Improvement of Algebraic attacks for solving superdetermined Minrank instances

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MinRank

DAGS

Conclusion

The MinRank problem in general

Computational MinRank

- `Input`: integers $r, m, n \in \mathbb{N}$, and K matrices $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_K \in \mathbb{F}_q^{m \times n}$
- `Output`: $(x_1, \ldots, x_K) \in \mathbb{F}_q$, not all zero, such that

$$\text{Rank}\left(\sum_{i=1}^{K} x_i \boldsymbol{M}_i\right) \leqslant r.$$

- This is exactly the decoding problem for matrix codes,
- NP-complete problem (Buss, Frandsen, Shallit 1999),
- used to cryptanalyse various multivariate and code-based cryptosystems.

Modeling MR: $\text{Rank}(M_{\vec{x}}) \leqslant r$ with $M_{\vec{x}} = \sum_{i=1}^{K} x_i M_i$

- Kipnis-Shamir modeling 1999 (hyp: last r columns of $M_{\vec{x}}$ are independent)

$$M_{\vec{x}} \begin{pmatrix} I_{n-r} \\ -R \end{pmatrix} = 0_{m\times(n-r)}, \qquad R \in \mathbb{F}_q^{r\times(n-r)} \qquad \text{(KS)}$$

- Minors modeling (Analysis by Faugère-Safey El Din-Spaenlehauer 2010)

$$\text{Minors}_{r+1}(M_{\vec{x}}) = 0 \qquad \text{(Minors)}$$

Hyp: it is sufficient to consider $|M_{\vec{x}}|_{J,T} = 0$ with $\{n-r+1..n\} \subset T$.

- Support Minors modeling 2020, $\vec{m}_j = (M_{\vec{x}})_{j,*}$

$$\text{Minors}_{r+1}\begin{pmatrix} & \vec{m}_j \\ R & I_r \end{pmatrix} = 0 \qquad \forall j \in \{1..m\}. \qquad \text{(SM)}$$

Ideals and Algebraic varieties

Links between the 3 modelings? (no hypothesis on the parameters)

Proposition

$$\langle \mathrm{KS} \rangle = \langle \mathrm{SM} \rangle$$
$$\langle \mathrm{Minors} \rangle \subseteq \langle \mathrm{KS} \rangle \cap \mathbb{F}_q[\vec{\boldsymbol{x}}]$$

Ideals and Algebraic varieties

Lemma

KS is included in SM.

Proof.

For all $j \in \{1..m\}$, $\ell \in \{1..n-r\}$ we have (Laplace expansion along the first row):

$$\left|\begin{pmatrix} & \vec{\boldsymbol{m}}_j \\ \boldsymbol{R} & \boldsymbol{I}_r \end{pmatrix}\right|_{*,\{\ell\}\cup\{n-r+1..n\}} = (\boldsymbol{M}_{\vec{\boldsymbol{x}}})_{j,\ell} - \sum_{i=1}^{r} (\boldsymbol{M}_{\vec{\boldsymbol{x}}})_{j,i+n-r} \boldsymbol{R}_{i,\ell} = \left(\boldsymbol{M}_{\vec{\boldsymbol{x}}} \begin{pmatrix} \boldsymbol{I}_{n-r} \\ -\boldsymbol{R} \end{pmatrix} \right)_{j,\ell}.$$

□

Lemma

$\langle KS \rangle$ contains Minors.

Proof.

- We write any $\boldsymbol{A} = \begin{pmatrix} \overset{n-r}{\leftrightarrow} & \overset{r}{\leftrightarrow} \\ \boldsymbol{A}^1 & \boldsymbol{A}^2 \end{pmatrix}$
- $vec_{col}(\boldsymbol{A})$ is a vector formed by all columns of $\boldsymbol{A}$ put one after the other,
- $\vec{\boldsymbol{v}}\boldsymbol{A}\vec{\boldsymbol{e}}^{\mathsf{T}} = \vec{\boldsymbol{v}}\left(\sum_i e_i \boldsymbol{A}_{*,i}\right) = (\vec{\boldsymbol{e}} \otimes \vec{\boldsymbol{v}})vec_{col}(\boldsymbol{A})$

Let $\boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2) = (\underbrace{0}_{j \notin J}, \dots, \underbrace{\left|\boldsymbol{M_{\vec{x}}}^2\right|_{J \setminus \{j\},*}}_{j \in J})_{j=1..m}$ for any $J \subset \{1..m\}$ of size $r+1$.

Then $\boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2)\vec{\boldsymbol{a}}^{\mathsf{T}} = \left|\vec{\boldsymbol{a}}^{\mathsf{T}}\boldsymbol{M_{\vec{x}}}^2\right|_{J,*}$ for any $\vec{\boldsymbol{a}}$, hence $\boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2)\boldsymbol{M_{\vec{x}}}^2 = 0$.

For any $1 \leqslant i \leqslant n-r$ we get

$$\vec{\boldsymbol{e}}_i \otimes \boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2) \underbrace{vec_{col}\left(\boldsymbol{M_{\vec{x}}}\begin{pmatrix}\boldsymbol{I}_{n-r} \\ -\boldsymbol{R}\end{pmatrix}\right)}_{=vec_{col}(\boldsymbol{M_{\vec{x}}}^1 - \boldsymbol{M_{\vec{x}}}^2\boldsymbol{R}) \in \mathrm{KS}} = \underbrace{\boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2)\boldsymbol{M_{\vec{x}}}^1\vec{\boldsymbol{e}}_i^{\mathsf{T}}}_{=\left|\boldsymbol{M_{\vec{x}}}\right|_{J,\{i\}\cup\{n-r+1..n\}}} - \underbrace{\boldsymbol{V}_J(\boldsymbol{M_{\vec{x}}}^2)\boldsymbol{M_{\vec{x}}}^2}_{=0}\boldsymbol{R}\vec{\boldsymbol{e}}_i^{\mathsf{T}}$$

Lemma

$\langle KS \rangle$ contains SM.

Proof.

- $(\vec{e}_\ell \otimes \boldsymbol{Y}) vec_{row}(\boldsymbol{X}) = vec_{row}(\vec{e}_\ell \boldsymbol{X} \boldsymbol{Y}^\mathsf{T}) = vec_{row}(\boldsymbol{X}_{\ell,*} \boldsymbol{Y}^\mathsf{T})$
- $\vec{a} \boldsymbol{V}_J(\boldsymbol{M}^\mathsf{T})^\mathsf{T} = \left| \begin{pmatrix} \vec{a} \\ \boldsymbol{M} \end{pmatrix} \right|_{J,*}$ for any $\vec{a}$

For any $1 \leqslant \ell \leqslant m$ and $J \subset \{1..n-r\}$ of size $r+1$ we get

$$(\vec{e}_\ell \otimes \boldsymbol{V}_J(\boldsymbol{R}^\mathsf{T})) \underbrace{vec_{row}\left(\boldsymbol{M}_{\vec{x}} \begin{pmatrix} \boldsymbol{I}_{n-r} \\ -\boldsymbol{R} \end{pmatrix}\right)}_{\in \mathrm{KS}} = (\boldsymbol{M}_{\vec{x}})_{\ell,*} \begin{pmatrix} \boldsymbol{I}_{n-r} \\ -\boldsymbol{R} \end{pmatrix} \boldsymbol{V}_J(\boldsymbol{R}^\mathsf{T})^\mathsf{T}$$

$$= \left| \begin{pmatrix} (\boldsymbol{M}_{\vec{x}}{}^1)_{\ell,*} \\ \boldsymbol{R} \end{pmatrix} \right|_{*,J}$$

$\square$

Lemma
$\langle KS \rangle$ contains SM.

Proof.
- $(\vec{\boldsymbol{e}}_\ell \otimes \boldsymbol{Y})vec_{row}(\boldsymbol{X}) = vec_{row}(\vec{\boldsymbol{e}}_\ell \boldsymbol{X}\boldsymbol{Y}^\top) = vec_{row}(\boldsymbol{X}_{\ell,*}\boldsymbol{Y}^\top)$
- $\vec{\boldsymbol{a}}\boldsymbol{V}_J(\boldsymbol{M}^\top)^\top = \left|\begin{pmatrix}\vec{\boldsymbol{a}}\\ \boldsymbol{M}\end{pmatrix}\right|_{J,*}$ for any $\vec{\boldsymbol{a}}$

For any $1 \leqslant \ell \leqslant m$ and $J \subset \{1..n-r\}$ of size $d+1$ and $T \subset \{1..r\}, \#T = d$, $J' = J \cup ((\{1..r\} \setminus T) + n - r)$, $\#J' = r+1$ we get

$$(\vec{\boldsymbol{e}}_\ell \otimes \boldsymbol{V}_J({\boldsymbol{R}^\top}_{*,T}))\underbrace{vec_{row}\left(\boldsymbol{M}_{\vec{\boldsymbol{x}}}\begin{pmatrix}\boldsymbol{I}_{n-r}\\ -\boldsymbol{R}\end{pmatrix}\right)}_{\in \mathrm{KS}} = (\boldsymbol{M}_{\vec{\boldsymbol{x}}})_{\ell,*}\begin{pmatrix}\boldsymbol{I}_{n-r}\\ -\boldsymbol{R}\end{pmatrix}\boldsymbol{V}_J({\boldsymbol{R}^\top}_{*,T})^\top$$

$$= \left|\begin{pmatrix}(\boldsymbol{M}_{\vec{\boldsymbol{x}}})_{\ell,*}\\ \boldsymbol{R} \quad \boldsymbol{I}_r\end{pmatrix}\right|_{*,J'}$$

□

Computational point of view

KS and SM produce the same ideal, not the same computations.

Gröbner basis computation on KS with the Normal selection strategy

- Eq. SM are produced from KS by multiplying by $\boldsymbol{R}$ variables at degree $(1, r+1)$ in $\vec{\boldsymbol{x}}, \boldsymbol{R}$ after a degree fall.

Gröbner basis computation on SM with the Normal selection strategy

- Eq. KS are included in SM, $\rightarrow$ many syzygies when multiplying by monomials in $\boldsymbol{R}$.
- When multiplying by monomials in $\vec{\boldsymbol{x}}$ of degree r, we have degree falls and equations of degree $(r+1, 0)$ (Minors).

$\rightarrow$ compute with SM, but multiply only by $\vec{\boldsymbol{x}}$ variables. Expect regular behavior up to degree $r+1$.

Solving SM with the Plücker coordinates

Equations, $0 \leqslant d \leqslant r$, $\#\mathscr{E}(d) = m\binom{n-r}{d+1}\binom{r}{d}$

$$\mathscr{E}(d) \triangleq \left\{ E_{J,T,\ell} \triangleq \vec{\boldsymbol{e}}_\ell \boldsymbol{M}_{\vec{\boldsymbol{x}}} \begin{pmatrix} \boldsymbol{I}_{n-r} \\ -\boldsymbol{R} \end{pmatrix} \boldsymbol{V}_J(\boldsymbol{R}_{T,*})^\mathsf{T} : \begin{smallmatrix} \forall J \subset \{1..n-r\}, \#J = d+1, \\ \forall T \subset \{1..r\}, \#T = d, \\ \forall \ell \in \{1..m\} \end{smallmatrix} \right\}.$$

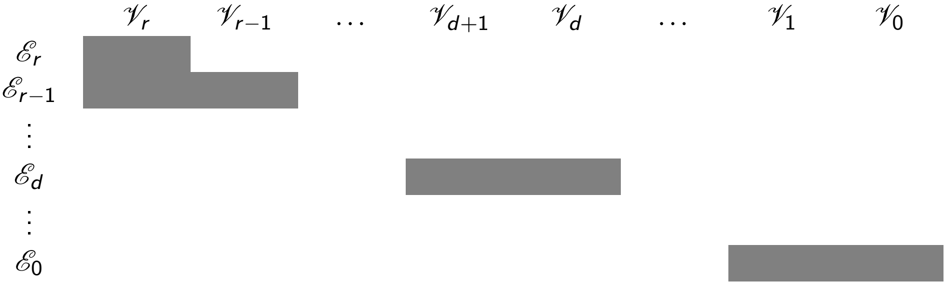
$$E_{J,T,\ell} = \left| \begin{pmatrix} \vec{\boldsymbol{m}}_\ell \\ \boldsymbol{R} \ \boldsymbol{I}_r \end{pmatrix} \right|_{*,T'} \quad \text{with } T' = J \cup (\{n-r+1..n\} \setminus (T+n-r)) \subset \{1..n\}$$

$$= \sum_{s \notin T} \left(\sum_{i=1}^{K} (\boldsymbol{M}_i^2)_{\ell,s} x_i \right) |\boldsymbol{R}|_{T \cup \{s\}, J} + \sum_{j \in J} \left(\sum_{i=1}^{K} (\boldsymbol{M}_i^1)_{\ell,j} x_i \right) |\boldsymbol{R}|_{T, J \setminus \{j\}}.$$

Variables, $0 \leqslant d \leqslant r$, $\#V(d) = K\binom{n-r}{d}\binom{r}{d}$

$$\mathscr{V}(d) \triangleq \{x_i |\boldsymbol{R}|_{T,J}\}_{i=1..K, \#J=d, \#T=d}, \qquad \mathscr{V}(r+1) \triangleq \emptyset.$$

Linearization: shape of the Macaulay matrix



- Degree fall: whenever $\#\mathscr{E}_d \geqslant \#\mathscr{V}_{d+1}$, i.e. $m(d+1) \geqslant K(r-d)$. $\rightarrow$ superdetermined MinRank instances
- End of computation (1 sol): whenever $\#\mathscr{E}_d \geqslant \#\mathscr{V}_{d+1} + \#\mathscr{V}_d - 1$.
- End of computation (1 sol): whenever $\sum_{d=0}^{r} \#\mathscr{E}_d \geqslant \sum_{d=0}^{r} \#\mathscr{V}_d - 1$, i.e. $m\binom{n}{r+1} \geqslant K\binom{n}{r} - 1$. Almost $m(n-r) \geqslant K(r+1)$
- Better linear exponent than for a random matrix.

Improvements

When linearization works too well:

- if $m\binom{n}{r+1} \gg K\binom{n}{r} - 1$, consider "punctured" codes (i.e. $n' < n$ columns) (but keep 1 solution).

When linearization does not work: $m\binom{n}{r+1} < K\binom{n}{r} - 1$, almost $m(n-r) < K(r+1)$

- use hybrid approach:
  - perform exhaustive search on k variables $\vec{\boldsymbol{x}}$ to get $m\binom{n}{r+1} \geqslant (K-k)\binom{n}{r}$,
  - perform exhaustive search on a columns of $\boldsymbol{R}$ to get $m\binom{n-a}{r+1} \geqslant (K-ma)\binom{n-a}{r} - 1$, almost $m(n-r) \geqslant K(r+1) - mar$ (we also get ma linear equations in $\vec{\boldsymbol{x}}$, see `https://arxiv.org/abs/2208.05471`!)
- Solve SM at higher degree b (multiplication by $\vec{\boldsymbol{x}}$ only).

Numerical values compared to Verbel et all, PQCrypto 2019

m	n	K	r	$\frac{m(n-r)}{K(r+1)}$	n_{eq}	n_{vars}	n_{rows} in PQcrypto 19
10	10	10	2	2.6	1,200	450	1,530
10	5	10	2	1	100	100	
10	10	10	3	1.75	2,100	1,200	20,240
10	7	10	3	1	350	350	
10	10	10	4	1.2	2,520	2,100	38,586
10	9	10	4	1	1,260	1,260	
10	10	10	5	**0.8**	2,100	2,520	341,495
10	10	10	5	$b=2$	14,400	13,860	
10	10	10	6	$b=6$	427,350	420,420	$> 2,035,458$

Table: Size of matrices on SM for a minrank instance with $K=10$ matrices of size $m \times n$, for various r. n can be decreased by puncturing the matrices to get a speedup. The results at $b=1$ have been verified experimentally on random instances.

Attack on DAGS by Barelli and Couvreur 2018

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DAGS Scheme

- KEM,
- quasi-dyadic alternant codes,
- submitted to the first round of the NIST PQ standardization process,
- attack by Barelli and Couvreur (Asiacrypt 2018): finding a secret code,
- it's a Minrank problem!

DAGS attack as a Minrank problem

Find a sub-code of the invariant public code such that:

$$\begin{pmatrix} \boldsymbol{I}_d & \boldsymbol{U} \end{pmatrix} \boldsymbol{G}_{inv} \star \boldsymbol{H}_{pub} \cdot \boldsymbol{V}^{\mathsf{T}} = 0.$$

with

- $\boldsymbol{U} \in \mathbb{F}^{(k_0-c)\times c}$,
- $\boldsymbol{G}_{inv} = (\boldsymbol{I}_{k_0} \; \boldsymbol{G}) \otimes \mathbb{1}_{2^\gamma}$ and $\boldsymbol{G} \in \mathbb{F}_{q^2}^{k_0\times(n_0-k_0)}$ public invariant matrix,
- $\boldsymbol{H}_{pub} = \begin{pmatrix} * & \boldsymbol{I}_{n_0-k_0} \otimes (1, 0_{2^\gamma}) \end{pmatrix}$ is a compact form of the public parity-check matrix,
- $\boldsymbol{V} = \vec{\tau} \otimes \mathbb{1}_{2^\gamma} + \sum_{i=1}^{\gamma-1} b_i \mathbb{1}_{n_0} \otimes \vec{\boldsymbol{e}}_i \in \mathbb{F}^{2^\gamma(n_0)}$ is a vector of unknowns $\vec{\tau} = (\tau_1, \dots, \tau_{n_0})$ and $(b_1, \dots, b_{\gamma-1})$.

Minrank version

$$\left(\sum_{i=1}^{k_0} \tau_i \boldsymbol{M}_i + \sum_{j=k_0+1}^{n_0-1} \tau_j \boldsymbol{M}_j + \sum_{i=1}^{\gamma-1} b_i \boldsymbol{H}_i\right) \begin{pmatrix} \boldsymbol{I}_{k_0-c} \\ \boldsymbol{U}^\mathsf{T} \end{pmatrix} = 0 \qquad (1)$$

$$\text{with } \boldsymbol{M}_i = \begin{pmatrix} 0_{i-1} & (\boldsymbol{G}_{\{i\},*})^\mathsf{T} & 0_{k_0-i} \end{pmatrix} \; \forall 1 \leqslant i \leqslant k_0$$

$$\boldsymbol{M}_{j+k_0} = \begin{pmatrix} 0_{j-1} \\ (\boldsymbol{G}_{*,\{j\}})^\mathsf{T} \\ 0_{n_0-k_0-j} \end{pmatrix} \; \forall 1 \leqslant j \leqslant n_0 - k_0$$

$$\boldsymbol{H}_i = \left(\boldsymbol{H}_{pub}(\boldsymbol{I}_{n_0} \otimes \vec{\boldsymbol{e}}_i{}^\mathsf{T})\right)_{*,\{1..k_0\}} \; \forall 1 \leqslant i \leqslant \gamma - 1$$

Non generic matrices

Proposition

For the DAGS minrank modeling, the part of the Macaulay matrix associated to rows $\mathscr{E}(d)$ and columns $\mathscr{V}(d+1)$ has

- $(n_0 - k_0)\binom{k_0-c}{d+1}\binom{c}{d}$ *rows,*
- $(n_0 - k_0 - 1 + c + \gamma - 1)\binom{k_0-c}{d+1}\binom{c}{d+1}$ *columns,*
- *rank* $\min\left(N_{rows}, \binom{k_0-c}{d+1}\left((n_0-k_0)\binom{c-1}{d} + \binom{c}{d+1}d\right)\right)$.

Reducing the number of variables: puncturing the code on a_0 columns $\rightarrow$ k_0 replaced by $k_0 - a_0$

Optimal attack on DAGS parameters

Security Level	q	n_0	k_0	γ	c	$k_0 - a_0 - c$	Matrix size	Rank	Time
DAGS_1 (128)	2^5	52	26	4	4	4	1456×2520	1322	3.5s
DAGS_3 (192)	2^6	38	16	4	4	5	2772×4284	2540	8.8s
DAGS_5 (256)	2^6	33	11	2	2	3	220×310	194	0.0s

Table: DAGS original sets of parameters, optimal attack, SM modeling

Conclusion

- better understanding of the algebraic systems associated to the MinRank problem, and why SM can perform better than KS or Minors,
- Plücker coordinates $r_T \leftrightarrow |\boldsymbol{R}|_{*,J}$,
- It is possible to use Minrank to attack cryptosystems in Hamming code-based crypto!

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