Classification of Unitary RCFTs with Two Primaries and $c < 25$

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Based on:

"Classification of Unitary RCFTs with Two Primaries and $c < 25$",
Sunil Mukhi and Brandon Rayhaun, to appear.

Background:

"Towards a Classification of Two-Character Rational Conformal Field Theories",
A. Ramesh Chandra and Sunil Mukhi,
arXiv:1810.09472.

"Curiosities above $c = 24$",
A. Ramesh Chandra and Sunil Mukhi,
arXiv:1812.05109.

And previous work:

"On 2d Conformal Field Theories with Two Characters",
Harsha Hampapura and Sunil Mukhi,
JHEP 1601 (2106) 005, arXiv: 1510.04478.

"Cosets of Meromorphic CFTs and Modular Differential Equations",
Matthias Gaberdiel, Harsha Hampapura and Sunil Mukhi,
JHEP 1604 (2016) 156, arXiv: 1602.01022.

"Reconstruction of conformal field theories from modular geometry on the torus",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Nucl. Phys. B318 (1989) 483.

"On the classification of rational conformal field theories",
Samir D. Mathur, Sunil Mukhi and Ashoke Sen,
Phys. Lett. B213 (1988) 303.

Outline

- Two-dimensional CFT has been around for a long time. In its modern form, it dates to the classic work of [Belavin–Polyakov-Zamolodchikov 1984].

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- The former is generally considered well-understood, while the latter is still quite mysterious – one attempts to bound its properties using bootstrap arguments (cf. talk of Nathan Benjamin in the Workshop).
- Without disputing this distinction, today I will present some perspectives on rational CFT (RCFT) that are perhaps new, and may shed a different light on it.

- The partition function of any 2d CFT is:

$$Z(\tau,\bar{\tau}) = \operatorname{tr} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}}, \quad q = e^{2\pi i \tau}$$

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- In an RCFT the partition function takes the form:

$$Z(\tau,\bar{\tau}) = \sum_{i=0}^{n-1} |\chi_i(\tau)|^2$$

where $\chi_i(\tau)$ are a set of n generalised characters

$$\chi_i(q) = \mathrm{tr}_i\, q^{L_0-\frac{c}{24}}$$

Here, tr_i is the trace over holomorphic descendants of the ith primary under the full chiral algebra.

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- In RCFT this is true if and only if the characters are vector-valued modular functions (VVMF):

$$\chi_i(\gamma\tau) = \sum_{j=0}^{n-1} M_{ij}(\gamma)\chi_j(\tau), \quad \gamma \in \mathrm{SL(2,Z)}$$

with $M^\dagger M = 1$.

- Characters have a q-expansion:

$$\chi_i(q) = q^{-\frac{c}{24}+h_i}(a_{i,0} + a_{i,1}q + a_{i,2}q^2 + \cdots)$$

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- They are holomorphic in the interior of moduli space but can diverge on the boundary $q \to 0$.
- There exist continuous families of VVMF's for which the $a_{i,m}$ are not positive, or integral, or even rational.
- A necessary (but not sufficient) condition for VVMF's to describe the characters of a CFT is:

$$a_{i,m} = \text{non-negative integer } \forall\, i, m : \text{“admissible character”}$$

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- These facts are exemplified in "meromorphic" CFT's, but those are often thought of as outliers in the space of RCFT.
- I will argue that these statements are true more generally, and also that meromorphic CFT do not outlie at all.

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- Then one can try to classify CFT within the same restrictions.
- Here are some known complete classifications of CFT with restrictions:

- For $c < 1$, all RCFT's are classified as Virasoro minimal models [BPZ 1984, Cappelli-Itzykson-Zuber 1987].

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- For $c = 8, 16, 24$, all meromorphic RCFT's (those having one primary – the identity) are classified [Goddard-Olive 1984, Schellekens 1992].

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- At $c = 1$ it is conjectured that all theories are given by a single compact free boson at rational values of R^2 [Ginsparg 1988, Dijkgraaf-Verlinde-Verlinde 1988].
- For $c = 8, 16, 24$, all meromorphic RCFT's (those having one primary – the identity) are classified [Goddard-Olive 1984, Schellekens 1992].
- In particular, Schellekens proposed that there are 71 meromorphic CFT with $c = 24$. His results have been rigorously confirmed in the mathematics literature [Møller-Scheithauer 2021].

- In this talk I will present the complete classification of unitary RCFT with two primaries $\{\mathbf{1}, \Phi\}$ with Φ real, and $c < 25$ [In collaboration with Brandon Rayhaun, to appear].

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- There are no restrictions on the chiral algebra or anything else.
- The result is a set of 121 theories.

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- The most general such equation is:

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- This is holomorphic and modular invariant if $\phi_2(\tau), \phi_4(\tau)$ are holomorphic and modular of weight $2, 4$ respectively.
- The two independent solutions of an SL(2,Z) invariant MLDE form a pair of VVMF's $\chi_0(\tau), \chi_1(\tau)$ under SL(2,Z), and vice versa.

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- The holomorphic modular bootstrap of [Mathur-Mukhi-Sen 1988] is a programme to scan the parameter space and look for values that lead to admissible characters.

- All our papers that year were greatly inspired by the works of [Dijkgraaf-Verlinde-Verlinde 1988, Verlinde 1988].

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- For any given ℓ there is a finite basis of functions of the Eisenstein series E_4, E_6 from which ϕ_2, ϕ_4 are built.
- Thus for any fixed ℓ, the differential equation has finitely many parameters.

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- We solved this equation and found a finite set of admissible characters. Remarkably all of them could be associated with a CFT based on compact simple Kac-Moody algebras $\mathrm{KM}_{r,k}$ or generalisations thereof:

$$A_0, A_{1,1}, A_{2,1}, G_{2,1}, D_{4,1}, F_{4,1}, E_{6,1}, E_{7,1}, E_{7.5,1}, E_{8,1}$$

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- The red ones in the above list either have negative fusion rules [Kawasetsu-IVOA] or more/less than two primaries – irrelevant for this talk.

- Thus the solutions with exactly two primaries are: $A_{1,1}, G_{2,1}, F_{4,1}, E_{7,1}$ with central charges $1, \frac{14}{5}, \frac{26}{5}, 7$. This is a complete classification for $\ell = 0$.

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- In this case, identification with CFT is more difficult. It was carried out in [Gaberdiel-Hampapura-Mukhi 2016] using a variant of the coset construction of RCFT's [Goddard-Kent-Olive 1984,1985].
- In this variant one considers cosets of meromorphic CFT $\mathcal{M}$ by affine theories KM:

$$\mathcal{C} = \frac{\mathcal{M}}{\text{KM}}$$

- A meromorphic CFT $\mathcal{M}$ has a partition function of the form:

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- Admissible meromorphic characters exist only at $c = 8n$ for any positive integer n. We will be interested in the $c = 24$ case.

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- However there are just 71 CFT's [Schellekens 1992].
- These include free bosons on 24 even unimodular lattices and a finite number of generalisations involving orbifolding.
- With one exception (the Monster CFT), these theories correspond to special modular invariant combinations of Kac-Moody characters for non-simple Lie algebras.

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- These quotients have Wronskian index $\ell = 2$. There are 13 such quotient theories (found by deleting one factor of the KM algebra), all with $16 < c < 24$. They can be shown to exhaust all two-character CFT with $\ell = 2$.

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- Thus we easily find four $\ell = 4$ theories with two characters:

$$E_{8,1}A_{1,1}, \quad E_{8,1}G_{2,1}, \quad E_{8,1}F_{4,1}, \quad E_{8,1}E_{7,1}$$

and central charges $8 < c < 16$.

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- Thus we easily find four $\ell = 4$ theories with two characters:

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- As Kac-Moody algebras, the commutants are respectively $A_{1,1}D_{14,1}$ and $B_{12,1}$.

- But there is a puzzle. The central charge of $A_{1,1}D_{14,1}$ is $15 = 16 - 1$. However the central charge of $B_{12,1}$ is $\frac{25}{2} \neq 16 - \frac{14}{5}$ leaving a deficit of $\frac{7}{10}$.

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- We learn an important lesson: there are 2-character extensions of direct sums of both Kac-Moody and Virasoro modules. Many of these appear in our classification.
- Thus we have found 6 two-primary theories with $\ell = 4$. From MLDE one finds three more admissible characters, but they have $c > 25$ so we can ignore them.

Outline

- We classified all two-primary CFT with Wronskian index $\ell = 0, 2, 4$ and $c < 25$. MLDE's were essential in providing the admissible characters.

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- The first result is that all admissible characters with $\ell = 6N$ are linear combinations of solutions of the original $\ell = 0$ MMS equation, having integral but not always positive coefficients.
- We called these quasi-characters. Let's look at an example:

- A_1 series of quasi-characters with $\ell = 0$:

$$c = 6n + 1,\ h = \frac{2n+1}{4}, \quad n \in \mathbb{Z}$$

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$$\chi_0 = q^{-\frac{1}{24}}(1 + 3q + 4q^2 + 7q^3 + 13q^4 + \cdots)$$

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- Quasi-characters cannot directly describe a CFT: a degeneracy of -245 is not physically sensible.

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- So we can add the $n=0$ and $n=4$ quasi-characters and still get a VVMF:

$$\begin{aligned}\chi_i &= \mathcal{N}\chi_i^{n=0} + \chi_i^{n=4} \\ &= q^{-\frac{25}{24}}\left(1 + (\mathcal{N} - 245)q + (3\mathcal{N} + 142640)q^2 + \cdots\right)\end{aligned}$$

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- The sum is an admissible character for all $\mathcal{N} \geq 245$. It has $(c, h) = (25, \frac{5}{4})$. From the Riemann-Roch theorem on the Wronskian we have:

$$\ell = \frac{c}{2} - 6h + 1 = 6$$

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- In this way, all admissible two-characters sets are generated.
- For three or more characters there are only partial results [Mukhi-Poddar-Singh 2020].

- For $\ell = 0$, the two relevant series of quasi-characters for us are:

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- Similarly for $\ell = 2$ the relevant series are:

$$c = 6n - 1, \qquad c = \frac{2(6n-1)}{5},\ n \neq 1 \bmod 5$$

  The first series contains $c = 24$ cosets by $A_{1,1}, E_{7,1}$, with $c = 23, 17$, while the second contains $c = 24$ cosets by $G_{2,1}, F_{4,1}$ with $c = \frac{106}{5}, \frac{94}{5}$. The rest are quasi-characters.

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- Finally, quasi-characters for $\ell = 4$ are $E_{8,1}$ times quasi-characters for $\ell = 0$.

Outline

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- For a unitary theory with positive c, h this implies that:

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  so for $c < 25$ we only need to study admissible characters with $\ell \leq 12$.
- Using quasi-characters, one finds that at $\ell = 6, 10, 12$ there are no admissible characters with $c < 25$. That leaves $\ell = 0, 2, 4, 8$.

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- From these, one can derive the would-be fusion rules using the Verlinde formula:

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- As first highlighted in [MMS 1988], all admissible characters do not lead to consistent fusion rules. For example one can find some for which $\mathcal{N}_{ijk}$ is a negative integer.

- Exchanging the characters and re-identifying the exponents as: $\alpha_0 = -\frac{c}{24}, \alpha_1 = -\frac{c}{24} + h$ can be seen to restore positivity of $\mathcal{N}_{ijk}$.

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- However this makes h negative if it was previously positive, and thereby renders the theory non-unitary.
- In this way one rules out many of the admissible characters – they cannot correspond to CFT.

- After imposing consistency of fusion rules, we find the quasi-characters have a total of 12 allowed SL(2,Z) representations. These correspond to:

$$\ell = 0 : A_{1,1},\ E_{7,1},\ G_{2,1},\ F_{4,1}$$
$$\ell = 2 : \text{cosets of } c = 24 \text{ by } A_{1,1},\ E_{7,1},\ G_{2,1},\ F_{4,1}$$
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- It follows that every two-primary CFT with $\ell = 8$ is a coset of a meromorphic theory by one of $A_{1,1}, G_{2,1}, F_{4,1}, E_{7,1}$.
- The central charges resulting from such cosets are:

$$17, \tfrac{94}{5}, \tfrac{106}{5}, 23$$

- It thus only remains to compute all possible embeddings. We already covered the "trivial" ones where a factor in the numerator is deleted and we get $\ell = 2$.

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- Now we must classify all non-trivial embeddings, which indeed give $\ell = 8$.
- This is a rather complex exercise involving Dynkin and embedding indices, so I will skip the details.
- We find multiple theories for each of the above values of c, for a total of 98 theories.

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  - Some theories have complete Kac-Moody algebras and others have incomplete ones together with minimal models, the latter being one of $c = \frac{7}{10}, \frac{4}{5}, \frac{1}{2} \oplus \frac{7}{10}$.
  - There are theories with the same c but different conformal dimension h, and also multiple theories with the same (c, h). For example we find:

$$\begin{aligned} 2 \text{ theories with } (c,h) &= \left(\tfrac{106}{5}, \tfrac{8}{5}\right) \\ 27 \text{ theories with } (c,h) &= \left(\tfrac{106}{5}, \tfrac{3}{5}\right) \end{aligned}$$

Theories

No.	Theory	c	h	ℓ	Subalgebra	d
1	$A_{1,1}$	1	1/4	0	$A_{1,1}$	2
2	$G_{2,1}$	14/5	2/5	0	$G_{2,1}$	7
3	$F_{4,1}$	26/5	3/5	0	$F_{4,1}$	26
4	$E_{7,1}$	7	3/4	0	$E_{7,1}$	56
5	$E_{8,1}A_{1,1}$	9	1/4	4	$A_{1,1}E_{8,1}$	2
6	$E_{8,1}G_{2,1}$	54/5	2/5	4	$G_{2,1}E_{8,1}$	7
7	$F_{4,1}E_{8,1}$	66/5	3/5	4	$F_{4,1}E_{8,1}$	26
8	$D^{+}_{16,1}/G_{2,1}$	66/5	3/5	4	$B_{12,1}L_{7/10}$	26
9	$E_{7,1}E_{8,1}$	15	3/4	4	$E_{7,1}E_{8,1}$	56
10	$D^{+}_{16,1}/A_{1,1}$	15	3/4	4	$D_{14,1}A_{1,1}$	56
11	$\mathbf{S}(D_{10,1}E^{2}_{7,1})/(E_{7,1} \hookrightarrow E_{7,1})$	17	5/4	2	$D_{10,1}E_{7,1}$	1632
12	$\mathbf{S}(A_{17,1}E_{7,1})/(E_{7,1} \hookrightarrow E_{7,1})$	17	5/4	2	$A_{17,1}$	1632
13	$E^{2}_{8,1}A_{1,1}$	17	1/4	8	$E^{2}_{8,1}A_{1,1}$	2
14	$\mathbf{S}(D_{16,1}E_{8,1})/(E_{7,1} \hookrightarrow E_{8,1})$	17	1/4	8	$D_{16,1}A_{1,1}$	2
15	$\mathbf{S}(C_{8,1}F^{2}_{4,1})/(F_{4,1} \hookrightarrow F_{4,1})$	94/5	7/5	2	$C_{8,1}F_{4,1}$	4794
16	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(F_{4,1} \hookrightarrow F_{4,1})$	94/5	7/5	2	$E_{7,2}B_{5,1}$	4794
17	$\mathbf{S}(E^{4}_{6,1})/F_{4,1}$	94/5	2/5	8	$E^{3}_{6,1}L_{4/5}$	1
18	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(F_{4,1} \hookrightarrow E_{6,1})$	94/5	2/5	8	$A_{11,1}D_{7,1}L_{4/5}$	1
19	$\mathbf{S}(D_{10,1}E^{2}_{7,1})/(F_{4,1} \hookrightarrow E_{7,1})$	94/5	2/5	8	$D_{10,1}E_{7,1}A_{1,3}$	3
20	$\mathbf{S}(A_{17,1}E_{7,1})/(F_{4,1} \hookrightarrow E_{7,1})$	94/5	2/5	8	$A_{17,1}A_{1,3}$	3
21	$E^{3}_{8,1}/F_{4,1} \cong G_{2,1}E^{2}_{8,1}$	94/5	2/5	8	$E^{2}_{8,1}G_{2,1}$	7
22	$\mathbf{S}(D_{16,1}E_{8,1})/(F_{4,1} \hookrightarrow E_{8,1})$	94/5	2/5	8	$D_{16,1}G_{2,1}$	7
23	$\mathbf{S}(E_{6,3}G^{3}_{2,1})/G_{2,1}$	106/5	8/5	2	$E_{6,3}G^{2}_{2,1}$	15847
24	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(G_{2,1} \hookrightarrow G_{2,1})$	106/5	8/5	2	$D_{7,3}A_{3,1}$	15847
25	$\mathbf{S}(D_{6,2}C_{4,1}B^{2}_{3,1})/(G_{2,1} \hookrightarrow B_{3,1})$	106/5	3/5	8	$D_{6,2}C_{4,1}B_{3,1}L_{7/10}$	1
26	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(G_{2,1} \hookrightarrow B_{3,1})$	106/5	3/5	8	$A_{9,2}A_{4,1}L_{7/10}$	1
27	$\mathbf{S}(D^{6}_{4,1})/G_{2,1}$	106/5	3/5	8	$D^{5}_{4,1}L_{1/2}L_{7/10}$	2
28	$\mathbf{S}(A^{4}_{5,1}D_{4,1})/(G_{2,1} \hookrightarrow D_{4,1})$	106/5	3/5	8	$A^{4}_{5,1}L_{1/2}L_{7/10}$	2
29	$\mathbf{S}(D_{8,2}B^{2}_{4,1})/G_{2,1}$	106/5	3/5	8	$D_{8,2}B_{4,1}U_{1}L_{7/10}$	3
30	$\mathbf{S}(C^{2}_{6,1}B_{4,1})/(G_{2,1} \hookrightarrow B_{4,1})$	106/5	3/5	8	$C^{2}_{6,1}U_{1}L_{7/10}$	3
31	$\mathbf{S}(A^{2}_{7,1}D^{2}_{5,1})/(G_{2,1} \hookrightarrow D_{5,1})$	106/5	3/5	8	$A^{2}_{7,1}D_{5,1}A_{1,2}L_{7/10}$	4
32	$\mathbf{S}(C_{8,1}F^{2}_{4,1})/(G_{2,1} \hookrightarrow F_{4,1})$	106/5	3/5	8	$C_{8,1}F_{4,1}A_{1,8}$	5
33	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(G_{2,1} \hookrightarrow B_{5,1})$	106/5	3/5	8	$E_{7,2}A^{2}_{1,1}F_{4,1}L_{7/10}$	5
34	$\mathbf{S}(E_{7,2}B_{5,1}F_{4,1})/(G_{2,1} \hookrightarrow F_{4,1})$	106/5	3/5	8	$E_{7,2}B_{5,1}A_{1,8}$	5
35	$\mathbf{S}(D^{4}_{6,1})/G_{2,1}$	106/5	3/5	8	$D^{3}_{6,1}B_{2,1}L_{7/10}$	6
36	$\mathbf{S}(A^{2}_{9,1}D_{6,1})/(G_{2,1} \hookrightarrow D_{6,1})$	106/5	3/5	8	$A^{2}_{9,1}B_{2,1}L_{7/10}$	6
37	$\mathbf{S}(C_{10,1}B_{6,1})/(G_{2,1} \hookrightarrow B_{6,1})$	106/5	3/5	8	$C_{10,1}A_{3,1}L_{7/10}$	7
38	$\mathbf{S}(E^{4}_{6,1})/G_{2,1}$	106/5	3/5	8	$E^{3}_{6,1}A_{2,2}$	8
39	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(G_{2,1} \hookrightarrow D_{7,1})$	106/5	3/5	8	$A_{11,1}B_{3,1}E_{6,1}L_{7/10}$	8
40	$\mathbf{S}(A_{11,1}D_{7,1}E_{6,1})/(G_{2,1} \hookrightarrow E_{6,1})$	106/5	3/5	8	$A_{11,1}D_{7,1}A_{2,2}$	8

No.	Theory	c	h	ℓ	Subalgebra	d
41	$\mathbf{S}(D^{3}_{8,1})/G_{2,1}$	106/5	3/5	8	$D^{2}_{8,1}B_{4,1}L_{7/10}$	10
42	$\mathbf{S}(E_{8,2}B_{8,1})/G_{2,1}$	106/5	3/5	8	$E_{8,2}D_{5,1}L_{7/10}$	11
43	$\mathbf{S}(A_{15,1}D_{9,1})/(G_{2,1} \hookrightarrow D_{9,1})$	106/5	3/5	8	$A_{15,1}B_{5,1}L_{7/10}$	12
44	$\mathbf{S}(D_{10,1}E^{2}_{7,1})/(G_{2,1} \hookrightarrow D_{10,1})$	106/5	3/5	8	$B_{6,1}E^{2}_{7,1}L_{7/10}$	14
45	$\mathbf{S}(D_{10,1}E^{2}_{7,1})/(G_{2,1} \hookrightarrow E_{7,1})$	106/5	3/5	8	$D_{10,1}E_{7,1}C_{3,1}$	14
46	$\mathbf{S}(A_{17,1}E_{7,1})/(G_{2,1} \hookrightarrow E_{7,1})$	106/5	3/5	8	$A_{17,1}C_{3,1}$	14
47	$\mathbf{S}(D^{2}_{12,1})/G_{2,1}$	106/5	3/5	8	$D_{12,1}B_{8,1}L_{7/10}$	18
48	$E^{3}_{8,1}/G_{2,1} \cong F_{4,1}E^{2}_{8,1}$	106/5	3/5	8	$E^{2}_{8,1}F_{4,1}$	26
49	$\mathbf{S}(D_{16,1}E_{8,1})/(G_{2,1} \hookrightarrow D_{16,1})$	106/5	3/5	8	$B_{12,1}E_{8,1}L_{7/10}$	26
50	$\mathbf{S}(D_{16,1}E_{8,1})/(G_{2,1} \hookrightarrow E_{8,1}) \cong D^{+}_{16,1}F_{4,1}$	106/5	3/5	8	$D_{16,1}F_{4,1}$	26
51	$\mathbf{S}(D_{24,1})/G_{2,1}$	106/5	3/5	8	$B_{20,1}L_{7/10}$	42
52	$\mathbf{S}(A^{24}_{1,1})/A_{1,1}$	23	7/4	2	$A^{23}_{1,1}$	32384
53	$\mathbf{S}(A^{4}_{3,2}A^{4}_{1,1})/A_{1,1}$	23	7/4	2	$A^{4}_{3,2}A^{3}_{1,1}$	32384
54	$\mathbf{S}(A_{5,3}D_{4,3}A^{3}_{1,1})/A_{1,1}$	23	7/4	2	$A_{5,3}D_{4,3}A^{2}_{1,1}$	32384
55	$\mathbf{S}(A_{7,4}A^{3}_{1,1})/A_{1,1}$	23	7/4	2	$A_{7,4}A^{2}_{1,1}$	32384
56	$\mathbf{S}(D_{5,4}C_{3,2}A^{2}_{1,1})/A_{1,1}$	23	7/4	2	$D_{5,4}C_{3,2}A_{1,1}$	32384
57	$\mathbf{S}(D_{6,5}A^{2}_{1,1})/A_{1,1}$	23	7/4	2	$D_{6,5}A_{1,1}$	32384
58	$\mathbf{S}(C_{5,3}G_{2,2}A_{1,1})/A_{1,1}$	23	7/4	2	$C_{5,3}G_{2,2}$	32384
59	$\mathbf{S}(A^{12}_{2,1})/A_{1,1}$	23	3/4	8	$A^{11}_{2,1}U_{1}$	2
60	$\mathbf{S}(D^{2}_{4,2}B^{4}_{2,1})/A_{1,1}$	23	3/4	8	$D^{2}_{4,2}C^{3}_{2,1}A_{1,1}$	2
61	$\mathbf{S}(A^{2}_{5,2}B_{2,1}A^{2}_{2,1})/(A_{1,1} \hookrightarrow A_{2,1})$	23	3/4	8	$A^{2}_{5,2}C_{2,1}A_{2,1}U_{1}$	2
62	$\mathbf{S}(A^{2}_{5,2}B_{2,1}A^{2}_{2,1})/(A_{1,1} \hookrightarrow B_{2,1})$	23	3/4	8	$A^{2}_{5,2}A_{1,1}A^{2}_{2,1}$	2
63	$\mathbf{S}(A_{8,3}A^{2}_{2,1})/A_{1,1}$	23	3/4	8	$A_{8,3}A_{2,1}U_{1}$	2
64	$\mathbf{S}(E_{6,4}C_{2,1}A_{2,1})/(A_{1,1} \hookrightarrow B_{2,1})$	23	3/4	8	$E_{6,4}A_{1,1}A_{2,1}$	2
65	$\mathbf{S}(E_{6,4}C_{2,1}A_{2,1})/(A_{1,1} \hookrightarrow A_{2,1})$	23	3/4	8	$E_{6,4}C_{2,1}U_{1}$	2
66	$\mathbf{S}(A^{8}_{3,1})/A_{1,1}$	23	3/4	8	$A^{7}_{3,1}A_{1,1}U_{1}$	4
67	$\mathbf{S}(D^{2}_{5,2}A^{2}_{3,1})/A_{1,1}$	23	3/4	8	$D_{5,2}A_{3,1}A_{1,1}U_{1}$	4
68	$\mathbf{S}(E_{6,3}G^{3}_{2,1})/A_{1,1}$	23	3/4	8	$E_{6,3}G^{2}_{2,1}A_{1,3}$	4
69	$\mathbf{S}(A_{7,2}C^{2}_{3,1}A_{3,1})/(A_{1,1} \hookrightarrow A_{3,1})$	23	3/4	8	$A_{7,2}C^{2}_{3,1}A_{1,1}U_{1}$	4
70	$\mathbf{S}(A_{7,2}C^{2}_{3,1}A_{3,1})/(A_{1,1} \hookrightarrow C_{3,1})$	23	3/4	8	$A_{7,2}C_{3,1}B_{2,1}A_{3,1}$	4
71	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(A_{1,1} \hookrightarrow G_{2,1})$	23	3/4	8	$D_{7,3}A_{3,1}A_{1,3}$	4
72	$\mathbf{S}(D_{7,3}A_{3,1}G_{2,1})/(A_{1,1} \hookrightarrow A_{3,1})$	23	3/4	8	$D_{7,3}G_{2,1}A_{1,1}U_{1}$	4
73	$\mathbf{S}(C_{7,2}A_{3,1})/A_{1,1}$	23	3/4	8	$C_{7,2}A_{1,1}U_{1}$	4
74	$\mathbf{S}(A^{6}_{4,1})/A_{1,1}$	23	3/4	8	$A^{5}_{4,1}A_{2,1}U_{1}$	6
75	$\mathbf{S}(C^{4}_{4,1})/A_{1,1}$	23	3/4	8	$C^{3}_{4,1}C_{3,1}$	6
76	$\mathbf{S}(D_{6,2}C_{4,1}B^{2}_{3,1})/(A_{1,1} \hookrightarrow C_{4,1})$	23	3/4	8	$D_{6,2}C_{3,1}B^{2}_{3,1}$	6
77	$\mathbf{S}(D_{6,2}C_{4,1}B^{2}_{3,1})/(A_{1,1} \hookrightarrow B_{3,1})$	23	3/4	8	$D_{6,2}C_{4,1}B_{3,1}A_{1,2}A_{1,1}$	6
78	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(A_{1,1} \hookrightarrow A_{4,1})$	23	3/4	8	$A_{9,2}A_{2,1}B_{3,1}U_{1}$	6
79	$\mathbf{S}(A_{9,2}A_{4,1}B_{3,1})/(A_{1,1} \hookrightarrow B_{3,1})$	23	3/4	8	$A_{9,2}A_{4,1}A_{1,2}A_{1,1}$	6
80	$\mathbf{S}(D^{6}_{4,1})/A_{1,1}$	23	3/4	8	$D^{5}_{4,1}A_{1,1}A_{1,1}A_{1,1}$	8
81	$\mathbf{S}(A^{4}_{5,1}D_{4,1})/(A_{1,1} \hookrightarrow A_{5,1})$	23	3/4	8	$A^{3}_{5,1}A_{3,1}D_{4,1}U_{1}$	8

No.	Theory	c	h	ℓ	Subalgebra	d
82	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{4,1})$	23	3/4	8	$\mathsf{A}_{5,1}^4\mathsf{A}_{1,1}^3$	8
83	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{5,1})$	23	3/4	8	$\mathsf{E}_{6,2}\mathsf{C}_{4,1}\mathsf{A}_{5,1}$	8
84	$\mathbf{S}(\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{5,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{5,1})$	23	3/4	8	$\mathsf{E}_{6,2}\mathsf{C}_{5,1}\mathsf{A}_{3,1}\mathsf{U}_1$	8
85	$\mathbf{S}(\mathsf{E}_{7,3}\mathsf{A}_{5,1})/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{E}_{7,3}\mathsf{A}_{3,1}\mathsf{U}_1$	8
86	$\mathbf{S}(\mathsf{A}_{6,1}^4)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{A}_{6,1}^3\mathsf{A}_{4,1}\mathsf{U}_1$	10
87	$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
88	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{6,1})$	23	3/4	8	$\mathsf{C}_{6,1}\mathsf{C}_{5,1}\mathsf{B}_{4,1}$	10
89	$\mathbf{S}(\mathsf{C}_{6,1}^2\mathsf{B}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{4,1})$	23	3/4	8	$\mathsf{C}_{6,1}^2\mathsf{B}_{2,1}\mathsf{A}_{1,1}$	10
90	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{7,1})$	23	3/4	8	$\mathsf{A}_{7,1}\mathsf{A}_{5,1}\mathsf{D}_{5,1}^2\mathsf{U}_1$	12
91	$\mathbf{S}(\mathsf{A}_{7,1}^2\mathsf{D}_{5,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{5,1})$	23	3/4	8	$\mathsf{A}_{7,1}^2\mathsf{A}_{3,1}\mathsf{A}_{1,1}$	12
92	$\mathbf{S}(\mathsf{D}_{9,2}\mathsf{A}_{7,1})/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{9,2}\mathsf{A}_{5,1}\mathsf{U}_1$	12
93	$\mathbf{S}(\mathsf{A}_{8,1}^3)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{A}_{8,1}^2\mathsf{A}_{6,1}\mathsf{U}_1$	14
94	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{8,1})$	23	3/4	8	$\mathsf{C}_{7,1}\mathsf{F}_{4,1}^2$	14
95	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	23	3/4	8	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}\mathsf{C}_{3,1}$	14
96	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{5,1})$	23	3/4	8	$\mathsf{E}_{7,2}\mathsf{B}_{3,1}\mathsf{A}_{1,1}\mathsf{F}_{4,1}$	14
97	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{F}_{4,1})$	23	3/4	8	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{C}_{3,1}$	14
98	$\mathbf{S}(\mathsf{D}_{6,1}^4)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{6,1}^3\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
99	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{9,1})$	23	3/4	8	$\mathsf{A}_{9,1}\mathsf{A}_{7,1}\mathsf{D}_{6,1}\mathsf{U}_1$	16
100	$\mathbf{S}(\mathsf{A}_{9,1}^2\mathsf{D}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{6,1})$	23	3/4	8	$\mathsf{A}_{9,1}^2\mathsf{D}_{4,1}\mathsf{A}_{1,1}$	16
101	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{C}_{10,1})$	23	3/4	8	$\mathsf{C}_{9,1}\mathsf{B}_{6,1}$	18
102	$\mathbf{S}(\mathsf{C}_{10,1}\mathsf{B}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{B}_{6,1})$	23	3/4	8	$\mathsf{C}_{10,1}\mathsf{B}_{4,1}\mathsf{A}_{1,1}$	18
103	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{E}_{6,1}^3\mathsf{A}_{5,1}$	20
104	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{11,1})$	23	3/4	8	$\mathsf{A}_{9,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1}\mathsf{U}_1$	20
105	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{7,1})$	23	3/4	8	$\mathsf{A}_{11,1}\mathsf{D}_{5,1}\mathsf{A}_{1,1}\mathsf{E}_{6,1}$	20
106	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{6,1})$	23	3/4	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{A}_{5,1}$	20
107	$\mathbf{S}(\mathsf{A}_{12,1}^2)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{A}_{12,1}\mathsf{A}_{10,1}\mathsf{U}_1$	22
108	$\mathbf{S}(\mathsf{D}_{8,1}^3)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{8,1}^2\mathsf{D}_{6,1}\mathsf{A}_{1,1}$	24
109	$\mathbf{S}(\mathsf{E}_{8,2}\mathsf{B}_{8,1})/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{E}_{8,2}\mathsf{B}_{6,1}\mathsf{A}_{1,1}$	26
110	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{15,1})$	23	3/4	8	$\mathsf{A}_{13,1}\mathsf{D}_{9,1}\mathsf{U}_1$	28
111	$\mathbf{S}(\mathsf{A}_{15,1}\mathsf{D}_{9,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{9,1})$	23	3/4	8	$\mathsf{A}_{15,1}\mathsf{D}_{7,1}\mathsf{A}_{1,1}$	28
112	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{10,1})$	23	3/4	8	$\mathsf{D}_{8,1}\mathsf{A}_{1,1}\mathsf{E}_{7,1}^2$	32
113	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	23	3/4	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{D}_{6,1}$	32
114	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{A}_{17,1})$	23	3/4	8	$\mathsf{A}_{15,1}\mathsf{E}_{7,1}\mathsf{U}_1$	32
115	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{7,1})$	23	3/4	8	$\mathsf{A}_{17,1}\mathsf{D}_{6,1}$	32
116	$\mathbf{S}(\mathsf{D}_{12,1}^2)/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{12,1}\mathsf{D}_{10,1}\mathsf{A}_{1,1}$	40
117	$\mathbf{S}(\mathsf{A}_{24,1})/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{A}_{22,1}\mathsf{U}_1$	46
118	$\mathbf{S}(\mathsf{E}_{8,1}^3)/\mathsf{A}_{1,1}\cong\mathsf{E}_{7,1}\mathsf{E}_{8,1}^2$	23	3/4	8	$\mathsf{E}_{8,1}^2\mathsf{E}_{7,1}$	56
119	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{D}_{16,1})$	23	3/4	8	$\mathsf{D}_{14,1}\mathsf{A}_{1,1}\mathsf{E}_{8,1}$	56
120	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{A}_{1,1}\hookrightarrow\mathsf{E}_{8,1})\cong\mathsf{D}_{16,1}^+\mathsf{E}_{7,1}$	23	3/4	8	$\mathsf{D}_{16,1}\mathsf{E}_{7,1}$	56
121	$\mathbf{S}(\mathsf{D}_{24,1})/\mathsf{A}_{1,1}$	23	3/4	8	$\mathsf{D}_{22,1}\mathsf{A}_{1,1}$	88

A close-up of a few entries:

13	$\mathsf{E}_{8,1}^2\mathsf{A}_{1,1}$	17	1/4	8	$\mathsf{E}_{8,1}^2\mathsf{A}_{1,1}$	2
14	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{E}_{7,1}\hookrightarrow\mathsf{E}_{8,1})$	17	1/4	8	$\mathsf{D}_{16,1}\mathsf{A}_{1,1}$	2
15	$\mathbf{S}(\mathsf{C}_{8,1}\mathsf{F}_{4,1}^2)/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{F}_{4,1})$	94/5	7/5	2	$\mathsf{C}_{8,1}\mathsf{F}_{4,1}$	4794
16	$\mathbf{S}(\mathsf{E}_{7,2}\mathsf{B}_{5,1}\mathsf{F}_{4,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{F}_{4,1})$	94/5	7/5	2	$\mathsf{E}_{7,2}\mathsf{B}_{5,1}$	4794
17	$\mathbf{S}(\mathsf{E}_{6,1}^4)/\mathsf{F}_{4,1}$	94/5	2/5	8	$\mathsf{E}_{6,1}^3\mathsf{L}_{4/5}$	1
18	$\mathbf{S}(\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{E}_{6,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{6,1})$	94/5	2/5	8	$\mathsf{A}_{11,1}\mathsf{D}_{7,1}\mathsf{L}_{4/5}$	1
19	$\mathbf{S}(\mathsf{D}_{10,1}\mathsf{E}_{7,1}^2)/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{7,1})$	94/5	2/5	8	$\mathsf{D}_{10,1}\mathsf{E}_{7,1}\mathsf{A}_{1,3}$	3
20	$\mathbf{S}(\mathsf{A}_{17,1}\mathsf{E}_{7,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{7,1})$	94/5	2/5	8	$\mathsf{A}_{17,1}\mathsf{A}_{1,3}$	3
21	$\mathsf{E}_{8,1}^3/\mathsf{F}_{4,1}\cong\mathsf{G}_{2,1}\mathsf{E}_{8,1}^2$	94/5	2/5	8	$\mathsf{E}_{8,1}^2\mathsf{G}_{2,1}$	7
22	$\mathbf{S}(\mathsf{D}_{16,1}\mathsf{E}_{8,1})/(\mathsf{F}_{4,1}\hookrightarrow\mathsf{E}_{8,1})$	94/5	2/5	8	$\mathsf{D}_{16,1}\mathsf{G}_{2,1}$	7
23	$\mathbf{S}(\mathsf{E}_{6,3}\mathsf{G}_{2,1}^3)/\mathsf{G}_{2,1}$	106/5	8/5	2	$\mathsf{E}_{6,3}\mathsf{G}_{2,1}^2$	15847
24	$\mathbf{S}(\mathsf{D}_{7,3}\mathsf{A}_{3,1}\mathsf{G}_{2,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{G}_{2,1})$	106/5	8/5	2	$\mathsf{D}_{7,3}\mathsf{A}_{3,1}$	15847
25	$\mathbf{S}(\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}^2)/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{3,1})$	106/5	3/5	8	$\mathsf{D}_{6,2}\mathsf{C}_{4,1}\mathsf{B}_{3,1}\mathsf{L}_{7/10}$	1
26	$\mathbf{S}(\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{B}_{3,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{B}_{3,1})$	106/5	3/5	8	$\mathsf{A}_{9,2}\mathsf{A}_{4,1}\mathsf{L}_{7/10}$	1
27	$\mathbf{S}(\mathsf{D}_{4,1}^6)/\mathsf{G}_{2,1}$	106/5	3/5	8	$\mathsf{D}_{4,1}^5\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
28	$\mathbf{S}(\mathsf{A}_{5,1}^4\mathsf{D}_{4,1})/(\mathsf{G}_{2,1}\hookrightarrow\mathsf{D}_{4,1})$	106/5	3/5	8	$\mathsf{A}_{5,1}^4\mathsf{L}_{1/2}\mathsf{L}_{7/10}$	2
29	$\mathbf{S}(\mathsf{D}_{8,2}\mathsf{B}_{4,1}^2)/\mathsf{G}_{2,1}$	106/5	3/5	8	$\mathsf{D}_{8,2}\mathsf{B}_{4,1}\mathsf{U}_1\mathsf{L}_{7/10}$	3

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- Since we are aiming for a complete classification, we have to resolve one last issue: when the Schellekens theory has multiple copies of the same algebra, it is not clear that embedding the denominator algebra in different factors is equivalent.
- This is so if the copies are permuted by outer automorphisms, but may not be true otherwise.

Outline

Conclusions and Outlook

- Some future directions:

- Some future directions:
  - Some $c = 25$ theories are known [Chandra-Mukhi 2019] by embedding in $c = 32$ lattices theories. It is surely possible to enlarge this set, though a complete classification may be intractable (because there are at least 10^9 even unimodular lattices in 32d).

- Some future directions:
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Thank you

De allerbeste, Erik en Herman!