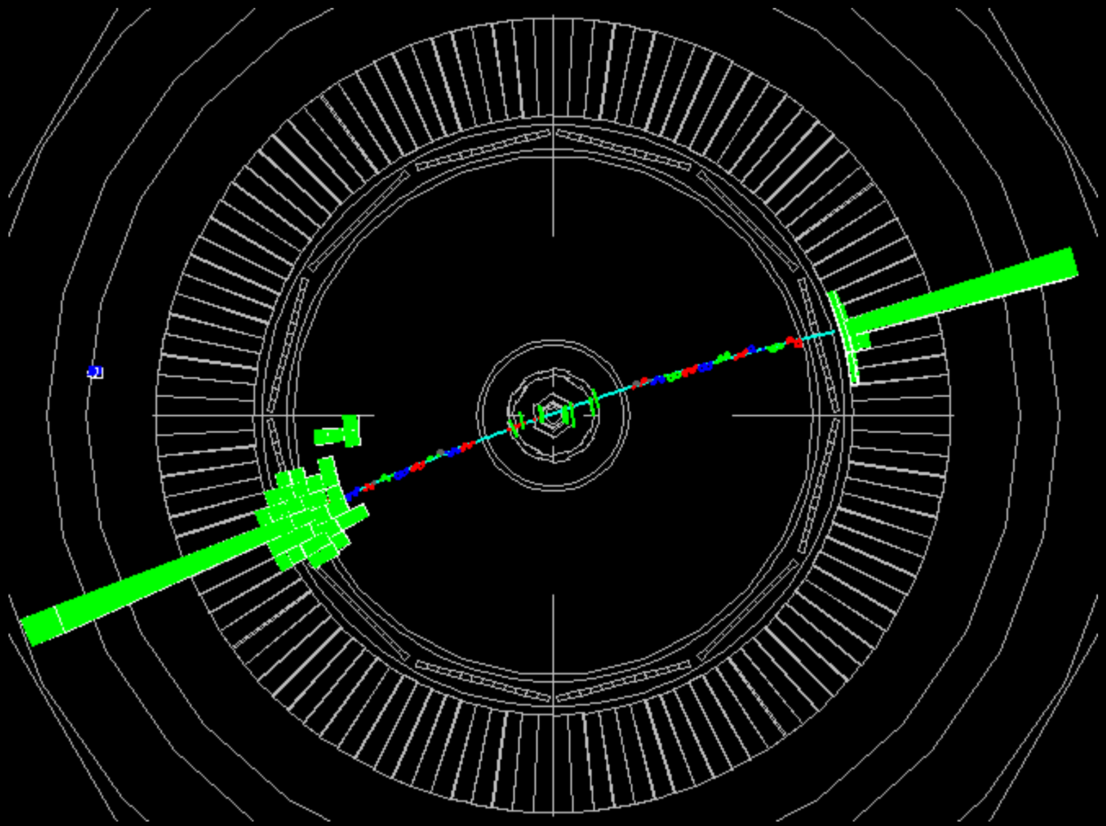
The golden rule of Fermi



Panos Christakoglou

Nikhef and Utrecht University

Particle decays

- Consider a number of independent particles, each having probability λ to decay
- The number of decayed particle within dt is given by

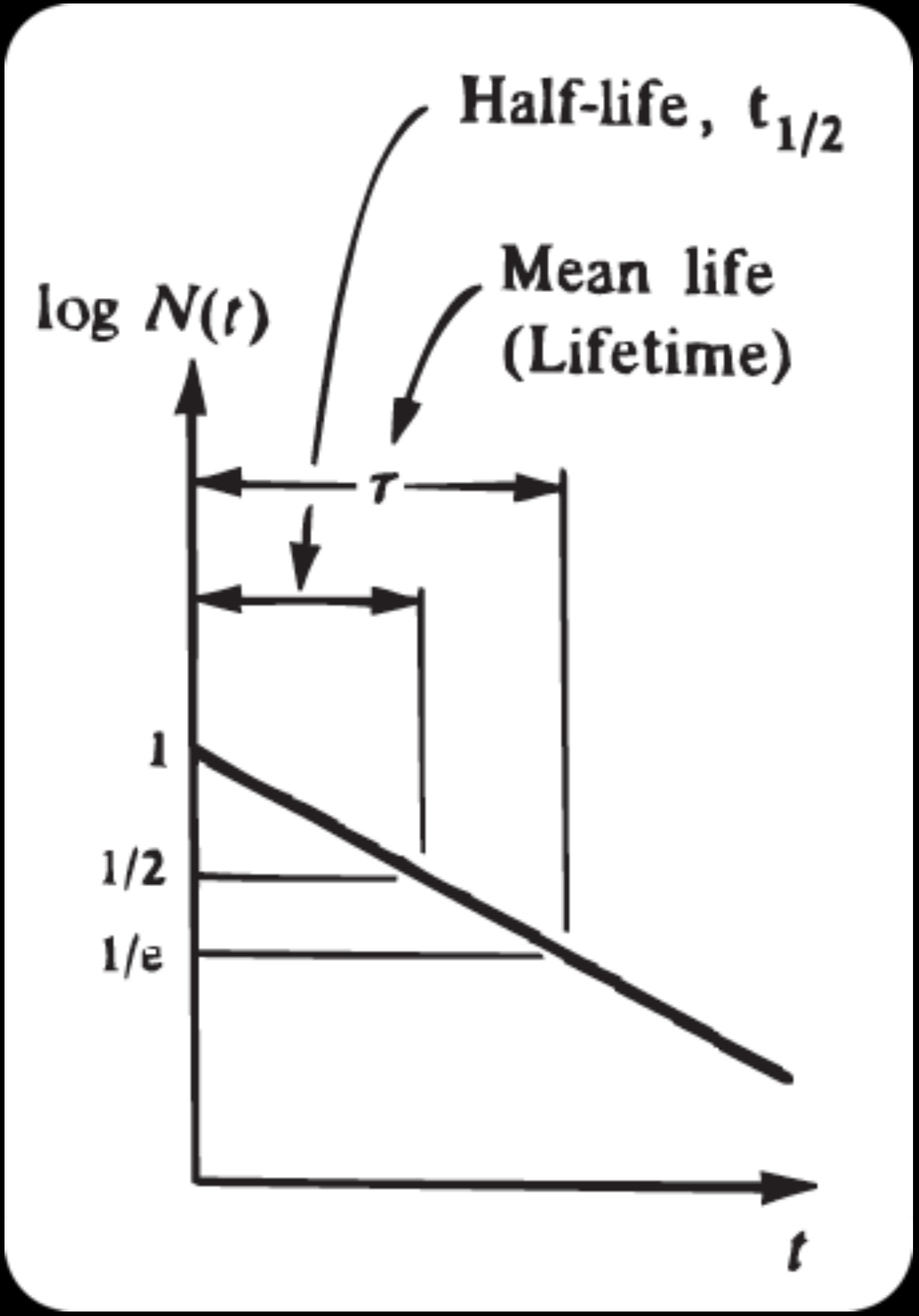
$$dN = -\lambda N(t)dt$$

$$\int_{N_0}^{N(t)} NdN = \int_0^t -\lambda dt$$

$$N(t) = N_0 e^{-\lambda t}$$

- Decay half life ($t_{1/2}$): the time it takes for half of the sample of particles to decay
- Mean lifetime (τ): The average time a particle exists before decaying

$$\tau = \frac{1}{\lambda} = \frac{t_{1/2}}{\ln 2}$$



Particle decays (cont.)

- The wave function of a particle at rest is given by

$$\psi(t)=\psi(0)e^{-iEt/\hbar}$$

- If the energy is real the probability of finding the particle is not time-dependent

$$|\psi(t)|^2=|\psi(0)|^2$$

- Allow the particle to decay, one has to introduce an imaginary part to the energy, such that

$$E=E_0-i\frac{\Gamma}{2}$$

- The probability of finding the particle becomes then

$$|\psi(t)|^2=|\psi(0)|^2e^{-\Gamma t/\hbar}$$

  - which agrees with the decay law for

$$\Gamma=\lambda\hbar$$

- The wave function is then given by

$$\psi(t)=\psi(0)e^{-iE_0t/\hbar}e^{-\Gamma t/2\hbar}$$

Particle decays (cont.)

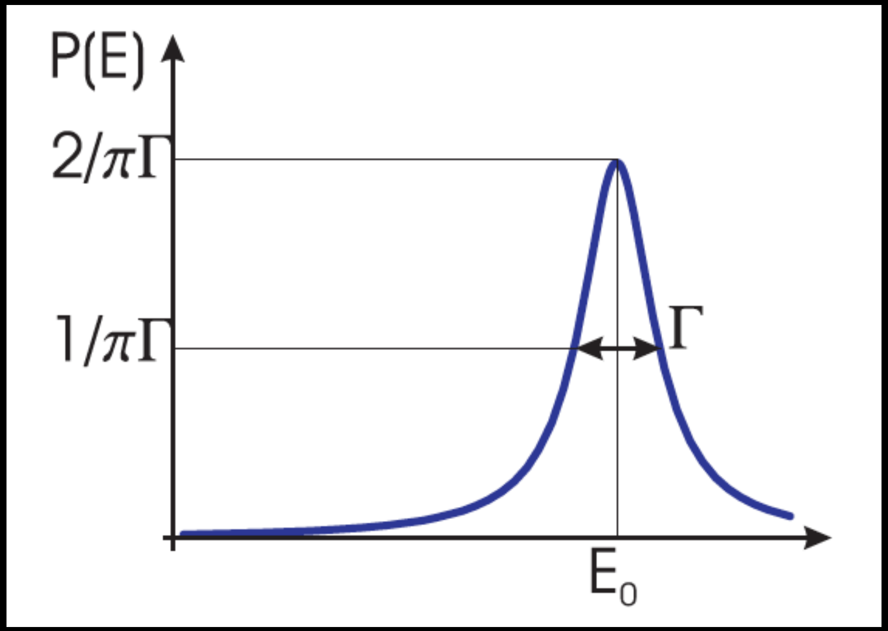
- The probability density of finding a particle with energy E is given by

$$P(E)=(const)\frac{\hbar^2}{2\pi}\frac{|\psi(0)|^2}{(E-E_0)^2+(\Gamma/2)^2}$$

$$\int_{-\infty}^{\infty}P(E)dE=1\longrightarrow(const)=\frac{\Gamma}{\hbar^2|\psi(0)|^2}$$

$$P(E)=\frac{\Gamma}{2\pi}\frac{1}{(E-E_0)^2+(\Gamma/2)^2}$$

- The energy of a decaying particle is not sharp but has a width ➡ **natural line width**
- The shape is called Breit-Wigner



- Γ is the full width half maximum (FWHM) $\tau\Gamma=\hbar$

Particle decays (cont.)

Particle	Mass (MeV/c^2)	Main Decays	Decay Energy (MeV)	Lifetime (sec)	Class
μ	106	$e\nu\bar{\nu}$	105	2.2×10^{-6}	W
$\pi^\pm$	140	$\mu\nu$	34	2.6×10^{-8}	W
π^0	135	$\gamma\gamma$	135	8.7×10^{-17}	EM
η	549	$\gamma\gamma,\pi\pi\pi$	549	6.3×10^{-19}	EM
ρ	769	$\pi\pi$	489	4.3×10^{-24}	H
n	940	$pe^-\bar{\nu}$	0.8	0.90×10^{3}	W
Λ	1116	$p\pi^-,n\pi^0$	39	2.6×10^{-10}	W
Δ	1232	$N\pi$	159	6×10^{-24}	H
$D^\pm$	1869	$\overline{K^0}+\cdots$		9.2×10^{-13}	W
D^0	1865	$K^\pm+\cdots$		4.3×10^{-13}	W
^{8}Be*	3726	2α	3	6×10^{-22}	H

- ✓ Strong interactions: ~10^{-23} s
- ✓ E/M interactions: ~10^{-18} s
- ✓ Weak interactions: ~10^{-10} s

Scattering theory

- Almost everything we know about nuclear and atomic physics has been discovered by scattering experiments,
  - Rutherford's discovery of the nucleus,
  - the discovery of sub-atomic particles (such as quarks), etc.

- In low energy physics, scattering phenomena provide the standard tool to explore solid state systems
  - neutron, electron, x-ray scattering, etc.

- As a general topic, it therefore remains central to any advanced course on quantum mechanics.

Scattering phenomena

- In an idealised scattering experiment, a sharp beam of particles (A) of definite momentum k are scattered from a localised target (B).

- As a result of collision, several outcomes are possible:

$$A+B \longrightarrow \begin{cases} A+B & \textit{(elastic)} \\ A+B+C \ \ (or \ \ A+B^{*}) & \textit{(inelastic)} \\ C & \textit{(absorption)} \end{cases}$$

- In high energy and nuclear physics, we are usually interested in deep inelastic processes.

- To keep our discussion simple, we will focus on elastic processes in which both the energy and particle number are conserved although many of the concepts that we will develop are general.

Differential cross section

- Both classical and quantum mechanical scattering phenomena are characterized by the scattering cross section, σ
- Consider a collision experiment in which a detector measures the number of particles per unit time, $N d\Omega$, scattered into an element of solid angle $d\Omega$ in direction (θ,φ).
- This number is proportional to the incident flux of particles, j_i, defined as the number of particles per unit time crossing a unit area normal to direction of incidence.
- Collisions are characterised by the differential cross section defined as the ratio of the number of particles scattered into direction (θ,φ) per unit time per unit solid angle, divided by incident flux

$$\frac{d\sigma}{d\Omega} = \frac{N}{j_i}$$

Total cross section

- ✓ From the differential, we can obtain the total cross section by integrating over all solid angles

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \sin\theta \, d\theta \, \frac{d\sigma}{d\Omega}$$

- ✓ The cross section, which typically depends sensitively on energy of incoming particles, has dimensions of area and can be separated into $\sigma_{elastic}$, $\sigma_{inelastic}$, $\sigma_{absorption}$, and σ_{total}.

- ✓ In the following, we will focus on elastic scattering where internal energies remain constant and no further particles are created or annihilated, e.g. low energy scattering of neutrons from protons.

- ✓ Let us consider classical scattering theory

Scattering phenomena: classical theory

- In classical mechanics, for a central potential, V(r), the angle of scattering is determined by the impact parameter b(θ)

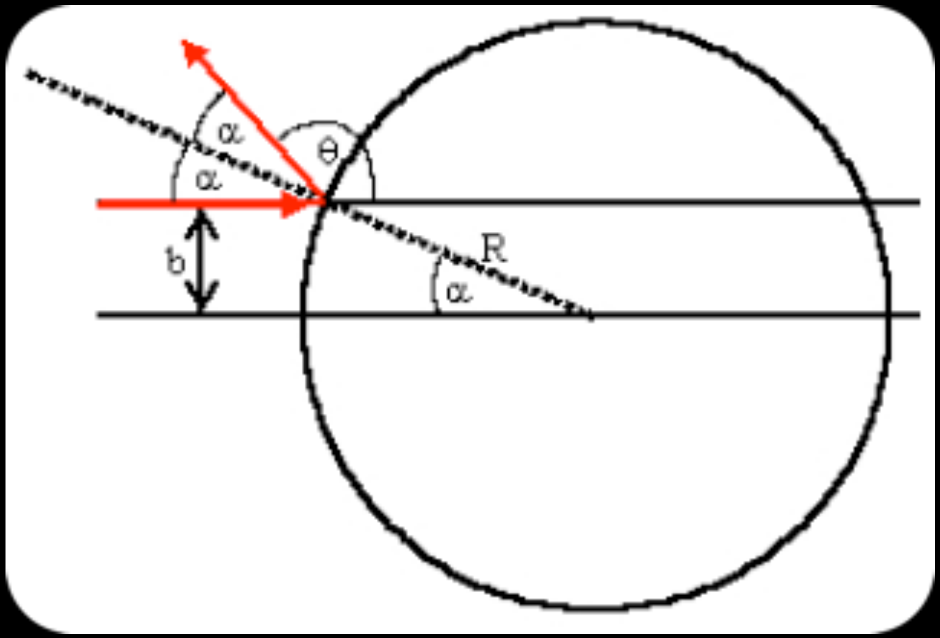
- The number of particles scattered per unit time between θ and (θ + dθ) is equal to the number incident particles per unit time between b and b + db.

- Therefore, for incident flux j_I, the number of particles scattered into the solid angle dΩ =2πsinθ dθ per unit time is given by

$$N\Omega = 2\pi \sin\theta \, d\theta N = 2\pi b\, db\, j_i$$

$$\frac{d\sigma}{d\Omega} \equiv \frac{N}{j_i} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

Scattering phenomena on a hard sphere



- For elastic scattering from a hard (impenetrable) sphere the differential cross section is given by

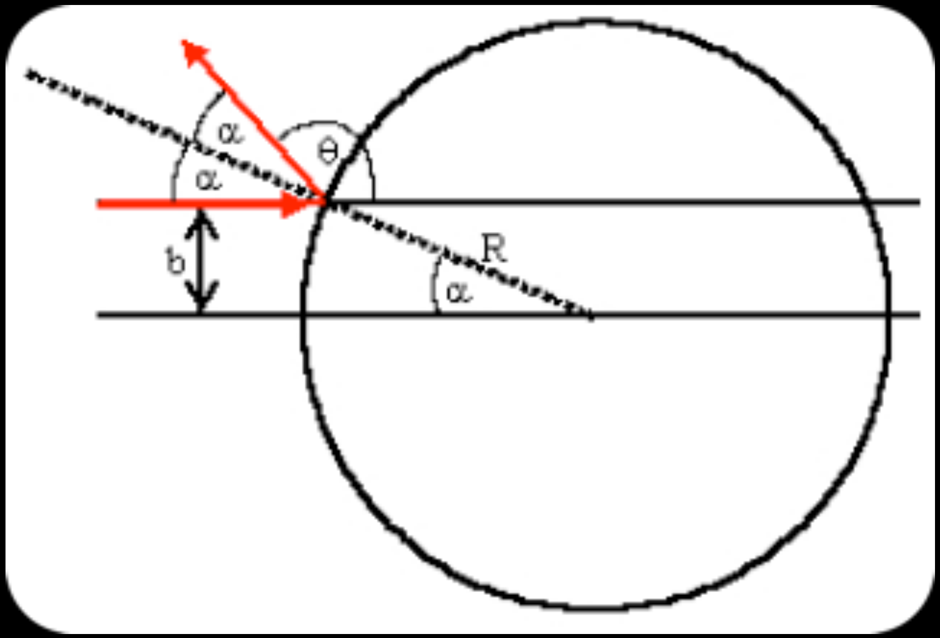
$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$

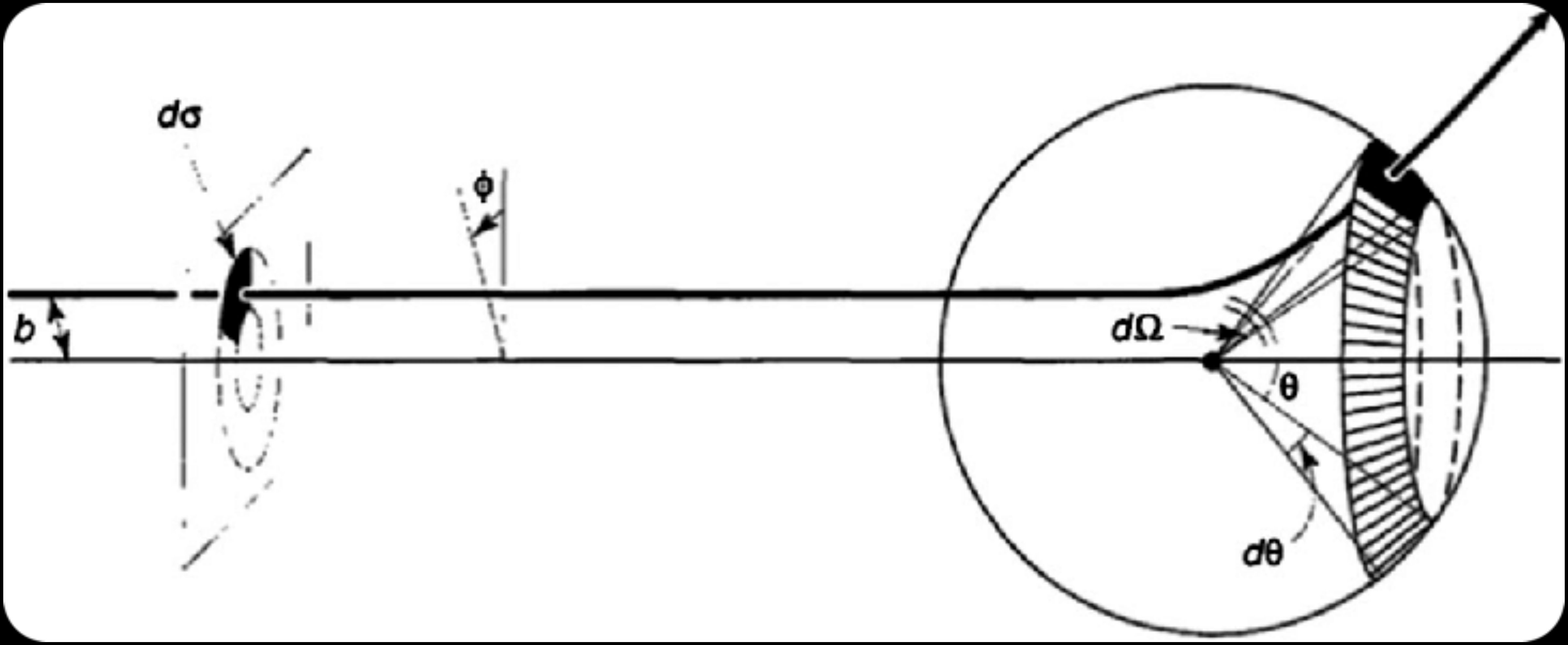
- And thus the total scattering cross section is

$$\int \frac{d\sigma}{d\Omega} d\Omega = \pi R^2$$

  - The projected area of the sphere

Scattering phenomena on a hard sphere





Panos.Christakoglou@nikhef.nl

The golden rule of Fermi

- For a given process, the physical quantity we would like to calculate in particle physics is its cross-section
- Experimentally this is done by placing detectors covering given stereo angles and measuring the particles emitted by such process
- Theoretically, this can be done with the help of the golden rule of Fermi
  - Two additional ingredients are needed
    - The amplitude (or matrix element) of a process
      - It contains all dynamical information originating from our theory (e.g. QED)
    - The phase space available (the density of final states)

$$\Gamma_{if} = \frac{2\pi}{\hbar} |M_{if}|^2 \rho(E)$$

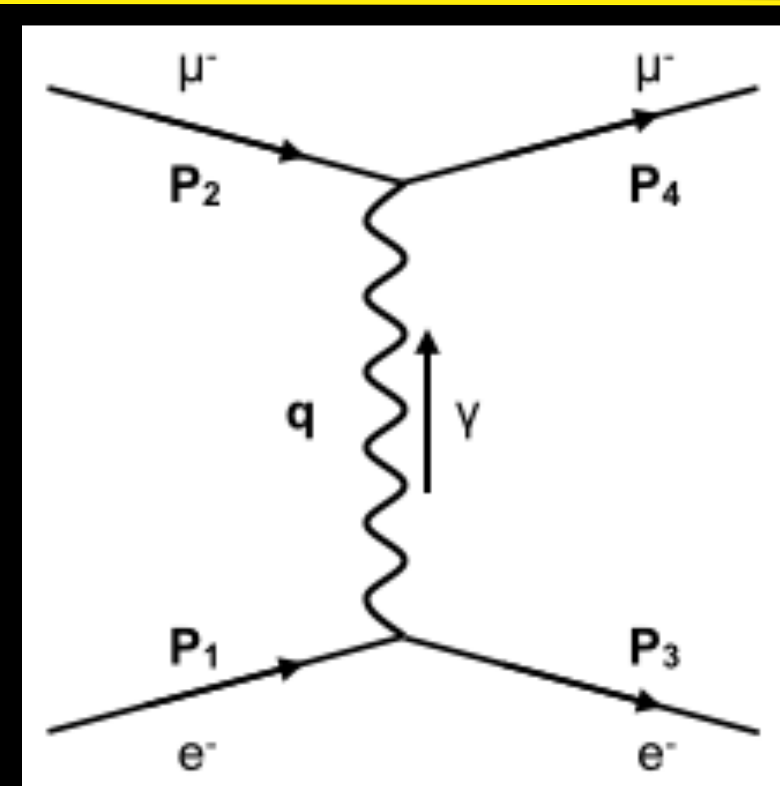
The golden rule of Fermi

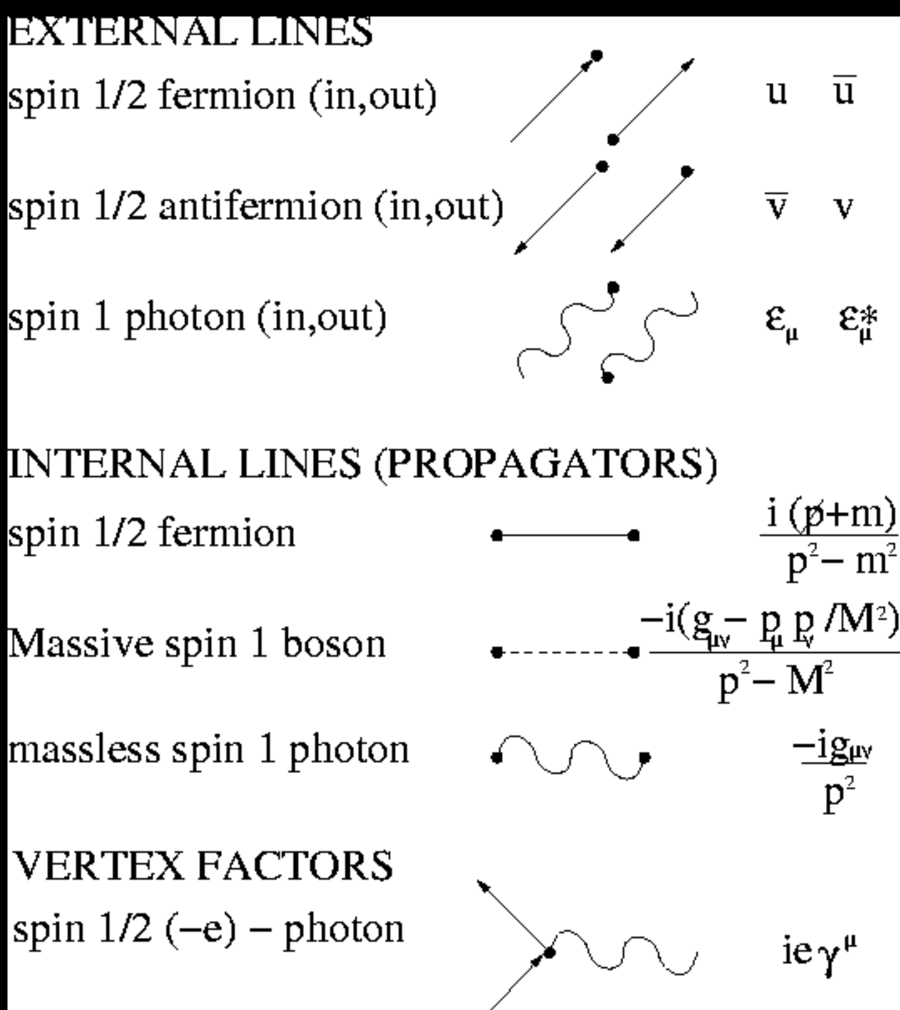
- The matrix element is calculated using Feynman diagrams and the corresponding Feynman rules
  - Specific rules for each of the gauge theories of the Standard Model
- The phase space factor is purely kinematic
  - It depends on the masses, 4-momenta of the particles involved in the process
  - It reflects the fact that an interaction is more likely to happen if there is enough “free space” in the final state
    - e.g. of enough “free space” in the final state: a decay of a heavy particle into a set of light daughters
    - Counterexample: $n \rightarrow p + e^{-} + \overline{\nu}_{e}$

$$\Gamma_{if} = \frac{2\pi}{\hbar} |M_{if}|^{2} \rho(E)$$

Feynman rules for QED

- A tool to describe interactions and calculate processes (e.g. cross-sections)
- Convenient set of rules to calculate the matrix element and the density of state factor





Following the Feynman rules for QED we have:

$$\int \left[\overline{u}(\mathbf{P_3})(ig_e\gamma^\mu)u(\mathbf{P_1})\right]\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left[\overline{u}(\mathbf{P_4})(ig_e\gamma^\nu)u(\mathbf{P_2})\right]\cdot\left[(2\pi)^4\delta^4(\mathbf{P_1}-\mathbf{P_3}-\mathbf{q})\right]\cdot\left[(2\pi)^4\delta^4(\mathbf{P_2}-\mathbf{P_4}+\mathbf{q})\right]\cdot\frac{d^4q}{(2\pi)^4}$$

$$= ig_e^2(2\pi)^4\int \overline{u}(\mathbf{P_3})\gamma^\mu u(\mathbf{P_1})\cdot\frac{g_{\mu\nu}}{q^2}\cdot\overline{u}(\mathbf{P_4})\gamma^\nu u(\mathbf{P_2})\cdot\delta^4(\mathbf{P_1}-\mathbf{P_3}-\mathbf{q})\cdot\delta^4(\mathbf{P_2}-\mathbf{P_4}+\mathbf{q})d^4q$$

But $\mathbf{P_1}-\mathbf{P_3}=\mathbf{q}$ and the previous can be written:

$$ig_e^2(2\pi)^4\overline{u}(\mathbf{P_3})\gamma^\mu u(\mathbf{P_1})\cdot\frac{g_{\mu\nu}}{(\mathbf{P_1}-\mathbf{P_3})^2}\cdot\overline{u}(\mathbf{P_4})\gamma^\nu u(\mathbf{P_2})\cdot\delta^4(\mathbf{P_1}+\mathbf{P_2}-\mathbf{P_3}-\mathbf{P_4})d^4q$$

To get the matrix element, one simply cancels the δ-function and gets rid of the imaginary factor:

$$M_{if}=\frac{-g_e^2}{(\mathbf{P_1}-\mathbf{P_3})^2}\left[\overline{u}(\mathbf{P_3})\gamma^\mu u(\mathbf{P_1})g_{\mu\nu}\overline{u}(\mathbf{P_4})\gamma^\nu u(\mathbf{P_2})\right]\Leftrightarrow$$

$$M_{if}=\frac{-g_e^2}{(\mathbf{P_1}-\mathbf{P_3})^2}\left[\overline{u}(\mathbf{P_3})\gamma^\mu u(\mathbf{P_1})\overline{u}(\mathbf{P_4})\gamma_\mu u(\mathbf{P_2})\right] \quad (3.2.1)$$

Feynman rules for QCD

- **Labeling:** We label every external line with the ingoing and outgoing momenta $\mathbf{P_1},\ldots,\mathbf{P_n}$, adding also an arrow indicating whether a particle is approaching or moving away from the vertex. If the diagram includes antiparticles, we still label them as particles but with the reverse direction of the arrow. We then label the 4-momenta for all internal lines $\mathbf{q_1},\ldots,\mathbf{q_j}$ and we give an arbitrary direction to the relevant arrow.
- **External lines:** Each external line contribute the following factors:

$$\text{Incoming quark} \to u^s \cdot c$$
$$\text{Outgoing quark} \to \overline{u}^s \cdot c^\dagger$$

$$\text{Incoming anti-quark} \to \overline{v}^s \cdot c^\dagger$$
$$\text{Outgoing anti-quark} \to v \cdot c$$

$$\text{Incoming gluon} \to \varepsilon_\mu \cdot a^\alpha$$
$$\text{Outgoing gluon} \to \varepsilon^*_\mu \cdot a^{\alpha *}$$

where u and v are the relevant Dirac spinors. In the previous c are the matrices that represent the colour:

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \text{for R}, \begin{pmatrix}0\\1\\0\end{pmatrix} \text{for G}, \begin{pmatrix}0\\0\\1\end{pmatrix} \text{for B}$$

and a are the 8-element column matrices, one for each gluon state (i.e. α goes from 1 to 8):

$$|1\rangle \equiv \begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix}, |2\rangle \equiv \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}, |3\rangle \equiv \begin{pmatrix}0\\0\\1\\0\\0\\0\\0\\0\end{pmatrix}, |4\rangle \equiv \begin{pmatrix}0\\0\\0\\1\\0\\0\\0\\0\end{pmatrix}, |5\rangle \equiv \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix}, |6\rangle \equiv \begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix}, |7\rangle \equiv \begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix}, |8\rangle \equiv \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

- **Vertices:** For each vertex we note down in the diagram the coupling constant factor $\approx g_s$. This factor is connected to the coupling constant via the equation

$$g_s = \sqrt{4\pi\alpha_s}$$

For a quark-gluon vertex (see fig. 3.4) the factor is of the form:

$$\frac{-ig_s}{2}\lambda^\alpha\gamma^\mu$$

where the parameters λ^α are the Gell-Man λ-matrices of SU(3).

For a 3-gluon vertex (see fig. 3.4) the factor is of the form:

$$-g_s f^{\alpha\beta\gamma}\left[g_{\mu\nu}(\mathbf{k_1}-\mathbf{k_2})_\rho + g_{\nu\rho}(\mathbf{k_2}-\mathbf{k_3})_\mu + g_{\rho\mu}(\mathbf{k_3}-\mathbf{k_1})_\nu\right]$$

,

where the factors $f^{\alpha\beta\gamma}$ are the structure constants of SU(3) and $\mathbf{k}_i$ are the 4-momenta of each internal line (with $i = 1,2,3$).

Finally, for a 4-gluon vertex (see fig. 3.4) the factor is of the form:

$$-ig_s^2\left[f^{\alpha\beta\eta}f^{\gamma\delta\eta}(g_{\mu\sigma}g_{\nu\rho}-g_{\mu\rho}g_{\nu\sigma}) + f^{\alpha\delta\eta}f^{\beta\gamma\eta}(g_{\mu\nu}g_{\sigma\rho}-g_{\mu\sigma}g_{\nu\rho}) + f^{\alpha\gamma\eta}f^{\delta\beta\eta}(g_{\mu\rho}g_{\nu\sigma}-g_{\mu\nu}g_{\sigma\rho})\right]$$

- **Propagators:** For each internal line, we give a factor of

$$q-\overline{q}: \frac{i(\not q + m)}{\mathbf{q}^2 - m^2}$$

$$\text{gluon}: -ig_{\mu\nu}\delta^{\alpha\beta}$$

where $\not q \equiv \gamma_\nu q^\nu$.

- **δ-functions and integration:** The remaining steps are identical as in the general rules described before.

Figure 3.4 presents the lines for the basic particles and anti-particles but also the propagators for the strong interactions.

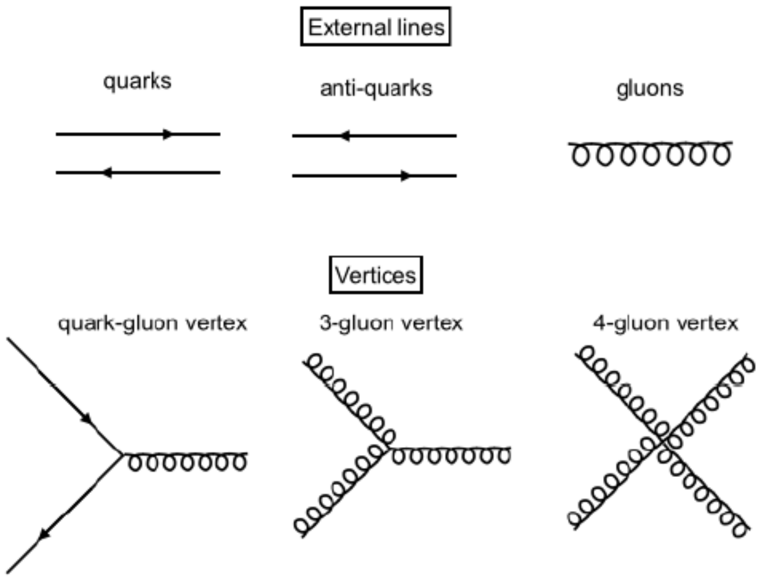


Fig. 3.4: The most characteristic lines for the Feynman diagrams in strong interactions.