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THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS WITH AND WITHOUT SUCTION

Ph.D THESIS

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Approximate boundary layer calculations using moment equations and compatibility conditions

4. Approximate boundary layer calculations using moment equations and compatibility conditions.

4.1. General.

For bodies with arbitrary pressure- and suction distributions the similarity and series solutions as discussed in chapter 3 can not be used. In this case only the finite-difference methods can be applied to provide accurate solutions of the boundary layer equations. In the past however use of these methods on a large scale has been prohibited by the large amount of work required. Therefore approximate methods have been used to a great extent and possibly they will continue to be used in the future for technical applications. An important class of these methods is based on the von Kármán-Pohlhausen technique. In these methods the requirement that the boundary layer equations should be satisfied for every fluid element within the boundary layer is abandoned. Instead a plausible form of the velocity profile is assumed. This expression contains a few parameters to be chosen in such a way as functions of x that certain moment equations and compatibility conditions are satisfied. This technique will be illustrated in section 4.2. for the well known Pohlhausen method. In later sections of this chapter some other methods will be mentioned.

4.2. Pohlhausen's method.

In 1921 Pohlhausen published a method [22] which allowed the approximate calculation of laminar boundary layers without suction using the momentum equation (2.16). This method was considerably simplified by Holstein and Bohlen in 1940 [50]; a description of the modified method may be found in chapter 12 of Schlichting's book [7]. In what follows the main characteristics of this method will briefly be discussed.
In the Pohlhausen method the boundary layer thickness δ is assumed to be finite. The velocity profile is approximated by the following quartic polynomial in $\eta = y/\delta$.

$$\bar{u} = u/U = a\,\eta + b\,\eta^2 + c\,\eta^3 + d\,\eta^4 \qquad 0 \leqslant \eta \leqslant 1 \qquad (4.1)$$

$$\bar{u} = 1 \qquad \eta \geqslant 1 \qquad (4.2)$$

The coefficients δ, a, b, c and d are functions of x to be determined from the following relations:

the momentum equation (2.16) for $v_o = 0$ (4.3)

the first compatibility condition (2.10) for $v_o = 0$ (4.4)

the boundary conditions

$$\eta = 0 : \quad \bar{u} = 0 \tag{4.5}$$

$$\eta = 1 : \quad \bar{u} = 1, \quad \frac{\partial \bar{u}}{\partial \eta} = \frac{\partial^2 \bar{u}}{\partial \eta^2} = 0 \tag{4.6}$$

Using the conditions (4.4), (4.5) and (4.6) the coefficients a, b, c and d can be expressed in terms of a parameter λ defined by

$$\lambda = \frac{\delta^2}{\nu} \frac{dU}{dx} \tag{4.7}$$

The parameter λ is then found as function of x from the momentum equation. Equation (4.1) for the velocity profile can be written in the form

$$\bar{u} = F(\eta) + \lambda \; G(\eta) \tag{4.8}$$

with (for $0 \leqslant \eta \leqslant 1$)

$$F(\eta) = 1 - (1 + \eta)(1 - \eta)^3 \tag{4.9}$$

and

$$G(\eta) = \frac{1}{6} \eta (1 - \eta)^3 \tag{4.10}$$

Holstein and Bohlen use the parameter

$$\Lambda_1 = \frac{\theta^2}{\nu} \frac{dU}{dx} \tag{4.11}$$

instead of λ . This is attractive since Λ_1 occurs in the momentum equation; Λ_1 is directly related to λ . The shape of the velocity profile depends only on λ , and hence on Λ_1. Therefore the non-dimensional quantities $\ell = \frac{\tau_o \theta}{\mu U}$ and $H = \frac{\delta^*}{\theta}$ can be considered as given functions of Λ_1. Then, using the abbreviations given in the list of symbols, the momentum equation (2.16) may be written in the form

$$\frac{d\bar{\theta}^2}{d\bar{x}} = \frac{F(\Lambda_1)}{\bar{U}} \tag{4.12}$$

It follows that the boundary layer calculation is reduced to the solution of an ordinary differential equation. The function $F(\Lambda_1)$ is universal for Pohlhausen's method and is given as fig. 4.1; a table of $F(\Lambda_1)$ may be found in [7].

Normally the calculation is started in a stagnation point where $\bar{U} = 0$. To avoid an infinite value for $\frac{d\bar{\theta}^2}{d\bar{x}}$ it is assumed that in the stagnation point also $F(\Lambda_1) = 0$. Then $\frac{d\bar{\theta}^2}{d\bar{x}}$ assumes the undetermined value $\frac{0}{0}$; it can be made determinate using ℓ' Hopital's rule (see [7] , chapter 12).

An inspection of fig. 4.1 shows that $F(\Lambda_1)$ has a zero for $\Lambda_1 = 0.0770$ which can be chosen to represent the stagnation point. Other important points in fig. 4.1 are $\Lambda_1 = 0$ (flat plate) and $\Lambda_1 = -0.1567$ (separation, $\tau_o = 0$).

Walz [51] has been the first to notice that equation (4.12) can be integrated directly when the relation between $F(\Lambda_1)$ and Λ_1 is of the form

$$F(\Lambda_1) = a_1 - b_1 \Lambda_1 \tag{4.13}$$

Using (4.13) the result of the integration is

$$\bar{U}\,\bar{\theta}^2 = \frac{a_1}{\bar{U}^{b_1-1}} \int_o^{\bar{x}} \bar{U}^{b_1-1}\, d\bar{x} \tag{4.14}$$

where $\bar{x} = 0$ corresponds to the stagnation point. A reasonable approximation of $F(\Lambda_1)$ is obtained for a = 0.470 and b = 6 (see fig. 4.1). From applications of Pohlhausens method it is known that the results are reasonably accurate for favourable pressure gradients ($\Lambda_1 > 0$). However, for adverse pressure gradients ($\Lambda_1 < 0$) the accuracy is rather poor; in general the method predicts separation too late (see [28], chapter 5).

4.3. Other methods using the momentum equation and compatibility conditions.

Following Pohlhausen many authors developed similar methods using other compatibility conditions or different expressions for the velocity

profile. In this section some of these methods will be briefly described. The treatment cannot possibly be exhaustive due to the large number of methods available. Only the methods, to be referred to later, will be mentioned; extensive reviews may be found in [7, 28, 29].
The characteristic features of the methods to be described are collected in table 4.1; in what follows some additional remarks on these methods will be made.

Timman's method. [52] In this method the velocity profile is chosen in such a way that the right asymptotic behaviour for large values of y is obtained. Slight modifications have been introduced by Zaat [53] and Nunnink [54].
Schlichting's method. [55] Here the velocity profile is chosen in such a way that two important cases with and without suction are represented with good accuracy. For these cases the flat plate without suction and the asymptotic suction boundary layer were selected. The expression for the velocity profile, given in table 4.1, reduces to the asymptotic suction profile for $K = 0$ and to $\bar{u} = \sin(\frac{\pi}{6}\eta)$ for $K = -1$; the sine function is used as approximation to Blasius' velocity profile. A disadvantage of the method is that no unique solution is obtained near separation; to overcome this difficulty Schlichting had to introduce a rather arbitrary separation criterion. A critical review of Schlichting's method has been given by Truckenbrodt [56] who at the same time developed a different method.
A new method. The present author designed a method which may be considered as a further development of Schlichting's method. Here a third velocity profile - namely the separation profile of Timman's method - is introduced into the general expression of the velocity profile. A detailed discussion of the new method will be given in chapter 5. The method will be referred to as the "momentum method".
Thwaites' method. [57] An interesting type of method, valid for the no-suction case, has been given by Thwaites. The momentum equation is used - in a form similar to (4.12) - to find the non-dimensional momentum loss thickness $\bar{\Theta}$. It was observed by Thwaites that for this calculation

no necessity exists to specify the velocity profile in advance; all that is needed is a function similar to $F(\Lambda_1)$ of the Pohlhausen method. To obtain this function Thwaites plotted $F(\Lambda_1)$ versus Λ_1 for available exact solutions of the boundary layer equations and selected a linear relationship of the type (4.13) to represent mean values. The values $a_1 = 0.45$ and $b_1 = 6$ give a reasonably good average of the exact solutions. By plotting ℓ and H versus Λ_1 for exact solutions and deducing average curves Thwaites was able also to specify ℓ and H as functions of Λ_1. From the first compatibility condition at the wall (2.10) it follows that for the no-suction case $\Lambda_1 = -m$. Hence, once $\bar{\theta}$ and Λ_1 are known from the momentum equation as functions of $\bar{x}$ also ℓ, m and H are known. Then, if needed, a velocity profile can be composed which has the right values for ℓ, H and m.
A slight modification of the method has been introduced by Curle and Skan [58]. Due to lack of exact solutions for cases with suction the method cannot be generalised easily to suction problems.

4.4. Methods using the kinetic energy equation in addition to the momentum equation.

The approximate methods, using only the momentum equation, described in section 4.3, do not always give an accurate description of the boundary layer especially near the separation point. To improve upon this, methods have been devised which use the kinetic energy equation (2.21) in addition to the momentum equation. Such methods have been given for instance by Walz [59], Tani [60], Wieghardt [32], Truckenbrodt [61] and most recently by Head [62, 63, 64]. Reviews of these methods may be found in [28] and [29].
The method of Head seems to be the most accurate. In this method the momentum equation (2.15), the kinetic energy equation (2.21) and the first compatibility condition (2.10) are used. A wide range of velocity profiles is defined graphically from which relations between the characteristic boundary layer parameters H, $\bar{H}$, $2D^*$, ℓ and m are derived. These relations are plotted in charts to be used for the boundary layer calculations. Available results of the method show a good agreement with

exact solutions. A disadvantage is the use of charts which makes it somewhat difficult to program the method for automatic computation.

4.5. Possible methods using moment equations of higher order.

It is a disadvantage of all the approximate methods mentioned so far that the accuracy can only be assessed by comparison with exact solutions. In the no-suction case a sufficient number of exact solutions is available for this purpose but the situation is different for suction boundary layers. In the latter case the number of available exact solutions is too small to provide a good check. Such a check is necessary however since a method which works well in the no-suction case will not necessarily be satisfactory in the case of suction. This is caused by the fact that for suction boundary layers a far larger variety of velocity profiles has to be included than in the case of no-suction.
A striking example is given by the Pohlhausen method. If in this method the momentum equation and compatibility condition are modified to include the effect of suction it is found that a complex boundary layer thickness is predicted for the asymptotic suction profile.
In order to acquire confidence in the approximate methods it should be possible to estimate their accuracy without making reference to exact solutions.
The improvement obtained by the use of the kinetic energy equation in addition to the momentum equation suggests that such a method might be constructed by using a whole series of moment equations as defined by equation (2.25) for $k = 0,1,2,..,K$. Then it can be expected that the results obtained converge to the exact solution for $K \to \infty$. As far as the author is aware no succesful method has been developed along these lines. The practical application of such a method will be cumbersome for large values of K. To see this let the velocity profile be defined by

$$\frac{u}{U} = \bar{u} = \sum_{n=0}^{N} a_n F_n(y) \qquad (4.15)$$

The δ_{k+2} occurring in the moment equations (2.25) then are algebraic expressions of degree k+2 in the coefficients a_n defined by equation

(2.26). The step by step solution of the moment equations requires the determination of the a_n once the δ_{k+2} are known for all values of k to be used. This leads to the solution of a set of non-linear algebraic equations in a_n. For K=0 essentially the Pohlhausen method appears which in its simplest form requires the solution of one quadratic equation. For K=1 a method like Head's is obtained, which requires the simultaneous solution of a quadratic and a cubic equation. In a method for which K=2 a quartic equation would be added, etc. This situation makes the application of this method difficult for large values of K.

To obtain a workable method the moments should be defined in such a way that the moment equations can be written in the form

$$\frac{d\ J_k}{dx} = M_k \qquad (4.16)$$

where the J_k are linear functions of the parameters specifying the velocity profile. In this case the step by step calculation requires only the solution of a set of linear algebraic equations.

In chapter 7 a method will be described which is designed along these lines. From applications of this method, to be given in chapter 8, it appears that the results converge to the exact solution when the number of moment equations is increased.

Table 4.1: Characteristic features of some approximate methods.

Author and ref.	Expression for the velocity profile	Definition of η	Compatibility conditions used
Timman, [52]	$\frac{u}{U} = 1-e^{-\eta^2} (b + d\eta^2 + ...) - \int_{\eta}^{\infty} e^{-\eta^2} (a+c\eta^2+...)d\eta$	$\eta = \alpha y$; α^{-1} is related to the boundary layer thickness	first (eq. 2.10), second (eq. 2.11), and to some extent the third (eq. 2.12)
Schlichting, [55]	$\frac{u}{U} = F_1(\eta) + K F_2(\eta)$; $F_1(\eta) = 1-e^{-\eta}$ $F_2(\eta) = F_1(\eta) - \sin(\frac{\pi}{6}\eta)$ for $0 \leq \eta \leq 3$ $F_2(\eta) = F_1(\eta) - 1$ for $\eta \geq 3$	$\eta = \frac{y}{\delta_1(x)}$; $\delta_1(x)$ is related to the boundary layer thickness	first (eq. 2.10)

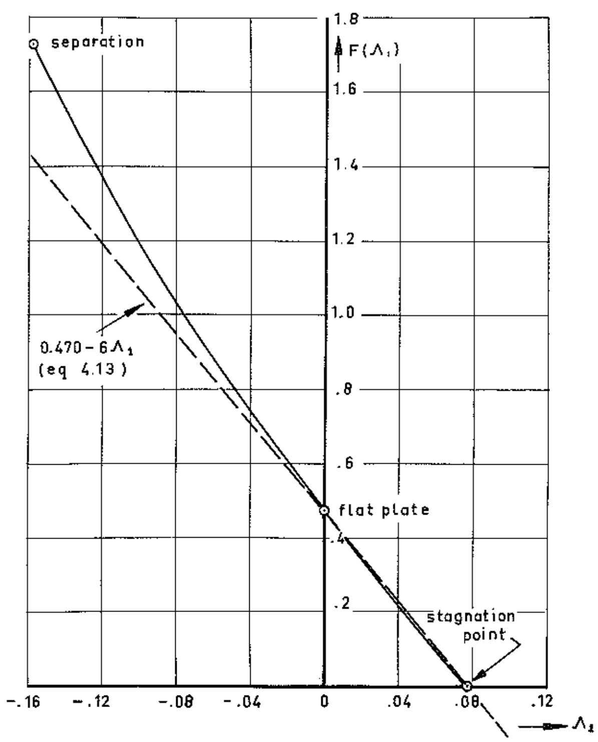

FIG. 4.1 : $F(\Lambda_1)$ FOR POHLHAUSEN'S METHOD.