem 6. Show that $g(x) := \sup_{y \in Y} f(x, y)$ is convex if f is convex in x for all $y \in Y$. Does Y have to be convex to ensure the convexity of $g(x)$ and why?

Problem 7. Show that if a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{-\infty, +\infty\}$ is convex, then ~~if and only if~~ each of the level sets:

$$S_y = \{x \in \mathbb{R}^n : \ f(x) \leq y\}$$

is convex for every $y \in \mathbb{R}$. Give an example showing that the converse is not necessarily true.