Homework 8-9 - CAO 2024

7. **Douglas Rachford primal-dual Symmetry** Consider the problem

$$\min_x \{h(x) + g(x)\} \tag{1}$$

and its Fenchel dual

$$\max_u \{-h^*(u) - g^*(-u)\}. \tag{2}$$

Recall that the Douglas Rachford operator $F_{DR}(w) = \frac{1}{2}(w + R_h(R_g(w)))$ where the (non-expansive) reflection operator R_h is defined by $R_h(x) = 2\,\mathrm{prox}_h(x) - x$. Assume $\mathcal{F}_{F_{DR}}$ is non-empty. Let $\overline{X} := \mathrm{argmin}_x\{h(x) + g(x)\}$ and $\overline{U} := \mathrm{argmax}_u\{-h^*(u) - g^*(-u)\}$.

(a) Show that

$$\overline{X} = \{\mathrm{prox}_h(\bar{w}) : \bar{w} \in \mathcal{F}_{F_{DR}}\} \text{ and } \overline{U} = \{\mathrm{prox}_{h^*}(\bar{w}) : \bar{w} \in \mathcal{F}_{F_{DR}}\}.$$

(b) Show that

$$\mathcal{F}_{F_{DR}} = \overline{X} + \overline{U}.$$

(c) Show that $R_f(x) + R_{f^*}(x) = 0$ for any $f \in \Gamma_0$.

(d) Let F^*_{DR} the D-R operator applied to the dual problem. Show that $F^*_{DR}(w) = F_{DR}(w)$ for all w.

8. **Tightness of the relation between ρ_F and K_F.** Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a $\frac{1}{2}$-averaged operator with $\mathcal{F}_F \neq \emptyset$. Define $\tilde{\rho}_F \in [0, 1]$ and $\tilde{K}_F \in \mathbb{R} \cup \{\infty\}$ as

$$\tilde{\rho}_F = \sup_x \frac{\mathrm{dist}(F(x), \mathcal{F}_F)}{\mathrm{dist}(x, \mathcal{F}_F)} \text{ and } \tilde{K}_F = \sup_x \frac{\mathrm{dist}(x, \{x : F(x) = x\})}{\|F(x) - x\|},$$

the tightest constants such that linear convergence, respectively the error-bound condition hold.

(a) Proof corrollary 1 (Sec 5 from the lecture notes) in this case, that is prove that

$$1 - \frac{1}{\tilde{K}_F} \leq \tilde{\rho}_F \leq 1 - \frac{1}{2\tilde{K}_F^2}. \tag{3}$$

(b) Consider the gradient descend operator $F_G(x) = x - \eta \nabla f(x)$. Compute $\tilde{K}_{F_G}$ and $\tilde{\rho}_{F_G}$ when $f(x) = \frac{1}{2}\|x\|^2$. Which side of (3) is tight?

(c) Let $\mathrm{rot}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the rotation by the angle θ. Assume θ is fixed. Let $F_\theta(x) = \frac{1}{2}(x + \mathrm{rot}_\theta(x))$ be the corresponding averaged operator. Compute $\tilde{K}_{F_\theta}$ and $\tilde{\rho}_{F_\theta}$. Which side of (3) is tight?

(d) Now consider $F(x) = \frac{1}{2}(x + \mathrm{rot}_{\|x\|}(x))$. Compute $\tilde{K}_F$ and $\tilde{\rho}_F$. Which side of (3) is tight?

9. **Douglas Rachford for quadratic optimization**. Let $m < n$. Let H be an $n \times n$ **positive definite** matrix. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank m (i.e. A has independent rows). And let $b \in \mathbb{R}^m$. We are interested on using Douglas Rachford to solve the *quadratic optimization problem*:

$$\min_{x:Ax \leq b} \frac{1}{2} x^T H x. \tag{QP}$$

Let $X := \{x : Ax \leq b\} \neq \emptyset$. Notice that problem (QP) can be written as $\min_x g(x) + \delta_X$, where $g : \mathbb{R}^n \to \mathbb{R}$ is defined by $g(x) = \frac{1}{2} x^T H x$.

In the next problems, express your solution (in simplified form) in terms of H, A and b.

(a) Compute prox_g and prox_{δ_X}.

(b) Write down the Douglas Rachford operator $F_{DR} : \mathbb{R}^n \to \mathbb{R}^n$.

(c) Find the fixed-points of F_{DR}

(d) When is F_{DR} averaged? (This implies the Douglas Rachford method converges with rate $O(1/\sqrt{k})$.)

(e) Give some (natural) condition(s) under which F_{DR} satisfies the error bound condition. This implies linear convergence rate for the Douglas Rachford. Give (a bound on) the rate of convergence.

10. **Another reflection based algorithm** This exercise uses the same notation as exercise 7. Let $R_{hg} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be defined by $R_{hg}(x, y) = (R_h(y), R_g(x))$.

(a) Show that R is nonexpansive.

Let $F_R(x, y) = \frac{1}{2}((x, y) + R_{hg}(x, y))$. We have then that F_R is averaged.

(b) What is the relation between $\mathcal{F}_{F_R}$ and $\text{argmin}_x\{h(x) + g(x)\}$?

(c) Give some (natural) condition(s) under which F_R satisfies the error bound condition. This implies linear convergence when using FPI on F, give (a bound on) the rate of convergence.