CIEM5110-2: FEM, lecture 5.2

Introduction to nonlinear material models

Iuri Rocha and Frans van der Meer

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Agenda for today

1. Nonlinear material models — solver interface
2. An overview of different strategies for modeling nonlinear materials
3. Illustration: 1D plasticity

CIEM5110-2 workshops and lectures

	(Theory)	SolidModel (1.2, 5.2)	FrameModel (4.1, 4.2)	TimoshenkoModel (2.1)
SolverModule	(2.2)	3.2	3.2	3.2
NonlinModule	(3.1, 5.2)	6.1	4.1 + 4.2 + 5.1	
ArclenModule	(4.2)		4.2	
LinBuckModule	(4.1)		4.1 + 5.1	
ModeShapeModule	(6.2)	7.1	8.2	
ExplicitTimeModule	(6.2)		7.2 + 8.2	

Recap — Linear FEM

This is the general discretized equilibrium equation:

$$\underbrace{\int_\Omega \mathbf{B}^T \boldsymbol{\sigma}\, \mathrm{d}\Omega}_{\mathbf{f}_{\text{int}}} = \underbrace{\int_\Omega \mathbf{N}^T \mathbf{b}\, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t}\, \mathrm{d}\Gamma}_{\mathbf{f}_{\text{ext}}}$$

Assuming linear elasticity, we could substitute $\boldsymbol{\sigma} = \mathbf{DBa}$ to get

$$\int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B}\, \mathrm{d}\Omega\, \mathbf{a} = \int_\Omega \mathbf{N}^T \mathbf{b}\, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t}\, \mathrm{d}\Gamma \qquad \Rightarrow \qquad \mathbf{Ka} = \mathbf{f}_{\text{ext}}$$

Linearity is assumed twice there

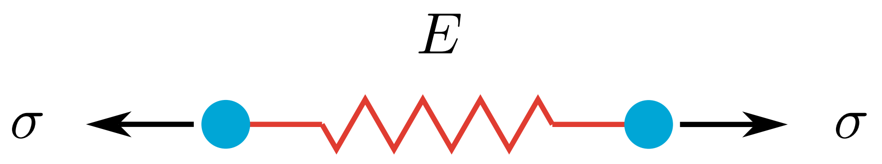
$\boldsymbol{\varepsilon} = \mathbf{Ba}$ (kinematic relation)

and

$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$ (constitutive relation)

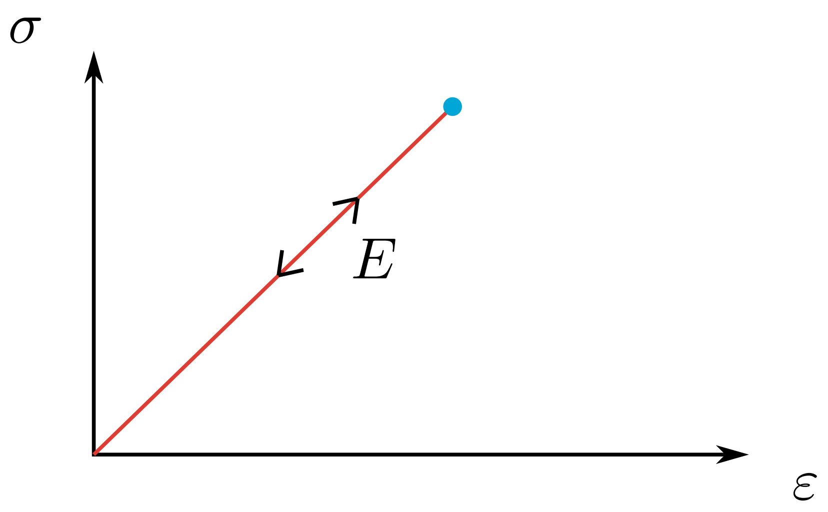
Recap — Elasticity

Constant stiffness, full reversibility



In Voigt notation in 2D plane stress:

$$\mathbf{D} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{E\nu}{1-\nu^2} & 0 \\ \frac{E\nu}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix}$$



In Voigt notation in 3D:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

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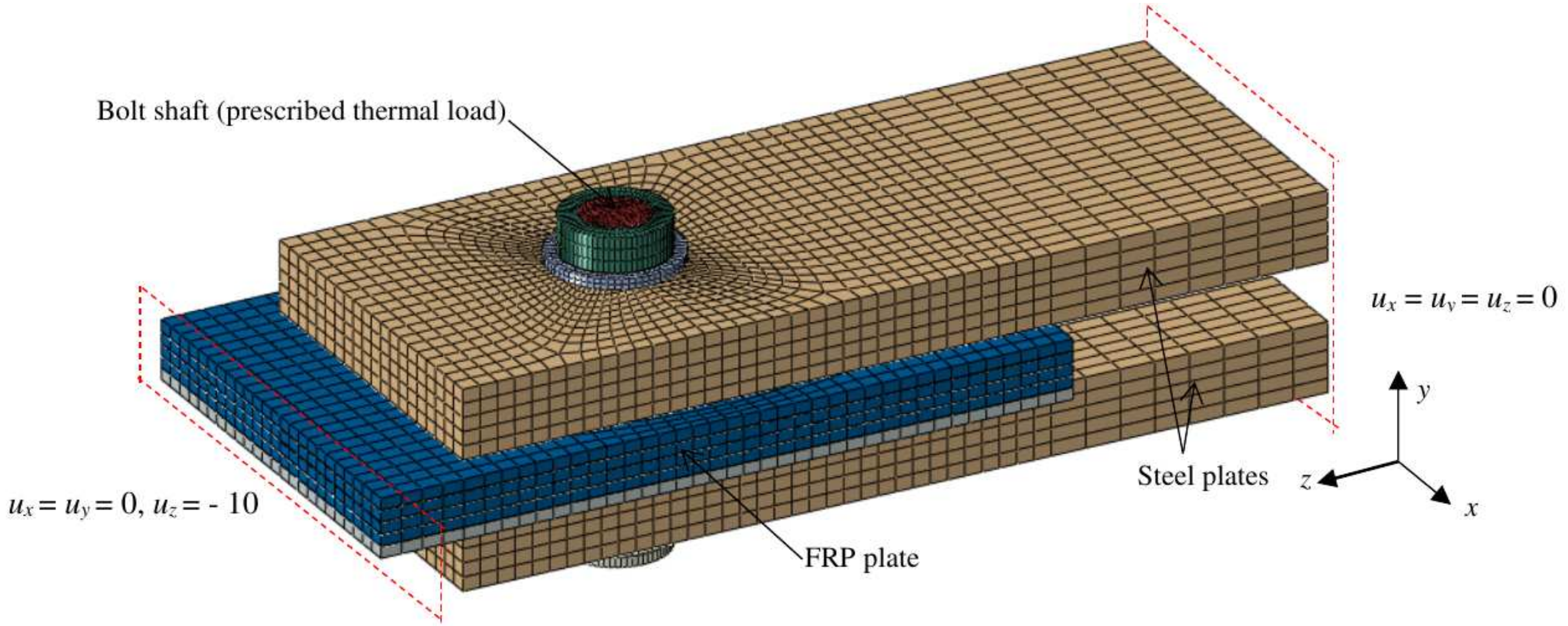
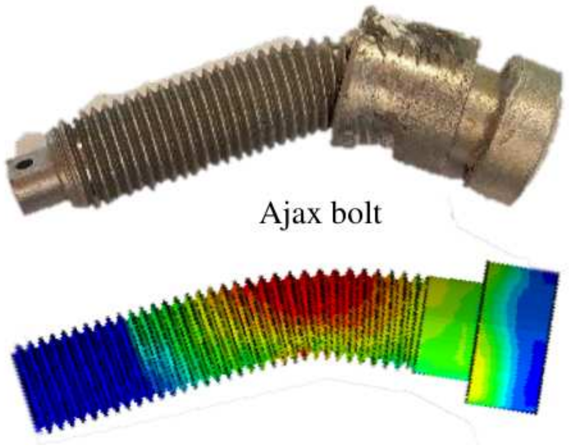
Example: Bolted joints, Fruzsina Csillag (2018)

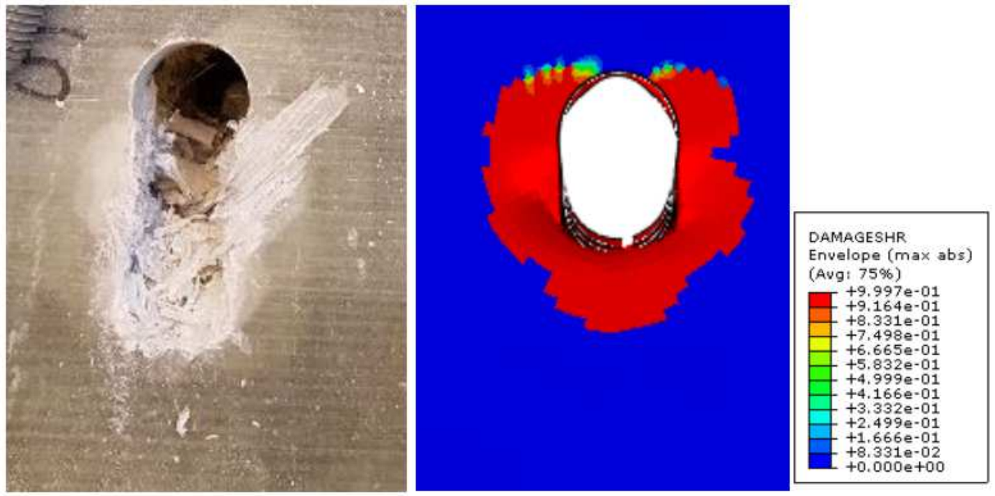
Objective: investigate the behavior of FRP-steel bolted connections



Setup



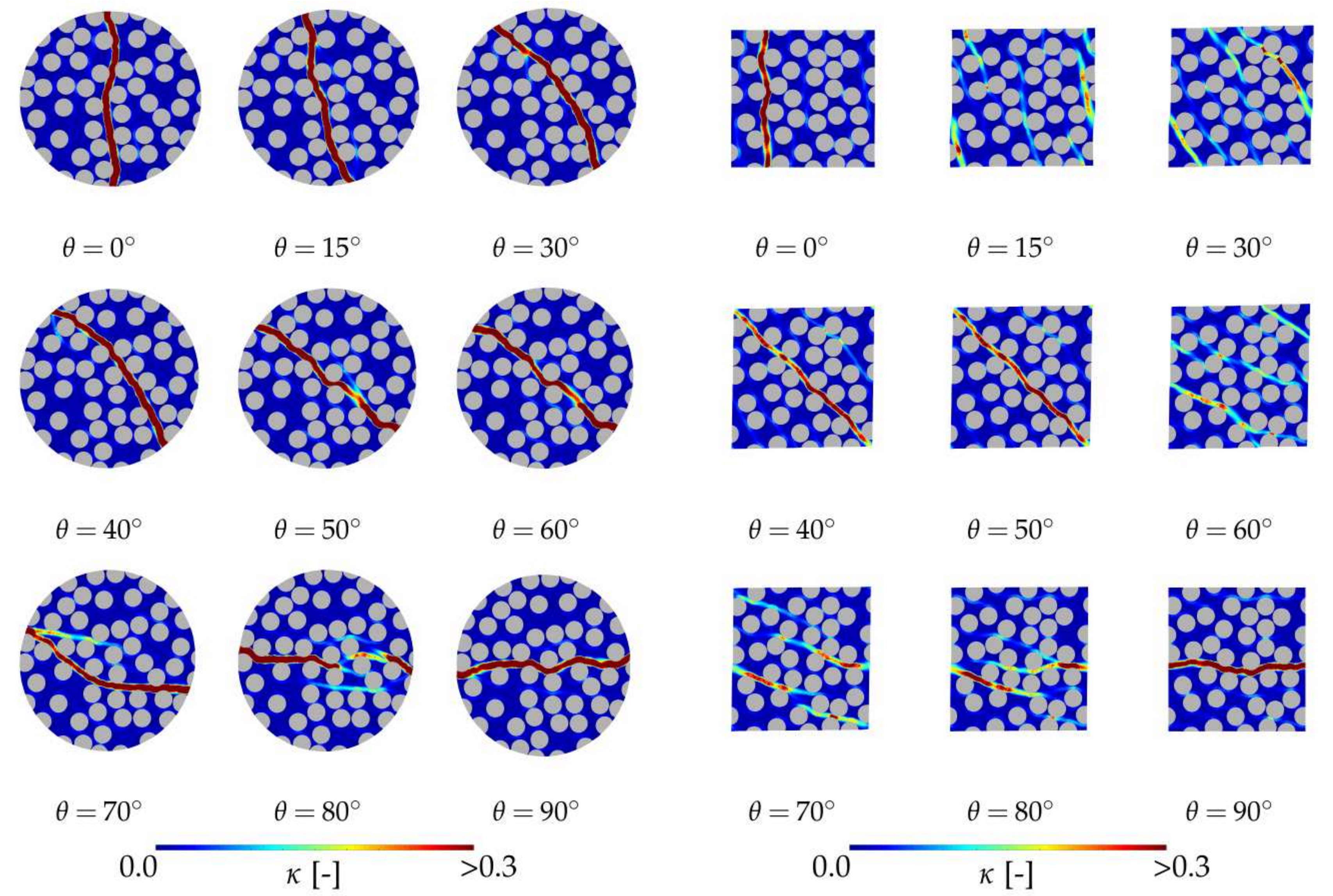
Deformation of the bolt



Damage of the FRP plate

Analysis type: nonlinear analysis (plasticity, damage, contact)

Example: Circular micromodels, Pieter Hofman (2021)

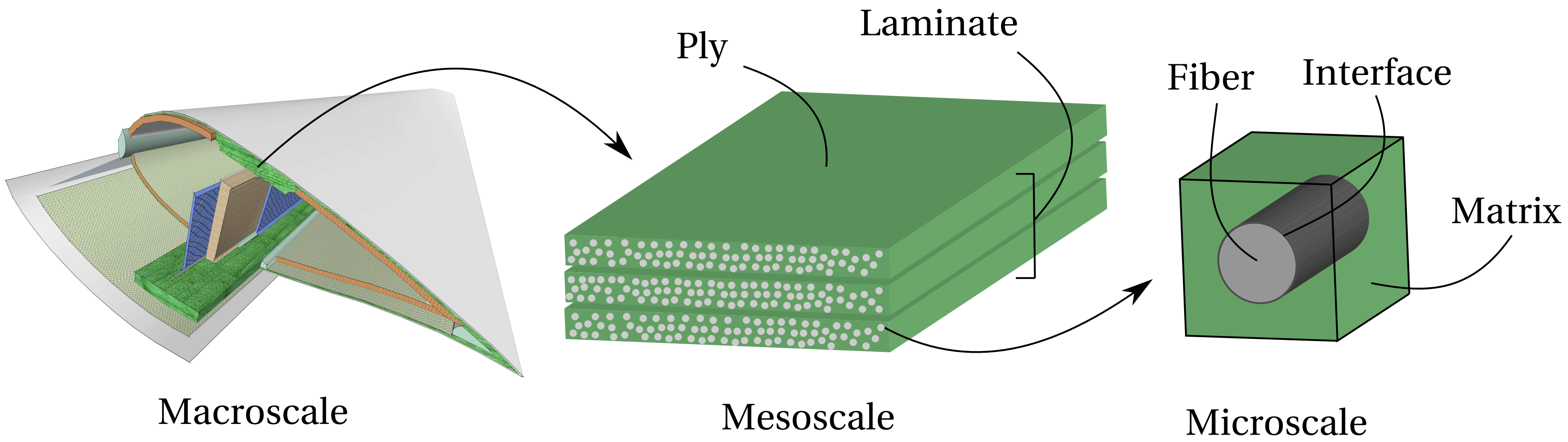


Objective:
make a micromodel that gives the same failure response in all directions

Analysis type:
material nonlinear analysis

Example: Aging of composites, Iuri Rocha (2014-2018)

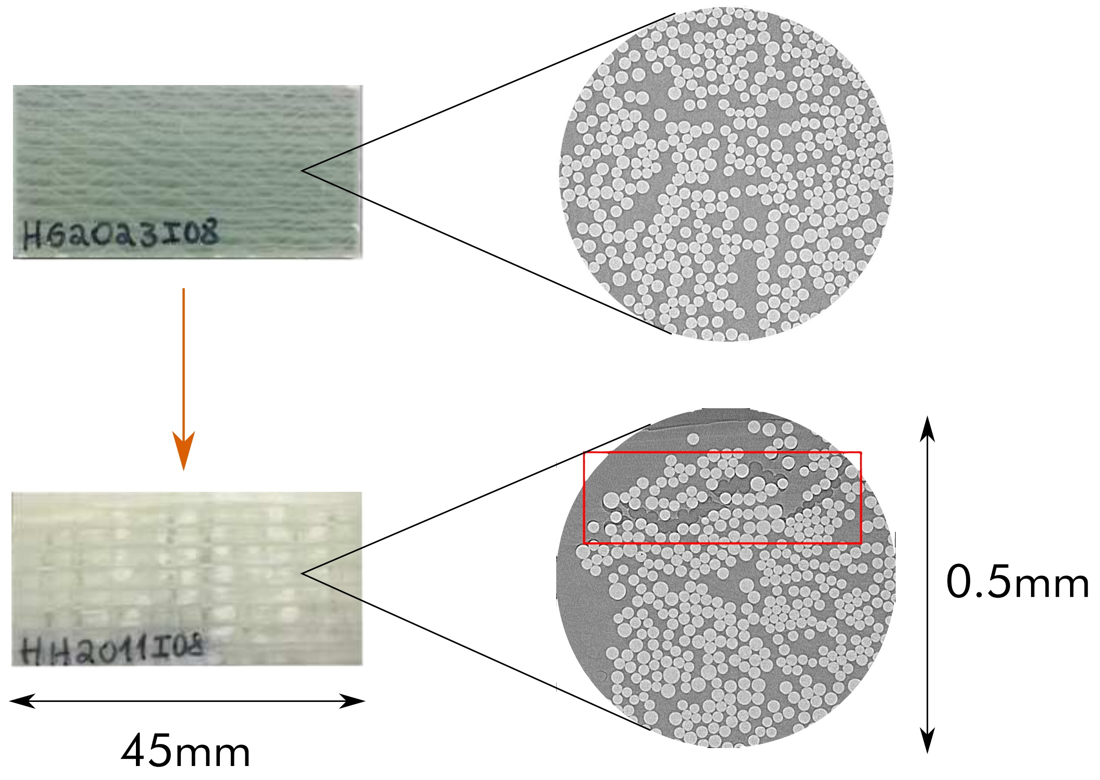
Making wind turbines last longer by understanding material behavior:



Example: Aging of composites, Iuri Rocha (2014-2018)

Making wind turbines last longer by understanding material behavior:

- Experimental observation: the material loses half of its strength after exposure to hot water



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Making wind turbines last longer by understanding material behavior:

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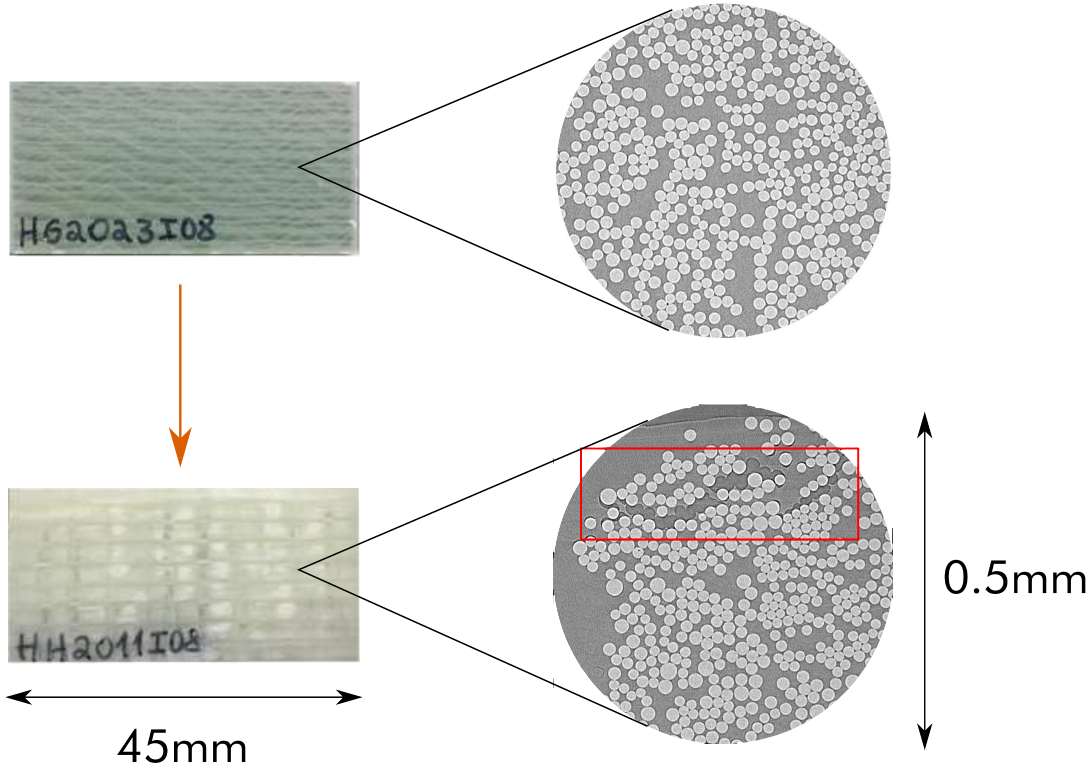
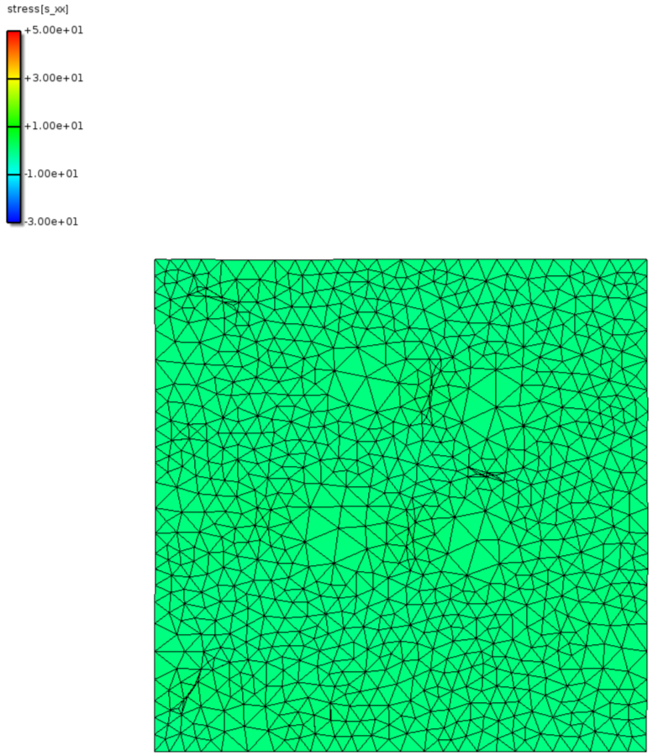
- Translating small scale phenomena back to the higher scale





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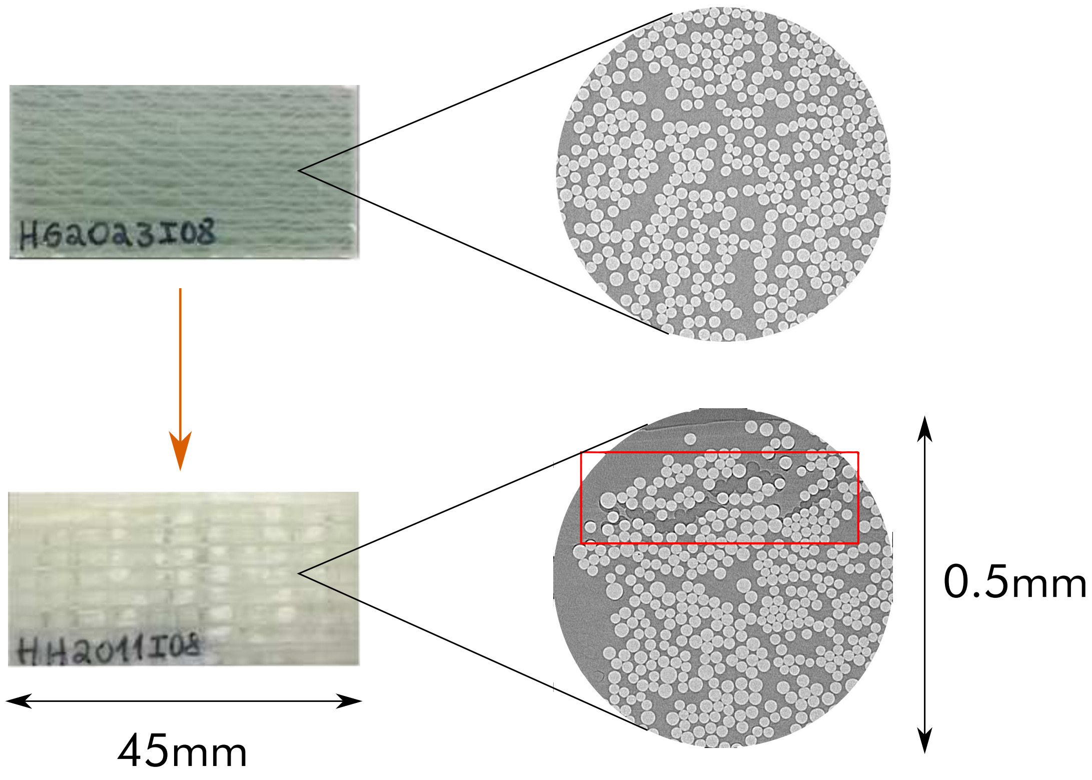

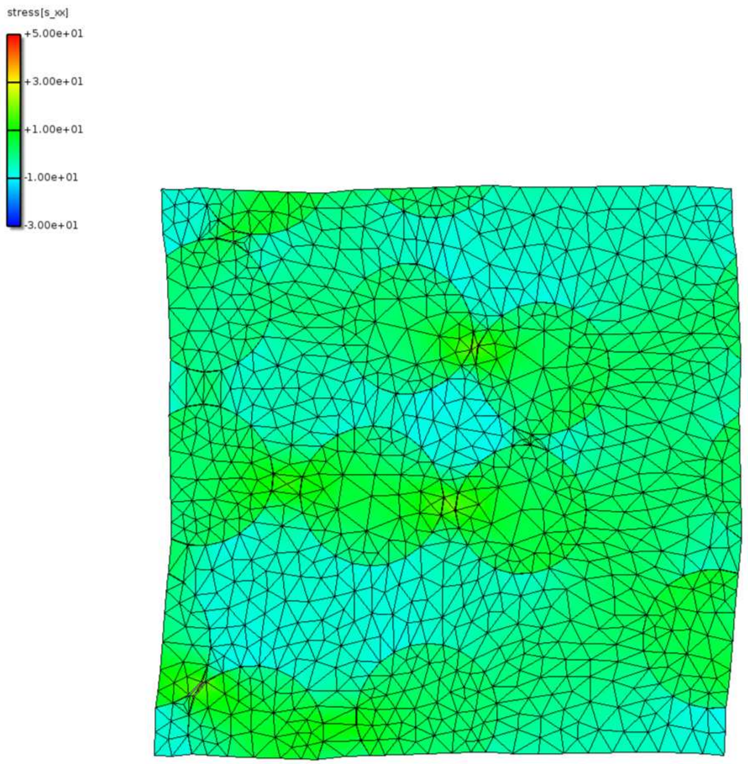

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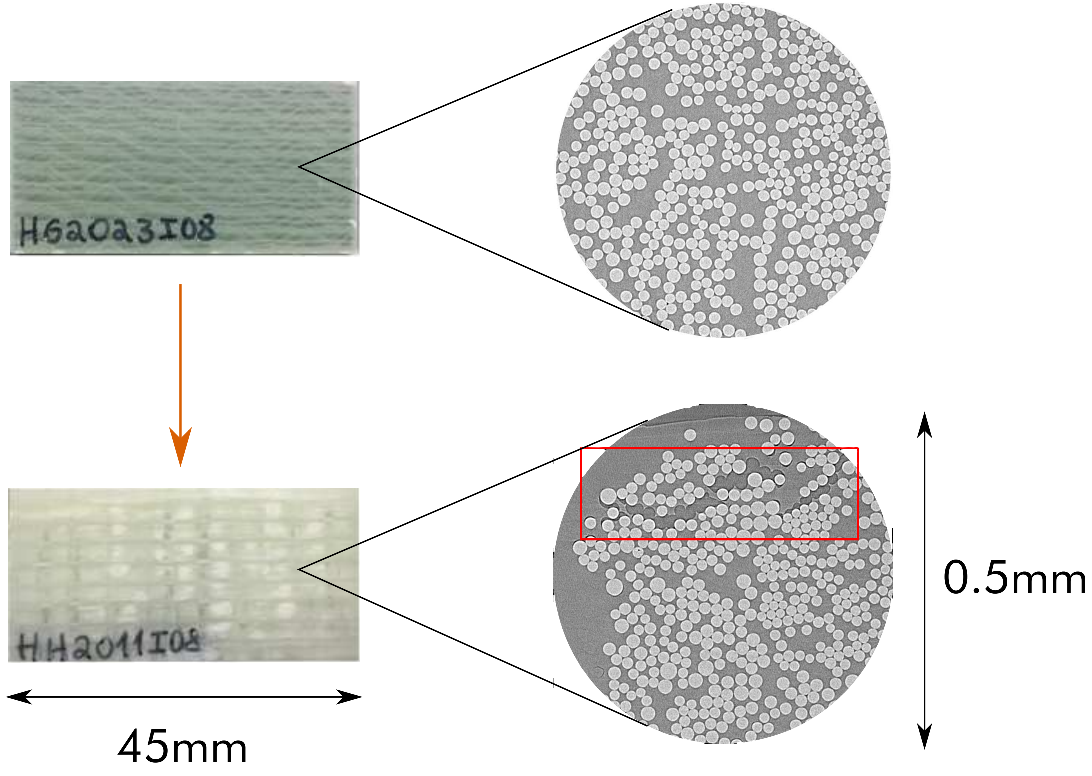

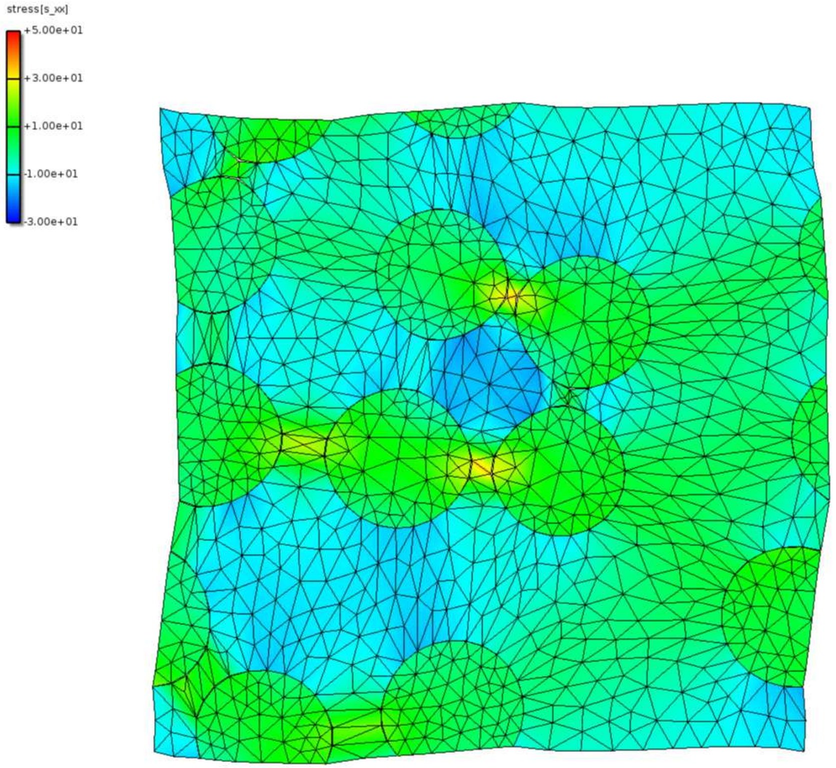

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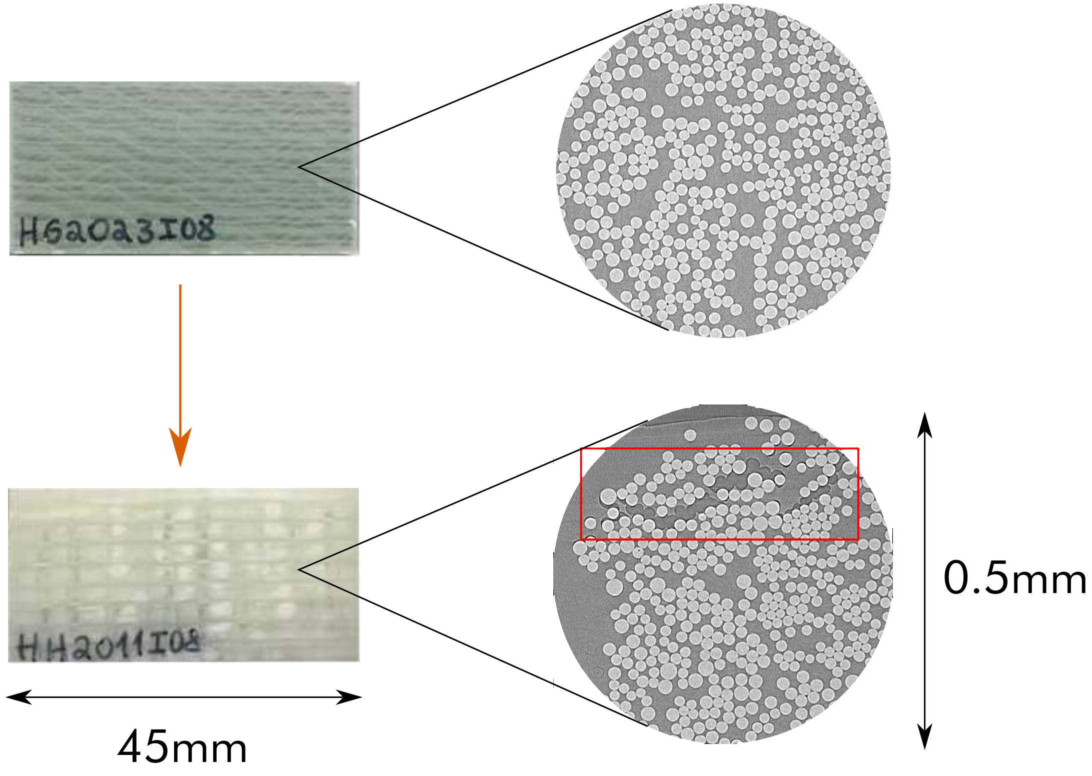

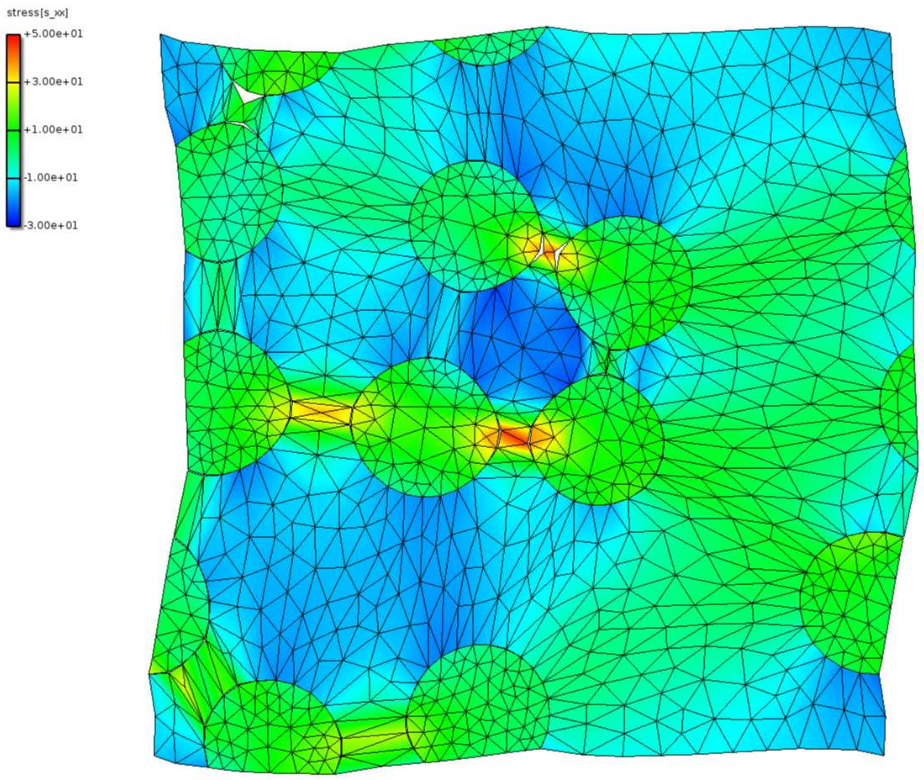

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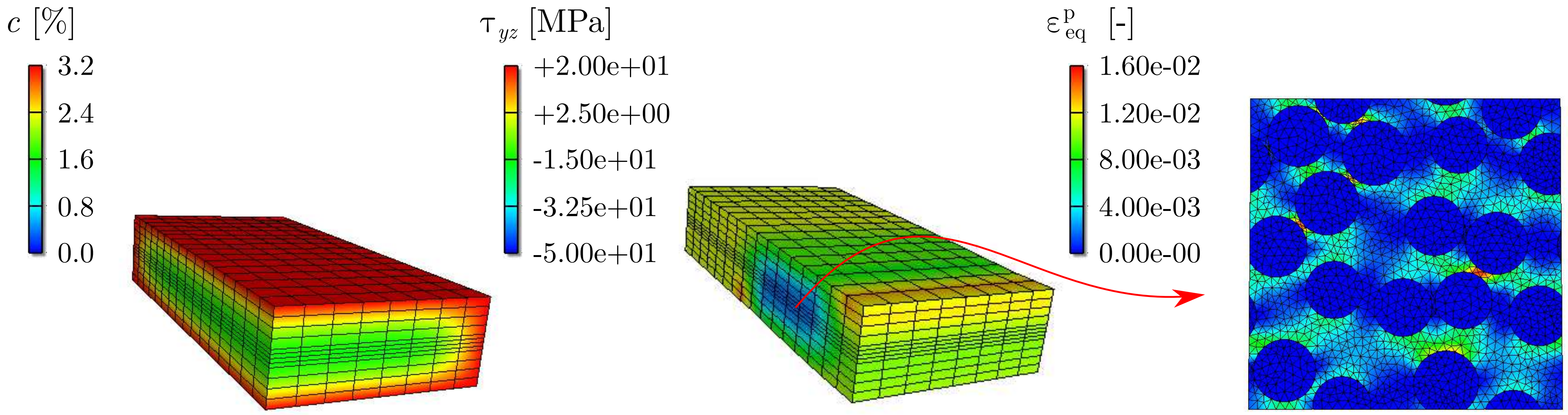

Recap — Nonlinear FEM

Discretized form:

$$\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

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Geometric nonlinearity:

$$\mathbf{B} = \mathbf{B}\left(\mathbf{a}\right)$$

Recap – Nonlinear FEM

Discretized form:

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Geometric nonlinearity:

$$\mathbf{B} = \mathbf{B}(\mathbf{a})$$

Material nonlinearity:

$$\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \mathbf{D} = \mathbf{D}(\mathbf{a})$$

Load control with a nonlinear material

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

Initialize $n \leftarrow 0$, $\mathbf{a}^0 \leftarrow \mathbf{0}$
while $n <$ number of time steps **do**
    Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
    Initialize new solution at old one: $\mathbf{a}^{n+1} \leftarrow \mathbf{a}^n$
    Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
    Evaluate first residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
    repeat
        Solve linear system of equations: $\mathbf{K}^{n+1}\Delta\mathbf{a} = \mathbf{r}$
        Update solution: $\mathbf{a}^{n+1} \leftarrow \mathbf{a}^{n+1} + \Delta\mathbf{a}$
        Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
        Evaluate residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
    until $|\mathbf{r}| <$ tolerance
    $n \leftarrow n + 1$
end while

Load control with a nonlinear material

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  $n \leftarrow n + 1$
end while

$$\mathbf{a}$$
$$\Downarrow$$
$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}$$
$$\Downarrow$$
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \qquad \mathbf{D} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$$
$$\Downarrow$$
$$\mathbf{f}_{\text{int}} = \int_\Omega \mathbf{B}^{\text{T}}\boldsymbol{\sigma}\,\text{d}\Omega \qquad \mathbf{K} = \int_\Omega \mathbf{B}^{\text{T}}\mathbf{D}\mathbf{B}\,\text{d}\Omega$$

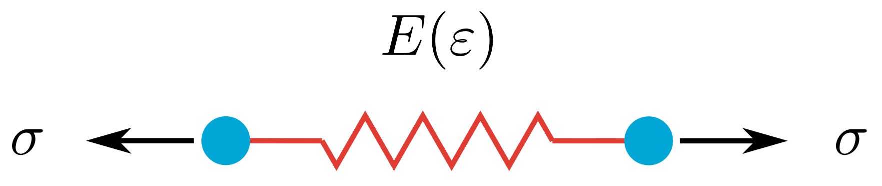
Hyperelasticity

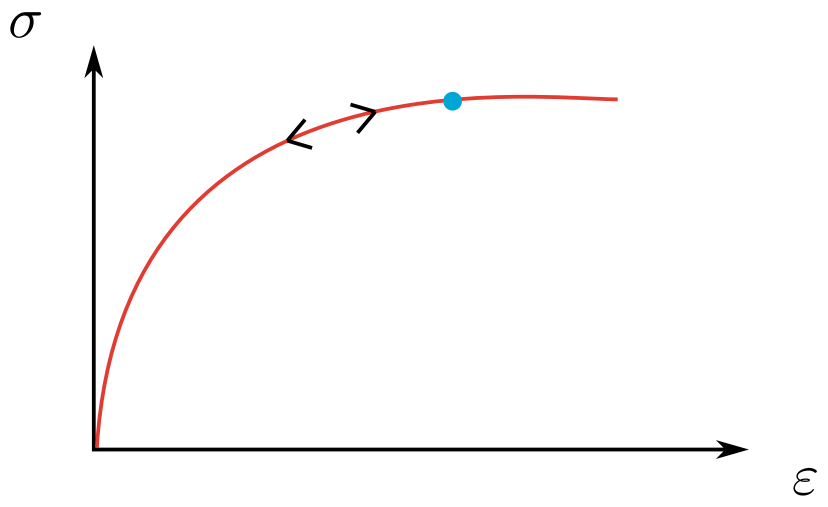
Analogous to a nonlinear spring

Popular for modeling large strains

Still fully reversible

Usually derived from a single scalar potential W





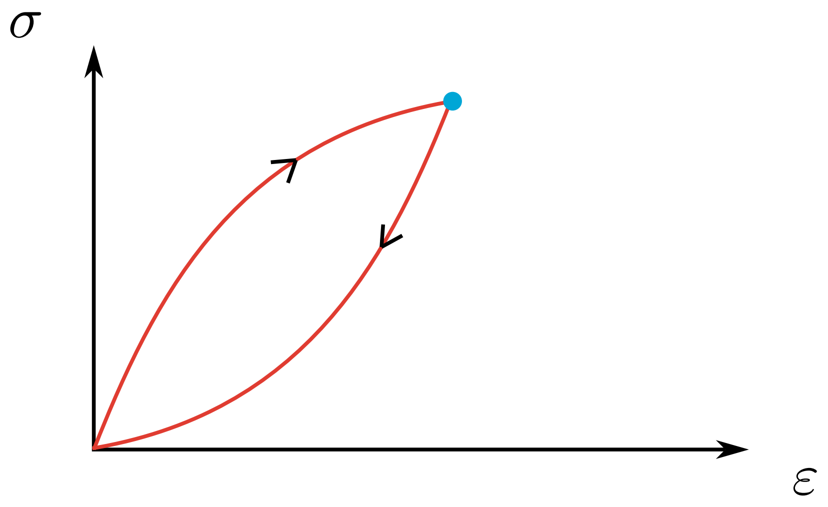
$$W(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \quad \Rightarrow \quad \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 W}{\partial \boldsymbol{\varepsilon}^2}$$

Viscoelasticity

Stiffness is time-dependent

Fully reversible response, but stiffer if loaded faster

An integral in time appears

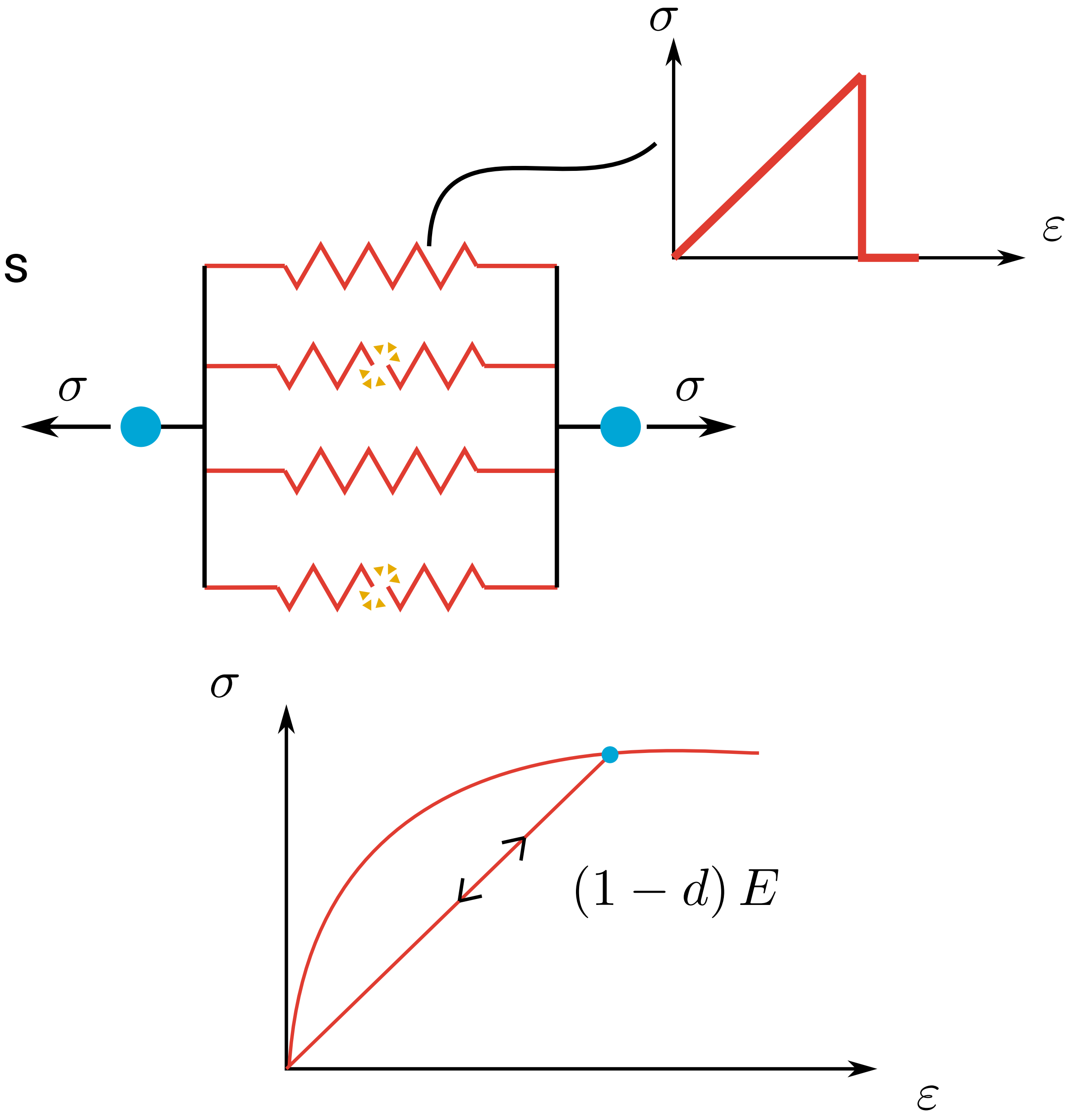


$$\boldsymbol{\sigma} = \mathbf{D}_{\infty}\boldsymbol{\varepsilon} + \int_0^t \mathbf{D}_{\mathrm{ve}}\left(t - \tilde{t}\right)\frac{\partial \boldsymbol{\varepsilon}\left(\tilde{t}\right)}{\partial \tilde{t}}\mathrm{d}\tilde{t}$$

Damage

Loss of load-carrying area modeled as loss of stiffness

$$A = (1 - d)\, A_0$$



$$\boldsymbol{\sigma} = (1 - d)\, \mathbf{D}^{\mathrm{e}} \boldsymbol{\varepsilon}$$

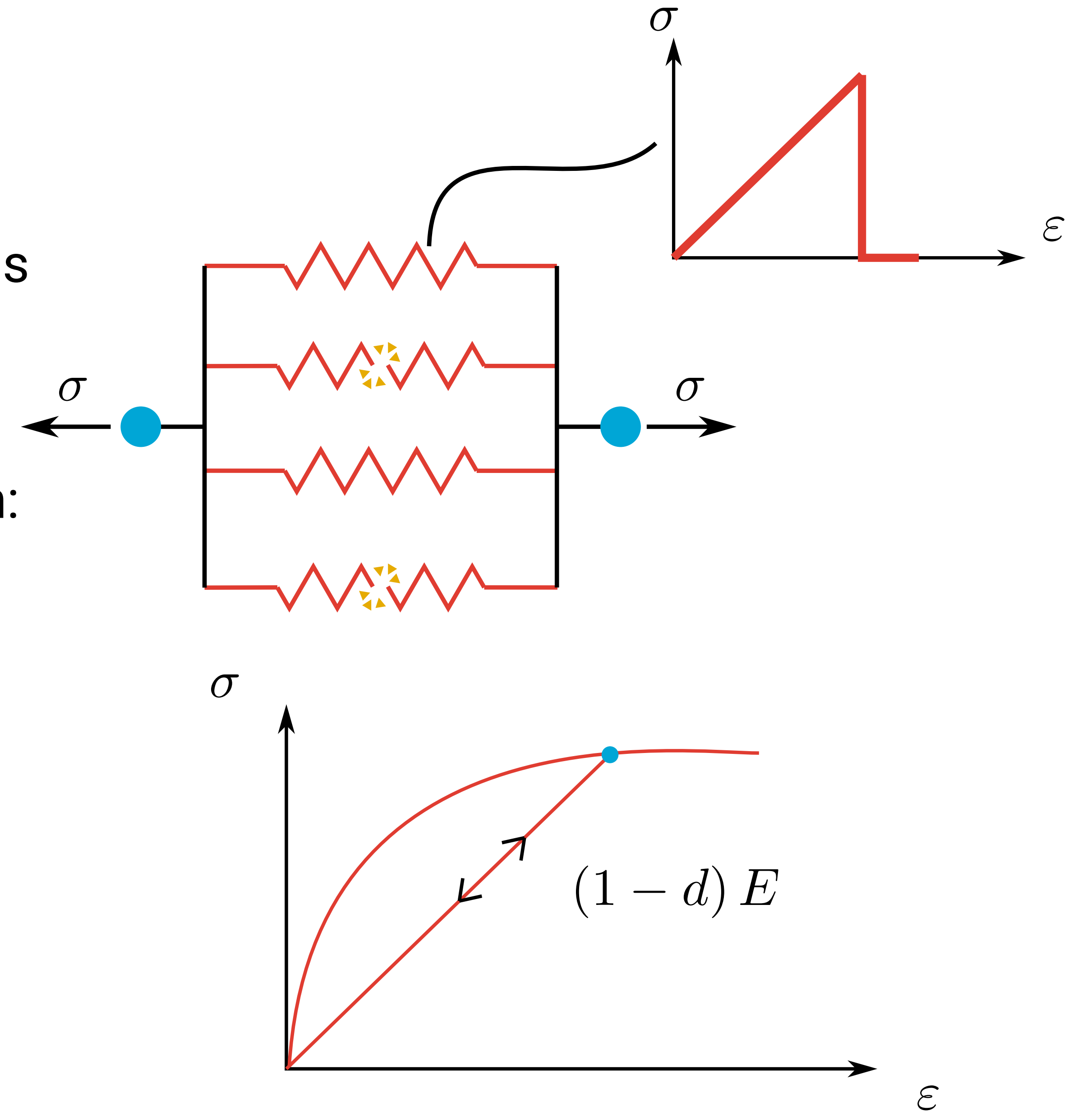
Damage

Loss of load-carrying area modeled as loss of stiffness

$$A = (1 - d)\, A_0$$

The damage d evolves according to a loading function:

$$f\,(\tilde{\varepsilon}, \alpha) = \tilde{\varepsilon} - \alpha, \quad f \leq 0, \dot{\alpha} \geq 0, f\dot{\alpha} = 0$$



$$\boldsymbol{\sigma} = (1 - d)\, \mathbf{D}^{\mathrm{e}} \boldsymbol{\varepsilon}$$

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Damage

Loss of load-carrying area modeled as loss of stiffness

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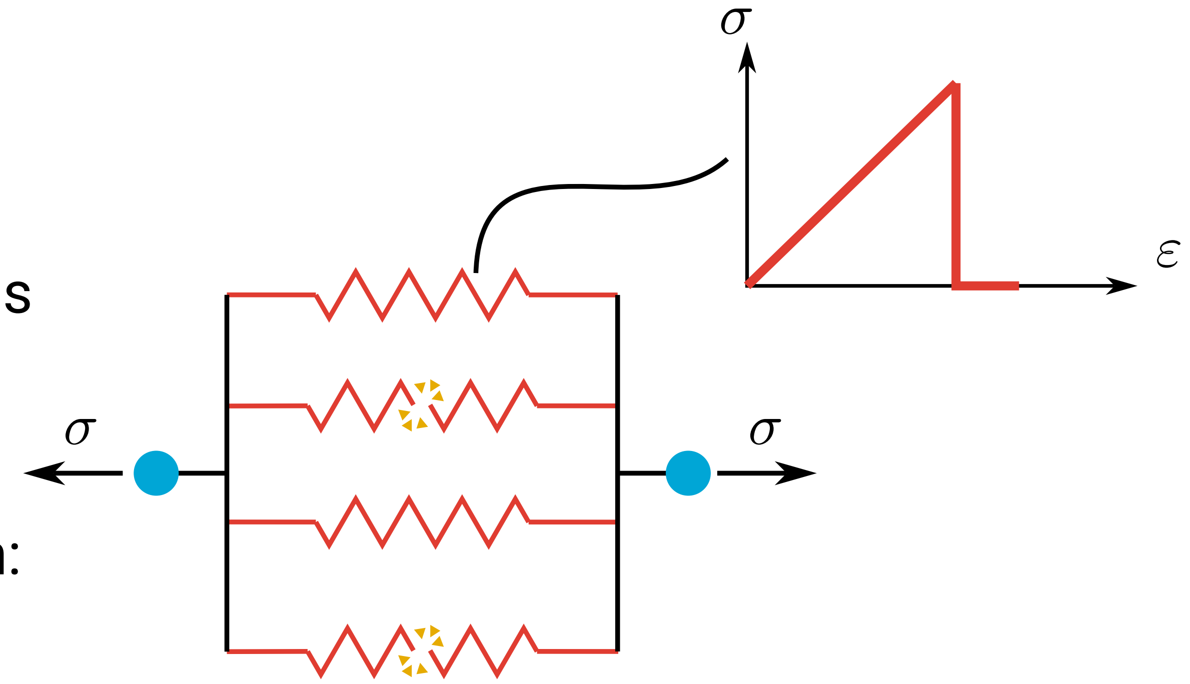
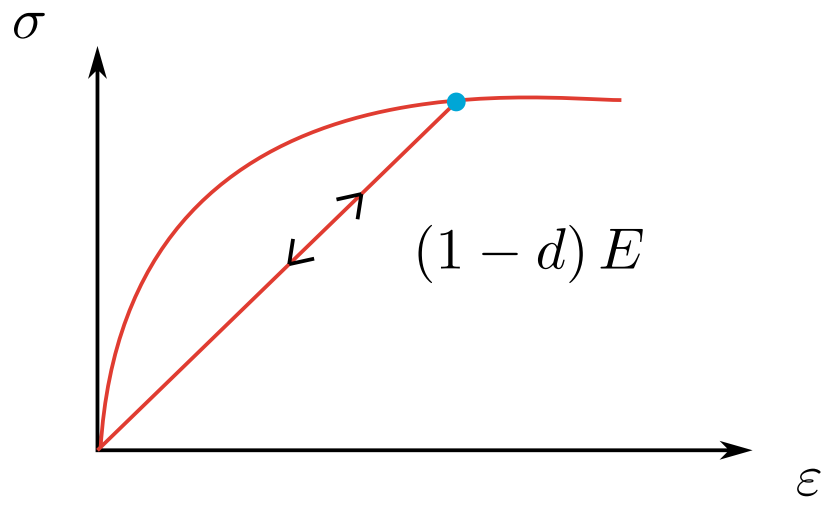
The damage d evolves according to a loading function:

$$f\left(\tilde{\varepsilon}, \alpha\right) = \tilde{\varepsilon} - \alpha, \quad f \leq 0, \dot{\alpha} \geq 0, f\dot{\alpha} = 0$$

and an evolution equation:

$$d = d\left(\alpha\right)$$





$$\boldsymbol{\sigma} = (1 - d)\, \mathbf{D}^{\mathrm{e}} \boldsymbol{\varepsilon}$$

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Displacement control with a history-dependent material

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

Initialize new solution at old one: $\mathbf{a}^{n+1} \leftarrow \mathbf{a}^{n}$

Update material model: $\left\{\boldsymbol{\sigma}^{n+1}, \mathbf{D}^{n+1}, \boldsymbol{\alpha}_{\text{new}}\right\} = \mathcal{M}\left(\boldsymbol{\varepsilon}^{n+1}, \boldsymbol{\alpha}_{\text{old}}\right)$

Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^{n+1} \mathrm{d}\Omega$, $\mathbf{K}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D}^{n+1} \mathbf{B} \mathrm{d}\Omega$

Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = \overline{\mathbf{a}}^{n+1} - \overline{\mathbf{a}}^{n}$

Evaluate first residual: $\mathbf{r} = -\mathbf{f}_{\text{int},f}^{n+1}$

repeat

  Solve linear system of equations: $\mathbf{K}^{n+1} \Delta \mathbf{a} = \mathbf{r}$

  Update solution: $\mathbf{a}^{n+1} \leftarrow \mathbf{a}^{n+1} + \Delta \mathbf{a}$

  Update material model: $\left\{\boldsymbol{\sigma}^{n+1}, \mathbf{D}^{n+1}, \boldsymbol{\alpha}_{\text{new}}\right\} = \mathcal{M}\left(\boldsymbol{\varepsilon}^{n+1}, \boldsymbol{\alpha}_{\text{old}}\right)$

  Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^{n+1} \mathrm{d}\Omega$, $\mathbf{K}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D}^{n+1} \mathbf{B} \mathrm{d}\Omega$

  Evaluate residual: $\mathbf{r} = -\mathbf{f}_{\text{int},f}^{n+1}$

  Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = 0$

until $|\mathbf{r}| <$ tolerance

Commit material history: $\boldsymbol{\alpha}_{\text{old}} \leftarrow \boldsymbol{\alpha}_{\text{new}}$

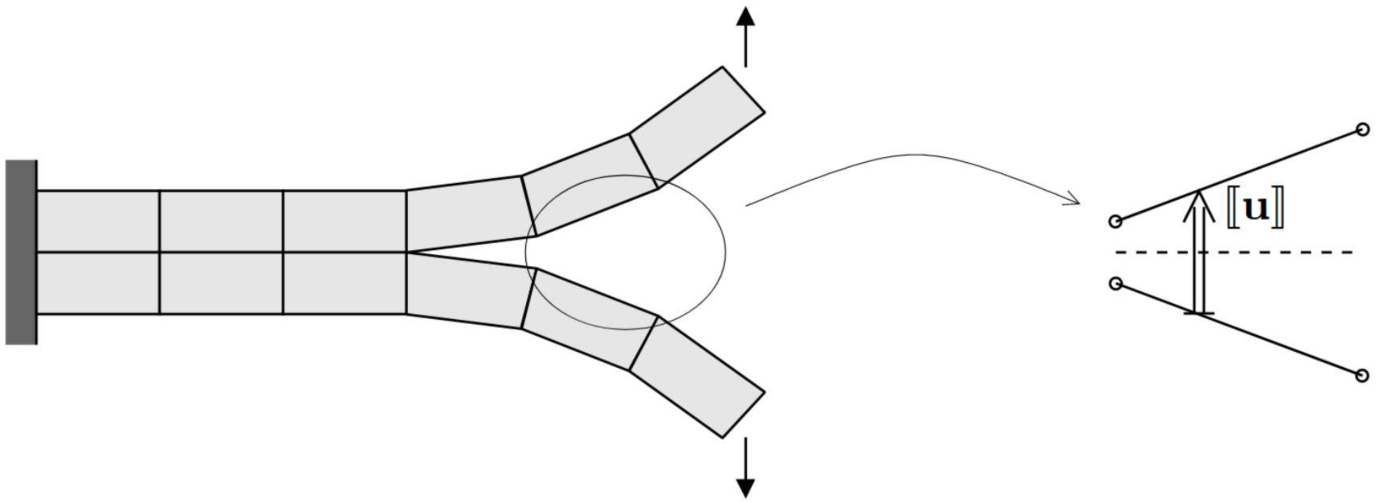
Discontinuous damage

Model an actual displacement discontinuity

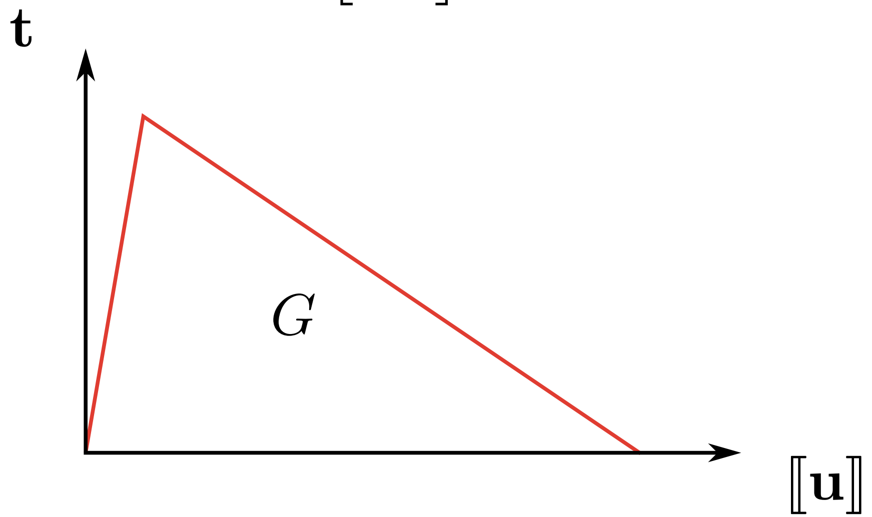
Traction-separation through a cohesive zone law

Special interface elements are needed

Similar to damage, but explicit link to energy dissipation



$$[\![\mathbf{u}]\!] = \mathbf{u}^{\text{top}} - \mathbf{u}^{\text{bot}} \quad \Rightarrow \quad \begin{bmatrix}\mathbf{N} & -\mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{a}^{\text{top}} \\ \mathbf{a}^{\text{bot}}\end{bmatrix} \quad \Rightarrow \quad [\![\mathbf{u}]\!] = \mathbf{N}_\Gamma \mathbf{a}_\Gamma$$

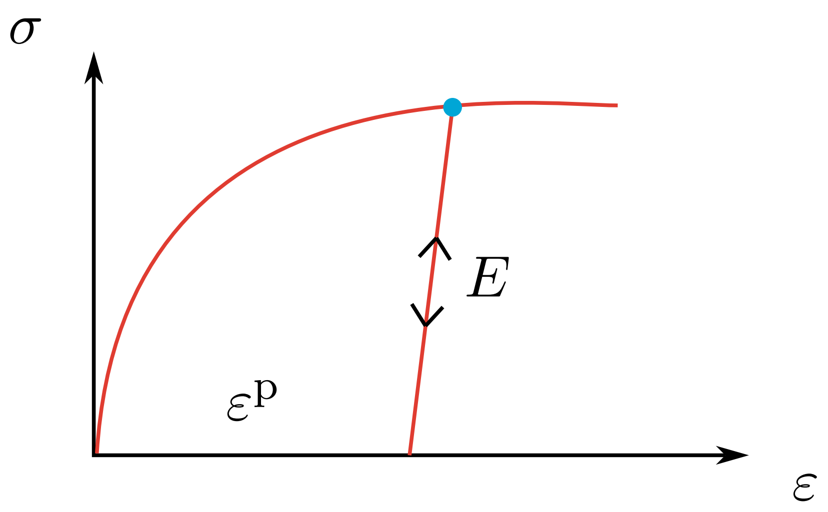


$$\mathbf{t} = \mathcal{T}\left([\![\mathbf{u}]\!]\right) \quad \Rightarrow \quad \mathbf{f}_{\text{int}}^{\Gamma} = \int_\Gamma \mathbf{N}_\Gamma^{\mathrm{T}} \mathbf{t} \, \mathrm{d}\Gamma$$

Plasticity

Deformations are split:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$$



$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right)$$

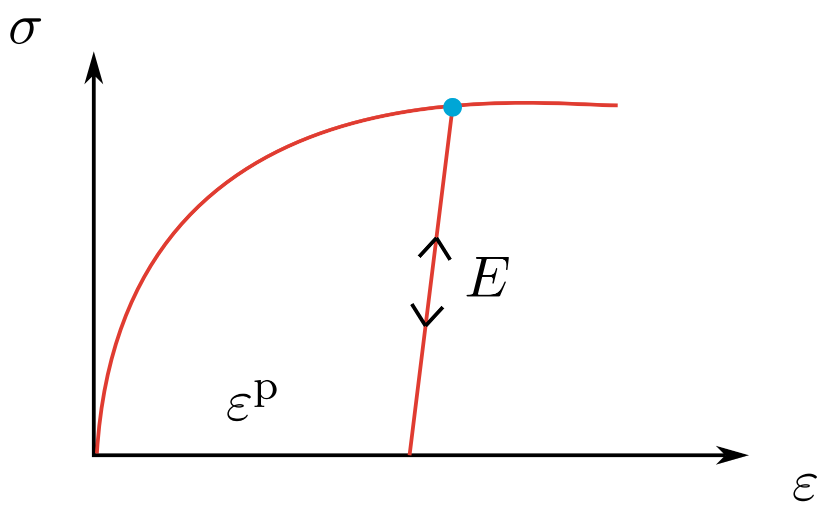
Plasticity

Deformations are split:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$$

A yield surface defines the plasticity threshold:

$$f\left(\boldsymbol{\sigma}\right) = \tilde{\boldsymbol{\sigma}} - \sigma_y\left(\varepsilon^{\mathrm{p}}_{\mathrm{acc}}\right)$$



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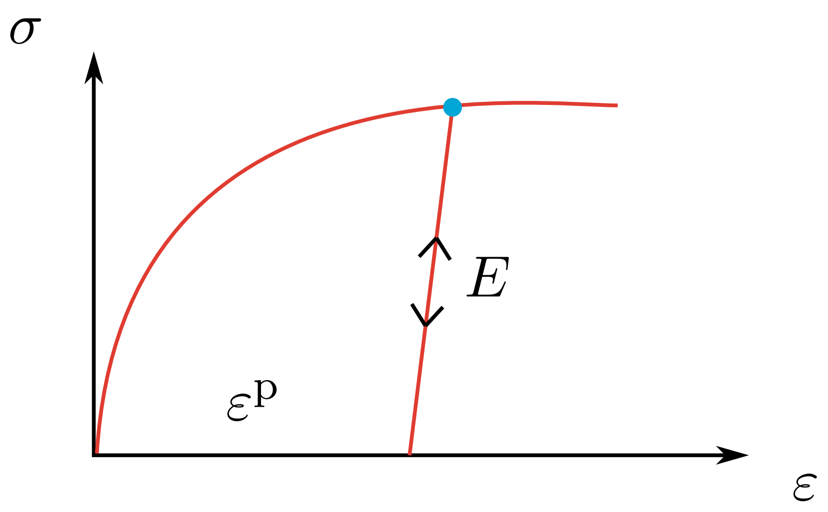
$$f\left(\boldsymbol{\sigma}\right) = \tilde{\boldsymbol{\sigma}} - \sigma_y\left(\varepsilon^{\mathrm{p}}_{\mathrm{acc}}\right)$$

Plastic flow occurs when the yield surface is pushed:

$$f = 0, \dot{f} = 0 \quad \Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\gamma}\mathbf{m}$$

We need to keep track of history variables $\boldsymbol{\alpha} = \left[\boldsymbol{\varepsilon}^{\mathrm{p}}, \varepsilon^{\mathrm{p}}_{\mathrm{acc}}\right]$



$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right)$$

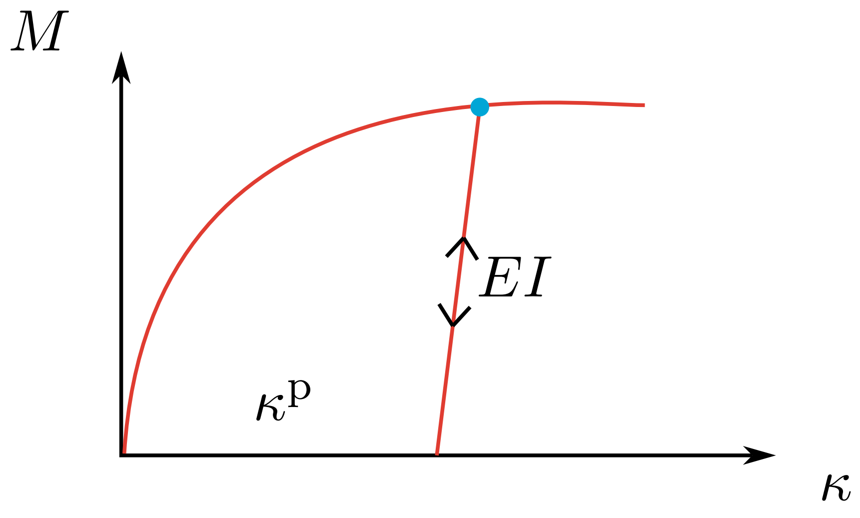
Hardening plasticity for bending moments

Curvatures are split:

$$\kappa = \kappa^{\mathrm{e}} + \kappa^{\mathrm{p}}$$

A yield point defines the plasticity threshold:

$$f\left(M, \kappa^{\mathrm{p}}\right) = |M| - M^{\mathrm{y}}\left(\kappa^{\mathrm{p}}_{\mathrm{acc}}\right), \quad \kappa^{\mathrm{p}}_{\mathrm{acc}} = \int_t |\dot{\kappa}^{\mathrm{p}}| \mathrm{d}t$$



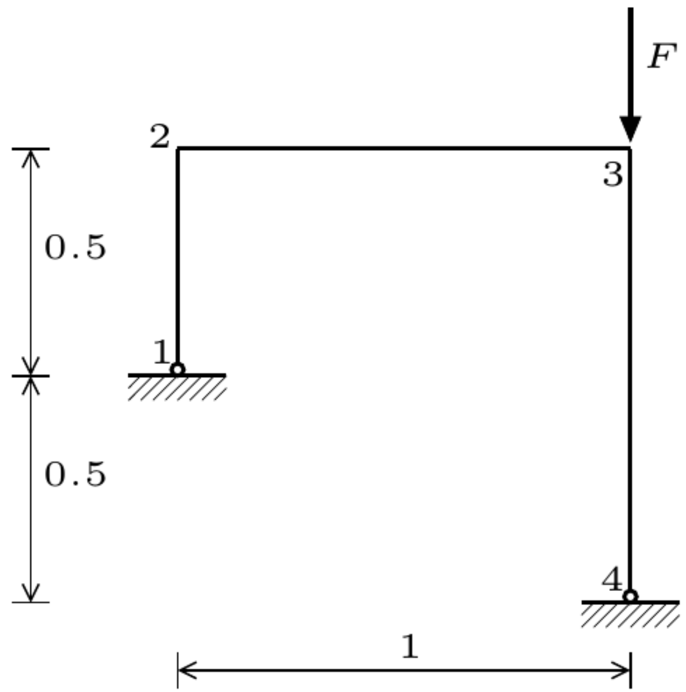
Plastic flow occurs when the yield surface is pushed:

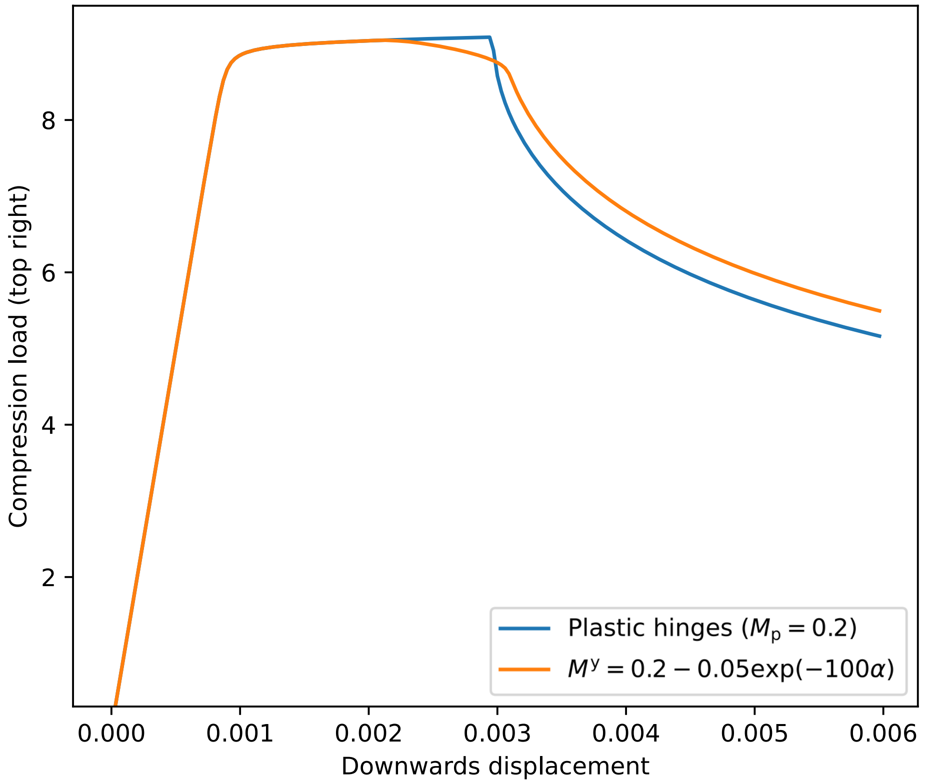
$$f = 0, \dot{f} = 0 \quad \Rightarrow \quad \Delta\kappa^{\mathrm{p}} = \Delta\gamma \, \mathrm{sign}\left(M\right)$$

History variables are $\boldsymbol{\alpha} = \left[\kappa^{\mathrm{p}}, \kappa^{\mathrm{p}}_{\mathrm{acc}}\right]$

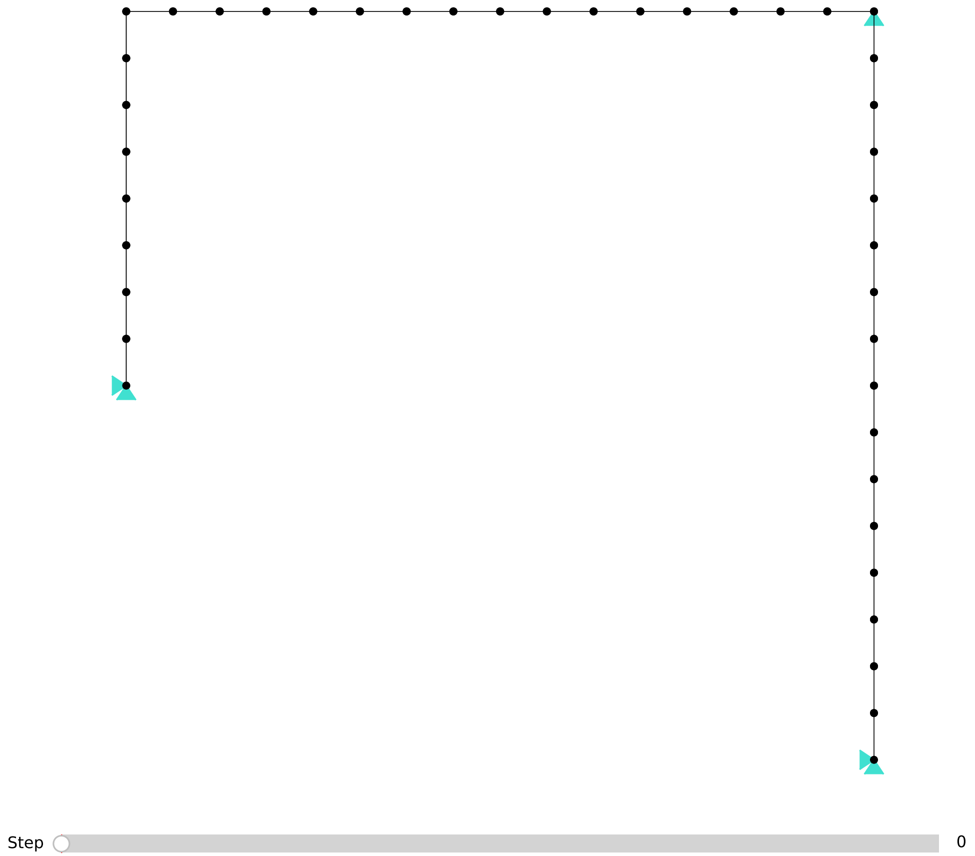
$$M = EI\left(\kappa - \kappa^{\mathrm{p}}\right)$$

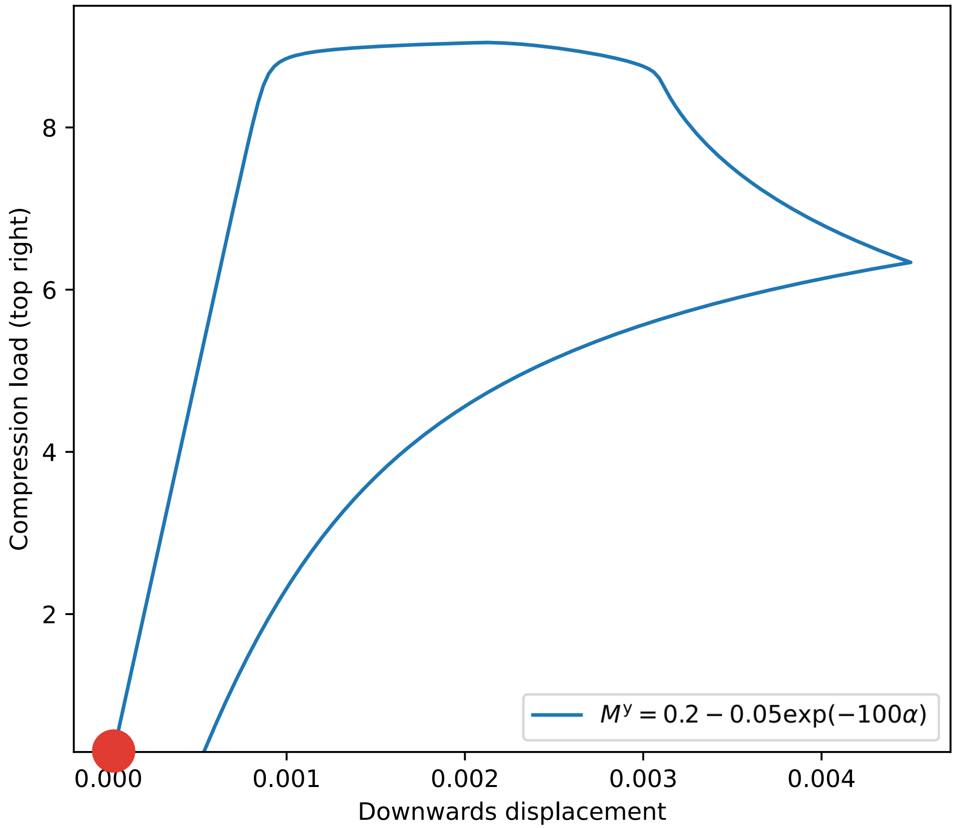
Hardening plasticity for bending moments — examples



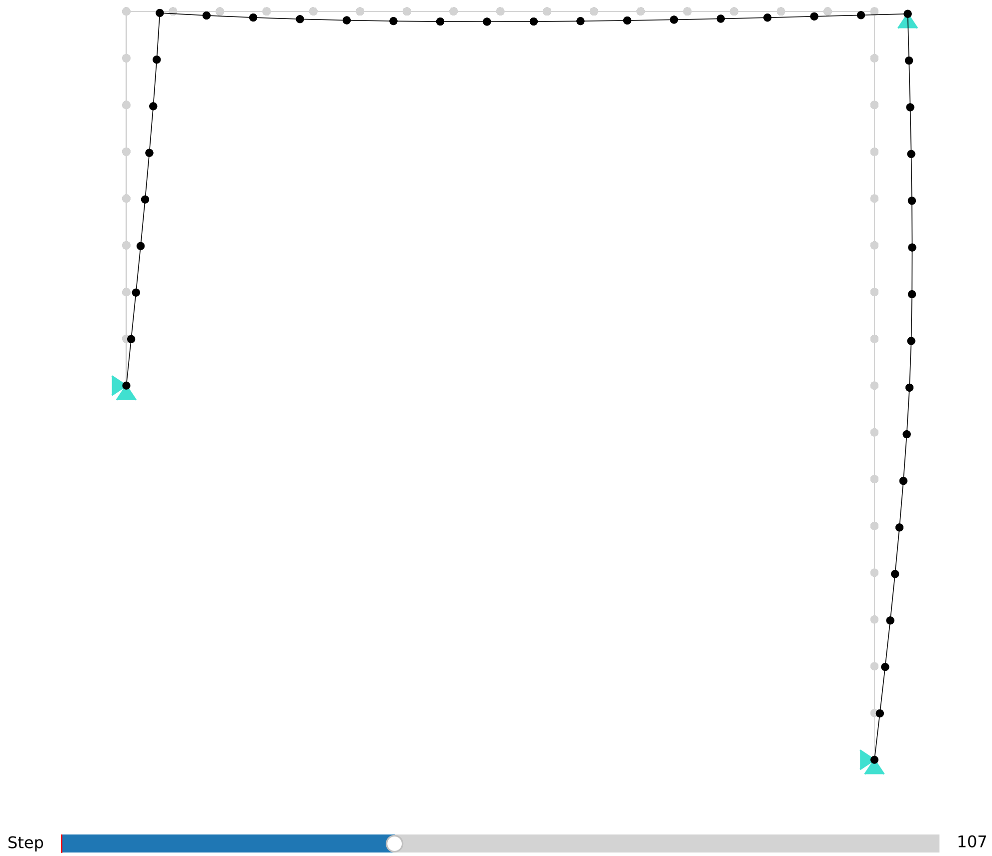


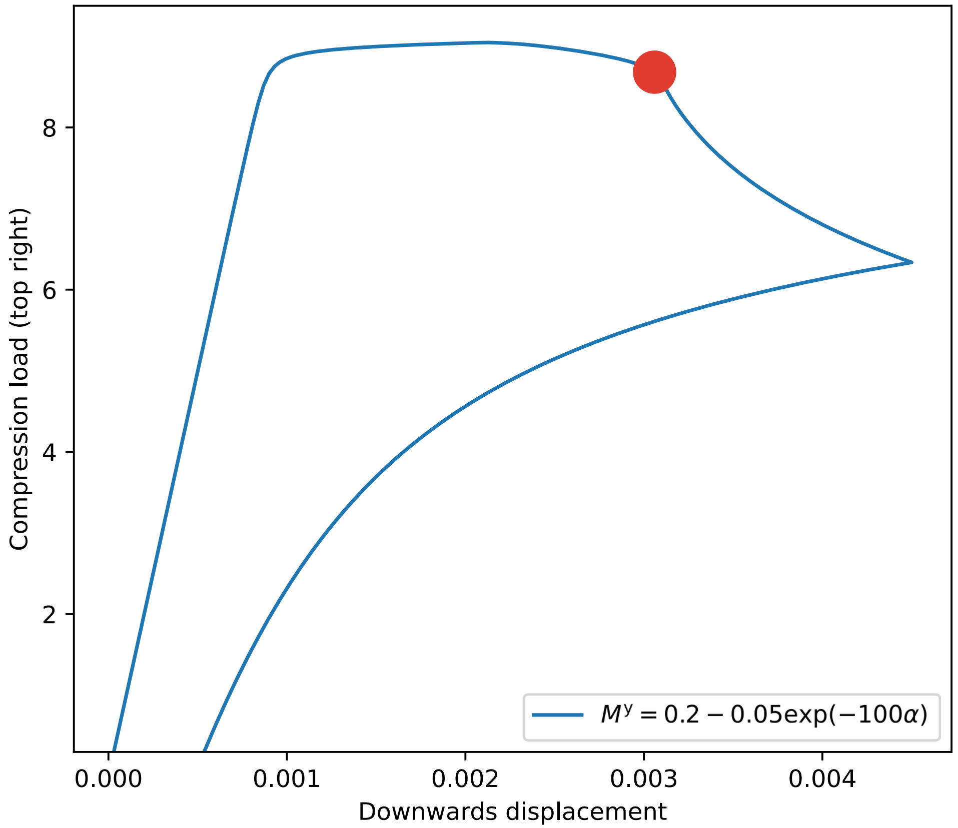
Hardening plasticity for bending moments — examples



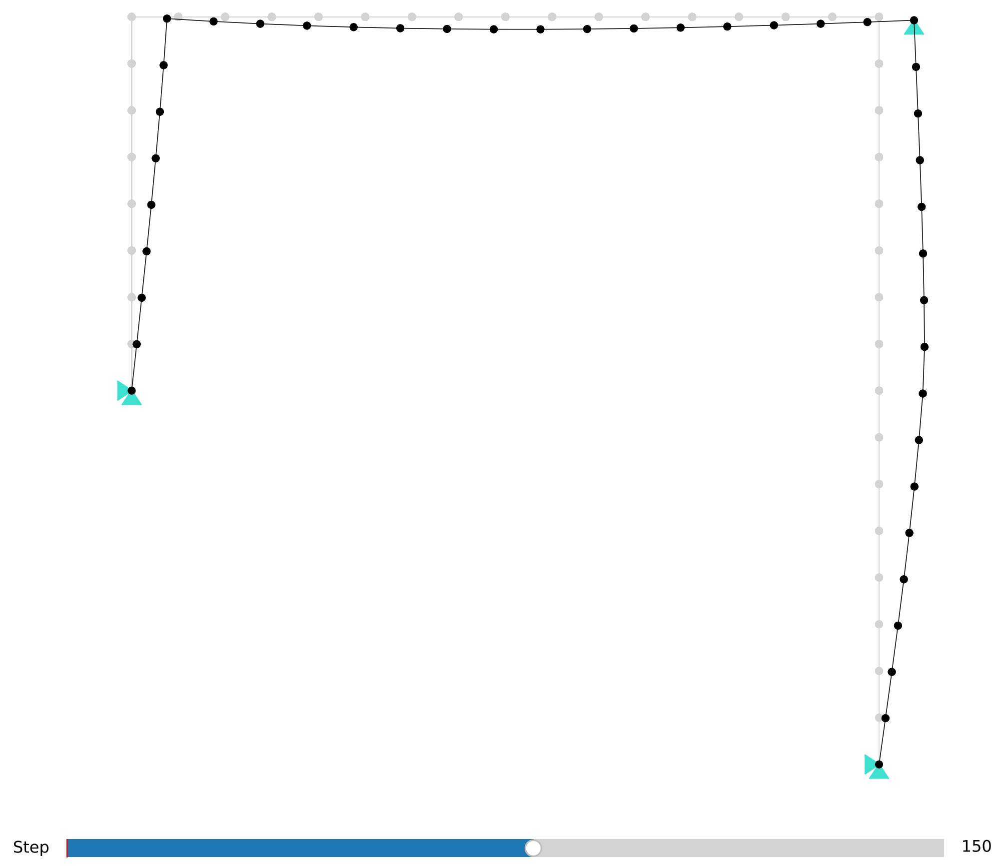


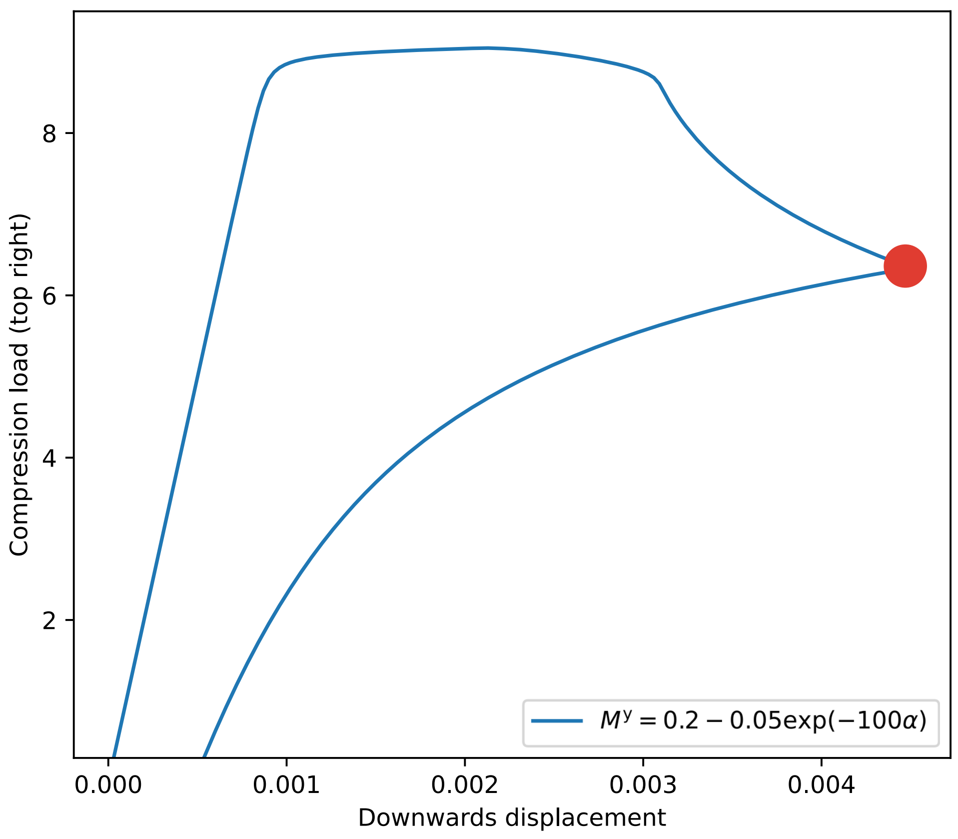
Hardening plasticity for bending moments — examples



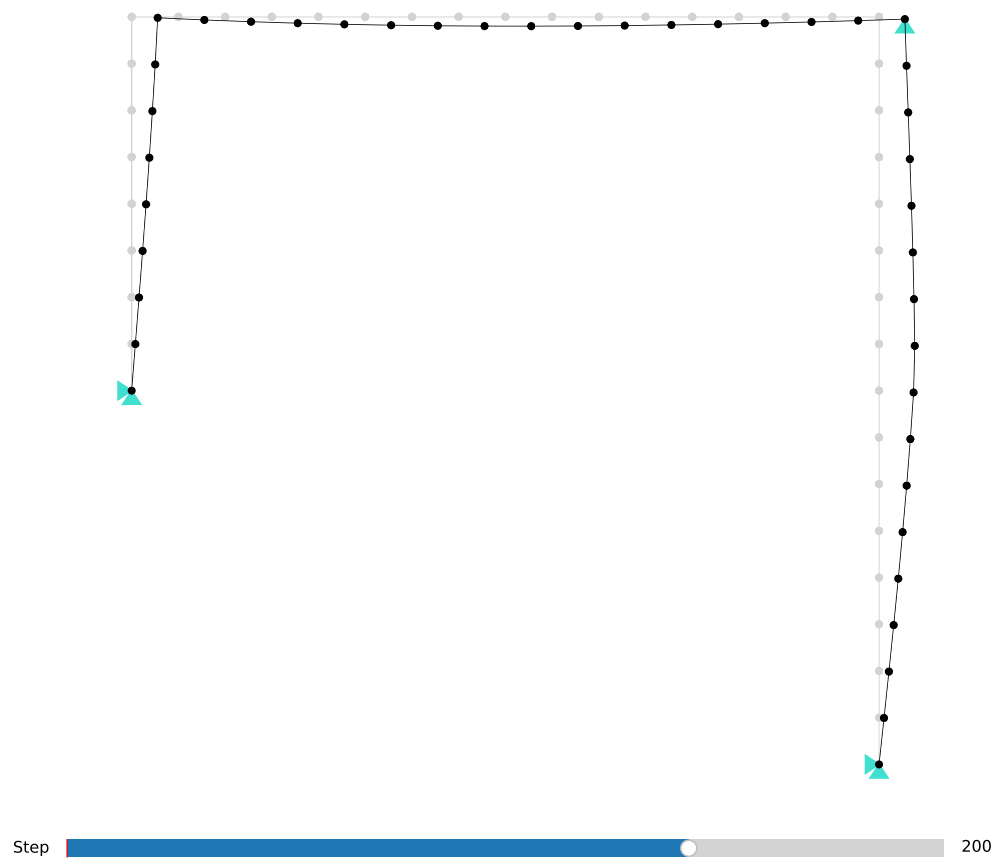


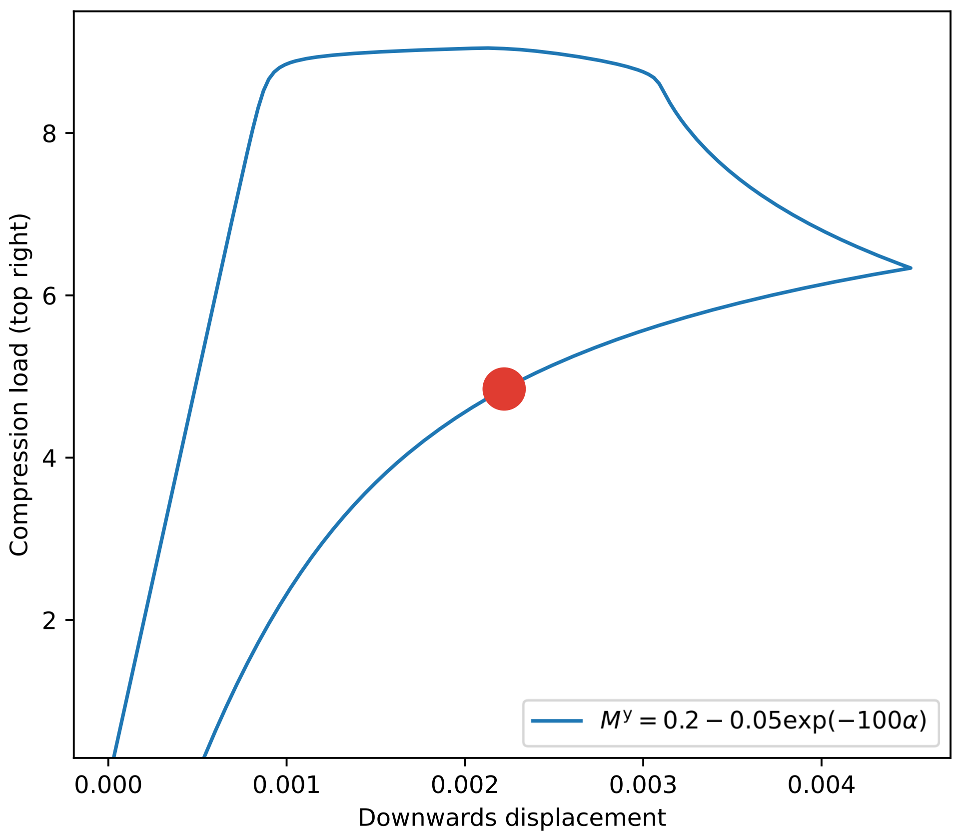
Hardening plasticity for bending moments — examples



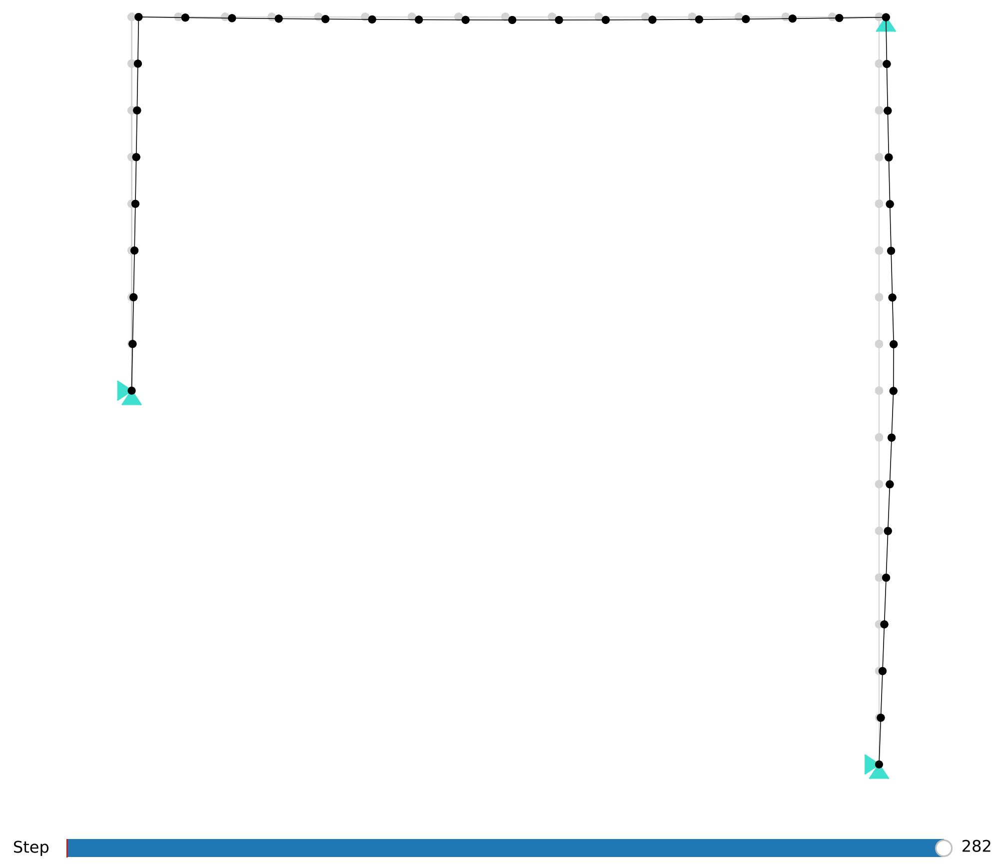


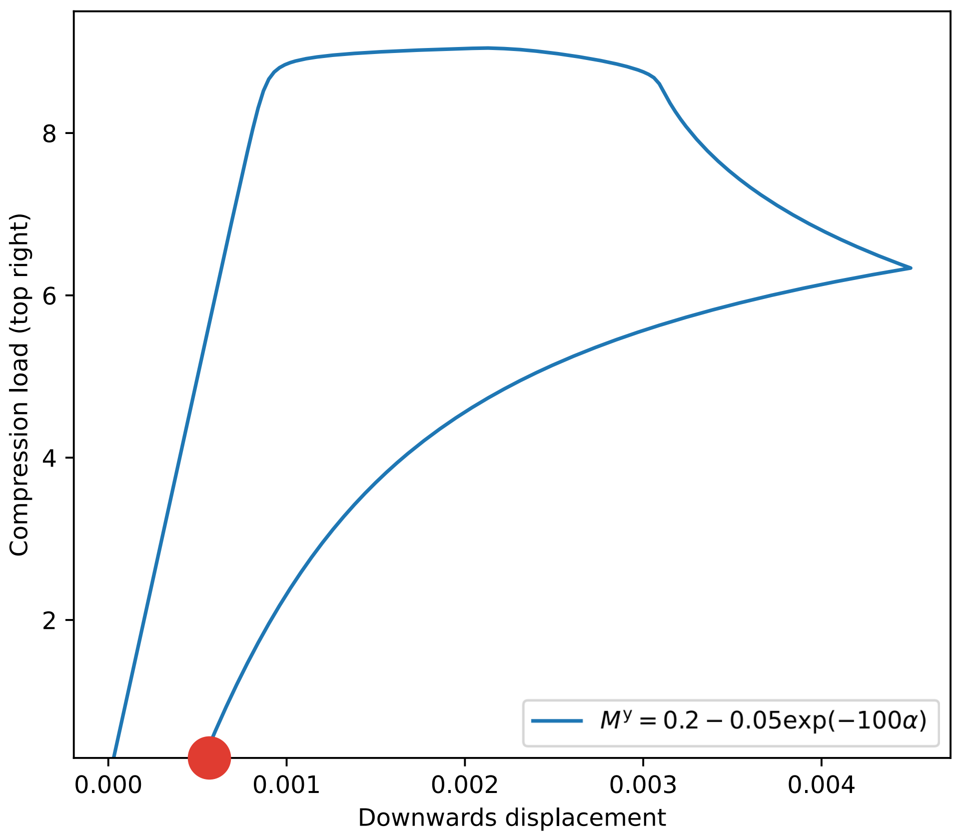
Hardening plasticity for bending moments — examples





Hardening plasticity for bending moments — examples





Recap and outlook

This short story on nonlinear FEM is now round:

- Sources of nonlinearity
- Dealing with geometric nonlinearities
- Dealing with material nonlinearities
- Solver strategies for nonlinear FEM

If you want to keep digging, this is just the beginning:

- CIEM5210-2: Advanced constitutive modeling
  - Computational plasticity, continuum and discrete damage, advanced path following methods
- CIEM1303 (upscaling techniques) and CIEM1301 (advanced computational mechanics):
  - Multiscale/multiphysics modeling, recent advances in nonlinear FEM