**TECHNOLOGICAL UNIVERSITY DELFT**

DEPARTMENT OF AERONAUTICAL ENGINEERING



Report VTH-124


# THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS WITH AND WITHOUT SUCTION

Ph.D THESIS


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DELFT
the NETHERLANDS

OCTOBER, 1965


## This PDF-file contains chapter 7:

## *A new multi-moment method to obtain solutions of the boundary layer equations*

# 7. A new multimoment method to obtain solutions of the boundary layer equations.

## 7.1. Introductory remarks.

It was shown in chapters 4 and 5 that - at least for cases without suction - reasonably accurate solutions of the boundary layer equations can be obtained by means of the von Kármán-Pohlhausen technique. In methods of this type a suitable expression for the velocity profile is used in combination with certain compatibility conditions and moments. Since these methods in themselves do not provide a check on their accuracy, it will only be possible to get an idea about their validity by applying them to boundary layer flows for which exact solutions are available. If such a method works well for a specific example it may reasonably be expected that the results for similar cases will also be sufficiently accurate. For widely different cases however the results may be entirely useless.

It may be expected that the accuracy of the Pohlhausen-type methods can be improved by increasing the number of parameters in the expression for the velocity profile. These extra parameters then have to be determined from additional compatibility conditions and/or moment equations. Since moment equations are relations between mean quantities in the boundary layer while compatibility conditions give relations between quantities at the wall or at the edge of the boundary layer only, it can be expected that the best results will be obtained from taking additional moment equations.

Increasing the number of moment equations leads to considerable difficulties for existing methods however; it has been explained already in section 4.5. that the difficulties arise from the fact that non-linear algebraic equations have to be solved.

Therefore if a workable Pohlhausen-type method, using many moment equations, is to be developed the moments should be defined in such a way that the moment equations reduce to relations of the form (4.16)

$$\frac{d\,J_k}{d\bar{x}} = M_k \tag{7.1}$$

where the $J_k$ are linear in the parameters specifying the velocity profile. Such a method will be described in the present chapter; the principle idea is outlined below.

It was shown in chapter 6 that for some special configurations a simple description of the boundary layer may be given in the "phase plane", where the shear $\tau$ is plotted versus the velocity component u parallel to the wall. For the case of inflow between impervious non-parallel plane walls and for the asymptotic suction boundary layer the relation between $\tau^2$ and u is given exactly by a simple polynomial (eqs 6.34 and 6.30-6.31 respectively). This observation suggested the idea to develop a kind of Pohlhausen method starting from the boundary layer equations written in a form, where $\bar{x} = \frac{x}{c}$ and $\bar{u} = \frac{u}{U}$ are the independent variables and $\bar{\tau}^2$ is the dependent variable. Here $\bar{\tau}$ is the non-dimensional shear stress to be defined by equations (7.11) and (7.20). The governing equation is a modified form of the well known Crocco equation [75] where $\tau$ is used instead of $\tau^2$ and where moreover compressible flow is assumed. In what follows the new equation will be called the "modified Crocco equation"; a slightly different form has been used by Schönauer [76, 77] to develop a finite difference method.
In the present method $\bar{\tau}^2$ will be approximated by a polynomial in $\bar{u}$ of degree N. Moments are obtained by multiplication of the modified Crocco equation with $\bar{u}^k$ for k = 0, 1, 2, ..... followed by integration over the interval $\bar{u}$ = 0 to $\bar{u}$ = 1. The method allows N to be increased by taking more moment equations without unduly complicating the procedure.
For special forms of the functions $\bar{U}(\bar{x})$ and $\bar{v}_o(\bar{x})$ solutions in series are possible; this series method shows many features similar to the exact series solutions discussed in section 3.2.
In the application of the method, to be discussed in subsequent chapters, the calculations were made on the Telefunken TR 4 computer of Delft Technological University.

## 7.2. The modified Crocco equation.

Crocco [75] was the first to introduce a form of the boundary layer equations in which x and u are used as independent variables and $\tau$ as dependent variable. This equation will be derived in the present section for the special case of incompressible flow with constant viscosity. The equations to be transformed are the boundary layer equation (2.7)

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = U \frac{dU}{dx} + \nu \frac{\partial^2 u}{\partial y^2} \tag{7.2}$$

and the continuity equation (2.5)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{7.3}$$

New independent variables $x^*$ and $y^*$ are introduced by

$$\left.\begin{aligned} x &= x^* \\ y &= y(x^*, y^*) \end{aligned}\right\} \tag{7.4}$$

From (7.4) it follows that

$$\left.\begin{aligned} \frac{\partial}{\partial x^*} &= \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \frac{\partial y}{\partial x^*} \\ \frac{\partial}{\partial y^*} &= \frac{\partial}{\partial y} \frac{\partial y}{\partial y^*} \end{aligned}\right\} \tag{7.5}$$

or

$$\left.\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial}{\partial x^*} - \frac{\partial}{\partial y} \frac{\partial y}{\partial x^*} \\ \frac{\partial}{\partial y} &= \frac{\dfrac{\partial}{\partial y^*}}{\dfrac{\partial y}{\partial y^*}} \end{aligned}\right\} \tag{7.6}$$

Crocco selects $y^*$ to be u; a partial differentiation with respect to $x^*$ then leaves u constant. Using (7.6) and noting that U does not depend on y it follows that

$$\left.\begin{aligned}
\frac{\partial u}{\partial x} &= -\frac{\partial u}{\partial y}\frac{\partial y}{\partial x^{*}} \\
\frac{\partial u}{\partial y} &= \frac{1}{\frac{\partial y}{\partial u}} \\
\frac{\partial v}{\partial y} &= \frac{\frac{\partial v}{\partial u}}{\frac{\partial y}{\partial u}} \\
\frac{dU}{dx} &= \frac{dU}{dx^{*}}
\end{aligned}\right\} \quad (7.7)$$

With $\tau = \mu \frac{\partial u}{\partial y}$ and introducing (7.7) into (7.2) and (7.3) the following equations are obtained

$$- u \frac{\partial y}{\partial x^{*}} + v = \frac{\mu}{\tau} U \frac{dU}{dx^{*}} + \frac{1}{\rho}\frac{\partial \tau}{\partial u} \quad (7.8)$$

$$- \frac{\partial y}{\partial x^{*}} + \frac{\partial v}{\partial u} = 0 \quad (7.9)$$

Finally v is eliminated from these equations by differentiating (7.8) with respect to u and subtracting (7.9) from the result. If in the resulting equation $x^{*}$ is again replaced by x and after some rearrangement, the following equation is obtained

$$-\rho \mu u \left(\frac{\partial \tau}{\partial x}\right)_u + \tau^2 \left(\frac{\partial^2 \tau}{\partial u^2}\right)_x - \rho \mu U \frac{dU}{dx}\left(\frac{\partial \tau}{\partial u}\right)_x = 0 \quad (7.10)$$

The subscripts u and x in equation (7.10) are added to emphasize that the differentiations have to be performed at constant u and x respectively. The equation is equivalent to Crocco's equation for the special case of constant $\rho$ and $\mu$.

Equation (7.10) will be transformed further by introducing non-dimensional quantities, defined as follows

$$\left.\begin{aligned}
&\bar{x} = \frac{x}{c} && \bar{\tau} = \frac{\tau\delta}{\mu U} \\
&\bar{y} = \frac{y}{\delta} && S = \bar{\tau}^2 \\
&\bar{u} = \frac{u}{U} && \alpha_1 = \bar{U}\,\bar{\delta}^2 \\
&\bar{U} = \frac{U}{U_\infty} && \lambda_1 = \bar{\delta}^2 \frac{d\bar{U}}{d\bar{x}} \\
&\bar{\delta} = \frac{\delta}{c}\sqrt{\frac{U_\infty c}{\nu}} &&
\end{aligned}\right\} \quad (7.11)$$

In (7.11) c and $U_\infty$ represent a constant reference lenth and velocity respectively; $\delta$ is a given function of $\bar{x}$ related to the boundary layer thickness. In section 7.3 the choice for $\delta$ will be specified.
In what follows $\bar{x}$ and $\bar{u}$ are taken as independent variables while S is the dependent variable. The transformation to the non-dimensional variables follows from the following equations.

$$\left.\begin{aligned}
\left(\frac{\partial}{\partial x}\right)_u &= \frac{\partial}{\partial \bar{u}}\left(\frac{\partial \bar{u}}{\partial x}\right)_u + \frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{x}}{\partial x}\right)_u \\
\left(\frac{\partial}{\partial u}\right)_x &= \frac{\partial}{\partial \bar{u}}\left(\frac{\partial \bar{u}}{\partial u}\right)_x + \frac{\partial}{\partial \bar{x}}\left(\frac{\partial \bar{x}}{\partial u}\right)_x
\end{aligned}\right\} \quad (7.12)$$

in which

$$\left.\begin{aligned}
&\left(\frac{\partial \bar{u}}{\partial x}\right)_u = -\frac{u}{U^2}\frac{dU}{dx} \quad ; \quad \left(\frac{\partial \bar{x}}{\partial x}\right)_u = \frac{1}{c} \\
&\left(\frac{\partial \bar{u}}{\partial u}\right)_x = \frac{1}{U} \quad ; \quad \left(\frac{\partial \bar{x}}{\partial u}\right)_x = 0
\end{aligned}\right\} \quad (7.13)$$

Introducing (7.11) and (7.13) into (7.10) leads to

$$\frac{\partial}{\partial \bar{x}}(\bar{u}\,S) = \frac{\left(\frac{d\alpha_1}{d\bar{x}} - 3\lambda_1\right)\bar{u}\,S + S\,S'' - \frac{1}{2}(S')^2 - \lambda_1(1-\bar{u}^2)S'}{\alpha_1} \quad (7.14)$$

In equation (7.14) and in the remainder of the present chapter primes denote differentiation with respect to $\bar{u}$. Equation (7.14) will be called "the modified Crocco equation". The boundary condition for (7.14) at the edge of the boundary layer is $S(1) = 0$. At the wall ($\bar{u} = 0$) a boundary condition is provided by the first compatibility condition (2.10); this will be discussed further in section 7.4.

If the requirement is made that at the edge of the boundary layer $1-\bar{u}$ tends to zero as $e^{-y}$ the shear stress $\bar{\tau}$ behaves like $1-\bar{u}$ for $\bar{u} \longrightarrow 1$ (compare section 6.4). Hence $S = \bar{\tau}^2$ tends to zero like $(1-\bar{u})^2$ for $\bar{u} \longrightarrow 1$; this leads to the following boundary conditions for equation (7.14) at $\bar{u} = 1$

$$S(1) = S'(1) = 0 \tag{7.15}$$

Equation (7.14) permits the calculation of $S(\bar{u})$ provided for a certain initial value of $\bar{x}$ the profile $S(\bar{u})$ is known and $\bar{U}$ and $v_o(\bar{x})$ are known as functions of $\bar{x}$.

## 7.3. A special choice for $\delta$.

Although $\delta$ may be any known function of x it is convenient to choose it in such a way that for some similar boundary layer flows the shear function S will not depend on $\bar{x}$ but only on $\bar{u}$. To point this out more clearly possible similar solutions of (7.14) will be studied first.

If S does not depend on $\bar{x}$ equation (7.14) reduces to

$$\left( \frac{d\alpha_1}{d\bar{x}} - 3\lambda_1 \right) \bar{u}S + SS'' - \tfrac{1}{2}(S')^2 - \lambda_1(1-\bar{u}^2)S' = 0 \tag{7.16}$$

This equation admits solutions independent of $\bar{x}$ only if both $\lambda_1$ and $\frac{d\alpha_1}{d\bar{x}}$ are constants. Hence it follows that

$$\alpha_1 = \bar{U}\,\bar{\delta}^2 = \alpha_2\,\bar{x} + \alpha_3 \tag{7.17}$$

$$\lambda_1 = \bar{\delta}^2 \frac{d\bar{U}}{d\bar{x}} = \text{constant} \tag{7.18}$$

where $\alpha_2$ and $\alpha_3$ are constants.

If $\alpha_2 \neq 0$ it is possible, without loss of generality, to make $\alpha_2 = 1$ and $\alpha_3 = 0$ by a trivial change of the variable $\bar{x}$.
Equation (7.17) then reduces to

$$\alpha_1 = \bar{U}\,\bar{\delta}^2 = \bar{x} \tag{7.19}$$

implying that

$$\bar{\delta} = \sqrt{\frac{\bar{x}}{\bar{U}}} \quad \text{or} \quad \frac{\delta}{x}\sqrt{\frac{Ux}{\nu}} = 1 \tag{7.20}$$

Elimination of $\bar{\delta}^2$ between (7.18) and (7.19) gives

$$\frac{\lambda_1}{\bar{x}}\,d\bar{x} = \frac{d\bar{U}}{\bar{U}} \tag{7.21}$$

and upon integration

$$\bar{U} = \text{constant} \,.\, \bar{x}^{\lambda_1} \tag{7.22}$$

Hence a first class of similar solutions, for which S becomes independent of $\bar{x}$, may be obtained for pressure distributions defined by

$$\bar{U} = \text{constant} \,.\, \bar{x}^{m_1} \tag{7.23}$$

where $m_1 = \lambda_1 = \text{constant}$ and $\delta$ is defined by equation (7.20). This is in agreement with the exact results discussed in section 3.1.
A second class of similar solutions is obtained from the case $\alpha_2 = 0$ which was hitherto excluded. This case may be shown to lead to

$$\bar{U} = \text{constant} \,.\, e^{\frac{\lambda_1 \bar{x}}{b_2}} \tag{7.24}$$

where both $\lambda_1$ and $b_2$ are constant. Hence also for $\alpha_2 = 0$ one of the results discussed in chapter 3 is regained; in this case $\delta$ is defined by $\alpha_1 = \bar{U}\,\bar{\delta}^2 = \text{constant}$.

In what follows always the definition (7.20) for $\delta$ will be used; the corresponding expressions for $\alpha_1$ and $\lambda_1$ then become

$$\alpha_1 = \bar{x} \quad ; \quad \lambda_1 = \bar{\delta}^2 \ \frac{d\bar{U}}{d\bar{x}} = \frac{\bar{x}}{\bar{U}} \ \frac{d\bar{U}}{d\bar{x}} \tag{7.25}$$

while equation (7.14) reduces to

$$\frac{\partial}{\partial \bar{x}} (\bar{u}\ S) = \frac{(1-3\lambda_1)\ \bar{u}\ S + SS'' - \frac{1}{2}(S')^2 - \lambda_1(1-\bar{u}^2)S'}{\bar{x}} \tag{7.26}$$

From the derivation of equation (7.26) it follows that for pressure distributions defined by equation (7.22) solutions may be found for which S is independent of $\bar{x}$.

## 7.4. The polynomial approximation for S and compatibility conditions of the modified Crocco equation.

It was shown in chapter 6 that for some special cases of similar boundary layer flows $S(\bar{u})$ is given by a simple polynomial. For instance for inflow between impervious non-parallel plane walls the following result was obtained (equation 6.34)

$$S = \bar{\tau}^2 = \frac{4}{3} + 2\ \bar{u} - \frac{2}{3}\ \bar{u}^3 \tag{7.27}$$

For the asymptotic suction profile equations (6.30) or (6.31) lead to

$$S = \bar{\tau}^2 = \text{constant}\ (1 - \bar{u})^2 \tag{7.28}$$

In the present method it will be attempted to obtain solutions of the modified Crocco equation (7.26) by assuming that in all cases S can be approximated by a polynomial expression of the following form

$$S = a_o + a_1\bar{u} + a_2\ \bar{u}^2 + \ldots\ldots\ldots + a_N\ \bar{u}^N \tag{7.29}$$

In (7.29) the coefficients $a_n$ are functions of $\bar{x}$ for general boundary layer flows and constants for the similar boundary layers defined by $\bar{U} = u_1\ \bar{x}^{m_1}$. Introducing (7.29) into (7.26) gives

$$\bar{x}\frac{\partial}{\partial \bar{x}}\left(\sum_{n=0}^{N} a_n \bar{u}^{n+1}\right) = d_o + d_1\bar{u} + d_2\bar{u}^2 + \dots + d_e\bar{u}^e \quad (7.30)$$

where $e = N+1$ for $N < 3$ and $e = 2N-1$ for $N \geqslant 3$.
In equation (7.30) the coefficients d are quadratic expressions in the $a_n$. The first few of these read as follows

$$d_o = 2\, a_o a_2 - \tfrac{1}{2}\, a_1^{\,2} - \lambda_1\, a_1 \quad (7.31)$$

$$d_1 = \quad a_o + 6\, a_o a_3 - \lambda_1\,(3\, a_o + 2\, a_2) \quad (7.32)$$

$$d_2 = \quad a_1 + 12\, a_o a_4 + 3\, a_1 a_3 - \lambda_1 (2\, a_1 + 3\, a_3) \quad (7.33)$$

If (7.30) is valid for all values of $\bar{u}$ the coefficients of equal powers of $\bar{u}$ in the left- and right-hand sides of the equation have to be equal. This leads to

$$0 = d_o \quad (7.34)$$

$$\bar{x}\frac{da_o}{d\bar{x}} = d_1 \quad (7.35)$$

$$\bar{x}\frac{da_1}{d\bar{x}} = d_2. \quad (7.36)$$

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or using (7.31), (7.32) and (7.33)

$$0 = 2\, a_o a_2 - \tfrac{1}{2}\, a_1^{\,2} - \lambda_1 a_1 \quad (7.37)$$

$$\bar{x}\frac{da_o}{d\bar{x}} = (1-3\,\lambda_1)a_o + 6\, a_o a_3 - 2\,\lambda_1 a_2 \quad (7.38)$$

$$\bar{x}\frac{da_1}{d\bar{x}} = a_1 + 12\, a_o a_4 + 3\, a_1 a_3 - 2\,\lambda_1 a_1 - 3\,\lambda_1 a_3 \quad (7.39)$$

Equations (7.37) to (7.39) are compatibility conditions at the wall for the modified Crocco equation. They are the analogues to the compatibility conditions discussed in chapter 2 for Prandtl's form of the boundary layer equations.

In the derivation of the modified Crocco equation the normal velocity v has been eliminated. In order to be able to specify a boundary condition on v at the wall and to discuss problems with suction, the normal velocity v has to be introduced again by means of the compatibility conditions of Prandtl's boundary layer equations.

The first and second of these compatibility conditions read (see section 2.3)

$$v_o \left(\frac{\partial u}{\partial y}\right)_o = U \frac{dU}{dx} + \nu \left(\frac{\partial^2 u}{\partial y^2}\right)_o \tag{7.40}$$

$$v_o \left(\frac{\partial^2 u}{\partial y^2}\right)_o = \nu \left(\frac{\partial^3 u}{\partial y^3}\right)_o \tag{7.41}$$

Writing these equations in terms of S and $\bar{u}$ leads to

$$-\lambda_2 \sqrt{S(o)} = \lambda_1 + \frac{S'(o)}{2} \tag{7.42}$$

$$-\lambda_2 \quad S'(o) = S''(o) \sqrt{S(o)} \tag{7.43}$$

in which $\lambda_2$ stands for

$$\lambda_2 = \frac{-v_o \delta}{\nu} = \frac{-v_o}{U} \sqrt{\frac{Ux}{\nu}} = \bar{v}_o . \bar{\delta} \tag{7.44}$$

while $\bar{v}_o$ is defined by

$$\bar{v}_o = \frac{-v_o}{U_\infty} \sqrt{\frac{U_\infty c}{\nu}} \tag{7.45}$$

If S(o), S'(o) and S''(o) are expressed in terms of the $a_n$'s equations (7.42) and (7.43) can be written in the form

$$a_1 = -2 \lambda_1 - 2 \lambda_2 \sqrt{a_o} \tag{7.46}$$

$$a_2 = \frac{-\lambda_2 a_1}{2\sqrt{a_o}} \tag{7.47}$$

Elimination of $\lambda_2$ from (7.46) and (7.47) again leads to (7.37). This means that the first and second compatibility condition of Prandtl's boundary layer equation include the first compatibility condition of the modified Crocco equation. This, of course, could be expected from the derivation of the modified Crocco equation.
Some further relations between the coefficients $a_n$ are obtained from the conditions (7.15) at the edge of the boundary layer. They lead to

$$\sum_{n=0}^{N} a_n = 0 \tag{7.48}$$

and

$$\sum_{n=1}^{N} n\, a_n = 0 \tag{7.49}$$

In what follows both conditions (7.48) and (7.49) at $\bar{u} = 1$ will be retained together with the equations (7.38), (7.46) and (7.47). In the next section these equations will be supplemented by some moment equations.

## 7.5. Moments of the modified Crocco equation.

In taking moments of the modified Crocco equation it should be tried - in order to fulfill the requirement set out in section 7.1 - to reduce the left-hand side of (7.26) to the form $\frac{dJ}{d\bar{x}}$ in which J is a linear combination of the $a_n$'s. Evidently such a relation can be obtained by multiplying the equation with some function $G(\bar{u})$ and integrating the resulting equation w.r.t. $\bar{u}$ from 0 to 1.
In what follows $G(\bar{u}) = \bar{u}^k$ will be used where k in turn takes the values 0,1,2,..., K. This leads to K+1 moment equations defined by

$$\bar{x}\, \frac{dJ_k}{d\bar{x}} = M_k \tag{7.50}$$

$$M_k = (1-3\lambda_1)J_k - \lambda_1 P_k + Q_k \tag{7.51}$$

$$J_k = \sum_{n=0}^{N} j_{k,n}\, a_n \tag{7.52}$$

$$P_k = \sum_{n=0}^{N} p_{k,n}\, a_n \tag{7.53}$$

$$Q_k = \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} q_{k,\ell,n}\, a_\ell a_n \tag{7.54}$$

$$\left.\begin{aligned}
j_{k,n} &= \frac{1}{k+n+2} && 0 \leqslant n \leqslant N \\
p_{k,0} &= 0 \\
p_{k,n} &= \frac{2n}{(k+n)(k+n+2)} && 1 \leqslant n \leqslant N \\
q_{k,0,0} &= q_{k,0,1} = 0 \\
q_{k,0,n} &= \frac{n(n-1)}{k+n-1} && 2 \leqslant n \leqslant N \\
q_{k,1,1} &= \frac{-\frac{1}{2}}{k+1} \\
q_{k,1,n} &= \frac{n(n-2)}{k+n} && 2 \leqslant n \leqslant N \\
q_{k,\ell,\ell} &= \frac{\frac{1}{2}\ell^2-\ell}{k+2\ell-1} && 2 \leqslant \ell \leqslant N \\
q_{k,\ell,n} &= \frac{n(n-1)-\ell n+\ell(\ell-1)}{k+\ell+n-1} && \begin{matrix} 2 \leqslant \ell \leqslant N \\ \ell < n \leqslant N \end{matrix}
\end{aligned}\right\} \tag{7.55}$$

## 7.6. Summary of the formulae to be used in the new calculation method.

This section summarizes the formulae derived in the preceding sections which have to be used in the new calculation method.
The flow outside the boundary layer is determined by $\bar{U}(\bar{x})$ and $\frac{d\bar{U}}{d\bar{x}}$ ; the suction distribution by $\bar{v}_o(\bar{x})$ with

$$\bar{v}_o = - \frac{v_o}{U_\infty} \sqrt{\frac{U_\infty c}{\nu}} \tag{7.56}$$

Furthermore the following definitions are used

$$\bar{\delta} = \sqrt{\frac{\bar{x}}{\bar{U}}} \tag{7.57}$$

$$\lambda_1 = \bar{\delta}^2 \frac{d\bar{U}}{d\bar{x}} = \frac{\bar{x}}{\bar{U}} \frac{d\bar{U}}{d\bar{x}} \tag{7.58}$$

$$\lambda_2 = \bar{v}_o . \bar{\delta} = - \frac{v_o}{U} \sqrt{\frac{Ux}{\nu}} \tag{7.59}$$

It should be noted that the pressure distribution only enters the calculation through $\lambda_1$ and $\lambda_2$ while the suction distribution only enters through $\lambda_2$. The shear stress function S is approximated by the polynomial

$$S = \sum_{n=0}^{N} a_n \bar{u}^n \tag{7.60}$$

where the coefficients $a_n$ are determined as functions of $\bar{x}$ from the following equations.

Compatibility conditions at the wall:

$$\bar{x} \frac{da_o}{d\bar{x}} = (1-3 \lambda_1)a_o + 6 a_o a_3 - 2 \lambda_1 a_2 \tag{7.61}$$

$$a_1 = -2 \lambda_1 - 2 \lambda_2 \sqrt{a_o} \tag{7.62}$$

$$a_2 = - \frac{\lambda_2 a_1}{2\sqrt{a_o}} \tag{7.63}$$

Conditions at the edge of the boundary layer $(\bar{u} = 1)$:

$$a_1 + 2 a_2 + 3 a_3 + \ldots\ldots\ldots + N a_N = 0 \tag{7.64}$$

$$a_o + a_1 + a_2 + a_3 + \ldots\ldots\ldots + a_N = 0 \tag{7.65}$$

Moment equations:

$$\bar{x} \frac{dJ_k}{d\bar{x}} = M_k \quad \text{for} \quad k = 0,1,2,\ldots\ldots,K \tag{7.66}$$

where the $M_k$ follow from equations (7.51) - (7.55). The total number of equations (7.61) to (7.66) thus obtained is K+6. These equations should yield the N+1 coefficients $a_n$; hence it follows that

$$K = N - 5 \tag{7.67}$$

Some further remarks on the compatibility conditions at the wall and the moments may be made here. The choice of the moments and the compatibility conditions to be used has been made in a rather arbitrary manner. No systematic investigation has been made of the best possible choice. However there is an argument in favour of the present choice which will be given now. The compatibility conditions (7.37) through (7.39) have been obtained by equating to zero the coefficients of terms with various powers of $\bar{u}$ in (7.30). The same result will be obtained by repeated differentiation of (7.30) w.r.t. $\bar{u}$ and putting $\bar{u} = 0$ in the resulting equations. A natural complement to these compatibility conditions would be a set of moment equations obtained by repeated integrations of (7.30) w.r.t. $\bar{u}$ from $\bar{u} = 0$ to $\bar{u}$ and putting $\bar{u} = 1$ in the final results. This has in fact been precisely achieved because it can be shown that the members of the present set of moment equations are linear combinations of the equations which appear upon repeated integration of (7.30). The number of compatibility conditions was taken as small as possible because it was felt that the moment equations might be more decisive for the mean boundary layer characteristics than the compatibility conditions. However, since in the first compatibility condition (7.62) the square root of $a_o$ already occurs it was decided to go on and to include (7.61) in the system which provides a differential equation from which $a_o$ can easily be calculated.

## 7.7. Step by step solution starting from given initial conditions.

In this section it is assumed that at some station $\bar{x} = \bar{x}_o$ starting values of the $a_n$'s are known (The determination of the starting values will be discussed in section 7.9). The $a_n$ downstream of $\bar{x}_o$ can be determined in the following way using one of the numerical methods for the integration of a system of ordinary differential equations.

From the given starting values at $\bar{x} = \bar{x}_o$ the value of $a_o$ at the next station is found using equation (7.61). Then $a_1$ and $a_2$ follow from (7.62) and (7.63) respectively. The values of the $J_k$ at the next station follow from equations (7.66). The only remaining problem is to find the $a_n$ for $n \geqslant 3$ from the values of the $J_k$. As the $J_k$ are linear relations in the $a_n$ (equation 7.52) this leads, in combination with the conditions (7.64) and (7.65) at the edge of the boundary layer, to the following set of linear equations.

$$\begin{aligned}
3a_3 + 4a_4 + 5a_5 + \ldots\ldots + N a_N &= -a_1 - 2a_2 = \beta_1 \\
a_3 + a_4 + a_5 + \ldots\ldots + a_N &= -a_o - a_1 - a_2 = \beta_2 \qquad (7.68) \\
\frac{a_3}{k+5} \quad \frac{a_4}{k+6} \quad \frac{a_5}{k+7} + \ldots\ldots + \frac{a_N}{k+N+2} &= J_k - \frac{a_o}{k+2} - \frac{a_1}{k+3} - \frac{a_2}{k+4} = \beta_{k+3}
\end{aligned}$$

where the coefficients $\beta_n$ have been introduced to denote the known right-hand sides. The last equation should be used for $k = 0,1,2,\ldots\ldots,N-5$. The left hand sides of (7.68) do not contain specific data of the boundary layer being calculated and therefore the coefficient matrix of the equations can be inverted once for all, for all values of N to be used. As the original matrix is very orderly built the inverse can easily be obtained by hand computation for increasing values of N. Denoting with $a_{ij}$ the coefficient of $a_{i+2}$ in the $j^{th}$ row of the set (7.68) the elements of the inverse matrix will be denoted by $A_{ij}$ where i and j assume the values $1,2,3,\ldots\ldots,N-2$. The results are given in table 7.1 for N=5 to 10. It is noticed that the elements of A become very large for large values of N. This is caused by the tendency of the coefficient matrix of (7.68) to become singular at large values of N. It may be noted that the tendency to singularity had no influence on the accuracy of the inverses presented because these were obtained by hand computation in the exact number of figures.

Solutions of (7.68) can now be given in the form

$$a_n = \sum_{j=1}^{N-2} A_{n-2,j}\, \beta_j \qquad (7.69)$$

in which $n \geqslant 3$.

When the value of N is increased, only the number of linear equations to be solved increases but the method remains in principle the same. Hence it may be conjectured that the approximate solution will approach the exact solution when N is successively increased.

A practical limit to the maximum permissible value of N is imposed however by the loss of significant figures which occurs in (7.69) for large values of N. This is due to the fact that both the $a_n$ and $\beta_j$ are of order 1 while the $A_{ij}$ are of a considerably larger order of magnitude (see table 7.1). This difficulty may be postponed to large values of N if the procedure outlined above is not applied to the $a_n$ and $J_k$ but only to the increments of these quantities.

When values at the initial station are denoted by a bar the increments follow from

$$\begin{aligned} \Delta a_n &= a_n - \overline{a_n} \\ \Delta J_k &= J_k - \overline{J_k} \\ \Delta \beta_j &= \beta_j - \overline{\beta_j} \end{aligned} \tag{7.70}$$

As both the initial and final values satisfy the linear equations (7.68) it follows that also the increments are determined by the equations (7.68) when the $a_n$, $J_k$ and $\beta_j$ are replaced by $\Delta a_n$, $\Delta J_k$ and $\Delta \beta_j$ respectively. Hence for $n \geqslant 3$ the $\Delta a_n$ are given by

$$\Delta a_n = \sum_{j=1}^{N-2} A_{n-2,j} \, \Delta \beta_j \tag{7.71}$$

or after separating the contributions of $\Delta a_o$, $\Delta a_1$, $\Delta a_2$ and $\Delta J_k$:

$$\begin{aligned} \Delta a_n = &- \Delta a_o \left[ A_{n-2,2} + \sum_{j=3}^{N-2} \frac{A_{n-2,j}}{j-1} \right] \\ &- \Delta a_1 \left[ A_{n-2,1} + A_{n-2,2} + \sum_{j=3}^{N-2} \frac{A_{n-2,j}}{j} \right] \\ &- \Delta a_2 \left[ 2\, A_{n-2,1} + A_{n-2,2} + \sum_{j=3}^{N-2} \frac{A_{n-2,j}}{j+1} \right] \\ &+ \sum_{j=3}^{N-2} A_{n-2,j} \, \Delta J_{j-3} \end{aligned} \tag{7.72}$$

The number of significant figures retained is made as large as possible by first evaluating the terms between brackets in (7.72).
Once the $\Delta a_n$ are obtained from (7.72) the values of $a_n$ at the next station follow from (7.70). These values can be used as starting values for the next step etc.
In all examples given in the present work the integration was performed by a third order Runge-Kutta method. It should be understood that in this method a full step is made up of some sub-steps and that the procedure outlined above has to be applied in each sub-step. As in certain applications of the method the largest permissible step length may vary considerably with $\bar{x}$ use was made of one of the Runge-Kutta formulae with self-adjusting step length given by Zonneveld [78]. These formulae provide explicit expressions for the last term of the Taylor series taken into account. The step length used is adjusted in such a way that the absolute value of this last term is equal to a certain tolerance, to be specified in the program.

## 7.8. Similar solutions for $\bar{U} = \bar{x}^{\lambda_1}$.

It was shown in section 7.3 that for $\bar{U} = \bar{x}^{\lambda_1}$ with constant $\lambda_1$ similar solutions may occur for which the $a_n$ are constants. From the compatibility conditions (7.62) and (7.63) it follows that also $\lambda_2$ should be constant. Therefore the permissible suction distribution for this class of similar boundary layers is given by

$$\bar{v}_o = \lambda_2 \, \bar{x}^{\frac{\lambda_1 - 1}{2}} \tag{7.73}$$

From the fact that the $a_n$ are constant for the similar solutions under consideration it follows that all terms with $\bar{x}\dfrac{d}{d\bar{x}}$ vanish from the equations (7.61) to (7.66). Hence the moment equations for this case reduce to $M_k = 0$ with $k = 0,1,2,\ldots, N-5$.
Since the resulting equations contain non-linear terms they are not easy to solve directly; however, solutions can easily be obtained by interpolation or iteration. In what follows two different procedures which were used for this purpose will be described.

It should be noted that due to the non-linearity multiple solutions may occur. One of these solutions is always $a_n \equiv 0$ which of course is an unrealistic one. In all the examples to be discussed in chapter 8, only one realistic solution occurred.

A procedure to obtain similar solutions by interpolation. If values for $a_o$, $a_7$, $a_8$, ...., $a_N$ are assumed, the compatibility conditions (7.61) to (7.63) with the additional condition $\bar{x} \frac{d}{d\bar{x}} = 0$ provide values for $a_1$, $a_2$ and $a_3$ while $a_4$ and $a_5$ can be expressed as a linear relation in $a_6$ using (7.64) and (7.65). In this way the moment equations $M_k = 0$ are reduced to quadratic equations in $a_6$; real roots of these equations provide, for each value of k, one or two values of $a_6$ for which $M_k = 0$. Repeating this procedure for other values of $a_o$, $a_6$, $a_7$, $a_8$, ... it is rather easy to find by interpolation values for the $a_n$ for which all compatibility conditions and moment equations are satisfied. The method outlined above works well for $N \leqslant 7$; for higher values of N however it becomes too complicated. Therefore an iterative procedure was designed which will be outlined in the remainder of the present section.

An iteration procedure to obtain similar solutions. At first it was attempted to use the step by step method of section 7.7 for this purpose by starting from guessed initial values and running the program for constant $\lambda_1$ and $\lambda_2$ until the $a_n$'s became constant. It was found that the required solutions were stable so that the proposed procedure was convergent. However, a large amount of computation was required to obtain the solutions with sufficient precision. It turned out that the solutions could be obtained much more rapidly by using the following iteration procedure.

For the similar solutions equations (7.61) through (7.66) reduce to:

$$\left.\begin{aligned}
E_1 &= (1-3\lambda_1)a_o + 6\,a_o a_3 - 2\lambda_1 a_2 &&= 0\\
E_2 &= a_1 + 2\lambda_1 + 2\lambda_2\sqrt{a_o} &&= 0\\
E_3 &= 2\,a_2\sqrt{a_o} + \lambda_2 a_1 &&= 0\\
E_4 &= a_1 + 2\,a_2 + 3\,a_3 + \dots + N\,a_N &&= 0\\
E_5 &= a_o + a_1 + a_2 + a_3 + \dots + a_N &&= 0\\
E_{6+k} &= M_k &&= 0
\end{aligned}\right\} \qquad (7.74)$$

The last equation written down in (7.74) has to be used for $k = 0,1,2,\dots, N-5$. The equations (7.74) can be solved using an iterative procedure equivalent to Newton's method for one equation.

If initial values $\bar{a}_n$ for all $a_n$ are assumed to be known, then also initial values $\bar{E}_i$ of the functions $E_i$ can be calculated. Now, the $\bar{E}_i$ will in general be different from zero; to make them zero the $a_n$ should be changed by amounts $\delta a_n$. For small variations the $E_i$ may be replaced by their Taylor series expansions using only terms up to and including those of the first degree. Hence, if $\frac{\partial \bar{E}_i}{\partial a_n}$ is denoted by $e_{in}$ and $\delta a_n$ by $t_n$ the equations (7.74) are replaced by

$$\left.\begin{aligned}
E_1 &= \bar{E}_1 + e_{10}\,t_o + e_{12}\,t_2 + e_{13}\,t_3 &&= 0\\
E_2 &= \bar{E}_2 + e_{20}\,t_o + e_{21}\,t_1 &&= 0\\
E_3 &= \bar{E}_3 + e_{30}\,t_o + e_{31}\,t_1 + e_{32}\,t_2 &&= 0\\
E_4 &= \bar{E}_4 + \sum_{n=0}^{N} e_{4n}\,t_n &&= 0\\
E_5 &= \bar{E}_5 + \sum_{n=0}^{N} e_{5n}\,t_n &&= 0\\
E_{6+k} &= \bar{E}_{6+k} + \sum_{n=0}^{N} e_{6+k,n}\,t_n &&= 0\\
& \qquad k = 0,1,2,\dots,N-5
\end{aligned}\right\} \qquad (7.75)$$

The derivatives $e_{in}$ are given by

$$\left.\begin{array}{lll}
e_{10} = 1-3\lambda_1 + 6\,\bar{a}_3 & e_{20} = \dfrac{\lambda_2}{\sqrt{\bar{a}_o}} & e_{30} = \dfrac{\bar{a}_2}{\sqrt{\bar{a}_o}} \\
e_{12} = -2\lambda_1 & & \\
e_{13} = 6\,\bar{a}_o & e_{21} = 1 & e_{31} = \lambda_2 \\
 & & e_{32} = 2\sqrt{\bar{a}_o} \\
e_{4n} = n \quad \text{for} \quad n = 0,1,2,\ldots, N & & \\
e_{5n} = 1 \quad \text{for} \quad n = 0,1,2,\ldots, N & & \\
e_{6+k,n} = \dfrac{\partial M_k}{\partial a_n} \quad \text{for} \begin{cases} k = 0,1,2,\ldots, N-5 \\ n = 0,1,2,\ldots, N \end{cases} & &
\end{array}\right\} \quad (7.76)$$

From equations (7.51) through (7.54) it follows that

$$\frac{\partial M_k}{\partial a_n} = (1-3\lambda_1)j_{k,n} - \lambda_1 p_{k,n} + \sum_{\ell=0}^{n} q_{k,\ell,n}\,\bar{a}_\ell + \sum_{\ell=n}^{N} q_{k,n,\ell}\,\bar{a}_\ell \quad (7.77)$$

It is convenient to select the initial values $\bar{a}_n$ in such a way that the compatibility conditions at the wall (equations 7.61 to 7.63) and the conditions at $\bar{u} = 1$ (7.64) and (7.65) are satisfied which means that $\bar{E}_i = 0$ for $i = 1,2,\ldots,5$. Then, denoting $\bar{E}_{6+k}$ by $\bar{M}_k$ the equations (7.75) reduce to the following set of linear algebraic equations.

$$\left.\begin{array}{ll}
e_{10}t_o + e_{12}t_2 + e_{13}t_3 & = 0 \\
e_{20}t_o + e_{21}t_1 & = 0 \\
e_{30}t_o + e_{31}t_1 + e_{32}t_2 & = 0 \\
\sum_{n=0}^{N} e_{4n}\,t_n & = 0 \\
\sum_{n=0}^{N} e_{5n}\,t_n & = 0 \\
\sum_{n=0}^{N} \dfrac{\partial M_k}{\partial a_n}\,t_n & = -\bar{M}_k \text{ for}
\end{array}\right\} \quad (7.78)$$

$k = 0,1,2,\ldots,N-5$

Solving these equations for the $t_n$ gives improved values for the $a_n$ from $a_n = \bar{a}_n + t_n$; these values for the $a_n$ can be used as new starting values $\bar{a}_n$ etc. It should be noted that throughout the iteration process $\bar{E}_1$ to $\bar{E}_5$ remain zero which means that the compatibility conditions are always satisfied. The iteration procedure serves to adjust the $a_n$ such that also the moment equations are satisfied.

## 7.9. Series solutions for special types of the functions U(x) and $v_o(x)$.

### 7.9.1. General remarks.

In section 7.7 it was assumed that for a certain initial value of $\bar{x}$ starting values for the $a_n$ were known. It will be shown now that these starting values can be determined from a series solution starting from $\bar{x} = 0$.

In the series solution a new variable z is used defined by

$$z = x^f \tag{7.79}$$

where f may be any real positive number. Series solutions for the coefficients $a_n$ will be obtained for those functions $\lambda_1$ and $\lambda_2$ which can be developed in power series in z of the following form

$$\lambda_1 = \sum_{p=0}^{\infty} \lambda_{1,p} \; z^p \tag{7.80}$$

$$\lambda_2 = \sum_{p=0}^{\infty} \lambda_{2,p} \; z^p \tag{7.81}$$

The expressions (7.80) and (7.81) correspond to special forms of the pressure- and suction distributions. They are sufficiently general however to be applied near $\bar{x} = 0$ for all problems likely to be encountered. The pressure- and suction distributions which lead to (7.80) and (7.81) will be obtained first.

From (7.80) and

$$\lambda_1 = \frac{\bar{x}}{\bar{U}} \frac{d\bar{U}}{d\bar{x}} = \frac{fz}{\bar{U}} \frac{d\bar{U}}{dz} = fz \frac{d}{dz} (\ln \bar{U}) \tag{7.82}$$

it follows that

$$\frac{\lambda_{1,0}}{z} + \lambda_{1,1} + \lambda_{1,2}\, z + \lambda_{1,3}\, z^2 + \ldots + \lambda_{1,p}\, z^{p-1} + \ldots = f \frac{d}{dz} (\ell n\, \bar{U}) \qquad (7.83)$$

Integration of (7.83) leads to

$$\bar{U} = \text{constant} \;.\; z^{\frac{\lambda_{1,0}}{f}} \left[ e^{\frac{\lambda_{1,1}}{f} z + \frac{1}{2} \frac{\lambda_{1,2}}{f} z^2 + \ldots + \frac{\lambda_{1,p}}{pf} z^p + \ldots} \right] \qquad (7.84)$$

Development of the exponential function in a power series in z results in the following expression for $\bar{U}$

$$\bar{U} = z^{\frac{\lambda_{1,0}}{f}} \left[ u_o + u_1 z + u_2\, z^2 + \ldots + u_p\, z^p + \ldots \right] \qquad (7.85)$$

In terms of $\bar{x}$ the permissible pressure distribution follows from (7.79) and (7.85)

$$\bar{U} = \bar{x}^{\lambda_{1,0}} (u_o + u_1\, \bar{x}^{-f} + u_2\, \bar{x}^{-2f} + \ldots + u_p\, \bar{x}^{\,pf} + \ldots) \qquad (7.86)$$

The permissible suction distribution then follows from (7.57), (7.59) and (7.86)

$$\bar{v}_o = \lambda_2\, \bar{x}^{\frac{\lambda_{1,o}-1}{2}} (u_o + u_1\, \bar{x}^{-f} + u_2\, \bar{x}^{\,2f} + \ldots)^{\frac{1}{2}} \qquad (7.87)$$

Noting that $\lambda_2$ is represented by the series (7.81) and developing the square root in (7.87) gives the following expression for the permissible suction distribution

$$\bar{v}_o = \bar{x}^{\frac{\lambda_{1,o}-1}{2}} (s_o + s_1\, \bar{x}^{-f} + s_2\, \bar{x}^{\,2f} + \ldots + s_p\, \bar{x}^{\,pf} + \ldots) \qquad (7.88)$$

In practical applications of the series method the coefficients $u_p$ and $s_p$ in (7.86) and (7.88) are given while the coefficients of the series (7.80) and (7.81) have to be determined. It is possible to derive universal formulae from which these coefficients can be calculated but they will not be given here. In the examples of the method to be discussed in chapter 8 the series developments of $\lambda_1$ and $\lambda_2$ will be given directly for each case.

The series solutions are obtained from equations (7.61) to (7.66) if in these equations $\bar{x}\frac{d}{d\bar{x}}$ is replaced by $f\,z\,\frac{d}{dz}$. The equations obtained in this way read as follows:

$$\left.\begin{aligned}
fz\frac{da_o}{dz} &= (1-3\lambda_1)\,a_o + 6\,a_o a_3 - 2\lambda_1 a_2\\
a_1 &= -2\lambda_1 - 2\lambda_2\sqrt{a_o}\\
a_2 &= -\frac{\lambda_2 a_1}{2\sqrt{a_o}}\\
\sum_{n=1}^{N} n\,a_n &= 0\\
\sum_{n=0}^{N} a_n &= 0\\
fz\frac{dJ_k}{dz} &= M_k \quad \text{for} \quad k = 0,1,2,\ldots, N-5
\end{aligned}\right\} \qquad (7.89)$$

In sections 7.9.2 to 7.9.4 the solutions in series of the equations (7.89) will be given.

## 7.9.2. Series expressions for some functions occurring in the theory.

For pressure- and suction distributions which conform to

$$\lambda_1 = \sum_{p=0}^{\infty} \lambda_{1,p}\, z^p \qquad (7.90)$$

and

$$\lambda_2 = \sum_{p=0}^{\infty} \lambda_{2,p}\, z^p \qquad (7.91)$$

Solutions of the equations (7.89) will be sought of the form

$$a_n = \sum_{p=0}^{\infty} a_{n,p}\, z^p \qquad (7.92)$$

To do this some functions occurring in the equations have to be expressed in the form of a power series in z. This will be done in the present section.

For $\sqrt{a_o}$ the following series is used

$$\sqrt{a_o} = \sum_{p=0}^{\infty} r_p \, z^p \tag{7.93}$$

where the $r_p$ follow from:

$$r_o = \sqrt{a_{o,o}} \tag{7.94}$$

$$r_p = \frac{a_{o,p}}{2\, r_o} - \sigma_p \quad \text{for} \quad p > 0 \tag{7.95}$$

in which for p even and $\geqslant$ 2

$$\sigma_p = \frac{(r_{p/2})^2}{2\, r_o} + \frac{1}{r_o} \sum_{i=1}^{\frac{p}{2}-1} r_i r_{p-i} \tag{7.96}$$

and for p odd and 1

$$\sigma_p = \frac{1}{r_o} \sum_{i=1}^{\frac{p-1}{2}} r_i r_{p-i} \tag{7.97}$$

The sums in (7.96) and (7.97) must be omitted when the upper bound is smaller than 1.
The series development (7.93) is not possible for $a_{o,o} = 0$; this means that the series method is not applicable in cases where the boundary layer starts at $\overline{x} = 0$ with a separation point. As such a boundary layer cannot easily be imagined this seems no real limitation of the method.

Other series expressions to be used are the following

$$\left.\begin{aligned}
J_k &= \sum_{p=0}^{\infty} J_{k,p} \, z^p \\
P_k &= \sum_{p=0}^{\infty} P_{k,p} \, z^p \\
Q_k &= \sum_{p=0}^{\infty} Q_{k,p} \, z^p \\
M_k &= (1-3\,\lambda_1)J_k - \lambda_1 P_k + Q_k = \sum_{p=0}^{\infty} M_{k,p} \, z^p
\end{aligned}\right\} \tag{7.98}$$

$$\left.\begin{aligned}
J_{k,p} &= \sum_{n=0}^{N} j_{k,n}\, a_{n,p} \\
P_{k,p} &= \sum_{n=1}^{N} P_{k,n}\, a_{n,p} \\
Q_{k,p} &= \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} \sum_{i=0}^{p} q_{k,\ell,n}\, a_{\ell,i}\, a_{n,p-i} = \\
&= \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} q_{k,\ell,n} \left(a_{\ell,o}\, a_{n,p} + a_{\ell,p}\, a_{n,o}\right) \\
&+ \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} q_{k,\ell,n} \sum_{i=1}^{p-1} a_{\ell,i}\, a_{n,p-i} \\
M_{k,p} &= J_{k,p} + Q_{k,p} - 3 \sum_{i=0}^{p} \lambda_{1,i}\, J_{k,p-i} \\
&- \sum_{i=0}^{p} \lambda_{1,i}\, P_{k,p-i}
\end{aligned}\right\} \tag{7.99}$$

The coefficients $j_{k,n}$, $p_{k,n}$ and $q_{k,\ell,n}$ follow from equations (7.55). If the series expressions given above are inserted into the equations (7.89) and the coefficients of successive powers p of z are equated to zero a set of algebraic equations is obtained for each value of p. Coefficients in these equations are determined by the $a_{n,j}$ for $j < p$ and by $\lambda_{1,j}$ and $\lambda_{2,j}$ for $j \leqslant p$. For $p = 0$ the set of equations contains non-linear terms and hence is not easy to solve directly; it will be discussed further in section 7.9.3. For $p > 0$ the equations are linear in the unknown $a_{n,p}$ and can be solved easily provided the solutions of the sets of order less than p are known. This will be discussed further in section 7.9.4.

### 7.9.3. The zero-order terms of the series solution.

For p=0 the following set of equations is obtained

$$\left.\begin{aligned}
(1-3\lambda_{1,0})a_{o,o} + 6\, a_{o,o}\, a_{3,o} - 2\lambda_{1,o}\, a_{2,o} &= 0 \\
2\lambda_{2,0}\sqrt{a_{o,o}} + a_{1,o} + 2\lambda_{1,o} &= 0 \\
2\, a_{2,0}\sqrt{a_{o,o}} + \lambda_{2,o}\, a_{1,o} &= 0 \\
a_{1,o} + 2\, a_{2,o} + \ldots + N\, a_{N,o} &= 0 \\
a_{o,o} + a_{1,o} + a_{2,o} + \ldots\ldots + a_{N,o} &= 0 \\
M_{k,o} = (1-3\lambda_{1,o})\, J_{k,o} - \lambda_{1,o}\, P_{k,o} + Q_{k,o} &= 0 \\
(\text{for } k = 0,1,2,\ldots,N-5) &
\end{aligned}\right\} \quad (7.100)$$

In the last equation $J_{k,o}$; $P_{k,o}$ and $Q_{k,o}$ follow from equations (7.99) for p = 0. The equations (7.100) are non-linear due to the occurrence of $\sqrt{a_{o,o}}$; $a_{o,o}\, a_{3,o}$ and the $Q_{k,o}$ which contain quadratic terms in the $a_{n,o}$. However, a comparison with equations (7.74) shows that the solution for p=0 corresponds to the similar solution for $\lambda_1 = \lambda_{1,o}$ and $\lambda_2 = \lambda_{2,o}$.
These similar solutions have been discussed already in section 7.8 and hence the solutions of (7.100) can be considered as known.
From the preceding remarks it follows that the present approximate method reproduces the result known from exact solutions (see chapter 3) in this respect that a boundary layer for which $\overline{U}(\overline{x})$ and $\overline{v}_o(\overline{x})$ are given by (7.86) and (7.88) with f=1, starts at $\overline{x} = 0$ as a similar boundary layer.

### 7.9.4. The terms of order $p > 0$ of the series solution.

For $p > 0$ the following set of equations is obtained

$$(fp-1 + 3\lambda_{1,o} - 6\, a_{3,o})a_{o,p} + 2\lambda_{1,o}\, a_{2,p} - 6\, a_{o,o}\, a_{3,p}$$
$$= -3\sum_{i=1}^{p} \lambda_{1,i}\, a_{o,p-i} + 6\sum_{i=1}^{p-1} a_{o,i}\, a_{3,p-i} - 2\sum_{i=1}^{p} \lambda_{1,i}\, a_{2,p-i}$$

$$\frac{\lambda_{2,o}}{r_o}\, a_{o,p} + a_{1,p} = 2\lambda_{2,o}\,\sigma_p - 2\lambda_{1,p} - 2\sum_{i=1}^{p} \lambda_{2,i}\, r_{p-i}$$

$$\left.\begin{aligned}
&\frac{a_{2,o}}{r_o}\ a_{o,p} + \lambda_{2,o}\ a_{1,p} + 2\ r_o\ a_{2,p} = \\
&\qquad = 2\ \sigma_p a_{2,o} - 2\sum_{i=1}^{p-1} a_{2,i}\ r_{p-i} - \sum_{i=1}^{p} \lambda_{2,i}\ a_{1,p-i} \\
&\sum_{n=1}^{N} n\ a_{n,p} = 0 \\
&\sum_{n=0}^{N} a_{n,p} = 0 \\
&\text{and for } k = 0,1,2,\ldots,\ N-5 \\
&(fp-1 + 3\lambda_{1,o}) \sum_{n=0}^{N} j_{k,n}\ a_{n,p} + \lambda_{1,o} \sum_{n=1}^{N} p_{k,n}\ a_{n,p} \\
&\quad - \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} q_{k,\ell,n}\ (a_{\ell,o}\ a_{n,p} + a_{n,o}\ a_{\ell,p}) = \\
&\quad = -3\sum_{i=1}^{p} \lambda_{1,i}\ J_{k,p-i} - \sum_{i=1}^{p} \lambda_{1,i}\ P_{k,p-i} + \\
&\quad + \sum_{\ell=0}^{N} \sum_{n=\ell}^{N} q_{k,\ell,n} \sum_{i=1}^{p-1} a_{\ell,i}\ a_{n,p-i}
\end{aligned}\right\} \quad (7.101)$$

From an inspection it follows that the equations are linear in the $a_{n,p}$ with coefficients depending only on p and the leading-edge conditions (These leading-edge conditions depend on $\lambda_{1,o}$; $\lambda_{2,o}$ and N; they are given by the quantities with second subscript equal to zero).
The conditions downstream of $\bar{x} = 0$ enter only through the right-hand sides of the equations. Therefore the coefficient matrix of the equations can be inverted once for all for given leading-edge conditions. For a specific example with the same leading-edge conditions the coefficients $a_{n,p}$ can then be obtained by simple multiplications of the right hand sides of the equations with universal constants obtained from the inverse matrix.
This is analogous to what happens in Görtler's method where universal functions are used which only depend on the leading-edge conditions.
In the present approximate method it is of no use to calculate and store the universal constants for a whole series of leading-edge conditions and values of p. It is easier to store the values of the $a_{n,o}$ corresponding

to given $\lambda_{1,o}$; $\lambda_{2,o}$ and N (which in fact corresponds to the solutions for p = 0) together with a machine-program which calculates the $a_{n,p}$ for successive values of $p > 0$.

7.9.5. Comparison of the new series method with existing series methods.

From the preceding sections it follows that the series solution for the present approximate method displays several features of the exact methods discussed in chapter 3. Some important points are listed below.

Both in the exact and approximate methods the zero-order terms correspond to one of the similar solutions. In the exact methods this solution follows from a non-linear ordinary differential equation whereas in the approximate method a set of non-linear algebraic equations has to be solved.
For the exact methods further terms of the series solution are obtained from linear ordinary differential equations in which the coefficients depend on the leading-edge conditions only. The full solution is obtained by multiplication of the universal functions with constants depending on the pressure- and suction distributions downstream of $\bar{x} = 0$. In the approximate method the terms of higher order follow from a set of linear algebraic equations. The coefficient matrix of these equations only depends upon the leading-edge conditions and the order of the terms to be found.
An advantage of the present approximate method above the exact series methods is, that the calculation of higher order terms is so simple that it can be done anew for each example to be calculated whereas in the exact methods a considerable number of universal functions has to be tabulated.
A further advantage of the present series method is that the same quantities are used as in the step by step method discussed in section 7.7. Hence the series method need only be used near $\bar{x} = 0$ to start the calculation. As soon as the series is no longer sufficiently convergent the boundary layer calculation may be continued using the step by step method. In existing series methods however, an entirely different method

is used for the continuation in regions where the series is not sufficiently convergent. Such a continuation is always necessary near separation.

## 7.10. Calculation of some characteristic boundary layer parameters from the coefficients $a_n$.

In the present method the boundary layer calculation is reduced to the determination in terms of $\bar{x}$ of the coefficients $a_n$ in the polynomial expression (7.29). The familiar boundary layer parameters can easily be calculated from these coefficients. The related formulae will be summarised in the present section.

Once the $a_n$ are known the shear stress may be calculated from (7.29). Then the velocity profile follows from

$$\bar{y} = \frac{y}{\delta} = \frac{y}{x}\sqrt{\frac{Ux}{\nu}} = \int_0^{\bar{u}} \frac{d\bar{u}}{\bar{\tau}} \qquad (7.102)$$

The parameters $\delta^*$, $\theta$ and  as defined by (2.18), (2.19) and (2.22) are given by

$$\left.\begin{aligned}
\frac{\delta^*}{x}\sqrt{\frac{Ux}{\nu}} &= \frac{\delta^*}{\delta} = \int_0^1 \frac{(1-\bar{u})}{\bar{\tau}}\, d\bar{u} \\
\frac{\theta}{x}\sqrt{\frac{Ux}{\nu}} &= \frac{\theta}{\delta} = \int_0^1 \frac{\bar{u}(1-\bar{u})}{\bar{\tau}}\, d\bar{u} \\
\frac{\varepsilon}{x}\sqrt{\frac{Ux}{\nu}} &= \frac{\varepsilon}{\delta} = \int_0^1 \frac{\bar{u}(1-\bar{u}^2)}{\bar{\tau}}\, d\bar{u}
\end{aligned}\right\} \qquad (7.103)$$

The integrals in (7.103) can be found numerically using Simpson's rule for instance. The integrals have to be evaluated with some care near $\bar{u} = 1$ because $\bar{\tau} \longrightarrow 0$ for $\bar{u} \longrightarrow 1$. Therefore, in the examples to be discussed in chapter 8, the integrals were calculated using Simpson's rule from $\bar{u} = 0$ to $\bar{u} = 0.99$. For $0.99 \leqslant \bar{u} \leqslant 1.00$ the integrals were calculated as follows.

Because $\bar{\tau} \rightarrow 0$ like $1-\bar{u}$ for $\bar{u} \rightarrow 1$ (see section 7.2) the following approximation for $\bar{\tau}$ may be made in the interval $0.99 \leqslant \bar{u} \leqslant 1$

$$\bar{\tau} = 100\ \bar{\tau}_{0.99}\ (1-\bar{u}) \tag{7.104}$$

Now, using (7.104) the integrals in (7.102) and (7.103) can be found analytically for $0.99 \leqslant \bar{u} \leqslant 1$ and hence the equations reduce to

$$\left.\begin{aligned}
\frac{y}{x}\sqrt{\frac{Ux}{\nu}} &= \int_0^{0.99} \frac{d\bar{u}}{\bar{\tau}} - \frac{1}{100\ \bar{\tau}_{0.99}}\ \ln\ 100(1-\bar{u}) \\
\frac{\delta^*}{x}\sqrt{\frac{Ux}{\nu}} &= \int_0^{0.99} \frac{(1-\bar{u})d\bar{u}}{\bar{\tau}} + \frac{10^{-4}}{\bar{\tau}_{0.99}} \\
\frac{\theta}{x}\sqrt{\frac{Ux}{\nu}} &= \int_0^{0.99} \frac{\bar{u}(1-\bar{u})d\bar{u}}{\bar{\tau}} + \frac{0.995\ 10^{-4}}{\bar{\tau}_{0.99}} \\
\frac{\varepsilon}{x}\sqrt{\frac{Ux}{\nu}} &= \int_0^{0.99} \frac{\bar{u}(1-\bar{u}^2)d\bar{u}}{\bar{\tau}} + \frac{1.985033\ 10^{-4}}{\bar{\tau}_{0.99}}
\end{aligned}\right\} \tag{7.105}$$

Once the integrals have been calculated all parameters of interest can easily be found.

## 7.11. Some related methods known from the literature.

In the literature two methods are found which have some features in common with the present method. However, for so far known to the author they have not been worked out in as much detail as the present method. The first one is due to Trilling [79] who starts from Crocco's equation in the form (7.10), the compatibility condition at the wall (7.42) and the condition at the edge of the boundary layer

$$\bar{\tau} = 0 \quad \text{for} \quad \bar{u} = 1 \tag{7.106}$$

Furthermore the following approximation for $\tau$ is used

$$\tau = \tau_o + \tau_1\bar{u} + \tau_2\bar{u}^2 + \dots + \tau_6\bar{u}^6 \tag{7.107}$$

Substituting (7.107) into (7.10) and (7.42) and using (7.106) leads to an ordinary differential equation for $\tau_o(x)$ which contains the known functions $v_o(x)$, $\frac{dU}{dx}$ and their derivatives with respect to x. The application of the method seems rather cumbersome; only one example has been given in [79].

The second method has been designed by Dorodnitsyn [80]. In his method the von Kármán-Pohlhausen momentum equation (2.15) is used together with some related moment equations of the type 2.14 . The resulting equations are written in terms of $\tau$, $\frac{1}{\tau}$ and $\bar{u}$.
Then, solutions of the equations are sought of the form

$$\frac{1}{\tau} = \frac{1}{(1-\bar{u})} \quad (a_o + a_1\bar{u} + a_2\bar{u}^2 + \ldots..) \tag{7.108}$$

$$\tau = (1-\bar{u}) \quad (b_o + b_1\bar{u} + b_2\bar{u}^2 + \ldots..) \tag{7.109}$$

The coefficients $a_i$ and $b_i$ in (7.108) and (7.109) are expressed in the values of $\tau$ at some equidistant values of $\bar{u}$.
It is shown in [80] that a good agreement is obtained for the similar boundary layers corresponding to $\bar{U} = u_1 \bar{x}^{m_1}$.

Table 7.1: $A_{i,j}$ for different values of N.

$$N = N_1 \qquad \begin{bmatrix} A_{11} & A_{12} & \cdots\cdots & A_{1,N_1-2} \\ A_{21} & & & \\ ---- & & & \\ A_{N_1-2,1} & & & A_{N_1-2,N_1-2} \end{bmatrix}$$

$$N = 5 \qquad \begin{bmatrix} +2.5 & -27.5 & +105 \\ -6 & +60 & -210 \\ +3.5 & -31.5 & +105 \end{bmatrix}$$

$$N = 6 \qquad \begin{bmatrix} -5 & +100 & +1680 & -2520 \\ +21 & -399 & -5880 & +9072 \\ -28 & +504 & +6720 & -10584 \\ +12 & -204 & -2520 & +4032 \end{bmatrix}$$

$$N = 7 \qquad \begin{bmatrix} +8.75 & -271.25 & +13230 & -44100 & +34650 \\ -56 & +1680 & -70560 & +241920 & -194040 \\ +126 & -3654 & +136080 & -476280 & +388080 \\ -120 & +3360 & -113400 & +403200 & -332640 \\ +41.25 & -1113.75 & +34650 & -124740 & +103950 \end{bmatrix}$$

$$N = 8 \qquad \begin{bmatrix} -14 & +616 & +70560 & -388080 & +665280 & -360360 \\ +126 & -5418 & -529200 & +2993760 & -5239080 & +2882880 \\ -420 & +17640 & +1512000 & -8731800 & +15523200 & -8648640 \\ +660 & -27060 & -2079000 & +12196800 & -21954240 & +12355200 \\ -495 & +19800 & +1386000 & -8232840 & +14968800 & -8494200 \\ +143 & -5577 & -360360 & +2162160 & -3963960 & +2265120 \end{bmatrix}$$

$$N = 9 \qquad \begin{bmatrix} +21 & -1239 & +291060 & -2328480 & +6486480 & -7567560 & +3153150 \\ -252 & +14616 & -2910600 & +23950080 & -68108040 & +80720640 & -34054020 \\ +1155 & -65835 & +11434500 & -96049800 & +277477200 & -332972640 & +141891750 \\ -2640 & +147840 & -22869000 & +195148800 & -570810240 & +691891200 & -297297000 \\ +3217.5 & -176962.5 & +24774750 & -214053840 & +632431800 & -772972200 & +334459125 \\ -2002 & +108108 & -13873860 & +121080960 & -360720360 & +443963520 & -193243050 \\ +500.5 & -26526.5 & +3153150 & -27747720 & +83243160 & -103062960 & +45090045 \end{bmatrix}$$