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THEORETICAL AND EXPERIMENTAL INVESTIGATIONS OF INCOMPRESSIBLE LAMINAR BOUNDARY LAYERS WITH AND WITHOUT SUCTION

Ph.D THESIS

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This PDF-file contains chapter 3:

Some known methods to obtain solutions of the boundary layer equations

3. Some known methods to obtain accurate solutions of the boundary layer equations.

3.1. Similar solutions.

3.1.1. General.

The partial differential equations (2.5) and (2.7) can be reduced to one ordinary differential equation for "similar" boundary layer flows. Here similarity means that the velocity profiles for all different stations x can be reduced to a single curve by rescaling the y and u variables with scaling factors depending on x. Well known examples are the flat plate boundary layer flow (Blasius, 1908, [33]) and the plane stagnation point (Hiemenz, 1921, [34]). A detailed discussion of the occurrence of similar boundary layers may be found in Goldstein [35], Mangler [36,37], Schlichting [7] (chapter 8) and [29] (chapter V).

It was found that similarity is only possible if the velocity U at the edge of the boundary layer is given by one of the following three expressions.

$$U = u_1 \, x^{m_1} \tag{3.1}$$

$$U = u_1 \, x^{-1} \tag{3.2}$$

$$U = u_1 \, e^{u_2 x} \tag{3.3}$$

where u_1, u_2 and m_1 are constants.

Expression (3.1) corresponds to the potential flow in the neighbourhood of the vertex of a wedge with an angle $\pi\beta$ where

$$\beta = \frac{2\, m_1}{m_1+1} \tag{3.4}$$

The related boundary layer flows have been calculated by Hartree; they will be discussed further in section 3.1.2.

In potential flow, (3.2) corresponds to a line source or-sink and hence the related boundary layer flow is that between non-parallel plane walls. It will be discussed extensively in chapter 6.

Finally (3.3) describes the potential flow through a channel with curved walls (Goldstein [35], Mangler [36,37]); it will not be discussed further in the present work.

3.1.2. Hartree's boundary layer flows for $U = u_1 x^{m_1}$.

For the wedge-type similar flows $U = u_1 x^{m_1}$ the proper non-dimensional variables are η and $f(\eta)$ defined by

$$\eta = y \sqrt{\frac{m_1+1}{2} \frac{U}{\nu x}} \tag{3.5}$$

and
$$\psi(x,y) = \sqrt{\frac{2\nu u_1}{m_1+1}}\; x^{\frac{m_1+1}{2}} f(\eta) \tag{3.6}$$

In (3.6) ψ denotes the streamfunction which is related to the velocity components u and v by

$$u = +\frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{3.7}$$

From (3.5), (3.6) and (3.7) it follows that

$$u = U\, f'(\eta) \tag{3.8}$$

and
$$v = -\sqrt{\frac{m_1+1}{2}\,\nu\, u_1\, x^{m_1-1}}\left[f + \frac{m_1-1}{m_1+1}\,\eta\, f'\right] \tag{3.9}$$

where primes denote differentiation with respect to η.
With (3.7) the continuity equation (2.5) is already satisfied. The boundary layer equation (2.7) reduces to

$$f''' + f\,f'' + \beta(1 - f'^2) = 0 \tag{3.10}$$

The boundary conditions (2.8) and (2.9) lead to

$$\eta = 0: \quad f(o) = -v_o \sqrt{\frac{2}{m_1+1}\,\frac{x^{1-m_1}}{\nu\, u_1}} \quad ; \quad f'(o) = 0 \tag{3.11}$$

$$\eta \to \infty: \quad f' \longrightarrow 1 \tag{3.12}$$

Equation (3.10) was first given - in a slightly different form - by Falkner and Skan in 1930 [38] and is usually called after them. Special cases of (3.10) had been given earlier for the flat plate (β=0) by Blasius and for the stagnation point flow (β=1) by Hiemenz (see sections 3.1.3 and 3.1.4). Solutions of (3.10) for the no-suction case (f(o) = 0) were obtained by Hartree in 1937 [39] using a differential analyser. He found that for $\beta \geqslant 0$ the boundary conditions (3.11) and (3.12) specify a unique solution of (3.10) whereas for $\beta < 0$ an infinity of solutions exists, all satisfying the boundary conditions. This is illustrated in fig. 3.1 where velocity profiles are sketched which correspond to solutions of (3.10) satisfying 3.11. It is seen from the figure that for $\beta \geqslant 0$ there is only one solution for which (3.12) is fulfilled; for $\beta < 0$ all solutions satisfy (3.12). For negative values of β Hartree selected as the relevant solution that one which satisfied the extra condition (see fig. 3.1):

" $f' = \frac{u}{U} \longrightarrow 1$ as fast as possible without making an overshoot"

With this choice the skin friction considered as a function of β becomes continuous at β=0. For $\beta = -0.198838$ the solution determined in this way gives f''(o) = 0 indicating that a boundary layer occurs which is on the verge of separation at all values of x.
Subsequent to Hartree's work many investigations have been made of the characteristic features of solutions of (3.10). An extensive review may be found in [29], chapter V.21. The extra condition at the edge of the boundary layer introduced by Hartree to obtain a unique solution has become known as the "Hartree condition". A mathematical justification for its use has been described recently by Goldstein [40].
It follows from (3.11) that in the case of suction similar solutions can be found if the suction distribution $v_o(x)$ is chosen in such a way that f(o) is constant. The permissible suction distribution follows then from

$$-v_o(x) = \text{constant} \; . \; x^{\frac{m_1-1}{2}} \qquad (3.13)$$

Solutions of (3.10) for the case of suction have been obtained by various authors. A review of work in this field may be found in [29], chapter V.21.

3.1.3. Blasius' solution for the flat plate without suction.

For the flat plate $\frac{dU}{dx} = 0$ and hence β=0; this reduces the Falkner-Skan equation (3.10) to the well known Blasius equation

$$f''' + f\,f'' = 0 \tag{3.14}$$

Boundary conditions for solutions of (3.14) in the no-suction case are

$$\eta = 0 : \quad f = f' = 0 \tag{3.15}$$

$$\eta \rightarrow \infty : \quad f' \longrightarrow 1 \tag{3.16}$$

The solution of (3.14) has been given already by Blasius in 1908 [33]. Improved solutions were given later by Töpfer, Hartree, Howarth, Smith and others. (see [29]).
Experimental observations of the flat plate boundary layer were made by Burgers and van der Hegge Zijnen in 1924 [41] and later by Hansen [42]. These investigations fully confirmed the validity of Blasius' solution at least for not too high values of the Reynolds number $\frac{Ux}{\nu}$. Fig. 3.2 shows the boundary layer velocity profile according to this theory as compared with results of a recent experimental investigation in the low speed wind tunnel of the Department of Aeronautical Engineering at Delft Technological University (unpublished).
According to the theory the shearing stress at the wall is given by

$$\frac{\tau_o}{\rho U^2} = 0.33206 \sqrt{\frac{\nu}{Ux}} \tag{3.17}$$

Upon integration of (3.17) x = 0 to x = c the friction drag coefficient c_{d_f} of one side of a plate with unit span and length c is found to be

$$c_{d_f} = \frac{\int_o^c \tau_o dx}{\frac{1}{2}\rho U^2 c} = \frac{1.3282}{\sqrt{\frac{Uc}{\nu}}} \tag{3.18}$$

Experimental results for the friction drag, taken from the measurements already referred to, are given as fig. 3.3.
It should be noted that for airfoil sections with unit span the drag

coefficient is defined as the drag due to both the upper and the lower surface divided by $\frac{1}{2}\rho U^2 c$ where c is the chord length of the airfoil. Hence, if the flat plate is considered as an airfoil section with zero thickness the drag coefficient should be given twice the value following from (3.18).

3.1.4. The plane stagnation point without suction.

For a plane stagnation point U varies linearly with x as

$$U = u_1\, x \tag{3.19}$$

and hence $\beta = 1$. In this case (3.10) reduces to

$$f''' + f\, f'' + 1 - f'^2 = 0 \tag{3.20}$$

with boundary conditions (3.15) and (3.16). This equation was obtained and solved in 1911 by Hiemenz [34]; later investigations were made by Hartree, Smith and many others (see section 3.1.2.). Results of these calculations will be given in chapter 8.

3.1.5. The asymptotic suction boundary layer.

A very special similar solution is given by the asymptotic suction boundary layer. Experimentally this layer is expected to occur far from the leading edge of a porous flat plate with constant suction velocity v_o (note that v_o is negative for suction).
Assuming $\frac{\partial}{\partial x} = 0$ in (2.5) and (2.7) it is easily found that

$$\frac{u}{U} = 1 - e^{\frac{v_o y}{\nu}} \tag{3.21}$$

The solution (3.2.1) is due to Griffith and Meredith [8]. The special feature of this boundary layer is that the velocity profiles at different values of x are not only similar but even identical. (see also section 8.11). From (3.21) it is easily found that for this case

$$\frac{-v_o \delta^*}{\nu} = 1 \;;\quad \frac{-v_o \theta}{\nu} = 0.5 \;;\; H = 2 \text{ and } \ell = \frac{\tau_o \theta}{\mu U} = 0.25$$

3.2. Solutions in series.

3.2.1. General.

In section 3.1 it was indicated that for a number of special functions $U(x)$ and $v_o(x)$ similar boundary layers are obtained for which the governing equations (2.5) and (2.7) are reduced to one ordinary differential equation. For more general functions $U(x)$ and $v_o(x)$ it is possible to obtain solutions of the boundary layer equations by solving a series of ordinary differential equations. In what follows some examples will be given.

3.2.2. Blasius' series.

For blunt-nosed bodies which are symmetrical with respect to the direction of the oncoming flow the velocity U at the edge of the boundary layer can be developed in a power series of the form

$$\bar{U} = u_1 \bar{x} + u_3 \bar{x}^3 + u_5 \bar{x}^5 + \ldots\ldots = \sum_{n=0} u_{2n+1} \bar{x}^{2n+1} \tag{3.22}$$

The coefficients u_{2n+1} depend on the shape of the body. In (3.22) $\bar{x} = x/c$ and $\bar{U} = U/U_\infty$ where c and U_∞ are a constant reference length and -speed respectively. Using a non-dimensional wall distance η defined by

$$\eta = y \sqrt{\frac{u_1}{\nu c}} \tag{3.23}$$

the streamfunction ψ can be written in the form

$$\psi = (u_1 \nu c)^{\frac{1}{2}} \sum_{n=0} \bar{x}^{2n+1} F_{2n+1}(\eta) \tag{3.24}$$

If, using (3.7) the expressions (3.22), (3.23) and (3.24) are introduced into the boundary layer equation (2.7) and the coefficients of the various powers of $\bar{x}$ are equated to zero a sequence of ordinary differential equations for the functions F_{2n+1} is obtained. These equations can be solved in succession giving F_1, F_3, ... etc. The procedure indicated above has been given by Blasius in 1908 [33].
The equation for F_1 is found to be non-linear and identical to equation

(3.20) given by Hiemenz for the plane stagnation point flow. Hence it follows that the boundary layer on any symmetrical blunt-nosed body starts at the leading-edge as the plane stagnation point flow discussed in section 3.1.4.

The functions F_{2n+1} for $n > 0$ are obtained from linear differential equations in which the coefficients are determined by the functions F_{2k+1} with $k < n$. Hiemenz showed [34] that the solution of the differential equations for F_1 and F_3 can be made independent of u_1 and u_3 by introducing new functions f_1 and f_3 defined by

$$
\begin{aligned}
F_1 &= f_1 \\
F_3 &= 4\,\frac{u_3}{u_1}\,f_3
\end{aligned}
\qquad (3.25)
$$

Later Howarth showed that all the functions F_{2n+1} can be written as sums of universal functions which are independent of the u_{2n+1} and hence can be calculated once for all. Calculations were made by Howarth, Frössling, Ulrich and most recently by Tifford. At present the functions are available up to and including $n = 5$; hence six terms of the series (3.24) can be determined. References to the investigations mentioned above and abstracts of Tiffords tables may be found in Curle [28], chapter 2; (see also [7] and [29]).

A similar procedure can be used for the non-symmetrical case when also even powers of $\bar{x}$ occur in the power series development of U (Howarth, [43]). However, only very few of the universal functions have been calculated.

3.2.3. Series solution from a cusped leading-edge.

The procedure given by Blasius for bodies with a blunt leading-edge can be generalised to bodies with any wedge-shaped leading edge for which

$$
U = u_1\,\bar{x}^{m_1}\left[1 + a_1\,\bar{x} + a_2\,\bar{x}^2 + \ldots\ldots\right] \qquad (3.26)
$$

In this case also it may be expected that the boundary layer calculation is reduced to the solution of a series of ordinary differential equations. The first of these equations then would be the Falkner-Skan equation (3.10) with β determined by m_1 according to (3.4). Hence, it may be assumed that the boundary layer on a body with a cusped leading-edge, for which $m_1 = \beta = 0$, would start like the Blasius flat plate boundary layer.

As far as the author knows, this case has not been worked out in as much generality and detail as the Blasius series. Only for the special case $\overline{U} = 1 - \overline{x}^j$ calculations have been made by Howarth [44] for j = 1 and by Tani [45] for j = 2, 4 and 8. Their results show indeed that the first differential equation of the series thus obtained is the Blasius equation (3.14); the remaining equations are linear. Results of the calculations by Howarth and Tani will be given in chapter 8.

3.2.4. Görtler's series method.

The most refined application of the series method available up till now is due to Görtler [46, 47, 48, 49] ; see also Schlichting [7], chapter 9. This method can be applied to any wedge-shaped leading-edge; it contains the blunt and cusped leading-edge as special cases.

Görtler introduces new variables ξ and η by

$$\xi = \frac{1}{\nu} \int_0^x U \, dx \tag{3.27}$$

$$\eta = yU \left(2\nu \int_0^x U \, dx\right)^{-\frac{1}{2}} \tag{3.28}$$

The streamfunction ψ is written in the form

$$\psi(x,y) = \nu \sqrt{2\xi} \;\; F(\xi, \eta) \tag{3.29}$$

Introduction of (3.27) through (3.29) in the boundary layer equation (2.7) leads to the following differential equation for the non-dimensional streamfunction $F(\xi, \eta)$:

$$F_{\eta\eta\eta} + FF_{\eta\eta} + \beta(\xi)\left[1 - F_\eta^{\;2}\right] = 2\xi\left[F_\eta F_{\xi\eta} - F_\xi F_{\eta\eta}\right] \tag{3.30}$$

In this equation $\beta(\xi)$ depends on the given pressure distribution according to

$$\beta(\xi) = \frac{2 \frac{dU}{dx} \int_0^x U \, dx}{U^2} \tag{3.31}$$

Boundary conditions for (3.30) in the no-suction case are:

$$\eta = 0 : \quad F = 0 \quad ; \quad F_\eta = 0 \tag{3.32}$$

$$\eta \rightarrow \infty : \quad F_\eta \rightarrow 0 \tag{3.33}$$

For the similar boundary layers corresponding to $U = u_1 \, x^{m_1}$ the function $\beta(\xi)$ becomes a constant; eq. (3.30) then reduces to the Falkner-Skan equation (3.10). For functions U(x) of the form

$$U = x^{m_1} \left[s_o + s_{\frac{1}{2}} \, x^{\frac{m_1+1}{2}} + s_1 \, x^{m_1+1} + s_{\frac{3}{2}} \, x^{\frac{3}{2}(m_1+1)} + \ldots\ldots \right] \tag{3.34}$$

with $s_o \neq 0$, $m_1 \neq -1$ the function $\beta(\xi)$ is given by

$$\beta(\xi) = \beta_o + \beta_{\frac{1}{2}} \xi^{\frac{1}{2}} + \beta_1 \, \xi^1 + \beta_{\frac{3}{2}} \, \xi^{\frac{3}{2}} + \ldots\ldots \tag{3.35}$$

The coefficients in the right-hand side of (3.35) follow from the coefficients of (3.34); especially β_o is given by

$$\beta_o = \frac{2 \, m_1}{m_1+1} \tag{3.36}$$

Görtler assumes the following series for the streamfunction $F(\xi, \eta)$

$$F(\xi, \eta) = F_o(\eta) + F_{\frac{1}{2}}(\eta) \, \xi^{\frac{1}{2}} + F_1(\eta) \, \xi^1 + F_{\frac{3}{2}}(\eta) \, \xi^{\frac{3}{2}} + \ldots \tag{3.37}$$

When this expression is substituted into (3.30) and the coefficients of various powers of ξ are equated to zero a series of differential equations is obtained. The first of these equations is non-linear and contains only F_o and β_o; it is identical to the Falkner-Skan equation (3.10) for $\beta = \beta_o$. Hence it follows that for bodies allowing an expansion

(3.34) for $U(x)$ the boundary layer at $x = 0$ starts as one of Hartree's boundary layers. The differential equations for the functions F of higher order are linear, the coefficients of the equations depend on the F's of lower order. It was shown by Görtler that the functions F can be split up into universal functions which depend only on β_o. Hence these functions can be tabulated once for all for each value of the leading-edge angle $\pi \beta_o$. For $\beta_o = 0$ and $\beta_o = 1$ the universal functions are available to calculate F_n for $n = 0, 1, \ldots, 5$; for $\beta_o = 1$ sufficient functions have been calculated to form F_n for $n = 0, \frac{1}{2}, 1, \ldots, 2$ [48]. With the aid of the tabulated functions boundary layer calculations can easily be made for $U(x)$ conforming to (3.34); however in general the results are not sufficiently accurate near a separation point (section 3.2.5).

The series method was extended by Görtler to the case of suction in [49]. For pressure distributions given by

$$U = x^{m_1} \left[s_o + s_1 x^{m_1+1} + s_2 x^{2(m_1+1)} + \ldots \right] \tag{3.38}$$

the permissible suction distribution follows from

$$v_o = x^{\frac{m_1-1}{2}} \left[\sigma_o + \sigma_1 x^{m_1+1} + \sigma_2 x^{2(m_1+1)} + \ldots \right] \tag{3.39}$$

The number of universal functions to be calculated is becoming very large in the case of suction and - for so far the author knows - these calculations have not yet been performed. Therefore this method will not be discussed further. Results of Görtler's method for some cases without suction will be mentioned in chapter 8 of the present work.

3.2.5. Disadvantages of the series methods.

For slender bodies like airfoil sections it is impossible to represent U by one of the series (3.22) or (3.34) with a resonable number of terms. Hence the series methods can not be used for practical boundary layer calculations. Of course they remain useful for small values of x to start the calculation near the leading-edge.

Another disadvantage is the following. Even if an expansion (3.22), (3.26) or (3.34) is possible with a small number of terms the number of differential equations to be solved is in principle infinitely large. The relatively small number of universal functions which is available, is in general sufficient for boundary layer calculations not too near separation. However, when separation is approached more functions are needed to obtain sufficient accuracy. Therefore in this region the available series methods have to be supplemented by different calculation methods.

3.3. Finite difference methods.

Various authors devised methods to solve the boundary layer equations with finite difference methods. A discussion of these methods is outside the scope of the present work however. Reviews of available methods may be found in [29]. Where they are available results obtained from these methods will be used in chapter 8 to test the accuracy of the new calculation methods.

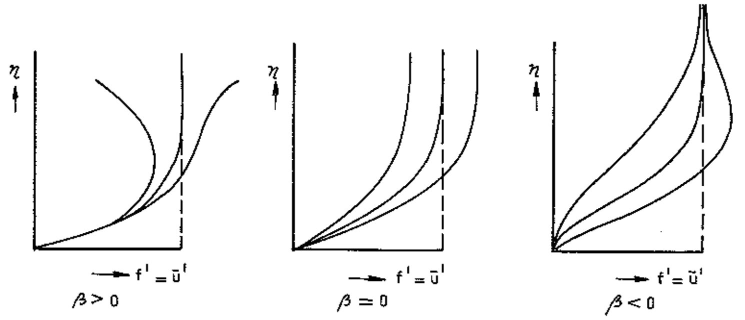

FIG. 3.1: TYPICAL SOLUTIONS OF THE FALKNER-SKAN EQUATION (3.10)

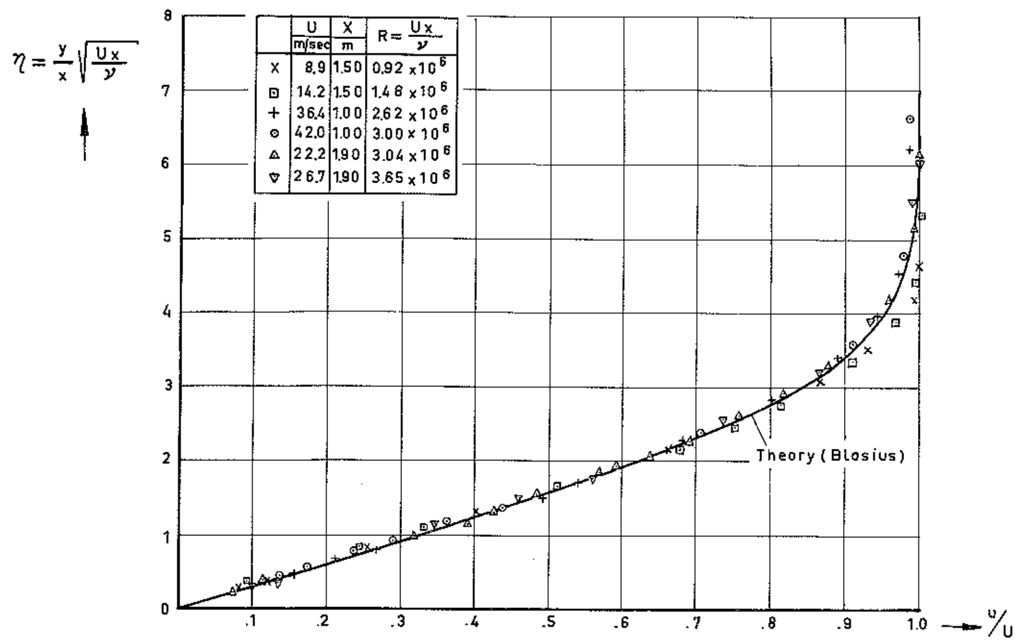

FIG. 3.2: THE VELOCITY PROFILE IN THE LAMINAR BOUNDARY LAYER ON A FLAT PLATE WITHOUT SUCTION.

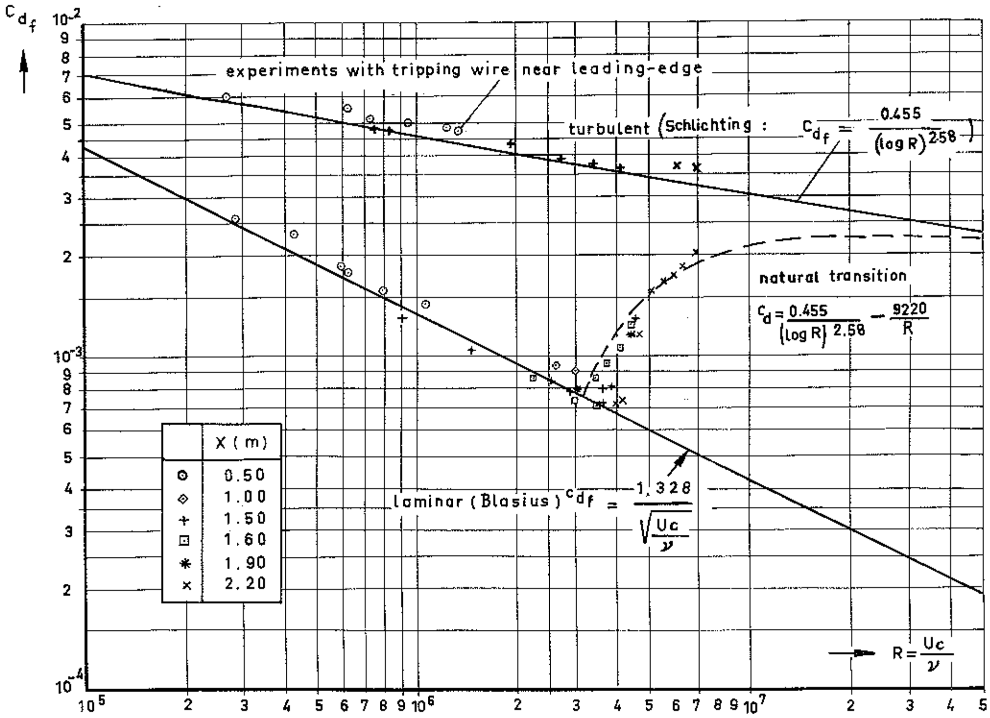

FIG. 3.3 : SKIN FRICTION COEFFICIENT FOR THE FLAT PLATE WITHOUT SUCTION.