TECHNOLOGICAL UNIVERSITY DELFT

DEPARTMENT OF AERONAUTICAL ENGINEERING

Report VTH-124

THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS WITH AND WITHOUT SUCTION

Ph.D THESIS

J.L. van INGEN

DELFT
the NETHERLANDS

OCTOBER, 1965

This PDF-file contains chapter 2:

The equations for two-dimensional laminar boundary layer flows

2. The equations for two-dimensional laminar boundary layer flows.

2.1. The Navier-Stokes equations.

Two-dimensional flows of an incompressible viscous fluid are governed by the Navier-Stokes equations and the continuity equation. Omitting body forces the equations may be written in cartesian coordinates as follows (see [7], chapter 3).

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{p}{x} + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial y}{\partial y} = -\frac{1}{\rho}\frac{p}{y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.3}$$

Equations (2.1) and (2.2) are the equations of motion in x and y direction respectively; (2.3) is the continuity equation. The notation is as usual: u and v are the velocity components in x and y direction, p is the pressure, ρ the density and ν the coefficient of kinematic viscosity. Throughout the present work ρ and ν are assumed to be constant. At the surface of a body placed in the flow the relative velocity vanishes. This leads to the usual boundary conditions that the normal and tangential components of the relative velocity vanish at the surface. In the present investigation problems with suction and blowing are considered so that a small normal component of the relative velocity at the surface will be allowed.

The Navier-Stokes equations are difficult to solve for flows around bodies of arbitrary shape. In a few cases where the geometry of the problem is very simple exact solutions show that for high values of the Reynoldsnumber $\frac{U_\infty c}{\nu}$ the effect of viscosity is confined to a narrow region near the surface called the boundary layer and a region behind the body called the wake. Within the boundary layer the relative velocity component tangential to the surface rises very fast from zero at the wall to a nearly constant value at a small distance from the wall.

This observation led Prandtl in 1904 [27] to his boundary layer theory which simplifies the Navier-Stokes equations by expressing the fact that there is a boundary layer of which the thickness is small compared to the body length.

2.2. Prandtl's boundary layer equations.

Prandtl's simplification of the Navier-Stokes equations leads to the following set of equations for the case of steady flow along a plane wall:

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \tag{2.4}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.5}$$

Here x and y are taken along and normal to the wall respectively. Eq.(2.4) is the boundary layer equation and results from (2.1). The equation of motion in y-direction (2.2) leads to the result that within the boundary layer $\frac{\partial p}{\partial y}$ can be neglected and hence for steady flow p only depends on x. The continuity equation (2.3) remains unchanged (2.5). A discussion of the boundary layer equations may be found in the books by Schlichting [7], Curle [28] and also in [29].

It can be shown that (2.4) and (2.5) are valid also for a two-dimensional curved body provided the radius of curvature is large compared to the boundary layer thickness and no rapid changes of curvature occur ([7], chapter 7). For curved bodies an orthogonal curvilinear coordinate system (x,y) should be used where x and y are measured parallel and normal to the wall respectively (fig. 2.1).

Outside the boundary layer the velocity gradient $\frac{\partial u}{\partial y}$ can be neglected and hence (2.4) reduces to:

$$U \frac{dU}{dx} = - \frac{1}{\rho} \frac{dp}{dx} \tag{2.6}$$

Using this, (2.4) may be written as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{2.7}$$

Together with the continuity equation (2.5) this equation determines the development of the boundary layer flow downstream of an initial station $x = x_o$ when the velocity profile at $x = x_o$ is known. Solutions of (2.5) and (2.7) are subject to the following boundary conditions:

$$y = 0: \qquad u = 0, \quad v = v_o(x) \tag{2.8}$$

$$y \rightarrow \infty: \qquad u \longrightarrow U \tag{2.9}$$

In boundary layer theory the velocity U at the edge of the boundary layer is assumed to be known either from a calculation using potential flow theory or from measurements.

The boundary conditions (2.8) imply that no oblique suction or blowing is considered.

Although the boundary layer equations are much simpler than the full Navier-Stokes equations, they can only be solved exactly for special types of the functions $U(x)$ and $v_o(x)$. Some of the available exact solutions will be reviewed in chapters 3 and 8.

The application of finite difference methods to obtain accurate numerical solutions has been limited in the past due to the large amount of work required. However, due to the introduction of high speed digital computors this situation has changed, so that now a number of accurate solutions has been made available. Some of these solutions will be discussed in chapter 8. In what follows both the exact solutions and accurate finite difference solutions of the boundary layer equations will be denoted as "exact" solutions. Approximate methods of solution have found a wide application in the past due to the difficulty of obtaining "exact" solutions. An important approximate method was introduced by Pohlhausen in 1921 [22] (see also [7] chapter 4). In methods of this type the boundary layer equations are not satisfied from point to point but relations are sought which fulfil certain more simple formulae derived from (2.5) and (2.7). Some of these formulae will be described in the remaining sections of the present chapter.

It should be stated in advance that these equations do not provide information which goes beyond the boundary layer equations; they only give a part of the information contained in the boundary layer equations in a different form.

2.3. Compatibility conditions of the boundary layer equations.

If the boundary conditions at the wall (2.8) are substituted into the boundary layer equation (2.7) the following result is obtained

$$v_o \left(\frac{\partial u}{\partial y}\right)_o = U \frac{dU}{dx} + \nu \left(\frac{\partial^2 u}{\partial y^2}\right)_o \tag{2.10}$$

where the subscript 0 denotes values at the wall (y = 0). Equation (2.10) is called the first compatibility condition at the wall; it relates the curvature of the velocity profile at the wall to the shear stress, pressure gradient and suction velocity. Compatibility conditions of higher order can be obtained by repeated differentiation of (2.7) with respect to y and using (2.5) and (2.8). The second compatibility condition thus obtained reads

$$v_o \left(\frac{\partial^2 u}{\partial y^2}\right)_o = \nu \left(\frac{\partial^3 u}{\partial y^3}\right)_o \tag{2.11}$$

and the third is found to be

$$\left(\frac{\partial u}{\partial y}\right)_o \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial y}\right)_o + v_o \left(\frac{\partial^3 u}{\partial y^3}\right)_o = \nu \left(\frac{\partial^4 u}{\partial y^4}\right)_o \tag{2.12}$$

2.4. Moments of the boundary layer equations.

The boundary layer equation (2.7) can be written symbolically as

$$F(x,y) = 0 \tag{2.13}$$

It follows that solutions of the boundary layer equations satisfy equations of the type:

$$\int_0^\infty F(x,y)\; G(x,y)dy = 0 \tag{2.14}$$

Where G(x,y) may be any function subject to the condition that the integral (2.14) exists. Relations of the form (2.14) are called moments of the boundary layer equations. Since a wide class of functions G may be used many different moments can be obtained.

Von Kármán's momentum equation. With $G(x,y) = 1$ the well known von Kármán momentum equation is found ([7], chapter 8). This equation can be written in the forms

$$\frac{d}{dx}(U^2\theta) = \frac{\tau_o}{\rho} + v_o U - U\delta^* \frac{dU}{dx} \tag{2.15}$$

and

$$\frac{U\theta}{\nu}\frac{d\theta}{dx} + (2+H)\frac{\theta^2}{\nu}\frac{dU}{dx} - \frac{v_o\theta}{\nu} = \frac{\tau_o\theta}{\mu U} \tag{2.16}$$

with τ_o, δ^*, θ and H denoting respectively

$$\tau_o = \mu\left(\frac{\partial u}{\partial y}\right)_o = \rho\nu\left(\frac{\partial u}{\partial y}\right)_o = \text{wall shear stress} \tag{2.17}$$

$$\delta^* = \int_o^\infty (1 - \frac{u}{U})dy = \text{displacement thickness} \tag{2.18}$$

$$\theta = \int_o^\infty \frac{u}{U}(1 - \frac{u}{U})dy = \text{momentum loss thickness} \tag{2.19}$$

$$H = \frac{\delta^*}{\theta} = \text{shape factor of the velocity profile} \tag{2.20}$$

Equation (2.15) was first obtained by von Kármán [30] as an equation expressing the momentum balance in the boundary layer. Later Pohlhausen [22] gave the derivation referred to above.

The kinetic-energy equation. With $G(x,y) = u$ equation (2.14) leads to:

$$\frac{d}{dx}(U^3\mathcal{E}) = v_o U^2 + D \tag{2.21}$$

(see for instance [7], chapter 8 and 13). Equation (2.21) is called the kinetic energy equation, while $\mathcal{E}$ and D denote the energy-loss thickness and dissipation integral. They are defined by

$$\mathcal{E} = \int_o^\infty \frac{u}{U}\left[1 - (\frac{u}{U})^2\right] dy \tag{2.22}$$

and

$$D = 2\nu\int_o^\infty \left(\frac{\partial u}{\partial y}\right)^2 dy \tag{2.23}$$

Usually from $\mathcal{E}$ and Θ a second shape factor $\bar{H}$ is defined by

$$\bar{H} = \frac{\mathcal{E}}{\Theta} \tag{2.24}$$

The kinetic-energy equation expresses the balance between mechanical energy and heat developed through frictional forces; it was first given by Leibenson [31] and later by Wieghardt [32].

Other moment equations can be derived for instance by taking $G(x,y) = u^k$ with $k > 1$. The resulting expression becomes

$$\frac{d}{dx}\left[U^{k+2}\ \delta_{k+2}\right] = v_o U^{k+1} + (k+1)\ U \frac{dU}{dx} \int_o^\infty \frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^{k-1}\right] dy - \nu (k+1) \int_o^\infty u^k \frac{\partial^2 u}{\partial y^2}\ dy \tag{2.25}$$

in which

$$\delta_{k+2} = \int_o^\infty \frac{u}{U}\left[1 - \left(\frac{u}{U}\right)^{k+1}\right] dy \tag{2.26}$$

It can easily be shown that equation (2.25) reduces to the momentum equation (2.15) for $k = 0$ and to the kinetic energy equation (2.21) for $k = 1$.

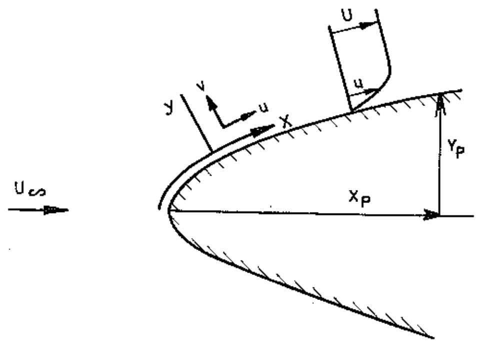

FIG. 2.1: COORDINATE SYSTEEM FOR BOUNDARY LAYER THEORY.