Model-based Control of Large-scale Baggage Handling Systems

Leveraging the Theory of Linear Positive Systems for Robust Scalable Control Design

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Proefschrift

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To my dearest Vida

Contents

Preface

This work was driven by the quest to develop model-based control algorithms for control of large-scale systems with the focus on large-scale baggage handling system. The main problem addressed in this thesis has overlaps with the problem of routing and scheduling of vehicles in automated warehousing, which is a vast field of research at the intersection of control theory & optimization of dynamical systems and operations research. Researchers from both fields have studied similar problems, albeit with a different scope and objective, for applications such as route planning of Automated Guided Vehicles (AGV).

What makes this work different is the especial attention paid to three key design features of the developed control solution, namely scalability, optimal performance, and robustness, which are necessary to apply the developed control algorithm in practice. In fact, these three features of the control algorithm guided me in making key decisions with respect to the choice of modeling framework and the control approach, and, hence, had significant impact on the composition of this thesis. For example, in Chapter 2, the choice for modeling the movement of vehicle as continuous flows rather than considering individual movement of vehicles was made to arrive at a tractable optimization problem that scales linearly with the size of problem. In addition, in the same chapter, model-predictive control was chosen as the main control approach as it enables one to incorporate the model in the control design with the aim of achieving a defined optimal performance. The robustness requirement led me to first write Chapter 3 to develop a model-based approach to deal with unmodeled disturbances in an optimal manner. My concerns with scalability of the solution developed in Chapter 3 subsequently led to the work of Chapter 4, where specific tools were developed for linear positive systems, of which baggage handling system in an example. Using the tools developed in Chapter 4, the original problem of Chapter 3 was solved in a scalable manner. In addition, the work of Chapter 4 incorporates the model-based design of Chapter 2, guaranteeing the optimal performance of the controlled system. Therefore, Chapter 4 marks the culmination of efforts made in its preceding chapters with the aim of achieving a robust scalable model-based control design for baggage handling systems.

Even though the subject of Chapter 5 was not initially among the main research questions driving my work, the work done in Chapter 4 on *robustly positively invariant sets* for discrete-time linear positive systems and the interesting properties of positive system intrigued me to study the theory of positive systems in more depth. In doing so, I gravitated towards the controllability problem for linear positive systems as it was surprisingly very different from the same problem for linear system. This led to the work of Chapter 5 on studying the geometry of reachable subsets from the origin for discrete-time linear positive systems.

This thesis may be of interest to those working on advanced model-based control design approaches for large-scale logistic system, and to those who see scalability as an inseparable part of their approach. In addition, enthusiasts of the theory of positive system and those curious about how certain properties of linear positive systems render certain control problems of linear system much simpler may find reading this work useful.

Yashar Zeinaly
Utrecht, June 2022

Acknowledgements

I am finally here, writing the last piece of my PhD dissertation, the piece I was quite frankly looking forward to all along. On my first day of PhD, I had to call the police to get me out of the 3ME building because I had confidently answered "yes" to my officemate, Dang Doan, when he asked me if I was sure I had access to the 3ME building after the business hours. Little did I know that there was a big difference between obtaining my access card and having the proper access rights installed on it. This marked the beginning of the excitement, drama, struggle, and joy that ensued during the next 4 years. Now, looking back, I would like to briefly reflect upon my time in DCSC, and thank the ones who supported, encouraged, and guided me throughout this journey.

First and foremost, I would like to express my deepest gratitude to my promoter Prof. Bart De Schutter for his endless support and his supervision. This thesis would not have been possible without his constant support, his valuable input on my research, and his insightful comments. I am especially thankful to him for showing extreme patience with me, for challenging my ideas and playing the devil's advocate, and for nudging me in the right direction whenever things didn't seem to work well. Moreover, I highly appreciate him making himself truly available for his students in the times of need.

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I very much enjoyed working together with Amir and Anna Sadowska as the teacher assistances of the *Control Theory* course. Thanks for your constructive collaboration. In addition, my appreciation goes to the students of the *Control Theory* course, who kept me very motivated by asking questions and triggering

interesting discussions. Without any doubt, my experience as the teacher assistant for this course was one of the best memories, if not the best memory, of my time as a PhD candidate in DCSC.

Huge thanks go to my dear officemate in my first two years: Amol, Dang, and Alfredo, who assisted me with practical matters in my first week, Ismini, Nicolas, Alfredo, Bart Kersbergen for making a fun work place, for intriguing discussions on society, politics and economy, and for DCSC game nights. I would also like to thank my DCSC colleagues Marco, Renshi, Le, Majid, Dieky, Elisabeth, Edwin, Farid, Hans, Iliya, and Ruxandra for intense foosball matches and for creating nice work atmosphere. I want to express my gratitude to Sadegh, Farzaneh, and Pirooz for their friendship and hospitality. My dear friends Amir, Sasan, and Patricio, thanks for your friendship. I consider myself lucky to have known you guys. I would like to express my sincere gratitude to my officemate-turned-great-friend Max Potters. Max, you have been a wise fun colleague, a patient listener, and a highly trusted friend, with whom I could share personal matters. I would like to thank you for keeping me motivated and for being a fantastic officemate.

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My wholehearted gratitude goes to my father, who was always there for me when I needed support and who encouraged me to persist in continuing the journey I had started. Even though he didn't live to see the completion of my PhD, his battle with cancer against all odds truly inspired me to finish this thesis. Thanks for everything dad.

There is no way I can thank you enough my dearest Vida. Navigating my way through the PhD years would have been impossible if it had not been for your unconditional love, endless support, and tremendous sacrifices. You have always been by my side even when it seemed the *world* was against me. You encouraged me the most, and helped me find the light at the end of the tunnel. Thanks for taking the journey of life with me and for giving us the most precious gift of all, our little *Nolan*.

Yashar Zeinaly
Utrecht, June 2022

1

Introduction

"I have had my results for a long time; but I do not yet know how I am to arrive at them."

Carl Friedrich Gauss, 1777 – 1855

This thesis is presented as a collection of papers, either published, accepted for publication, or under review. The papers discussed in this thesis cover two main categories of topics: i) model-based control design of large-scale Baggage Handling Systems (BHSs), and ii) positive linear systems and their applications in model-based control of large-scale baggage handling systems. The papers in the first part discuss modeling of BHSs and several Model Predictive Control (MPC) approaches for BHSs along with their computational complexity. The publications in the second part collectively rely on tools and methods from the theory of linear positive systems to develop a robust MPC approach for linear positive systems with applications to BHSs, and take a deeper dive into the theory of linear positive systems by providing a characterization for the infinite-time and the finite-time reachable subsets of discrete-time linear positive systems.

1.1. Model-Based Control of BHSs

The interest in large-scale BHSs has recently increased due to the increase in demand for air travel and cargo shipment, airport competition to attract passengers and airlines, and labor cost savings. Despite the recent downturn in the air travel demand due to the COVID-19 pandemic travel restrictions, the International Air Transportation Association predicts an average annual passenger growth rate of 3.3% [1], based on which 2.3 and 3.4 billion passengers worldwide are estimated to fly respectively in 2021 and 2022 with the recovery to the 2019 level taking place in 2024. In addition, robust growth of cargo demand is expected for 2021 and

2022, respectively, at 7.9% and 13.2% above the 2019 level [2]. The unexpected increase in cargo demand is most likely due to congested supply chains, which have caused a temporary shift of freight from sea to air. With the increasing demand, baggage handling service quality becomes a key factor for passenger and airline satisfaction, which, together with a reduction of expensive manpower, are crucial for long-term success and sustained business of airports.

Addressing the challenges mentioned above necessitates design and development of highly automated BHSs of a larger scales that are able to cope with the increased demand. More sophisticated designs, however, pose new challenges with respect to operational cost and efficiency. With modern baggage handling systems of large scale, it is of paramount importance to develop a systematic scheme to operate the system as efficiently as possible for two main reasons: i) to reduce both energy and operational costs to maintain affordability, ii) to fully exploit the capability of the sophisticated design. In order to develop design methods that are applicable to real-world BHSs, we focus on computational complexity and scalability for large-scale BHSs. Hence, the aim is not to develop the "best" performing MPC scheme, but rather the one that is also applicable to a large-scale BHS.

Chapter 2 of this thesis proposes a modeling framework for BHSs, in which the BHS is considered as a network of origin, transition, and destination nodes interconnected by links. The movement of pieces of baggage along the network links is modeled by baggage accumulation on the links driven by link-to-link baggage flows. The model developed in this chapter also takes into account the link-to-link flow travel time, which is generally a function of baggage accumulation on specific links.

Based on how the link-to-link travel time is treated, three classes of MPC [3, 4] approaches are proposed, namely a Nonlinear Programming (NLP) based approach, an Iterative Linear Programming (ILP) based approach, and a Linear Programming (LP) based approach. Given full information on the current and future baggage demand, the MPC approaches decide on modulating link-to-link baggage flows such that an optimal balance between timely delivery of pieces of baggage to the destination nodes and the overall energy consumption is achieved while guaranteeing the operational constraints. The recursive feasibility of the optimization problem and the asymptotic stability of the closed-loop system are either validated empirically via simulation or assured by employing sufficiently long prediction horizons.

The performance and computational complexity of the three MPC approaches are compared among each other and against a heuristic method commonly used for control of BHSs. It is shown that the MPC-based approaches can outperform the heuristic state-of-the art method, with further performance enhancement achieved for a longer prediction horizon. Furthermore, while the NLP approach provides the most accurate predictions, the optimization problem quickly becomes intractable for increasing values of the prediction horizon or for larger BHSs. The LP approach is the most efficient one, at the cost of low accuracy of predictions, especially for large prediction horizons. The ILP approach offers a good balance between accuracy of predictions and complexity of the optimization problem for long prediction horizons. Since it benefits from accurate predictions over a long horizon, it outperforms the heuristic method the most while being the most scalable to more complex BHS networks.

It is worth mentioning that there is a rich body of literature on Autonomous Mobile Robot (AMR) and Automated Guided Vehicle (AGV) based material handling and sorting systems [5–11]. A good overview on the application of AMRs in intralogistics is provided in [10]. For the AGV-based systems, [5, 6] provide a good review of the literature on the design and control of AGV systems for manufacturing, warehousing, and transport of material. While there is overlap between the problem investigated in Chapter 2 of this thesis and the three main category of problems in AMR- or AGV-based systems, namely route planning, scheduling of tasks, and dispatching, there are notable differences that does not allow direct application of those methods to the control of DCV-based BHSs.

These difference pertain to the objectives, complexity of solutions, and algorithms. The DCV-based BHSs control problem we discuss in Chapter 2, combines "light" versions of the three problem categories for AMR-based material handling systems. For example, in the approach of Chapter 2, we do not plan a route for each individual DCV, but we actually plan a time-based distribution of DCV flows. This is due to the fact that the main control objectives in Chapter 2, unlike AMR- or AGV-based systems, are not minimizing throughput time, finding the shortest path, minimizing travel time, or minimizing travel distance, but rather constrained time-based distribution of DCV flows in the network with minimum energy consumption. In addition, while the scheduling problem requires complex solutions for AMR-based manufacturing systems with multiple jobs and machine centers, the DCV-based BHS system only involves loading and unloading of DCVs, which makes the scheduling problem much simpler. The multi-objective MPC-based control algorithm developed in Chapter 2 can be implemented efficiently in a centralized manner for large-scale BHSs with several hundred DCVs. In addition, it provides a standard way of incorporating time-varying (dynamic) baggage demand profiles. In contrast, centralized routing and scheduling algorithms for AMR-based systems are practically feasible only for systems with a small number of vehicles. For large-scale systems, decentralized control structures are preferred, which come at the cost of sub-optimal control performance.

In Chapter 3, we focus on the predictability of the baggage demand profile and we consider the situations in which the baggage demand is only partially known in contrast to Chapter 2, where full knowledge of the current and future baggage demand was assumed. In this framework, the unpredictable baggage demand is considered as an additive disturbance to the system. We propose a two-level control approach consisting of i) a top-level controller, designed for the nominal system[1], that generates an optimal nominal control input based on full information of the baggage demand, and ii) a bottom-level controller, designed for the error system[2], that stabilizes the error system. Hence, the actual trajectory converges to the nominal trajectory driven by the top level controller. In addition to closed-loop stability of the error system, the bottom-level controller can optionally be designed

[1]The nominal system is the system without disturbance.

[2]The error considered here is the difference between actual trajectory and the nominal trajectory.

to achieve a defined input-output optimal performance.

Assuming that the baggage demand is composed of a nominal part and a small additive disturbance part, the modeling framework and the MPC approaches developed in Chapter 2 are used to develop an ILP-based MPC controller at the top level, which generates nominal link-to-link flows for the nominal component of the baggage demand. A state feedback controller is designed for the error system to minimize the L_2 gain of the error system from the disturbance input to the error output, hence minimizing the impact of disturbances on the nominal trajectory in the L_2-norm sense and providing robustness against perturbations caused by deviations of the demand from its nominal value. Using the bounded real lemma [12–15], the problem of searching for the feedback gain $\boldsymbol{K}$ minimizing the induced L_2 gain of the closed-loop error system can be expressed in terms of linear matrix inequalities (LMIs) [16], which results in a convex optimization problem. The final control input applied to the system is then $\boldsymbol{u}_{\text{mpc}} + \boldsymbol{K}\boldsymbol{e}$, where $\boldsymbol{u}_{\text{mpc}}$ and $\boldsymbol{e}$ are, respectively, the output of the MPC controller and the error. An important aspect of this approach is that the state and control constraints of the overall system, derived from operational constraints of the BHS, are ultimately handled by the MPC controller. For this purpose, assuming a disturbance input with a known bound on the L_2 norm, a positively invariant ellipsoid over the state space of error system is calculated offline. This state ellipsoid and the corresponding ellipsoid over the control input space can then be used to "tighten" the state and control constraints of the nominal MPC design such that the combined output of the controllers and the resulting system trajectory does not violate the control and state constraints.

In cases where the positively invariant ellipsoid associated with the unconstrained L_2-gain-optimal closed-loop error system turns out to be too "large" for the top level MPC controller to handle, a constrained approach is developed, where the L_2 gain of the error system in minimized subject to ellipsoidal state and control hard constraints. This approach is further extended by formulating the desired state and control constrains as soft constraints, hence avoiding infeasibility of the LMIs due to the overly restrictive state or control constraints. A small case study illustrates the implementation and performance of the suggested two-level control scheme.

1.2. Robust MPC for Linear Positive Systems

Chapter 4 discusses output-L_∞-norm-optimal feedback control design for linear positive systems, which is meant to be incorporated as the bottom-level controller in a tube-based MPC approach. For linear systems subject to additive disturbances, tube-based MPC [4, Chapter 3] is a low complexity robust MPC approach. It consists of a nominal MPC controller generating a nominal control input for the nominal (i.e., disturbance-free) system and a stabilizing state feedback controller governing the error system, which pertains to the difference between the uncertain system and the nominal system. Assuming a stable error system, the state trajectory of the disturbed system is contained in a bounded neighborhood of the nominal trajectory, which is called a tube. The combined goal of the controllers is to drive the tube center in a defined optimal manner while guaranteeing satisfaction of the constraints for the entire tube. This is achieved by ensuring that the nominal state and input trajectories satisfy appropriately "tightened" versions of the original constraints. The process of constraint tightening normally involves computation of the minimal robust invariant set, which is difficult to compute, especially for large-scale systems.

For stable linear positive systems, a robustly positively invariant hypercube containing the minimal robust invariant set can be obtained via a linear program, which can be used for constraint tightening in the tube-based MPC scheme. Based on this and given a disturbance set characterized by infinity-norm constraints (i.e., box constraints), we propose a state feedback design method for the error system that renders the closed-loop error system positive while minimizing the disturbance driven output of the error system in the L_∞ sense. For linear systems subject to additive disturbances, the L_∞ norm of output is a more general measure to capture the effect the of disturbances than the L_∞-induced norm (i.e., L_∞ gain) of the system as the L_∞ norm of the output provides a tighter bound on the "worst-case-scenario performance" of the system. We show that the joint problem of searching for the feedback gain $\boldsymbol{K}$ and the smallest robustly positively invariant hypercube $\mathbb{X}$ containing the minimal robust invariant set can be expressed as a linear program. Our problem formulation also allows for specifying a minimum size for $\mathbb{X}$ and for incorporation of infinity-norm hard and soft constraints on the control effort of the state feedback controller. Hence, in contrast to the approach of Chapter 3, where symmetric bounds on the state and the control input can only be imposed via ellipsoidal approximations, the approach developed in this chapter allows for direct inclusion of capacity constraints that arise in systems such as BHSs. In addition to providing a natural way of expressing the capacity constraints in BHSs, this method is better suited for large-scale BHSs since the conservatism introduced by the ellipsoidal approximations of the capacity constraints in the LMI-based approach may lead to infeasibility of the LMIs even though the original capacity constraints are feasible.

The developed method of state feedback design is applied, within a tube-based MPC scheme, to a BHS case study, where the nominal control input is generated by a linear-programming-based MPC controller. In contrast to Chapter 2, the design of the nominal MPC controller in this chapter explicitly enforces recursive feasibility and asymptotic stability by including an appropriate terminal constraint set and a terminal cost function, which are determined using linear programs. Therefore, the proposed tube-based control scheme is entirely a linear-programming-based approach, which scales well for large-scale BHSs.

1.3. Reachability of Linear Positive Systems

The class of positive dynamical systems, also known as nonnegative dynamical systems, generally refers to dynamical systems, the state trajectories of which reside in the positive orthant for nonnegative initial conditions and nonnegative inputs. This property, however, does not require the initial states or the control inputs to be nonnegative even though many real-word examples of positive systems are only defined for nonnegative inputs and nonnegative initial states. Positive systems arise in many applications such as econometrics, bio-chemical reactors, compartmental systems, and transportation systems. BHSs, as the main application area of this thesis, are prime examples. The variables in positive systems represent growth rates, concentration levels, mass accumulation, flows, etc. Obviously, variables of this nature can only assume nonnegative values. Many classical control problems such as H_∞, L_1, and L_∞ problems, which are the subject of Chapter 4, can be made significantly simpler for positive systems. Inspired by the theory of positive dynamical systems and its applicability to several application domains, this chapter focuses on a fundamental property of linear positive systems: *reachability*. Following prior work in the literature on the reachability of linear positive systems, reachability of discrete-time linear time-invariant positive systems is revisited. This topic is of particular interest since, due to the fact that linear positive systems are defined over a cone rather than over a linear subspace, their reachability is not in general equivalent to the reachability of linear systems.

We propose an alternative formulation of the reachability problem for discrete-time linear time-invariant positive systems, where the goal is to check whether a certain subset or subcone of the positive orthant can be reached from the origin in finite or infinite time, which is motivated from an application point-of-view and which is in contrast with the classical view of requiring the entire positive orthant to be reachable from the origin. The latter view of reachability leads to strong conditions, which are very difficult to satisfy in practice. To formulate our view of the reachability problem, first a characterization of the finite-time and infinite-time controllable subsets, which are the cones of all reachable states from the origin in finite time and infinite time, is given for a single-input positive system. We focus our attention to the geometry of the controllable subsets and to the cases where the finite or infinite controllable subsets are polyhedral (i.e., they can be expressed by a finite number of extremal rays). Then, necessary and sufficient conditions for polyhedrality of the controllable subsets are derived for a pair $(\boldsymbol{A}, \boldsymbol{b})$, with $\boldsymbol{A}$ and $\boldsymbol{b}$ being the system matrices. For a generic $\boldsymbol{b}$ without any imposed structure, it turns out that such conditions are characterized solely in terms of the spectrum of $\boldsymbol{A}$.

Finally, the reachability of a polyhedral target sub-cone or polyhedral target subset of the positive orthant is established by requiring all extremal rays of the target sub-cone or extremal points of the polyhedral subset to be included in the reachable finite-time or infinite-time controllable subsets, which leads to solving a set of constrained linear equations.

1.4. Organization

The organization of thesis is conceptually illustrated by Fig. 1.1, where the publication corresponding to each chapter is highlighted, and where relation of the thesis chapters to each other, in terms of common themes, is indicated.

Chapter 2 describes the model of BHSs, where various MPC-based approaches are discussed and their performance is evaluated using a detailed case study. For a BHS subject to additive disturbances, the proposed 2-level control scheme is introduced in Chapter 3, and the design of a constrained state feedback controller minimizing the L_2-induced-gain of the closed-loop system is discussed. Chapter 4 discusses the output-L_∞-norm-optimal state feedback design problem for discrete-time linear positive systems and implements the developed design approach in a tube-based MPC scheme for a BHS. In Chapter 5, we provide our view on the reachability of discrete-time linear time-invariant positive systems, where necessary and sufficient conditions for checking the reachability of a subset of the positive orthant are developed. Finally, Chapter 6 , concluding remarks are laid out and potential future research directions in continuation of this work are pointed out.

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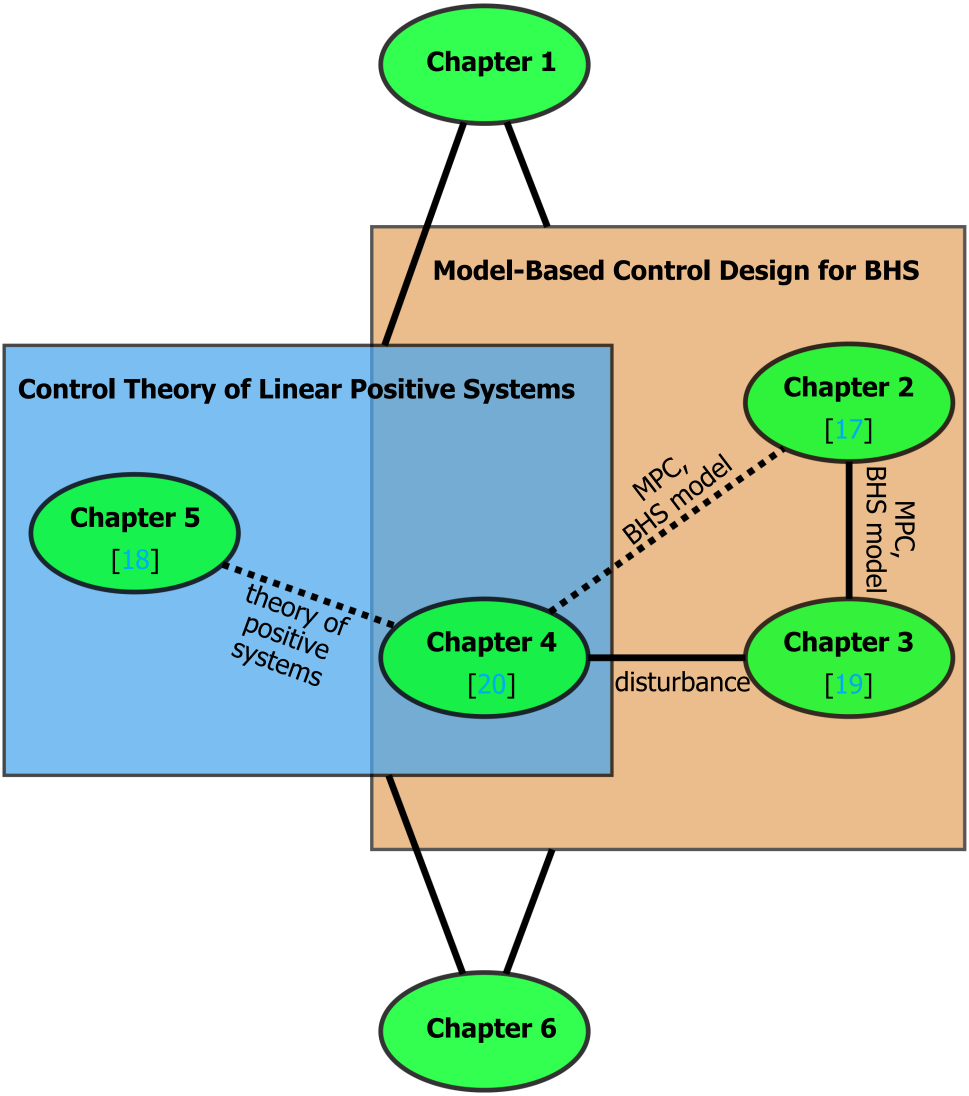

Figure 1.1: Schematic diagram of the thesis organization highlighting the publication used for each chapter along with the common themes between chapters. Solid links indicate that the chapters are strongly related with respect to the mentioned themes. Dashed links indicate that the chapters are weakly related with respect to the mentioned themes.

2

An Integrated Model Predictive Scheme for Baggage Handling Systems: Routing, Line Balancing, and Empty-Cart Management

"True optimization is the revolutionary contribution of modern research to decision processes."

George Bernard Dantzig, 1914 – 2005

This chapter proposes a new strategy for integrated control of destination-coded-vehicle-based baggage handling systems. Three main control issues in baggage handling systems, namely, routing and scheduling, empty-cart management, and line balancing, are identified and a combined control approach based on model predictive control is proposed to tackle these issues in an optimal way. It is shown that the control approach can be formulated as a linear programming problem, which can be solved very efficiently, and hence the proposed approach can be extended to large-scale baggage handling systems. We illustrate the applicability and performance of the proposed approach by a case study, and we compare the results with the state-of-the-art method currently used for baggage handling systems.

This chapter is based on [1].

2

2.1. Introduction

In the past decade, modern baggage handling systems [2, 3] (BHS) have been implemented in large airports to accommodate the rising demand in air travel. Such baggage handling systems are controlled by state-of-the-art techniques that are mostly tailor-made for a specific layout. However, with increasing demand, it becomes necessary to increase the efficiency and reliability of the baggage handling systems by utilizing a systematic controller design approach. Such a control approach should optimize the performance of the baggage handling system in terms of reliability and costs. A modern baggage handling system, an essential layout of which is depicted in Fig. 2.1, is composed of the following components [2]: Destination coded vehicles (DCVs), which are high-speed vehicles powered by linear induction motors transporting the pieces of baggage between various locations in the system. Each DCV can carry only one piece of baggage. The term DCV refers to both carts move powered by linear induction motors and passive tubs on modular conveyor elements, as illustrated in Fig. 2.2; loading stations, where the pieces of baggage are loaded onto DCVs after entering the system (either from the check-in desks or from the transfer flights); unloading stations, where the DCVs unload the pieces of baggage; the pieces of baggage are then transported to the planes; an early baggage storage (EBS), which is an automated storage/retrieval system used to temporarily store the loaded DCVs; a network of uni-directional tracks on which the DCVs travel. This network connects the loading stations, the unloading stations, and the EBS; and a switch controller at each junction that determines the path of the DCVs that pass through that junction. From a high-level control perspective, there are three main control challenges related to the baggage handling systems [3–5], namely, i) routing and scheduling of DCVs, ii) line balancing, and iii) empty-cart management. The routing problem is the problem of routing loaded DCVs from the loading stations to the unloading stations or to the EBS and the problem of routing the DCVs from the EBS to the unloading stations. Line balancing is the problem of dynamically assigning empty DCVs located at the unloading stations to the loading stations. Closely related to line balancing is empty-cart management, which is the problem of routing empty DCVs from the unloading stations, through the network, to their assigned loading stations.

The control problems in baggage handling system can, in general, be related to operation scheduling, flow shop scheduling, and production scheduling [6, 7], or to predictive routing and flow control [8, 9]. Recently, the application of model predictive control (MPC) [10–12] to supply chain management has been studied in the literature [13–15]. In [8] the authors propose a two-level decision making process: long-term strategic level based on offline branch and bound optimization and short-term tactical level based on MPC. In [14], the authors propose a MPC scheme based on quadratic programming (QP) optimization, and [13] proposes and MPC approach based on mixed integer linear programming (MILP). Our proposed approach here differs from the literature in the following ways: i) Unlike the supply chain networks where transport times are fixed or known functions of time, BHS involves highly variable queue-length-dependent transport delays, which needs special treatment when developing the model and defining the optimization

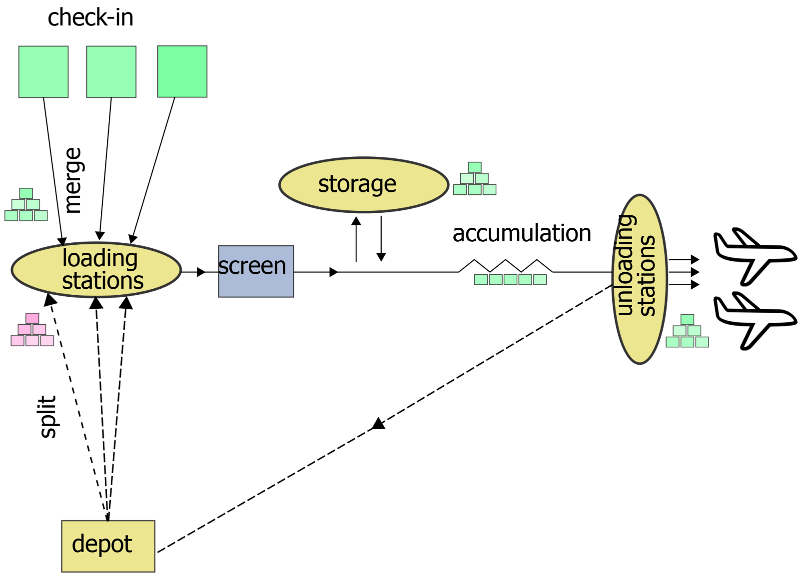

Figure 2.1: Basic configuration of a baggage handling system showing the loading stations, unloading stations, DCV storage (EBS), and the DCV depot. Vertical baggage and loaded-DCV stacks are assumed at the loading stations and the EBS, respectively. Vertical empty DCV stacks are assumed at the loading station and the central depot. In the rest of the network, loaded and empty DCVs accumulate along the network links.

problem. ii) To achieve timely arrival of DCVs within their time windows as well as possible, we introduce time-window constrains as soft, rather than hard, constraints in the objective function to ensure that the optimization problem remains feasible at all times. To this end, the objective function includes a time-weighted sum of DCV queues and DCV flows. In this way, we employ a different objective function from those in the literature, where travel times are often explicitly penalized and the time window requirements are imposed as optimization constraints. iii) For realtime control purposes, the proposed algorithm has to be computationally efficient for large-scale BHS. This makes MPC schemes with long prediction horizons based on MILP or nonlinear optimization practically unsuitable for large-scale BHS. Therefore, the solutions developed in the literature are not directly applicable to the baggage handling systems. To address this issue, we need to develop numerically-less-intensive schemes. In this chapter we have opted for an LP-based approach.

The state-of-the-art control method for baggage handling systems addresses the
routing problem by using look-up tables to control the switches at the junctions [2, 5]. These look-up tables are computed off-line for different system operation scenarios. However, such a control scheme cannot guarantee optimal performance of the system for complex network layouts. In addition, this control method addresses the empty-cart management by decoupling the empty-DCV and loaded-DCV

traffic flows using dedicated loops to transfer empty carts to loading stations. Moreover, it relies on heuristics for line balancing. An immediate shortcoming of such a control system is that baggage handling systems have to adopt simple design layouts (e.g., composed of few loops). This limits the achievable performance of the system in terms of flexibility and baggage throughput.

Recently, it has been shown [4, 5, 16, 17] that a systematically designed control system can cope with complex network layouts. In [16], a multi-agent control strategy has been proposed for conveyor-based baggage handling systems, where each bag is dynamically assigned a path using the shortest path algorithm. The authors of [16] have shown that their algorithm outperforms the conventional control scheme used in practice, but they have not included optimal performance in their problem formulation. Model-based control of DCV-based baggage handling systems is considered in [4], where an automated way of learning routing rules has been proposed to solve the routing problem. In comparison with [4], we discuss optimality of our approach as well as on-time arrival of DCVs to the unloading stations. In [5], a solution has been proposed for the routing problem based on model predictive control (MPC) [10–12]. In this approach, a dynamic sequence of optimal switch positions is assigned to each DCV in order to guide it along an optimal route to its destination. This approach guarantees optimal performance of the system, but it is computationally prohibitive for large-scale systems[1]. A more computationally efficient MPC-based approach has been proposed in [18], which arrives at a mixed integer linear programming (MILP) formulation of the problem. In [19], it has been shown that the problem in [18] can in fact be recast as a linear programming (LP) problem, which can be solved efficiently for large-scale systems [20]. Nevertheless, to the best of our knowledge, the previous works have only focused on a particular control issue of the baggage handling systems. The aim of this chapter is to propose an optimal integrated solution to routing and scheduling, empty-cart management, and line balancing.

We consider the following two criteria for an effective baggage handling system. First, the pieces of baggage should reach the assigned unloading stations within pre-specified time windows. Second, the cost of operating the system should be minimized. In order to achieve an overall optimal performance with respect to these criteria, in this chapter, we propose a control scheme based on MPC that addresses the aforementioned control problems in *one integrated approach*, rather than treating them as individual sub-problems. Moreover, based on the model we develop, we show that the resulting optimization problem is in general a nonlinear optimization problem the solution of which can be obtained using a general nonlinear programming (NLP) approach or a iterative linear programming (ILP) approach. We also propose a suboptimal solution based on linear programming. We then compare the performance of these three approaches to each other and to the state-of-the-art method used for BHS. In addition, we compare our proposed control approaches in terms of their computational complexity for the case study at hand.

[1] For instance, Amsterdam Schiphol airport operates 550 DCVs [2], and Denver International Airport operates about 4000 DCVs [3], which indicates real-life systems can indeed be large scale.

The rest of this chapter is organized as follows. In Section 2.2, we develop the dynamical model. In Section 2.3, we define the MPC optimization problem. A case study is presented in Section 2.4 to illustrate the performance of the proposed control approach for a given scenario and finally, Section 2.5 concludes the chapter.

(a) Carts powered by linear induction motors. (b) Passive tubs on modular conveyor elements.

Figure 2.2: Two types of DCVs used for transporting luggage within BHSs. Note that for both motorized carts and passive tubs power by modular conveyor belts one can assign a travel route within the network. Photos courtesy of Vanderlande.

2.2. Dynamical Model

2.2.1. Notation and Assumptions

The baggage handling system network can be seen as a directed graph, where nodes of the graph are composed of loading stations, unloading stations, junctions, and the EBS, and where the links represent the tracks of the system. The relation between the graph representation and the real network is rather symbolic. Not all components of DCV-based BHS are shown on the graph. However, the most important components of BHS, namely, the loading stations, the junctions, the unloading stations, and the EBS are present in our graph representation of the network. In

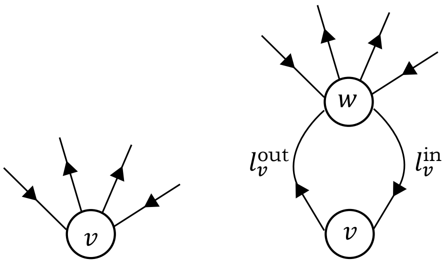

Figure 2.3: Physical node v, which represents a loading station, an unloading station, or the EBS (left) and its "extended" description (right)

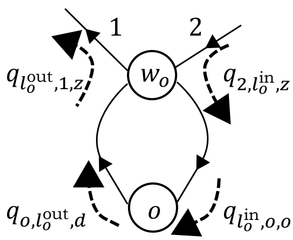

Figure 2.4: Flow variables associated with an extended loading station connected to external links 1 and 2.

our mathematical description of the system, as depicted in Fig. 2.3, we will replace physical loading stations, unloading stations, and the EBS with their "extended" description, which is a super node comprising two nodes, a unique virtual incoming link, and a unique virtual outgoing link. Please note that there is no virtual link associated with the junctions of the network, so the links connected to the junctions do represent the tracks. There are no DCV queues on the virtual links and the travel time on the virtual links represents the time needed for storing DCVs in their corresponding stacks or the time needed for loading pieces of baggage onto DCVs. Hence, each loading station and each unloading station, and the EBS has only one incoming link and one outgoing link. This considerably simplifies representation of the model. Hereafter, unless otherwise mentioned, we drop explicit reference to the "extended" prefix and use the term graph to refer to its extended version.The graph representation of the network is denoted as $G = (V, A)$, where $V = V_1 \cup V_2 \cup V_3 \cup \{v^*\}$ is the set of nodes composed of set V_1 associated with the loading stations, set V_2 associated with the intermediate nodes (i.e., junctions), set V_3 associated with the unloading stations, and the node v^* associated with the EBS. Moreover, A is the set of arcs composed of links, (i.e., physical tracks as well as virtual links) connecting the elements of V. In the sequel, we make the following assumptions regarding configuration of the network:

A1 Only loaded DCVs are dispatched from the loading stations.

A2 The baggage queues at the unloading stations are ignored. This is because we assume either destination nodes have sufficient capacity or the pieces baggage are immediately transported to the planes upon arrival.

A3 The movement of DCVs on the network is approximated by a continuous flow of DCVs.

A4 The DCV travel time on each link is an integer multiple of the sampling time Δt.

Assumption A3 is necessary for tractability of the control problem. Even though the number of DCVs is an integer in reality, for a fairly large number of DCVs, the movement of DCVs can be approximated by continuous flows. This is not very restrictive as the computed flows can then be realized as well as possible by a lower-level control loop that determines the optimal switching pattern for the switch controllers at the junctions. Assumption A4 allows us to arrive at a linear discrete-time model of the system. In this setup, we control the flows of DCVs within the network. The flows are indexed based on their destinations, enabling us to distinguish between loaded DCVs and empty DCVs, and also loaded or empty DCVs with different destinations among themselves. The DCV flows with an index $o \in V_1$ refer to empty-DCV flows whereas the DCV flows with an index $d \in V_3$ refer to loaded-DCV flows. Consequently, partial DCV queues associated with different destinations occur along the links of the network. The total DCV queue length along a link is then given as the sum of such partial queue lengths. The system is composed of baggage and empty-DCV vertical queues at the loading stations, empty-DCV vertical queues at the unloading stations, loaded-DCV queues at the EBS, and empty and loaded DCV queues along the links of the network, as shown in Fig. 2.4. Note that certain links of the network may carry both empty and loaded DCV flows. Moreover, the loading stations, the EBS, and the links have limited capacity. In our mathematical model, we make use of the following notation:

- For each node $v \in V$, L_v^{in} is the set of incoming links of v and L_v^{out} is the set of outgoing links of v.

- For each link $l = (v, w)$ of the network, $q_{l,p,z}$ is the flow from the end of link l to link $p \in L_w^{\text{out}}$, with destination z.

- For each destination z, L_z denotes to the set of links that are on some directed path to z.

- s_l is the length of link l. The speed of DCVs is denoted by v_{DCV}. Moreover Δt is the sampling time.

- For each $z \in V_3$, k_z^{open} and k_z^{close} mark, respectively, the beginning and the end of the time window of destination z, and $k_{v,z}^{\text{nom}}$ is the nominal travel time from $v \in V_1 \cup V_2 \cup \{v^*\}$ to z. Moreover, $k_{v,z}^1 = k_z^{\text{open}} - k_{v,z}^{\text{nom}}$ and $k_{v,z}^2 = k_z^{\text{close}} - k_{v,z}^{\text{nom}}$ are respectively the *relative* opening time-step and relative closing time-step of destination z as seen from v.

2.2.2. Model Description

Now we will derive the dynamical model of the baggage handling system in discrete time under the assumptions A1-A7. In the sequel, $\boldsymbol{x}(k)$, $k = 0, 1, \ldots$ denotes the value of $\boldsymbol{x}$ at time step k, and $\lceil x \rceil$ denotes the smallest integer bigger than or equal to x.

Loading Stations

For each loading station node o, let $l_o^{\text{in}} = (w_o, o)$ and $l_o^{\text{out}} = (o, w_o)$ respectively be the virtual incoming link and virtual outgoing link of o for some $w_o \in V \backslash V_1$. The control variables at each loading station are the flows of loaded DCVs, $q_{o,l_o^{\text{out}},d}$, from the DCV stack at o to l_o^{out} with destination $d \in V_3$, and the flow of empty DCVs, $q_{l_o^{\text{in}},o,o}$, from l_o^{in} to o with destination o. Fig. 2.4 illustrates how the flow variables are defined for the loading stations. For each outgoing link p of w_o, we need to impose the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned}
&q_{l_o^{\text{out}},p,z}(k) = 0, \forall\ z \in V_1 \\
&q_{l_o^{\text{out}},p,z}(k) = 0, \forall\ z \in V_3 \text{ s.t. } p \notin L_z \\
&q_{l_o^{\text{out}},p,z}(k) \geq 0, \forall\ z \in V_3 \text{ s.t. } p \in L_z.
\end{aligned} \tag{2.1}$$

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For each incoming link p of w_o, we impose the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned} q_{p,l_o^{\text{in}},z}(k) &= 0, \forall\ z \in V_3 \\ q_{p,l_o^{\text{in}},z}(k) &= 0, \forall\ z \in V_1 \backslash \{o\} \\ q_{p,l_o^{\text{in}},z}(k) &\geq 0, z = o. \end{aligned} \tag{2.2}$$

Note that constraint (2.1) implies that the loading station o can only send loaded-DCV flows to the unloading stations that are reachable from o. Constraint (2.2) implies that the loading station o can only accept empty-DCV flows that are designated for o. The evolution of DCV stack at loading station o, $\boldsymbol{x}_o(k)$, is then given by

$$\begin{aligned} \boldsymbol{x}_o(k+1) &= \boldsymbol{x}_o(k) + \Delta t\big(q_{l_o^{\text{in}},o,o}(k) - \sum_{d \in V_3} q_{o,l_o^{\text{out}},d}(k)\big) \\ 0 \leq \boldsymbol{x}_o(k) &\leq \boldsymbol{x}_{o,\max}, \end{aligned} \tag{2.3}$$

where $\boldsymbol{x}_{0,\max}$ is the maximum capacity of the DCV stack at loading station o. Let $\boldsymbol{x}_{o,d}^{\text{bag}}(k)$ be the length of the baggage queue, with destination $d \in V_3$, at loading station o. Then, $\boldsymbol{x}_{o,d}^{\text{bag}}(k)$ is described as

$$\begin{aligned} \boldsymbol{x}_{o,d}^{\text{bag}}(k+1) &= \boldsymbol{x}_{o,d}^{\text{bag}}(k) + \Delta t\big(Q_{o,d}(k) - q_{o,l_o^{\text{out}},d}(k)\big) \\ \boldsymbol{x}_{o,d}^{\text{bag}}(k) &\geq 0,\ \forall\ d \in V_3, \end{aligned} \tag{2.4}$$

where $Q_{o,d}(k)$ is the time varying baggage demand at loading station o that needs to be transported to destination d. The total length of baggage queue at node o is given by $\boldsymbol{x}_o^{\text{bag}}(k) = \sum_{d \in V_3} \boldsymbol{x}_{o,d}^{\text{bag}}(k)$. In order to guarantee that there is no DCV queue along the virtual outgoing link l_o^{out} and the virtual incoming link l_o^{in}, their inflow and outflow must be set equal, or equivalently

$$\begin{aligned} q_{o,l_o^{\text{out}},d}(k) &= \sum_{p \in L_{w_o}^{\text{out}}} q_{l_o^{\text{out}},p,d}(k + k_{l_o^{\text{out}}}),\ \forall\ d \in V_3 \\ q_{l_o^{\text{in}},o,o}(k + k_{l_o^{\text{in}}}) &= \sum_{p \in L_{w_o}^{\text{in}}} q_{p,l_o^{\text{in}},o}(k), \end{aligned} \tag{2.5}$$

where $k_{l_o^{\text{out}}}$ is the number of time steps required to load a piece of baggage onto the DCVs, and $k_{l_o^{\text{in}}}$ is the number of time steps that is required to store empty DCVs in the DCV stack.

Unloading Stations

For each unloading station node d, let $l_d^{\text{in}} = (w_d, d)$ and $l_d^{\text{out}} = (d, w_d)$ respectively be the virtual incoming link and virtual outgoing link of d for some $w_d \in V_2$. The control variables at each unloading station are the flows of empty DCVs, $q_{d,l_d^{\text{out}},o}$, from d to l_d^{out} with destination o, and the flows of loaded DCVs, $q_{l_d^{\text{in}},d,d}$, from l_d^{in} to d with destination d. For each outgoing link p of w_d, we impose the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned}
&q_{l_d^{\text{out}},p,z}(k) = 0, \forall\ z \in V_3,\\
&q_{l_d^{\text{out}},p,z}(k) = 0, \forall\ z \in V_1 \text{ s.t. } p \notin L_z \\
&q_{l_d^{\text{out}},p,z}(k) \geq 0, \forall\ z \in V_1 \text{ s.t. } p \in L_z.
\end{aligned} \tag{2.6}$$

For each incoming link p of w_d, we need to impose the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned}
&q_{p,l_d^{\text{in}},z}(k) = 0, \forall\ z \in V_1,\\
&q_{p,l_d^{\text{in}},z}(k) = 0, \forall\ z \in V_3 \backslash \{d\},\\
&q_{p,l_d^{\text{in}},z}(k) \geq 0, z = d.
\end{aligned} \tag{2.7}$$

Constraint (2.6) implies that unloading station d can only send empty-DCV flows to the loading stations that are reachable from d. Constraint (2.7) implies the unloading station d can only accept loaded-DCV flows the final destination of which is d. The evolution of the DCV stack at unloading station d is given by

$$\begin{aligned}
&\boldsymbol{x}_d(k+1) = \boldsymbol{x}_d(k) + \Delta t \big(q_{l_d^{\text{in}},d,d}(k) - \sum_{o \in V_1} q_{d,l_d^{\text{out}},o}(k)\big),\\
&0 \leq \boldsymbol{x}(k) \leq \boldsymbol{x}_{d,\max},
\end{aligned} \tag{2.8}$$

where $\boldsymbol{x}_{d,\max}$ is the maximum capacity of the DCV stack at unloading station d. Since, by definition, no DCV queues can occur along the virtual links of d, we have to set their inflow equal to their outflow, or equivalently

$$\begin{aligned}
q_{d,l_d^{\text{out}},o}(k) &= \sum_{p \in L_{w_d}^{\text{out}}} q_{l_d^{\text{out}},p,o}(k + k_{l_d^{\text{out}}}), \forall\ o \in V_1,\\
q_{l_d^{\text{in}},d,d}(k + k_{l_d^{\text{in}}}) &= \sum_{p \in L_{w_d}^{\text{in}}} q_{p,l_d^{\text{in}},d}(k),
\end{aligned} \tag{2.9}$$

where $k_{l_d^{\text{out}}}$, and $k_{l_d^{\text{in}}}$ are respectively the number of time steps that is required to release the DCVs stored in the DCV stack, and the number of time steps that is required to unload and store the DCVs in the DCV stack at the unloading station.

EBS

Let $l_{v^*}^{\text{out}} = (v^*, w^*)$ and $l_{v^*}^{\text{in}} = (w^*, v^*)$ be the virtual outgoing and incoming links of EBS, respectively. For the EBS node v^* and for each $d \in V_3$, the control variables are the outflows of loaded DCVs, $q_{v^*,l_{v^*}^{\text{out}},d}$, with destination d, and the inflows of

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loaded DCVs, $q_{l_{v^*}^{\text{in}},v^*,d}$, whose final destination is d. For each outgoing link p of w^*, we introduce the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned} &q_{l_{v^*}^{\text{out}},p,z}(k) = 0, \forall\ z \in V_1, \\ &q_{l_{v^*}^{\text{out}},p,z}(k) = 0, \forall\ z \in V_3 \text{ s.t. } v^* \notin L_z, \\ &q_{l_{v^*}^{\text{out}},p,z}(k) \geq 0, \forall\ z \in V_3 \text{ s.t. } v^* \in L_z. \end{aligned} \tag{2.10}$$

For each incoming link p of w^*, we impose the following constraints for all $k \in \mathbb{N}$:

$$\begin{aligned} &q_{p,l_{v^*}^{\text{in}},z}(k) = 0, \forall\ z \in V_1, \\ &q_{p,l_{v^*}^{\text{in}},z}(k) = 0, \forall\ z \in V_3 \text{ s.t. } v^* \notin L_z \\ &q_{p,l_{v^*}^{\text{in}},z}(k) \geq 0, \forall\ z \in V_3 \text{ s.t. } v^* \in L_z. \end{aligned} \tag{2.11}$$

Constraints (2.10) and (2.11) jointly imply that the EBS cannot accept or send out empty-DCV flows, and that it can only receive and send loaded-DCV flows the final destination of which is reachable from the EBS. The evolution of the loaded-DCV queue lengths at the EBS with final destination $d \in V_3$ is given as

$$\begin{aligned} &\boldsymbol{x}_{v^*,d}(k+1) = \boldsymbol{x}_{v^*,d}(k) + \Delta t\big(q_{l_{v^*}^{\text{in}},v^*,d}(k) - q_{v^*,l_{v^*}^{\text{out}},d}(k)\big), \\ &\boldsymbol{x}_{v^*,d}(k) \geq 0, \end{aligned} \tag{2.12}$$

The total length of the DCV queues at the EBS is, therefore, given by $\boldsymbol{x}_{v^*}(k) = \sum_{d\in V_3} \boldsymbol{x}_{v^*,d}(k)$ with the constraint $\boldsymbol{x}_{v^*}(k) \leq \boldsymbol{x}_{v^*,\max}$, where $\boldsymbol{x}_{v^*,\max}$ is the maximum capacity of EBS. The following guarantee that no queues occur along the virtual links of EBS:

$$\begin{aligned} q_{l_{v^*}^{\text{in}},v^*,d}(k + k_{l_{v^*}^{\text{in}}}) &= \sum_{p\in L_{w^*}^{\text{in}}} q_{p,l_{v^*}^{\text{in}},d}(k) \\ q_{v^*,l_{v^*}^{\text{out}},d}(k) &= \sum_{p\in l_{w^*}^{\text{out}}} q_{l_{v^*}^{\text{out}},p,d}(k + k_{l_{v^*}^{\text{out}}}), \end{aligned} \tag{2.13}$$

where $k_{l_{v^*}^{\text{in}}}$, and $k_{l_{v^*}^{\text{out}}}$ are respectively the number of time steps that is required to store loaded DCVs in the EBS, and the number of time steps that is required to release loaded DCVs from the EBS.

Links

For each real link $l = (v, w)$ and for each $z \in V_1 \cup V_3$, the controls are the empty and loaded DCV flows, $q_{l,p,z}$, from the link l to each of its outgoing links p with destination z. For each $p \in L_w^{\text{out}}$, the flows of DCVs from l to p must satisfy

$$\begin{aligned} &q_{l,p,z}(k) = 0, \forall z \in V_1 \cup V_3 \text{ s.t. } p \notin L_z, \\ &q_{l,p,z}(k) \geq 0, \text{otherwise}, \end{aligned} \tag{2.14}$$

for all $k \in \mathbb{N}$, which implies that the DCVs with final destination z can be sent from l to p if z is reachable from p. Let $F_{l,z}^{\text{in}}(k)$ be the sum of all DCV flows with destination z that enter link l, and let $F_{l,z}^{\text{out}}(k)$ be the sum of all DCV flows with destination z that leaves link l. We then have

$$\begin{aligned} F_{l,z}^{\text{in}}\big(k + k_l(k)\big) &= \sum_{p \in L_v^{\text{in}}} q_{p,l,z}(k), \\ F_{l,z}^{\text{out}}(k) &= \sum_{p \in L_w^{\text{out}}} q_{l,p,z}(k), \end{aligned} \tag{2.15}$$

where $k_l(k)$ is the number of travel time steps for DCV on the link l given by $k_l(k) = \left\lceil \frac{s_l - x_l(k) l_{\text{DCV}}}{v_{\text{DCV}} \Delta t} \right\rceil$, where $x_l(k) = \sum\limits_{z \in V_1 \cup V_3} x_{l,z}(k)$, $x_l(k) \leq x_{l,\max}$ is the total DCV queue length along link l with $x_{l,\max}$ being the maximum allowed queue length on link l and with $x_{l,z}(k)$ being the partial DCV queue length along link l associated with destination d described by

$$\begin{aligned} \boldsymbol{x}_{l,z}(k+1) &= \boldsymbol{x}_{l,z}(k) + \Delta t\big(F_{l,z}^{\text{in}}(k) - F_{l,z}^{\text{out}}(k)\big), \\ \boldsymbol{x}_{l,z}(k) &\geq 0. \end{aligned} \tag{2.16}$$

The equality and inequality constraints (2.1)-(2.16) define the set of feasible trajectories of the system. Let $\boldsymbol{x}(k) \in \mathbb{R}^n$ be the state vector at time step k the elements of which are the baggage queue lengths at the loading stations; the empty-DCV queues at the loading stations, at the unloading stations, and on the links; the loaded-DCV queues on the links and in the EBS; the empty- and loaded-DCV flows at the past time steps, where the number of past time steps for which we need to store the flow value depends on the capacity of the link. Let $\boldsymbol{u}(k) \in \mathbb{R}^{m_1}$ be the input vector the elements of which are empty- and loaded-DCV flows from the end of each link to its outgoing links; the inflow of empty DCVs from the virtual incoming link of loading stations to the loading stations and the outflow of empty DCVs from unloading stations to their virtual outgoing link; the flow of loaded DCVs to the EBS from its virtual incoming link, and from the EBS to its virtual outgoing link, and let $\boldsymbol{z}(k) \in \mathbb{R}^{m_2}$ be the disturbance vector the elements of which are $Q_{o,d}(k)$ for all origin and destination pairs. Note that we consider the baggage demand $Q_{o,d}(k)$ as a disturbance since it is an exogenous input that is fully measurable or predictable. We define the output vector $\boldsymbol{y}(k) \in \mathbb{R}^p$ as the collection of queue, i.e., $x_o(k)$, $x_d(k)$, $x_{v^*,d}(k)$, $x_{l,z}(k)$, $x_{o,d}^{\text{bag}}(k)$. Then, the system description can be expressed in the form

$$\begin{aligned} \boldsymbol{x}(k+1) &= A\big(\boldsymbol{x}(k)\big)\boldsymbol{x}(k) + B\boldsymbol{u}(k) + G\boldsymbol{z}(k), \\ \boldsymbol{y}(k) &= C\boldsymbol{x}(k), \\ E\boldsymbol{u}(k) &= F\boldsymbol{x}(k), \\ 0 \leq \boldsymbol{y}(k) &\leq \boldsymbol{y}_{\max}, \\ 0 \leq \boldsymbol{u}(k) &\leq \boldsymbol{u}_{\max}, \end{aligned} \tag{2.17}$$

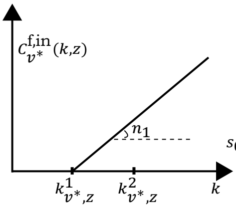

Figure 2.5: Weighting functions for EBS inflows.

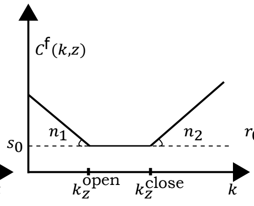

Figure 2.6: Weighting function for the inflows of the unloading stations.

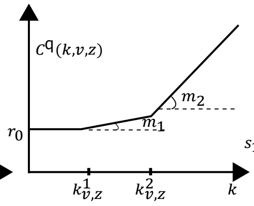

Figure 2.7: The general weighting function for the queue lengths.

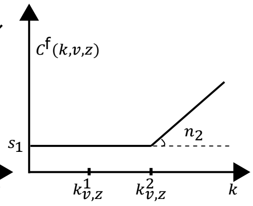

Figure 2.8: The general weighting function for the flows

for properly defined matrices A, B, C, E, F, and G. Note that initial state of the system 2.17, and the demand profile $Q_{o,d}(\cdot)$ have to be properly defined for the system to have solution for any $k \in \mathbb{N}$. Note that depending on the value of $\boldsymbol{x}(k)$, certain elements of $A\big(\boldsymbol{x}(k)\big)$ have non-zero values. Therefore, (2.17) *does not* define a linear discrete-time system with linear constraints.

2.3. MPC Problem Formulation

In this section, we use the dynamic model introduced in Section 2.2 within the context of MPC. At every time step, based on the current state of the system and a future prediction of baggage demands, a constrained finite horizon optimization problem will be solved yielding a sequence of optimal controls. According to the receding horizon policy, only the first step of this sequence is applied to the system, and this process is repeated at the next time step [10–12].

2.3.1. Objective Function

The aim of the control scheme is to assure delivery of pieces of baggage to the unloading stations within the given time windows. Imposing explicit constraints on delivery times would only be possible using a model that gives exact arrival times. However, due to its complexity, the resulting optimization problem would be intractable. Moreover, imposing explicit constraints on delivery times could lead to an infeasible optimization problem. Using the proposed flow model, the arrival time of DCVs to the unloading stations cannot be explicitly computed. Therefore, we include time-window constraints as a soft constraint in the objective function. Hence, the penalty functions penalize the flows and DCV queue lengths in such a way that the loaded DCVs are delivered to the unloading stations within their respective time windows as well as possible. Recall from Section 2.2 that $k_{v,z}^1$ and $k_{v,z}^2$ are respectively the relative opening and closing time steps of destination z, as seen from node v. For $v \in V_1 \cup V_2 \cup v^*$ and $z \in V_3$, the weighting function

$$C^{\mathrm{q}}(k,v,z) := \begin{cases} r_0 & \text{if } k \le k_{v,z}^1 \\ r_0 + m_1(k - k_{v,z}^1) & \text{if } k_{v,z}^1 < k \le k_{v,z}^2 \\ r_0 + m_1(k_{v,z}^2 - k_{v,z}^1) + m_2(k - k_{v,z}^2) & \text{if } k > k_{v,z}^2, \end{cases} \tag{2.18a}$$

depicted in Fig. 2.7, will be used in the penalty terms for the baggage queues in the loading stations, loaded DCV-queues in the EBS and along the links. For $v \in V_1 \cup V_2$ and $z \in V_3$, the weighting function

$$C^{\mathrm{f}}(k,v,z) := \begin{cases} s_1 & \text{if } k \leq k^2_{v,z} \\ s_1 + n_2(k - k^2_{v,z}) & \text{if } k > k^2_{v,z}, \end{cases} \tag{2.18b}$$

2

shown in Fig. and Fig. 2.8, will be used in the penalty terms associated with the loaded-DCV flows in the loading stations and along the links. For $z \in V_3$, the weighting functions

$$C^{\mathrm{f,in}}_{v^*}(k,z) := \begin{cases} 0 & \text{if } k \leq k^1_{v^*,z} \\ n_1(k - k^1_{v^*,z}) & \text{if } k > k^1_{v^*,z}, \end{cases} \tag{2.18c}$$

depicted in Fig. 2.5, and

$$C^{\mathrm{f,out}}_{v^*}(k,z) := C^{\mathrm{f,in}}_{v^*}(k,z) - n_1(k - k^1_{v^*,z}) \tag{2.18d}$$

will be respectively used to penalize the the EBS inflows and the EBS outflows. For $z \in V_3$, the weighting function

$$C^{\mathrm{f}}(k,z) := \begin{cases} s_0 - n_1(k - k^{\mathrm{open}}_z) & \text{if } k < k^{\mathrm{open}}_z \\ s_0 & \text{if } k^{\mathrm{open}}_z < k \leq k^{\mathrm{close}}_z \\ s_0 + n_2(k - k^{\mathrm{close}}_z) & k > k^{\mathrm{close}}_z, \end{cases} \tag{2.18e}$$

depicted in Fig. 2.6, will be used to penalize the empty-DCV outflows at unloading stations. In above, r_0, $m_2 > m_1$, s_0, s_1, n_1, and n_2 are strictly positive constants that determine the shape of the weighting function. Note that the relative magnitude of these constants is a design parameter that determines how much in-time delivery of DCVs is favored to energy consumption.

Using 2.18, we will define the following penalty terms, which penalize the loaded-DCV flows and queues. For loading station o, the penalty terms at time step k associated with the baggage queues and the loaded-DCV flows are defined as:

$$J^{\mathrm{bag}}_{\mathrm{LS}}(k) := \sum_{o \in V_1} \sum_{d \in V_3} C^{\mathrm{q}}(k,o,d) x^{\mathrm{bag}}_{o,d}(k), \tag{2.19}$$

$$J^{\mathrm{flow}}_{\mathrm{LS}}(k) := \sum_{o \in V_1} \sum_{d \in V_3} C^{\mathrm{f}}(k,o,d) q_{o,l^{\mathrm{out}}_o,d}(k).$$

For links $l = (v,w)$, the loaded-DCV queues and loaded-DCV flows are penalized as follows:

$$J^{\mathrm{DCV}}_{\mathrm{L}}(k) := \sum_{d \in V_3} \sum_{l=(v,w) \in L_d} C^{\mathrm{q}}(k,w,d) x_{l,d}(k), \tag{2.20}$$

$$J^{\mathrm{flow}}_{\mathrm{L}}(k) := \sum_{d \in V_3} \sum_{l=(v,w) \in L_d} \Big(C^{\mathrm{f}}(k,w,d) \sum_{p \in L^{\mathrm{out}}_w} q_{l,p,d}(k) \Big),$$

For the EBS, penalty term associated with the loaded-DCV queues is defined as:

$$J_{\mathrm{EBS}}^{\mathrm{DCV}}(k) := \sum_{d \in V_3} \left(C^{\mathrm{q}}(k, v^*, d) - r_0\right) x_{v^*,d}(k), \tag{2.21}$$

and the penalty terms associated with loaded-DCV flows in and out of the EBS are defined as:

$$J_{\mathrm{EBS}}^{\mathrm{inflow}}(k) := \sum_{d \in V_3} C_{v^*}^{\mathrm{f,in}}(k, d) q_{l_{v^*}^{\mathrm{in}}, v^*, d}(k), \tag{2.22}$$

$$J_{\mathrm{EBS}}^{\mathrm{outflow}}(k) := \sum_{d \in V_3} C_{v^*}^{\mathrm{f,out}}(k, d) q_{v^*, l_{v^*}^{\mathrm{out}}, d}(k),$$

For the unloading stations d, the penalty term associated with the inflow of loaded DCVs to d is defined as:

$$J_{\mathrm{US}}^{\mathrm{flow}}(k) := \sum_{d \in V_3} C^{\mathrm{f}}(k, d) q_{l_d^{\mathrm{in}}, d, d}(k). \tag{2.23}$$

One can observe that for $k \leq k_{o,d}^1$, as depicted in Fig. 2.7, we assign a constant weight r_0 to the baggage queues at the loading station and to the loaded-DCV queues along the links. With the choice of $s_0 \ll r_0$ this allows for early release of baggage, hence loaded-DCVs, into the network. These DCVs will move to the EBS since the loaded-DCV inflow of the unloading stations is highly penalized (see Fig. 2.6) and since the inflow of DCVs to the EBS inflicts no cost (see Fig. 2.5). Please note that, during this period, the DCVs will remain in the EBS since dispatching the DCVs is more expensive. For $k_{o,d}^1 < k \leq k_{o,d}^2$, the weighting functions for the baggage queues at the loading stations and for the loaded-DCV queues along the links and in the EBS increase with the constant slope m_1 to have more loaded DCVs released in the network. The weight of the loaded-DCV inflows to the EBS increases with the constant slope n_1 to prevent DCVs from entering the EBS. Moreover the inflow of loaded DCVs to the unloading station involves no cost. Hence, the released DCVs arrive at the specified unloading stations. For $k > k_{o,d}^2$ the weighting functions for the baggage queues at the loading stations and for the loaded-DCV queues along the links increase with the constant slope m_2. In addition the weighting functions of the loaded-DCV flows along the links and into the unloading stations increases with the constant slope n_2. Since the slope of the second part of the weighting functions is larger than the slope of the first part, the case of having loaded DCVs on the links, or having loaded-DCV flows arriving at the unloading stations during this time interval becomes expensive.

To take into account the energy consumption, we define the following penalty terms, which penalize the flow of empty DCVs. The cost of empty-DCV flows, $q_{l,p,o}(k)$ is given by

$$J^{\mathrm{e}}(k) := t_1 \sum_{o \in V_1} \sum_{l=(v,w) \in L_o} \sum_{p \in L_w^{\mathrm{out}}} q_{l,p,o}(k), \tag{2.24}$$

where we assign a constant weight $t_1 > 0$ to empty DCV flows to avoid arbitrary empty DCV flow circulations, which is in accordance with our objective of minimizing the energy consumption. We define

$$J_1(k) := J_{\mathrm{LS}}^{\mathrm{bag}}(k) + J_{\mathrm{LS}}^{\mathrm{DCV}}(k) + J_{\mathrm{L}}^{\mathrm{DCV}}(k) + J_{\mathrm{EBS}}^{\mathrm{DCV}}(k), \tag{2.25}$$

and

$$J_2(k) := J_{\mathrm{LS}}^{\mathrm{flow}}(k) + J_{\mathrm{L}}^{\mathrm{flow}}(k) + J_{\mathrm{US}}^{\mathrm{flow}}(k) + J_{\mathrm{EBS}}^{\mathrm{inflow}}(k) + J_{\mathrm{EBS}}^{\mathrm{outflow}}(k) \tag{2.26}$$

respectively as the total cost of baggage stacks and loaded DCV queues and the total cost of loaded DCV flows. The total cost function at time step k over the prediction horizon N_{p} is then given by

$$J_{N_{\mathrm{p}}}(k) := \frac{1}{J_{1,\mathrm{nom}}} \sum_{j=1}^{N_{\mathrm{p}}} J_1(k+j) + \frac{\alpha_1}{J_{2,\mathrm{nom}}} \sum_{j=0}^{N_{\mathrm{p}}-1} J_2(k+j) + \frac{\alpha_2}{J_{\mathrm{nom}}^{\mathrm{e}}(k)} \sum_{j=0}^{N_{\mathrm{p}}-1} J^{\mathrm{e}}(k)(k+j), \tag{2.27}$$

where $\alpha_1 > 0$, and $\alpha_2 > 0$ are constants indicating the relative importance of the respective component of the objective function, and where $J_{i,\mathrm{nom}}$, $i = 1, 2$ is the nominal value[2] of $J_i(k)$.

2.3.2. Linear Programming Approach

In general, finding the optimal flow values and optimal queue lengths is a nonlinear programming problem. To arrive at a linear programming formulation we first make the following additional assumption:

A-LP The queue lengths remain constant over the prediction horizon.

Assumption A-LP enables us to arrive at a linear programming formulation of the optimization problem. Under A-LP, the first equation in (2.17) can be written as

$$\boldsymbol{x}(k+j+1) = A\big(\boldsymbol{x}(k)\big)\boldsymbol{x}(k+j) + B\boldsymbol{u}(k+j) + G\boldsymbol{z}(k+j), \tag{2.28}$$

for $j = 1, \ldots, N_{\mathrm{p}} - 1$, which, together with constraints of (2.17), defines a linear discrete time-invariant system with linear constraints. Since (2.27) is weighted sum of the state variables and the input variables, it can be expressed as $J_{N_{\mathrm{p}}}(k) = \boldsymbol{F}_0^{\mathrm{T}}(k)\boldsymbol{z}_{N_{\mathrm{p}}}(k) + \boldsymbol{F}_1^{\mathrm{T}}(k)\boldsymbol{x}(k) + \boldsymbol{F}_2^{\mathrm{T}}(k)\boldsymbol{u}_{N_{\mathrm{p}}}(k)$ by successive substitution in (2.28). Here, $\boldsymbol{z}_{N_{\mathrm{p}}}(k) = [\boldsymbol{z}^{\mathrm{T}}(k), \ldots, \boldsymbol{z}^{\mathrm{T}}(k+N_{\mathrm{p}}-1)]^{\mathrm{T}} \in \mathbb{R}^{m_2 N_{\mathrm{p}}}$ is the predicted disturbance vector, $\boldsymbol{u}_{N_{\mathrm{p}}}(k) = [\boldsymbol{u}^{\mathrm{T}}(k), \ldots, \boldsymbol{u}^{\mathrm{T}}(k+N_{\mathrm{p}}-1)]^{\mathrm{T}} \in \mathbb{R}^{m_1 N_{\mathrm{p}}}$ is the control input sequence and $\boldsymbol{F}_0(k) \in \mathbb{R}^{m_2 N_{\mathrm{p}}}$, $\boldsymbol{F}_1(k) \in \mathbb{R}^{n}$, and $\boldsymbol{F}_2(k) \in \mathbb{R}^{m_1 N_{\mathrm{p}}}$ are coefficient vectors. Note that since the values of the weighting functions are known for $k, k+1, \ldots, k+N_{\mathrm{p}}-1$,

[2]The nominal value $J_{i,\mathrm{nom}}$, $i = 1, 2$ can be computed by averaging over $J_i(k)$ for $k = 1, \ldots, N$, where $J_i(k)$ is obtained based on simulating the system under the nominal input $\boldsymbol{u}_{\mathrm{nom}}(k)$ and the nominal baggage demand $\boldsymbol{z}_{\mathrm{nom}}(k)$, for $k = 1, \ldots, N$.

$\boldsymbol{F}_0(k)$, $\boldsymbol{F}_1(k)$, and $\boldsymbol{F}_2(k)$ are fully determined at time step k. Therefore, at every time step k, we solve an optimization problem of the form

$$\begin{aligned}
\min_{\boldsymbol{u}_{N_\mathrm{p}}(k)} \boldsymbol{F}_2^\mathrm{T}(k)\boldsymbol{u}_{N_\mathrm{p}}(k) \\
\text{subject to } A^{\mathrm{eq}}(k)\boldsymbol{u}_{N_\mathrm{p}}(k) = \boldsymbol{b}^{\mathrm{eq}}(k), \\
A^{\mathrm{ineq}}(k)\boldsymbol{u}_{N_\mathrm{p}}(k) \leq \boldsymbol{b}^{\mathrm{ineq}}(k), \\
0 \leq \boldsymbol{u}_{N_\mathrm{p}}(k) \leq \boldsymbol{u}_{\max},
\end{aligned} \tag{2.29}$$

where the matrices $A^{\mathrm{eq}}(k)$, and $A^{\mathrm{ineq}}(k)$ and the vectors $b^{\mathrm{eq}}(k)$, and $b^{\mathrm{ineq}}(k)$ are obtained based on (2.17), the current state of the system $x(k)$, and the predicted baggage demand $\boldsymbol{z}_{N_\mathrm{p}}(k) = [\boldsymbol{z}^\mathrm{T}(k), \ldots, \boldsymbol{z}^\mathrm{T}(k+N_\mathrm{p}-1)]^\mathrm{T} \in \mathbb{R}^{m_2 N_\mathrm{p}}$. This is an LP problem, which, among others, can be solved efficiently using the simplex, active-set, or interior point methods [21].

2.3.3. Iterative Linear Programming Approach

The prediction model (2.28) can yield inaccurate predictions. To remedy this problem and still arrive at a tractable formulation for the optimization problem, we propose an iterative version of (49): at each iteration, a problem of the form (2.29) is solved to find the optimal sequence of flow variables. This sequence is then used in the model (2.17) in forward simulation to compute the updated values for queue lengths resulting from the obtained flow values. The updated queue lengths are then used again in the LP problem formulation to find updated flow values. This process is repeated for a certain number of iterations. More formally, we replace the first equation in (2.17) by the prediction model

$$\boldsymbol{x}_{(i)}(k+j+1) = A\big(\boldsymbol{x}_{(i-1)}(k+j)\big)\boldsymbol{x}_{(i)}(k+j) + B\boldsymbol{u}_{(i)}(k+j) + G\boldsymbol{z}(k+j), \tag{2.30}$$

for $j = 0, \ldots, N_\mathrm{p}-1$ with the initial conditions $\boldsymbol{x}_{(0)}(k+j) = \boldsymbol{x}(k)$ for $j = 0, \ldots, N_\mathrm{p} - 1$, $\boldsymbol{x}_{(i)}(k) = \boldsymbol{x}(k)$ for $i = 1, 2, \ldots$, where i is the iteration index. Hence, at time step k, initializing the algorithm with with the initial predicted state [3] $\boldsymbol{x}_{N_\mathrm{p},(0)}(k) = [\boldsymbol{x}^\mathrm{T}(k), \ldots, \boldsymbol{x}^\mathrm{T}(k)]^\mathrm{T}$, the predicted state vector

$$\boldsymbol{x}_{N_\mathrm{p},(i)}(k) = [\boldsymbol{x}_{(i)}^\mathrm{T}(k), \ldots, \boldsymbol{x}_{(i)}^\mathrm{T}(k+N_\mathrm{p}-1)]^\mathrm{T}$$

at iteration i is computed based on the input sequence,

$$\boldsymbol{u}_{N_\mathrm{p},(i)}(k) = [\boldsymbol{u}_{(i)}^\mathrm{T}(k), \ldots, \boldsymbol{u}_{(i)}^\mathrm{T}(k+N_\mathrm{p}-1)]^\mathrm{T}$$

at iteration i, the predicted disturbance vector $\boldsymbol{z}_{N_\mathrm{p}}(k)$, and the predicted state vector at iteration $i-1$. Using (2.30), the objective function at time step k in iteration i

[3] If the optimal input sequence from the previous time step is availabe, it can be used to compute the initial predicted state.

be expressed as $J_{N_\mathrm{p},i}(k) = \boldsymbol{F}^\mathrm{T}_{0,(i-1)}(k)\boldsymbol{z}_{N_\mathrm{p}}(k) + \boldsymbol{F}^\mathrm{T}_{1,(i-1)}(k)\boldsymbol{x}(k) + \boldsymbol{F}^\mathrm{T}_{2,(i-1)}(k)\boldsymbol{u}_{N_\mathrm{p},(i)}(k)$. Therefore, at time step k and in iteration i, we solve an optimization problem of the following form:

$$\begin{aligned} \min_{\boldsymbol{u}_{N_\mathrm{p},(i)}(k)} \; & \boldsymbol{F}^\mathrm{T}_{2,(i-1)}(k)\boldsymbol{u}_{N_\mathrm{p},(i)}(k) \\ \text{subject to } & A^\mathrm{eq}_{(i-1)}(k)\boldsymbol{u}_{N_\mathrm{p},(i)}(k) = \boldsymbol{b}^\mathrm{eq}_{(i-1)}(k), \\ & A^\mathrm{ineq}_{(i-1)}(k)\boldsymbol{u}_{N_\mathrm{p},(i)}(k) \leq \boldsymbol{b}^\mathrm{ineq}_{(i-1)}(k), \end{aligned} \tag{2.31}$$

where the matrices $A^\mathrm{eq}_{(i-1)}(k)$, and $A^\mathrm{ineq}_{(i-1)}(k)$ and the vectors $b^\mathrm{eq}_{(i-1)}(k)$, and $b^\mathrm{ineq}_{(i-1)}(k)$ are obtained based on (2.30), and the constraints of (2.17) using the predicted state vector $\boldsymbol{x}_{N_\mathrm{p},(i-1)}(k)$ at iteration $i-1$, the current state of the system $x(k)$, and the predicted baggage demand $\boldsymbol{z}_{N_\mathrm{p}}(k)$. Note that even though we assume constant queue lengths at each inner iteration of (2.31), the variations in queue lengths over the prediction horizon are taken into account in the next inner iteration. Therefore, at the end of inner iterations, the computed flow values are the ones obtained having taken into account the effect of variations in the queue lengths over the prediction horizon.

2.3.4. Nonlinear Programming Approach

In this approach, without any simplifying assumptions, we make direct use of the model presented by (2.17) and the cost given by (2.27) in defining optimization problem of the following form:

$$\min_{\boldsymbol{u}_{N_\mathrm{p}}(k)=[\boldsymbol{u}^\mathrm{T}(k),\,\dots,\,\boldsymbol{u}^\mathrm{T}(k+N_\mathrm{p}-1)]^\mathrm{T}} J_{N_\mathrm{p}}(k) \text{ subject to (2.17)}, \tag{2.32}$$

which is a nonlinear nonconvex problem. Hence, one can use multi-start local nonlinear optimization algorithms (e.g., an SQP algorithm or interior-point methods [21]) or global optimization methods (e.g., pattern search and genetic algorithms [22]).

2.4. Case Study

First, we introduce a state-of-the-art (SOA) method, which is developed based on our understanding of the method currently used for control of BHS by a leading supplier of DCV-based baggage handling systems. In this method, the flow of loaded DCVs at each junction is dynamically assigned to its outgoing links based on the projected deviation of delivery times from the beginning of the time window of the destination. In this way, the DCVs are dynamically routed via links that make timely delivery possible while distributing the traffic in a smart way, hence avoiding congestion. The empty-DCV flows at each junction are assigned based on projected travel times, and the length of the baggage queues at the loading stations. Hence, more empty DCVs are sent via “faster” links to the loading station with longer baggage queues.

a) Routing and scheduling of loaded DCVs: at each junction of the network the flow from an incoming link l to an outgoing link p with destination d at current time $t = k\Delta t$ is obtained as

$$q_{l,p,d}(k) = \frac{1/|t - t_d^{\text{open}} - t_{l,p,d}(k)|}{\sum_j 1/|t - t_d^{\text{open}} - t_{l,j,d}(k)|} q_{l,\max},$$

where t_d^{open} is the opening time for the destination, $q_{l,\max}$ is the maximum flow of the link, and $t_{l,p,d}$ is the estimated travel time from the end of link l to destination d via link p. This travel time can be obtained using historical data or by computing $t_{l,p,d}(k) = t_p(k) + t^*_{l,p,d}$, where $t^*_{l,p,d}$ is the the travel time from l to d via the shortest path and $t_p(k)$ is the estimated clearance time of the queues given by

$$t_p(k) = \frac{x_p(k)}{q_{l,\max}} + \frac{\sum_{j=0}^{n_p-1} F_p^{\text{in}}(k-j)(n_p - j)}{\sum_{j=0}^{n_p-1} F_p^{\text{in}}(k-j)} + \frac{\Delta t}{q_{l,\max}} \sum_{j=0}^{n_p-1} F_p^{\text{in}}(k-j), \tag{2.33}$$

where $x_p(k)$ is the total queue length at the end of link p, $F_p^{\text{in}}(k)$ is the total inflow of DCVs to link p, and $n_p \Delta t$ is the DCV travel time from the beginning to the end of p. The first term on the right hand side of the (2.33) is the clearance time of current DCV queue at the end of link p. The second term determines the average time that the past DCV flows that are currently traveling on link p need to reach the end of the link, and the last term determines the clearance time of these flows once they have reached the end of the link.
b) Routing and scheduling of empty DCVs: at each junction of the network, the flow of empty DCVs from an incoming link l to an outgoing link p is with destination o is given as

$$q_{l,p,o}(k) = \frac{x_o^{\text{bag}}}{\sum_{j \in V_1} x_j^{\text{bag}}} \frac{1/t_{l,p,o}(k)}{\sum_j 1/t_{l,j,o}(k)} q_{l,\max},$$

where x_o^{bag} is the total baggage queue at loading station o, and $t_{l,p,d}$ is the estimated travel time from the end of link l to destination d that can be obtained using historical data or the aforementioned procedure via (2.33). For the network layout depicted in Fig. 2.9 with $V_1 = \{1, 9\}$, $V_3 = \{5, 13\}$, $V_2 = \{2, 3, 4, 6, 7, 10, 11, 12\}$, and $v^* = \{8\}$, we compare the the performance of the SOA, LP-MPC, ILP-MPC, and NLP-MPC against each other in terms of MPC-in-the-loop optimal cost and the computational burden under the demand scenario depicted in Fig. 2.10 using the parameters listed in Table 2.1. For the LP and ILP approaches, we use the CPLEX solver via TOMLAB toolbox for MATLAB, and for the NLP approach, we use the implementation of the interior-point algorithm in the MATLAB optimization toolbox. For each method, the total MPC-in-the-loop cost and the corresponding CPU time is listed in Table 2.2 for different prediction horizons using a dual core PC with Intel E8400 processor running at 3.00GHz and with 4GB of RAM. The reported CPU times are computed by averaging over the CPU times for all simulation time steps. It can be observed from Table 2.2 that while the computation time of the LP and ILP-MPC

Table 2.1: Closed-loop simulation parameters, including the total simulation time steps N_{sim}, destinations opening and closure time instants, MPC parameters, link capacities, and the initial value of the bags and empty DCVs at the loading stations. There are no DCVs initially in the other parts of the network.

MPC Parameters					
N_{p}	$(k_{\mathrm{U}_1}^{\mathrm{open}}, k_{\mathrm{U}_1}^{\mathrm{close}})$, $(k_{\mathrm{U}_2}^{\mathrm{open}}, k_{\mathrm{U}_2}^{\mathrm{close}})$	$\boldsymbol{u}_{\max}$[DCV/s]	$(\boldsymbol{x}_{o,\max}, \boldsymbol{x}_{d,\max}, \boldsymbol{x}_{v^*,\max}, \boldsymbol{x}_{l,\max})$[DCV]	(α_1, α_2)	(s_0, s_1, n_1, n_2), (r_0, m_1, m_2)
8	(60, 90), (80, 120)	6	(50, 100, 120, 26)	(1, 1)	(1, 1, 1, 2), (0.5, 1, 1, 2)
Closed-loop Simulation Parameters					
Δt[s]	$(x_{\mathrm{L}_1}(0), x_{\mathrm{L}_2}(0), x_{\mathrm{U}_1}(0), x_{\mathrm{U}_2}(0))$[DCV]	N_{sim}	v_{DCV}[m/s]	l_{DCV}[m]	
0.5	(20, 49, 90, 0)	150	10	1.5	

Table 2.2: Comparison of Closed-loop Performance and Computation Times

N_{p}	Opt. Scheme (Solver)	Iter./Multi-start Iter.	min. CPU Time [s]	Avg. CPU Time [s]	max. CPU Time	Optimal Cost
N/A	SOA	N/A	0.0015	0.0036	0.0411	16671.5
2	LP (CPLEX)	N/A	0.0056	0.0074	0.0411	8083.1
2	ILP (CPLEX)	2	0.0128	0.0237	0.7401	8083.1
2	ILP (CPLEX)	4	0.0249	0.0389	0.7416	8083.1
2	ILP (CPLEX)	6	0.0475	0.0530	0.0739	8083.1
2	NLP (interior-point)	3	99.6190	114.53	135.52	8083.1
4	LP (CPLEX)	N/A	0.0074	0.0094	0.0239	7982.1
4	ILP (CPLEX)	2	0.0150	0.0271	0.7664	7869.5
4	ILP (CPLEX)	4	0.0309	0.0445	0.7845	7820.5
4	ILP (CPLEX)	6	0.0453	0.0514	0.0913	7800.5
4	NLP (interior-point)	4	310.00	344.93	376.16	6813.4
6	LP (CPLEX)	N/A	0.0087	0.0197	0.6674	7471.3
6	ILP (CPLEX)	4	0.0375	0.0457	0.0967	5890.2
6	ILP (CPLEX)	8	0.0721	0.0852	0.1133	5574.2
6	NLP (interior-point)	8	419.62	450.15	486.42	4546.7
8	LP (CPLEX)	N/A	0.1212	0.1902	0.3299	5925.3
8	ILP (CPLEX)	6	0.8061	1.0185	1.4325	4949.9

approaches are comparable to the SOA method, they outperform the SOA in terms of controller-in-the-loop cost. This can also be observed by comparing Fig. 2.15 with Fig. 2.14 which illustrate the controller-in-the-loop performance of the two methods for the baggage demand profile depicted in Fig. 2.10. While with the ILP-MPC with three iterations, 70% of the baggage demand for destination $U_1 = 5$ and 80% of the baggage demand for destination $U_2 = 13$ arrive within the respective time window of destination, with SOA method, only 10% of demand to $U_1 = 5$ and 20% of demand to $U_2 = 13$ arrive within the time windows. One can observe from Fig. 2.13 and from Table 2.2 that the NLP approach outperforms the LP and ILP approaches, but its computational burden increases very sharply for high values of prediction horizon. Moreover, the the optimal cost of ILP approach converges to the NLP approach by increasing the number of iterations with far less computation effort. We also observe from Table 2.2 that the ILP approach outperforms the LP approach in terms of the closed-loop cost and that the difference between the closed-loop cost of the two increases for the increasing values of the prediction horizon. Fig. 2.11 compares the computation times of SOA, LP, and ILP methods for different values of N_{p}. It can be observed that the computation time of the LP approach is comparable with the SOA. More importantly, the computation time of LP and ILP methods increase linearly in the problem size. The computation time of ILP approach as a function of ILP iteration is depicted in Fig. 2.12.

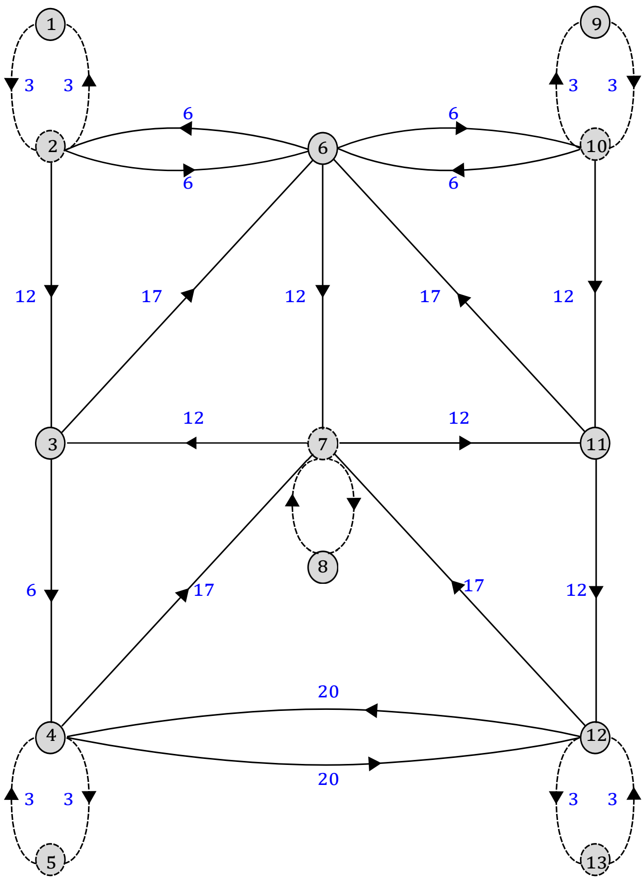

Figure 2.9: Schematic layout of the case study network with loading stations $\mathrm{L}_1 = 1$, and $\mathrm{L}_2 = 9$ and unloading stations $\mathrm{U}_1 = 5$, $\mathrm{U}_2 = 13$, and $\mathrm{EBS} = 8$. The nominal travel time of each link, based on the maximum DCV speed, is indicated in simulation time step.

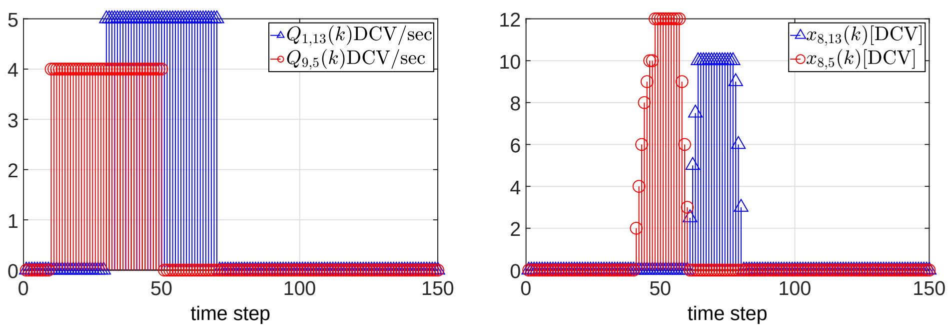

Figure 2.10: The baggage demand (left) and number of DCVs in the EBS (right).

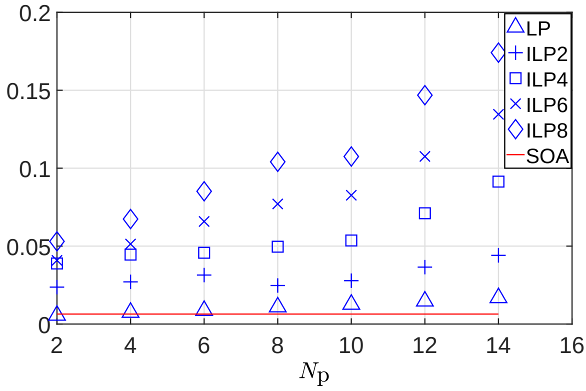

Figure 2.11: Computation time of LP and ILP in seconds as a function of N_{p}.

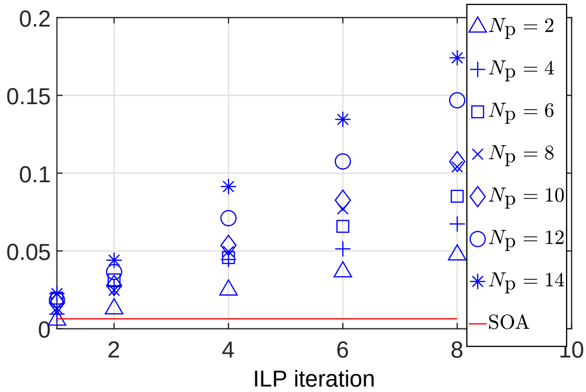

Figure 2.12: Computation time of the ILP in seconds as a function of ILP iterations

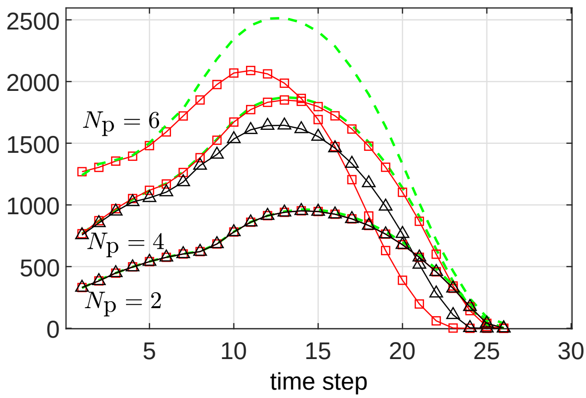

Figure 2.13: Predicted cost as a function of closed-loop time step using the LP approach (dashed line), the ILP approach (square markers), and the NLP approach (triangle markers) for different prediction horizons N_{p}.

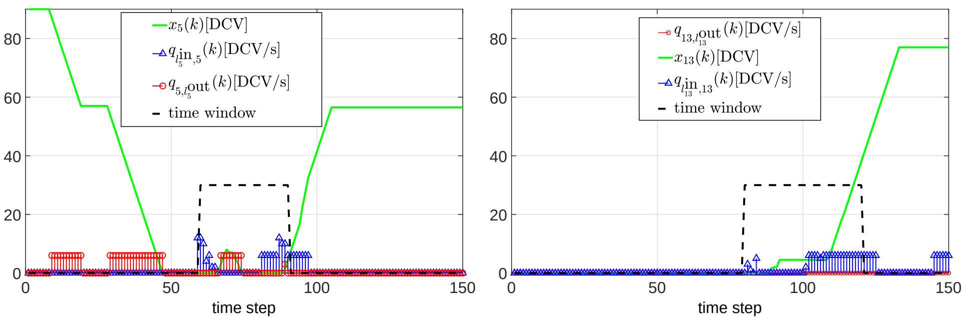

Figure 2.14: ILP-MPC-based control. Triangle: total DCV inflow. Square: DCV outflow. Solid line: DCV queue for unloading stations (left) U_1 and (right) U_2 with the associated time window of destination.

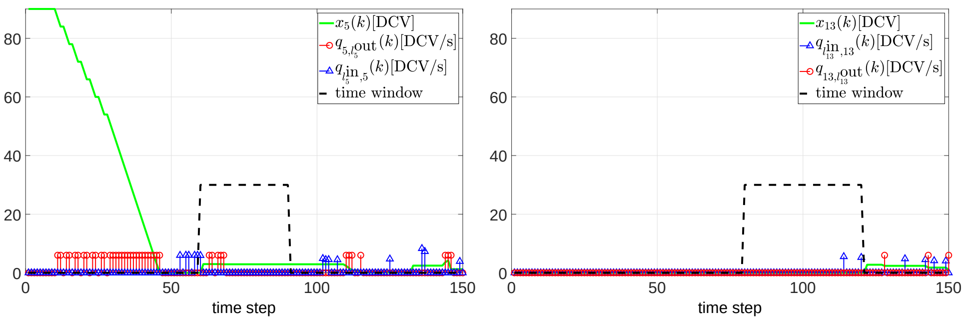

Figure 2.15: SOA control. Triangle: total DCV inflow. Square: DCV outflow. Solid line: DCV queue for unloading stations (left) U_1 and (right) U_2 with the associated time window of destination.

2.5. Conclusions and Future Work

In this chapter, we have revisited the problem of dynamic routing and scheduling of DCVs in baggage handling systems. We have jointly addressed the main control problems of DCV-based baggage handling systems, namely, routing and scheduling of DCVs, line balancing, and empty cart management. Our derived model was used as the prediction model within the MPC framework. The objective function was defined so as to penalize the deviation of baggage delivery time at the unloading stations from pre-specified time windows, and the energy consumption. We formulated the MPC problem as a nonlinear programming (NLP) problem, which, for the case study, was computationally very inefficient for large prediction horizons (e.g., $N_\mathrm{p} > 6$) due to the large-scale nature of the problem. Under some simplifying assumptions, we showed that the underlying optimization problem can be recast as a linear programming (LP) problem. However, the performance of the LP approach is suboptimal with respect to the NLP approach with significant performance loss for larger values of prediction horizons. To achieve a fast yet close-to-optimal solution, we have proposed an iterative linear programming (ILP) scheme that solves a sequence of LP problems to compute the optimal control sequence. We have illustrated that by using this method for an appropriate number of ILP iterations, it is possible to achieve a performance close to the NLP approach with significantly less computational burden. We also compared the performance of the our proposed approach to a state-of-the-art method (SOA) as is used in practice. We illustrated via simulations that our proposed method outperforms the SOA. The scalability of our approach was illustrated by numerical tests. Particularly, we showed that the required computation time increases linearly for increasing values of the prediction horizon and number of iterations.

For future work, we will compare the performance and computational requirements of our approach to an approach based on computing the exact travel time of each DCV in the network using a given distribution of passengers and baggage allowance for a predefined flight schedule. In addition, the trade-off between performance and computational efficiency of the ILP approach, especially for large-scale systems, will be analyzed in the future.

2

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3

Towards a Robust Multi-level Control Approach for Baggage Handling Systems

"It is my experience that proofs involving matrices can be shortened by 50% if one throws the matrices out."

Emil Artin, 1898 – 1962

This chapter revisits the routing problem in baggage handling systems. We propose a two-level control approach based on a model predictive controller at the top level and a constrained feedback controller at the bottom level that minimizes the L_2 gain of the closed-loop system. The model predictive control problem is recast as a linear programming problem and the constrained feedback controller design problem is formulated as minimization of a linear objective function subject to linear matrix inequalities. The effectiveness of the proposed method is illustrated by a case study.

3.1. Introduction

There has been a growing interest, in the last decade, in automated modern baggage handling systems for large airports. Such baggage handling systems have enabled big airports to achieve high throughput of passengers and cargo. The efficiency and reliability of baggage handling systems have improved over time by implementing more advanced control strategies. However, in order to meet the increasing demand for air travel and cargo shipment, we need more intelligent and

This chapter is based on [1].

reliable control methods than the currently available state-of-the-art methods. Modern baggage handling systems are composed of the following main components: i) loading stations, where the baggage demand originates. The pieces of baggage arrive at the loading stations either from a check-in desk or from a transfer flight, ii) unloading stations that are the final destination of the luggage and from where the pieces of baggage are boarded on to the planes, iii) a network of tracks that connect loading stations to unloading stations through junctions, iv) high-speed destination coded vehicles (DCV) that transport the pieces of baggage on the network from the loading stations to the unloading stations, v) switch controllers at the junctions that determine the route of DCVs. A complete description of the baggage handling system, the state-of-the-art control approaches, and the high-level control problems can be found in [2] and [3]. In this chapter, building on the work of [4], we develop a new approach for dynamic routing of DCVs within the network such that the pieces of baggage arrive at their destination within a given time window with minimum energy consumption. We also improve the robustness of our approach against variations in the baggage demand.

The proposed control structure is composed of a controller based on model predictive control (MPC) at the top level and a controller based on L_2 gain optimization at the bottom level. The MPC controller computes the nominal control input based on nominal prediction of the baggage demand such that the pieces of baggage arrive at their destination within a specified time window with minimal energy consumption. The L_2 based controller then minimizes the deviation of system trajectories from the nominal behavior due to unpredicted variations in the nominal predicted baggage demand. Fig. 3.1 depicts a schematic overview of the proposed two-level control approach.

The rest of the chapter is organized as follows. In Section 3.2, we present the dynamical model of the baggage handling system used for our control purposes. Section 3.3 and Section 3.4 describe the MPC approach and the L_2 optimization control approach, respectively. In Section 3.5, we explain how to combine these two controller approaches into a two-level control structure. In Section 3.6, we present a case study illustrating the performance of our proposed control scheme and finally Section 3.7 concludes the chapter.

3.2. Dynamical Model

The baggage handling system network can be seen as a directed graph $G = (V, A)$, where $V = O \cup I \cup D$ is the set of nodes composed of origin nodes O (i.e., loading stations), intermediate nodes I (i.e., junctions), and destination nodes D (i.e., unloading stations), and A is the set of arcs composed of links (i.e., tracks) connecting the nodes. The queue lengths are associated with the nodes and the control variables are defined at each node as the flows of DCVs from that node to its neighbor nodes. In a similar manner to the model in [5], the flows are indexed by their destination, enabling us to distinguish between baggage with different destinations. This is important as the baggage must end up in the right destination. Accordingly, at each node $v \in V$ there is a partial queue of DCVs associated with each destination $d \in D$. The following assumptions are made in the derivation of the model:

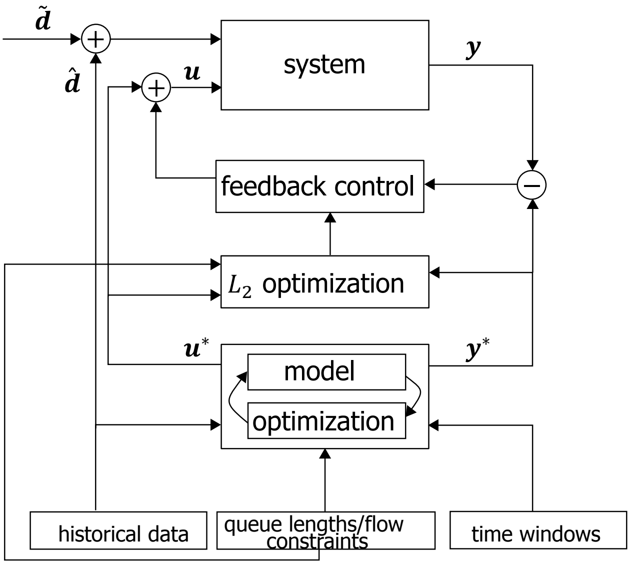

Figure 3.1: Schematic overview of the proposed two-level control approach, where $\boldsymbol{u}^*$ and $\boldsymbol{y}^*$ are the nominal input generated by the MPC controller and the resulting output trajectory, respectively, $\hat{\boldsymbol{d}}$ is the predicted baggage demand, $\tilde{\boldsymbol{d}}$ is the unpredictable deviation of baggage demand around $\hat{\boldsymbol{d}}$, and $\boldsymbol{u}$ and $\boldsymbol{y}$ are respectively the actual input and output of the system.

- A1 Each node in the network belongs to at least one directed path from an origin node (i.e., a loading station) to a destination node (i.e., an unloading station).
- A2 A DCV is present at the loading station whenever a piece of baggage arrives.
- A3 The movement of pieces of baggage on the network is approximated by a continuous flow of baggage.
- A4 At each node v, with exception of destination nodes, the DCVs stack up in vertical queues according to their destination. The queue lengths at destination nodes are considered to be zero. This is because we assume either destination nodes have unlimited capacity or there is no restriction on the outflow of destination nodes so the baggage are immediately taken to the planes upon arrival.
- A5 The DCV travel time on each link is an integer multiple of the sampling time Δt.

Assumption A1 guarantees that there are no redundant nodes in the network. By Assumption A2, the pieces of baggage are immediately dispatched from the loading stations as they arrive. Therefore, we do not need to distinguish between baggage flows and DCV flows within the system. Otherwise, we would need to take into account the movement of empty DCVs from the unloading stations to the loading stations. Assumption A3 is necessary for tractability of the control problem. Although the number of DCVs is an integer in reality, for a fairly large number of DCVs, the movement of DCVs can be approximated by continuous flows. This is not very restrictive as the computed flows can then be realized as well as possible by a lower-level control loop that determines the optimal switching pattern for the switch controllers at the junctions. The actual time required to travel from a node to another one depends on the length of the DCV queue at the end of the link connecting these nodes. However, if the queue lengths are sufficiently small compared to the length of the the links, the variation in the travel time is negligible. This is equivalent to having vertical queues at each node as stated in assumption A4. Assumption A5 allows us to arrive at a linear discrete-time model of the system.

We also make use of the following notation:

- The set of sending nodes of a node $v \in V$ defined as $V_v^{\text{send}} = \{w \in V \mid (w,v) \in A\}$, is the set of nodes that can send flow to node v .
- The set of receiving nodes of a node $v \in V$ defined as $V_v^{\text{recv}} = \{w \in V \mid (v,w) \in A\}$, is the set of nodes that can receive flow from node v.
- The set of all nodes that are on some directed path to a destination node $d \in D$ is V_d.
- For each destination node $d \in D$ and for each origin node $v \in O \cap V_d$, $Q_{v,d}(k)$ is the baggage inflow (demand) at v with destination d during the time interval $[k\Delta t, (k+1)\Delta t)$.

For each destination $d \in D$ and each $v \in V_d$ and each $w \in V_v^{\text{recv}} \cap V_d$, we define the control variable $q_{v,w,d}(k)$ that is the partial flow of DCVs with destination node d from node v to node w during the time interval $[k\Delta t, (k+1)\Delta t)$. Accordingly, $x_{v,d}(k)$ denotes the vertical queue length at node v associated with destination d. The set of feasible trajectories of the system is described by the following linear constraints in discrete time:

$$x_{v,d}(k+1) = x_{v,d}(k) + \Delta t (F_{v,d}^{\text{in}}(k) - F_{v,d}^{\text{out}}(k)) \tag{3.1a}$$

$$x_{v,d}(k) \geq 0 \tag{3.1b}$$

$$q_{v,w,d}(k) \geq 0 \tag{3.1c}$$

where $F_{v,d}^{\text{in}}(k)$ is the total inflow of DCVs to node v, associated with destination d, given by

$$F_{v,d}^{\text{in}}(k) = \begin{cases} Q_{v,d}(k) + \sum\limits_{w \in V_v^{\text{send}}} q_{w,v,d}(k - k_{w,v}) & \text{if } v \in V_d \cap O \\ \sum\limits_{w \in V_v^{\text{send}}} q_{w,v,d}(k - k_{w,v}) & \text{if } v \in V_d \cap (D \cup I) \\ 0 & \text{otherwise} \end{cases} \tag{3.2}$$

with $k_{w,v}\Delta t$ being the travel time[1] on the link (w,v), and $F_{v,d}^{\text{out}}(k)$ is the total outflow

[1]Assuming a constant speed for DCVs v_{DCV}, $k_{w,v}$ is given by $k_{w,v} = \frac{s_{w,v}}{\Delta t v_{\text{DCV}}}$, where $s_{w,v}$ is the length of link (w,v).

of DCVs from node v with destination d, given by

$$F_{v,d}^{\text{out}}(k) = \begin{cases} F_{v,d}^{\text{in}}(k) & \text{if } v \in V_d \cap D \\ \sum\limits_{w \in V_v^{\text{recv}}} q_{v,w,d}(k) & \text{if } v \in V_d \cap (O \cup I) \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

Equation (3.1a) describes the evolution of the queue lengths and (3.1b) constrains queue lengths to non-negative values. Likewise, (3.1c) guarantees non-negativity of the control variables (flows).

Let $\boldsymbol{x}(k)$ be the state vector that includes all queue lengths $x_{v,d}(k)$ and *delayed* samples of $q_{v,w,d}(k)$ with delay≥ 1. Let $\boldsymbol{u}(k)$ and $\boldsymbol{d}(k)$ be the control input vector that includes all control variables $q_{v,w,d}(k)$, and the demand vector composed of all individual demands $Q_{v,d}(k)$, respectively. Then (3.1) can be expressed by a constrained discrete-time linear system as

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B_1\boldsymbol{d}(k) + B_2\boldsymbol{u}(k) \tag{3.4a}$$

$$\boldsymbol{x}(k) \geq 0 \tag{3.4b}$$

$$\boldsymbol{u}(k) \geq 0 \tag{3.4c}$$

with properly defined matrices A, B_1, and B_2.

3.3. MPC Problem Formulation

The model presented in Section 3.2 is used as internal prediction model for the MPC approach. At time step k, given the current state of the system and an estimate of future baggage demand, this model is used to compute the trajectories of the system based on which a constrained optimal control problem is solved over a horizon yielding an optimal control sequence. The first element out of the optimal control sequence is then applied to the system according to the receding horizon policy and this process is then repeated at the next time step $k+1$ with new measurements [6].

The objective function must reflect the following performance criteria: i) the pieces of baggage assigned to a certain destination (unloading station) must reach the destination within a given time window, ii) the energy consumption of the system should be minimized. The time window represents the time duration in which the end point is ready to receive the luggage. It is undesirable to have the luggage arrive at the destination out of this time window. Indeed, if the pieces of luggage arrive too late, they will miss the flight. Too early arrival of the luggage at the destination point also might inflict a high storage cost on the operator. The energy consumption is associated with manipulating the actuators in the system and wear and tear inflicted on the actuators. There are two contributors to the energy consumption in the system: i) movements of DCVs in the system, which is related to the magnitude of DCV flows, and ii) variation in the DCV flows. This is particularly important when the DCV flows obtained here will be realized using switch controllers at each junction of the network. The variation in the flow then translates to switching frequency.

In order to achieve the aforementioned control objectives, we consider a cost function that is a weighted combination of four penalty terms that penalize the DCV queue lengths, DCV flows (control variables), and the variation of DCV flows. The cost associated with the DCV is defined as: The constrained linear model given in Section 3.2 cannot be used to determine the time instant at which a certain flow of baggage reaches to its destination explicitly. However, we can consider a cost function to indirectly penalize baggage arrival time deviation from a given time window. The cost function is composed of three penalty terms. The first penalty term penalizes the queue lengths being defined as

$$J_d^{\mathrm{tw}}(k) = \sum_{v \in V_d} C_{v,d}^{\mathrm{tw}}(k) x_{v,d}(k) \tag{3.5}$$

where $C_{v,d}^{\mathrm{tw}}(k)$ as illustrated in Fig. 3.2a is given as

$$C_{v,d}^{\mathrm{tw}}(k) = \begin{cases} 0 & \text{if } k + k_{v,d} \leq k_d^{\mathrm{open}} \\ c^{\mathrm{tw}}(k - k_d^{\mathrm{open}} + k_{v,d}) & \text{if } k_d^{\mathrm{open}} < k + k_{v,d} \leq k_d^{\mathrm{close}} \\ c^{\mathrm{tw}}(k_d^{\mathrm{close}} - k_d^{\mathrm{open}}) & \text{if } k + k_{v,d} > k_d^{\mathrm{close}} \end{cases} \tag{3.6}$$

where k_d^{open} and k_d^{close} are, respectively, the opening and the closing time steps of destination d and $k_{v,d}\Delta t$ is the expected travel time from node v to destination d under the current nominal operating conditions[2]. Note that since $C_{v,d}^{\mathrm{tw}}(k) = 0$ for $k \leq k_d^{\mathrm{open}} - k_{v,d}$, the queue lengths associated with destination d are not penalized before the destination is open, taking into account the DCVs travel time from v to d. During the time window of destination d, the weight associated with DCV queues increases linearly in time, hence, forcing the DCVs to move towards d.

The penalty term associated with the DCV flows is defined as:

$$J_d^{\mathrm{flow}}(k) = \sum_{v \in V_d} \sum_{w \in V_v^{\mathrm{recv}} \cap V_d} C_{v,d}^{\mathrm{flow}}(k) q_{v,w,d}(k) \tag{3.7}$$

with $C_{v,d}^{\mathrm{flow}}(k)$ as depicted in Fig. 3.2b being

$$C_{v,d}^{\mathrm{flow}}(k) = \begin{cases} -c_1^{\mathrm{flow}}(k - k_d^{\mathrm{open}} + k_{v,d}) & \text{if } k + k_{v,d} \leq k_d^{\mathrm{open}} \\ 0 & \text{if } k_d^{\mathrm{open}} < k + k_{v,d} \leq k_d^{\mathrm{close}} \\ c_2^{\mathrm{flow}}(k - k_d^{\mathrm{close}} + k_{v,d}) & \text{if } k + k_{v,d} > k_d^{\mathrm{close}} \end{cases} \tag{3.8}$$

Note that $C_{v,d}^{\mathrm{flow}}(k)$ is chosen in such a way that DCV flows to destination d are allowed during the time window of d. Higher values $C_{v,d}^{\mathrm{flow}}(k)$ outside of the time window prevent early or late DCV flows to the destination d. Moreover, in order to allow late DCVs to reach the destination, the slope of the third part of $C_{v,d}^{\mathrm{flow}}(k)$ is smaller than the slope of the first part. Now will will introduce the terms in the cost function that reflect the energy consumption in the network. We penalize all

[2]These can be obtained based on historical data for periods with similar conditions as the current one.

flows in the network in order to avoid indefinite circulation of DCVs throughout the network. Hence, we consider the following penalty term:

$$J^{\mathrm{e}}(k) = \sum_{d \in D} \sum_{v \in V_d} \sum_{w \in V_v^{\mathrm{recv}} \cap V_d} q_{v,w,d}(k) \tag{3.9}$$

In addition, we use the following penalty term to penalize the total variation of the control signal (i.e, flows), which reflects the wear and tear of the DCVs:

$$J^{\mathrm{sw}}(k) = \sum_{d \in D} \sum_{l \in V_d} \sum_{w \in V_v^{\mathrm{recv}} \cap V_d} |q_{v,w,d}(k) - q_{v,w,d}(k-1)| \tag{3.10}$$

The total cost at time step k is therefore given as

$$J(k) = \sum_{d \in D} J_d^{\mathrm{tw}}(k) + \alpha_1 \sum_{d \in D} J_d^{\mathrm{flow}}(k) + \alpha_2 J^{\mathrm{e}}(k) + \alpha_3 J^{\mathrm{sw}}(k) \tag{3.11}$$

where $\alpha_i > 0$ is a weight factor indicating the relative importance of the associated term in the objective function. The MPC performance index over the prediction horizon of N_{p} step is thus given as

$$J(k, N_{\mathrm{p}}) = \sum_{i=k}^{k+N_{\mathrm{p}}-1} J(i) \tag{3.12}$$

Now we would like to highlight the following remarks:

R1 The plots of Fig. 3.2a and Fig. 3.2b show respectively coefficients of the penalty terms (3.5) and (3.7), not the penalty terms themselves. In fact, at the given time step k and for a prediction horizon N_{p} the values of these coefficients are known for $k, \ldots, k + N_{\mathrm{p}} - 1$. Therefore, these coefficients have fixed values and hence the associated penalty terms (3.5) and (3.7) are linear in the control variable.

R2 By introducing some dummy variables according to standard techniques in optimization [7], terms of the form (3.10) can be recast as a linear programming problem with linear constraints.

Consider $\boldsymbol{u}(k)$, $\boldsymbol{x}(k)$, and $\boldsymbol{d}(k)$ as introduced in Section 3.2. At every time step k we solve the following optimization problem:

$$\begin{aligned}
&\min_{\overline{\boldsymbol{u}}(k)} F(k)\overline{\boldsymbol{u}}(k) \\
&\qquad \text{subject to } A_{\mathrm{ineq}}(k)\overline{\boldsymbol{u}}(k) \leq \boldsymbol{b}_{\mathrm{ineq}}(k) \\
&\qquad\qquad A_{\mathrm{eq}}(k)\overline{\boldsymbol{u}}(k) = \boldsymbol{b}_{\mathrm{eq}}(k)
\end{aligned} \tag{3.13}$$

where the vector $F(k)$ is defined based on the MPC objective function (3.11), as explained in Appendix 3.A, and the vector $\overline{\boldsymbol{u}}(k)$ includes the control inputs

$\boldsymbol{u}(k), \ldots, \boldsymbol{u}(k+N_\mathrm{p}-1)$ and the dummy variables mentioned in Remark R2. Moreover, $A_\mathrm{ineq}(k)$, and $A_\mathrm{eq}(k)$ are determined based on the constraints, and $\boldsymbol{b}_\mathrm{ineq}(k)$, and $\boldsymbol{b}_\mathrm{eq}(k)$ are constant vectors that depend on the current state $\boldsymbol{x}(k)$ and the demand values $\boldsymbol{d}(k), \ldots, \boldsymbol{d}(k+N_\mathrm{p}-1)$.

The optimization problem given by (3.13) is an LP problem, that can be solved efficiently with currently available solvers, e.g., MATLAB `linprog`.

3

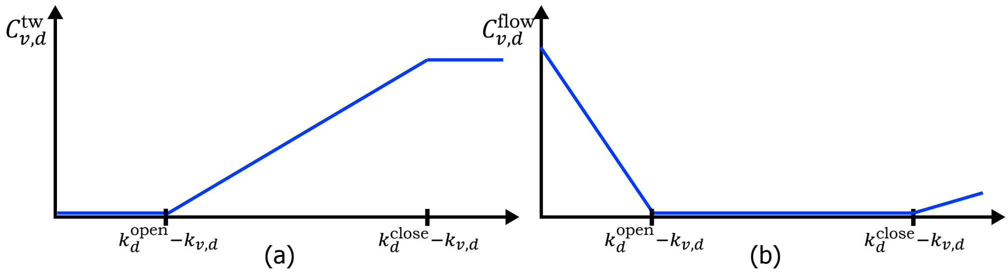

Figure 3.2: (a) the coefficient for the queue length penalty term; (b) the coefficient for the flow penalty term.

3.4. Feedback Control Problem Formulation

3.4.1. Problem Setup

Consider a discrete-time linear system

$$\boldsymbol{x}(k+1) = A\boldsymbol{x}(k) + B_1\boldsymbol{d}(k) + B_2\boldsymbol{u}(k) \tag{3.14a}$$

$$\boldsymbol{z}(k) = C_1\boldsymbol{x}(k) + D_1\boldsymbol{d}(k) + D_2\boldsymbol{u}(k) \tag{3.14b}$$

with full state feedback

$$\boldsymbol{u}(k) = K\boldsymbol{x}(k) \tag{3.15}$$

where the system matrices, $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{d} \in \mathbb{R}^{n_d}$, and $\boldsymbol{u} \in \mathbb{R}^m$ are those of (3.4a) and $\boldsymbol{z} \in \mathbb{R}^{n_z}$ is the controlled output vector. Assume that (A, B_2) is stabilizable and K is a stabilizing feedback gain. The L_2 gain of the closed-loop system is bounded by $\gamma > 0$ (i.e., $\sup\limits_{\boldsymbol{z}\neq 0} \frac{\|\boldsymbol{z}\|_2}{\|\boldsymbol{d}\|_2} \leq \gamma$) if and only if there exists a $P > 0$ such that [8], [9]

$$\begin{bmatrix} \mathcal{A}^\mathrm{T}P\mathcal{A} - P + \frac{1}{\gamma}\mathcal{C}^\mathrm{T}\mathcal{C} & \mathcal{A}^\mathrm{T}P\mathcal{B} + \frac{1}{\gamma}\mathcal{C}^\mathrm{T}\mathcal{D} \\ \mathcal{B}^\mathrm{T}P\mathcal{A} + \frac{1}{\gamma}\mathcal{D}^\mathrm{T}\mathcal{C} & \mathcal{B}^\mathrm{T}P\mathcal{B} + \frac{1}{\gamma}\mathcal{D}^\mathrm{T}\mathcal{D} - \gamma I \end{bmatrix} \leq 0 \tag{3.16}$$

where

$$\begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} A + B_2K & B_1 \\ C_1 + D_2K & D_1 \end{bmatrix} \tag{3.17}$$

or equivalently

$$\begin{bmatrix} -Q & \mathcal{A}Q & \mathcal{B} & 0 \\ Q\mathcal{A}^\mathrm{T} & -Q & 0 & Q\mathcal{C}^\mathrm{T} \\ \mathcal{B}^\mathrm{T} & 0 & -\gamma I & \mathcal{D}^\mathrm{T} \\ 0 & \mathcal{C}Q & \mathcal{D} & -\gamma I \end{bmatrix} \leq 0 \tag{3.18}$$

with $Q = P^{-1} > 0$.

Consider the problem of determining a feedback gain K that minimizes the L_2 gain of the closed-loop system. It is well-known [9] that with the transformation $Y = KQ$, the matrix inequality of (3.18) can be written as

$$\begin{bmatrix} -Q & AQ + B_2Y & B_1 & 0 \\ QA^{\mathrm{T}} + Y^{\mathrm{T}}B_2^{\mathrm{T}} & -Q & 0 & QC_1^{\mathrm{T}} + Y^{\mathrm{T}}D_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} & 0 & -\gamma I & D_1^{\mathrm{T}} \\ 0 & C_1Q + D_2Y & D_1 & -\gamma I \end{bmatrix} \leq 0 \tag{3.19}$$

with $Q > 0$.

Note that for the closed-loop system given by (3.14) and (3.15), (3.16) implies

$$\boldsymbol{x}^{\mathrm{T}}(k+1)P\boldsymbol{x}(k+1) - \boldsymbol{x}^{\mathrm{T}}(k)P\boldsymbol{x}(k) + \frac{1}{\gamma}\boldsymbol{z}^{\mathrm{T}}(\boldsymbol{k})\boldsymbol{z}(\boldsymbol{k}) \leq \gamma \boldsymbol{d}^{\mathrm{T}}(k)\boldsymbol{d}(k) \tag{3.20}$$

Now we define the ellipsoid $\varepsilon_\gamma := \{\boldsymbol{x}|\boldsymbol{x}^{\mathrm{T}}\frac{P}{\gamma}\boldsymbol{x} \leq 1\}$. Assuming $\boldsymbol{x}(0) = 0$, (3.20) yields

$$\boldsymbol{x}^{\mathrm{T}}(T)P\boldsymbol{x}(T) \leq \gamma \sum_{k=0}^{T-1} \boldsymbol{d}^{\mathrm{T}}(k)\boldsymbol{d}(k) < \gamma \sum_{k=0}^{\infty} \boldsymbol{d}^{\mathrm{T}}(k)\boldsymbol{d}(k) \tag{3.21}$$

for any $T \in \mathbb{N}$. Assuming[3] $\|\boldsymbol{d}\|_2^2 = 1$, we get

$$\boldsymbol{x}^{\mathrm{T}}(T)\frac{P}{\gamma}\boldsymbol{x}(T) < 1 \tag{3.22}$$

which shows that $\boldsymbol{x}(T) \in \varepsilon_\gamma$. Since (3.21) holds for all T, ε_γ contains the set of states that are reachable by a unit energy input signal $\boldsymbol{d}$ when the L_2 gain of the closed-loop system is bounded by γ.

3.4.2. Hard State Constraints

Now we consider the problem of searching for the feedback gain K that minimizes the L_2 gain of the closed-loop system subject to polytopic state constraints of the form

$$\boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x}(k) \leq 1, \ i = 1, \ldots, r. \tag{3.23}$$

To include the state constraints of (3.23), consider the polytope

$$\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^n | \boldsymbol{a}_i^{\mathrm{T}}\boldsymbol{x} \leq 1, i = 1, \ldots, r\} \tag{3.24}$$

associated with (3.23). We assume that $\mathcal{P}$ has the origin in its interior. To guaranty that (3.23) holds for all $k > 0$ with $x(0) = 0$, we must have $\varepsilon_\gamma \subseteq \mathcal{P}$ or equivalently [9]

$$\boldsymbol{a}_i^{\mathrm{T}}\gamma Q\boldsymbol{a}_i \leq 1, \ i = 1, \ldots, r \tag{3.25}$$

[3] It is always possible to scale $\boldsymbol{d}$ such that $\|\boldsymbol{d}\|_2^2 = 1$.

Therefore, the following optimization problem needs to be solved:

$$\min_{Q,Y,\gamma} \gamma \text{ subject to (3.19), (3.25), } Q > 0 \tag{3.26}$$

This problem is not jointly convex in γ and Q and Y. Moreover, it can be shown in a straightforward manner that the constraints of (3.18) and (3.25) do not satisfy the monotonicity property $G(Q,Y,\gamma_1) > G(Q,Y,\gamma_2)$ if $\gamma_1 > \gamma_2$, where $G < 0$ represents constraints (3.19) and (3.25) combined. Therefore, this problem cannot even be recast as a generalized eigenvalue problem, which is a class of quasiconvex optimization problems [9].

Now we will replace constraint (3.25) by a more conservative one that is convex in the optimization variables, in the following manner. Note that

$$\gamma Q = \frac{1}{4}(\gamma I + Q)^{\mathrm{T}}(\gamma I + Q) - \frac{1}{4}(\gamma I - Q)^{\mathrm{T}}(\gamma I - Q) \tag{3.27}$$

Obviously, $\gamma Q < \frac{1}{4}(\gamma I + Q)^{\mathrm{T}}(\gamma I + Q)$. Hence,

$$\frac{1}{4}\boldsymbol{a}_i^{\mathrm{T}}(\gamma I + Q)^{\mathrm{T}}(\gamma I + Q)\boldsymbol{a}_i \leq 1 \implies \boldsymbol{a}_i^{\mathrm{T}}\gamma Q\boldsymbol{a}_i < 1 \tag{3.28}$$

or equivalently expressed using the Schur complement

$$\begin{bmatrix} I & (\gamma I + Q)\boldsymbol{a}_i \\ \boldsymbol{a}_i^{\mathrm{T}}(\gamma I + Q) & 4 \end{bmatrix} > 0 \tag{3.29}$$

Clearly, this introduces conservatism as the feasibility set of (3.29) is a subset of the feasibility set of (3.25). This conservatism can be reduced if one can find a lower bound for $(\gamma I - Q)^{\mathrm{T}}(\gamma I - Q) \geq 0$ such that $(\gamma I - Q)^{\mathrm{T}}(\gamma I - Q) \geq \alpha^2 I$ or equivalently $\|\gamma I - Q\| \geq \alpha$ (in matrix norm sense) for some $\alpha > 0$. Then, instead of (3.29), one obtains

$$\begin{bmatrix} I & (\gamma I + Q)\boldsymbol{a}_i \\ \boldsymbol{a}_i^{\mathrm{T}}(\gamma I + Q) & 4 + \boldsymbol{a}_i^{\mathrm{T}}\alpha^2\boldsymbol{a}_i \end{bmatrix} > 0 \tag{3.30}$$

Therefore we consider (3.26) with (3.25) replaced by (3.29) or by (3.30). This is an eigenvalue problem [9], which is a convex optimization problem that can be solved with currently available LMI optimization toolboxes, e.g., MATLAB LMI toolbox, YALMIP [10], and CVX [11], [12].

3.4.3. Soft State Constraints

In the view of the proposed two-level control scheme, it makes more sense to replace the hard constraints of (3.25) by soft constraints due to the following observations: i) the constraints are mainly handled at the top level by the MPC controller, ii) if the constraints are too restrictive the conservative version of the original constraints as expressed by (3.29) may become infeasible, which is not desirable. As an alternative to the approach presented in Section 3.4.2, one can replace hard constraints by soft ones by considering a multi-objective optimization approach that penalizes the L_2 gain of the closed-loop system and, indirectly, the constraint violation at the same time. More precisely, we define the following optimization problem with the objective function that penalizes γ, and the volume of the ellipsoid ε_γ, which is proportional to $\sqrt{\det\ \gamma Q}$:

$$\min_{Q,Y,\gamma} c_\gamma \gamma + \log\big(\det(\gamma Q)\big) \text{ subject to } Q > 0 \text{ and } (3.19) \tag{3.31}$$

where $c_\gamma > 0$ is a weight factor. The magnitude of c_γ determines the trade-off between the L_2 gain and the volume of the ellipsoid that represents the set of reachable states. By minimizing the volume of ε_γ, we confine the set of reachable state from the origin. This indirectly minimizes constraint violation since the origin lies in the interior of polytope $\mathcal{P}$. However, this objective function is not convex in the optimization variables γ and Q. To mitigate this problem, instead of penalizing the volume of ε_γ, we penalize an upper bound on the length of semi-major axis of ε_γ, which is $\sqrt{\lambda_{\max}(\gamma Q)}$, where $\lambda_{\max}(\gamma Q)$ is the largest eigenvalue of γQ. It is clear from (3.27), that $\lambda_{\max}\big(\frac{1}{4}(\gamma I + Q)^{\mathrm{T}}(\gamma I + Q)\big)$ constitutes an upper bound on $\lambda_{\max}(\gamma Q)$. Then we get

$$\min_{Q,Y,\gamma} c_\gamma \gamma + \lambda_{\max}\big(\frac{1}{4}(\gamma I + Q)^{\mathrm{T}}(\gamma I + Q)\big) \text{ subject to } Q > 0 \text{ and } (3.19) \tag{3.32}$$

or equivalently

$$\min_{Q,Y,\gamma,\lambda} c_\gamma \gamma + \lambda \text{ subject to } (3.19),\ Q > 0,\ \begin{bmatrix} \lambda I & \gamma I + Q \\ \gamma I + Q & 4I \end{bmatrix} > 0 \tag{3.33}$$

This is an eigenvalue problem [9] that can be solved efficiently with currently available LMI solvers such as MATLAB LMI toolbox. Note that, by inspecting (3.27), the upper bound on $\lambda_{\max}(\gamma Q)$ can be made tighter if one can find an $\alpha > 0$ such that $(\gamma I - Q)^{\mathrm{T}}(\gamma I - Q) \geq \alpha^2 I$ or equivalently, in matrix norm[4] sense, $\|\gamma I - Q\| \geq \alpha$. Then the last constraint in (3.33) will be replaced by

$$\begin{bmatrix} (\lambda + \frac{\alpha^2}{4})I & \gamma I + Q \\ \gamma I + Q & 4I \end{bmatrix} > 0 \tag{3.34}$$

As an example, consider a discrete-time linear system given by

$$A = \begin{bmatrix} 0.1514 & 0.4377 & 0.7293 & 0.1839 \\ 0.3958 & 0.3999 & 0.7521 & 0.9368 \\ 0.9720 & 0.7636 & 0.8323 & 0.6137 \\ 0.7718 & 0.8639 & 0.4821 & 0.6050 \end{bmatrix},\ C_1 = I$$

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ -0.5881 \\ 0.2487 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.6954 \\ -0.2837 \\ -0.9723 \\ 0.6086 \end{bmatrix},\ D_1 = 0,\ D_2 = 0.$$

[4]For matrix norm, we use the definition $\|A\| = \sigma_{\max}(A)$, where $\sigma_{\max}(A)$ is the largest singular value of matrix A.

For c_γ taking values in the interval $[0.1, 100]$, Fig. 3.3 illustrates the trade-off between minimizing the L_2 gain and the length of the semi-major axis of ε_γ.

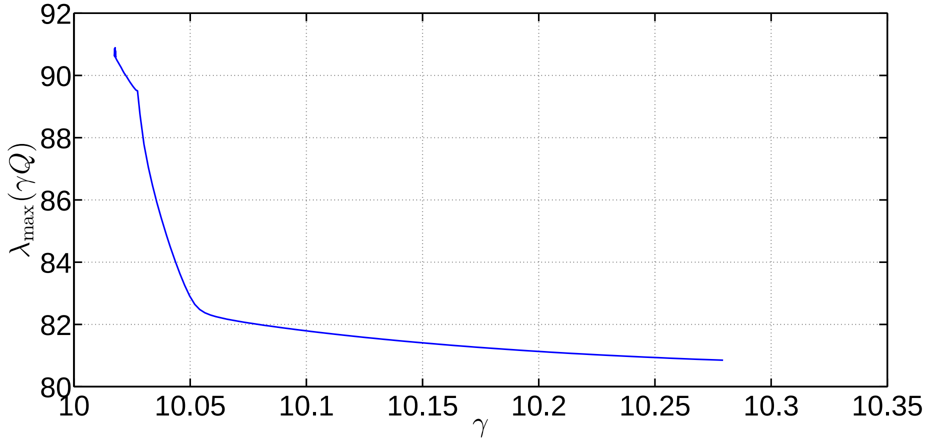

Figure 3.3: Trade-off curve between the optimal γ and the length of the semi-major axis of ε_γ.

3.5. Integration of MPC and Feedback Controllers

In this section, we briefly explain how the two control schemes presented in Sections 3.3 and 3.4 can be combined. The baggage demand at each origin node is composed of a base demand $\boldsymbol{d}^*$, which is assumed to be predictable over the prediction horizon N_p, and a small additive perturbation $\tilde{\boldsymbol{d}}$ around the base demand that cannot be predicted. Based on a future prediction of $\boldsymbol{d}^*$, the MPC controller computes the optimal DCV flows $\boldsymbol{u}^*$ and system trajectories $\boldsymbol{z}^*$ subject to flow and queue length constraints such that the DCVs arrive at their destinations with minimal energy consumption and with minimal deviation from the time windows. To minimize the adverse effect of $\tilde{d}$ on optimal system trajectories computed by the MPC controller, a feedback gain K minimizing $\frac{\|\tilde{\boldsymbol{z}}\|_2}{\|\tilde{\boldsymbol{d}}\|_2} = \frac{\|\boldsymbol{z}-\boldsymbol{z}^*\|_2}{\|\boldsymbol{d}-\boldsymbol{d}^*\|_2}$ based on the measurement $\boldsymbol{y}(k) - \boldsymbol{y}^*(k)$ is implemented along the MPC controller in the configuration depicted in Fig. 3.1. Therefore, the control law applied to the system at time step k is $\boldsymbol{u}(k) = \boldsymbol{u}^*(k) + K\big(\boldsymbol{y}(k) - \boldsymbol{y}^*(k)\big) = \boldsymbol{u}^*(k) + K\tilde{\boldsymbol{y}}(k)$.

When we impose constraints on the controlled output $\boldsymbol{z}(k) = \boldsymbol{z}^*(k) + \tilde{\boldsymbol{z}}(k)$, the constraints on $\tilde{\boldsymbol{z}}(k)$ depend on value of $\boldsymbol{z}^*(k)$. As a result, one needs to update the feedback gain K whenever the value of $\boldsymbol{z}^*(k)$ changes. This can be avoided if soft constraints as in Section 3.4.3 are used. Moreover, the MPC control law $\boldsymbol{u}^*$ does not have to be updated at every time instant $k\Delta t$. Particularly, if the base demand d^* is varying slowly with time, one can use a controller sampling time $m\Delta t$, with $m > 1$ being an integer number.

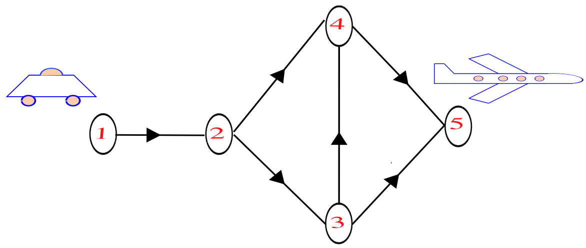

Figure 3.4: A layout of baggage handling system with one loading station and one unloading station. The length of each link in the network is 40 m.

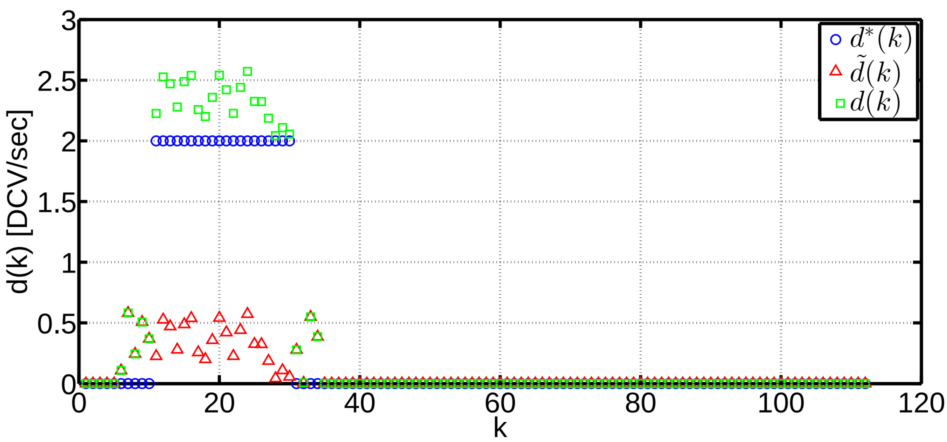

Figure 3.5: Base baggage demand,the perturbations on the base demand, and the actual demand at the loading station.

3.6. Case Study

In this section we present a case study to illustrate the performance of our proposed control approach for the baggage handling system. For the sake of simplicity, we consider a simple baggage handling system, the layout of which is depicted in Fig. 3.4. Here, the focus is to illustrate the effect of the feedback controller on suppressing the adverse effects of an unpredicted baggage demand on the behavior of the system. First, assuming that the demand is fully known, the optimal flows and optimal system trajectories are computed. Next, we consider some unpredictable random perturbations on the base demand and evaluate how closely our proposed two-level control approach can follow the optimal trajectory. For the two-level control approach, we have computed the feedback gain K based the approach of Section 3.4.3 using the MATLAB LMI toolbox. Table 3.1 lists the parameters used for the controller design and the closed-loop simulation. In Table 3.1, $\lambda^*_{\max}$ and $\gamma^*_{\min}$ denote, respectively, the *actual* values of $\lambda_{\max}(\gamma Q)$ and γ achieved by the closed-loop system whereas $\lambda^*_{\max}$ and $\gamma^*_{\min}$ denote those values obtained by solving (3.33).

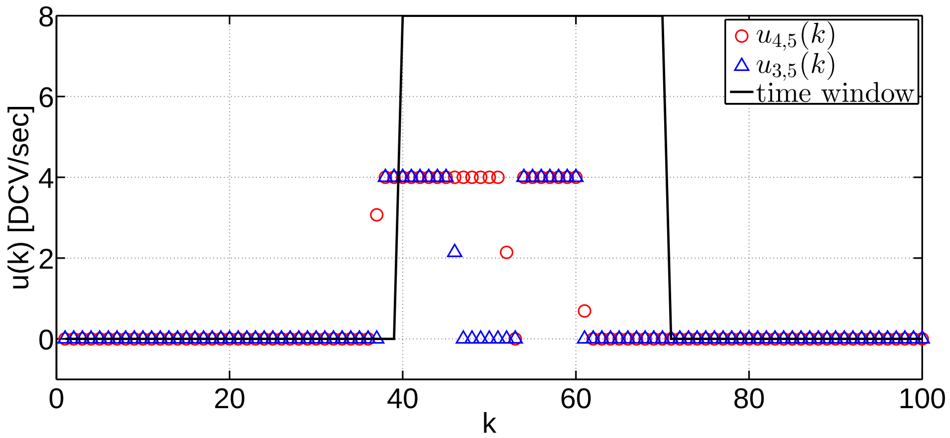

Figure 3.6: Optimal flows of DCVs at the unloading station. One can observe that most of the DCVs arrive at the unloading station within the specified time window.

Table 3.1: Controller design and simulation parameters

MPC Parameters				
N_p	time window	$\boldsymbol{u}_{\max}[\mathrm{DCV/s}]$	$(\alpha_1, \alpha_2, \alpha_3)$	$(c_1^{\text{flow}}, c_2^{\text{flow}}, c^{\text{tw}})$
12	$[40, 70]$	4	$(1, 125, 1)$	$(100, 0.2, 1)$
Feedback Controller Parameters				
c_γ	$\gamma^*_{\min}$	$\gamma_{\min}$	$\lambda^*_{\max}$	$\lambda_{\max}$
40	1.4420	1.5222	2.0001	3.8091
Closed-loop Simulation Parameters				
$\Delta t[\mathrm{s}]$	N_{sim}	x_0(initial condition)	demand perturbation	$v_{\text{DCV}}[\mathrm{m/s}]$
1.41	100	10	$\tilde{\boldsymbol{d}} \in \mathcal{U}(0,1), \|\boldsymbol{d}\|_2^2 = 4.02$	1.41

For the base demand $\boldsymbol{d}^*(k)$ depicted in Fig. 3.5, the optimal flows to the destination (node 5) are illustrated in Fig. 3.6 and the resulting optimal queue length at the origin node (node 1) is depicted in Fig. 3.7. It is clear from Fig. 3.6 that the optimal flows arrive at the destination within the desired time window. The perturbation on the base demand $\tilde{\boldsymbol{d}}(k) \in \mathcal{U}(0,1)$ is depicted in Fig. 3.5. It is obvious from Fig. 3.7 that in the presence of unpredictable demand perturbations, the two-level controller follows the optimal trajectory very closely whereas the MPC based approach deviates from the optimal trajectory.

3.7. Conclusions and Future Work

The routing problem in baggage handling systems was revisited. A flow-based model was derived for our control purposes, which are delivering the pieces of baggage at the unloading stations within a pre-specified time window, and minimizing the energy consumption. We proposed a multi-level control approach with an MPC controller at the top level and a constrained feedback controller at the

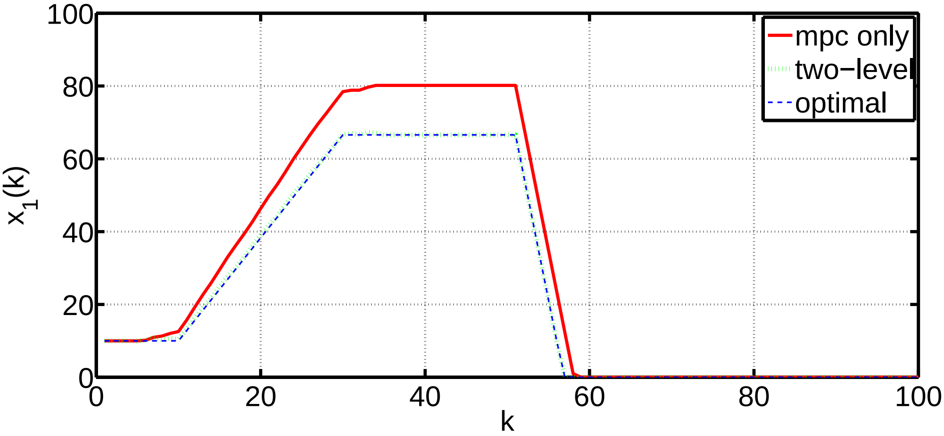

Figure 3.7: Queue lengths at node 1. One can observe that for the two-level control approach, the queue length at node 1 is only slightly affected by the disturbance.

bottom level that minimizes the L_2 gain of the closed-loop system. The idea was that based on some prior knowledge on the baggage demand, the MPC controller computes the optimal control inputs and system trajectories such that the pieces of baggage arrive at their destination within a desired time window and with minimal energy consumption. The feedback controller then would guarantee minimal deviation from this optimal trajectory in face of unknown perturbations on the baggage demand.

We showed that the MPC problem can be formulated as a linear programming problem. We proposed two methods to include state constraints in design procedure of the feedback controller that can be recast as LMI constraints. Using a simple case study, we showed the effectiveness of the proposed two-level control approach.

This approach should be extendable to large-scale system. However, for large-scale systems, the conservatism introduced by (3.28) may render the LMIs in (3.33) infeasible. Hence, one may need to find a tighter lower bound α in (3.34).

For future work, the scalability of the proposed two-level approach to large network layouts will be investigated. In addition, we will compare the performance of the two-level control approach with the MPC-based approach for larger network layouts and more elaborate scenarios. As a second extension to the current work, we will include non-polytopic state constraints as well as control signal constraints in the design procedure of the feedback controller.

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3

Appendix

3.A. Derivation of Linear Program

In this section we show how the MPC problem of linear system with a linear objective function can be formulated as a standard linear program. Consider the discrete-time linear time-invariant system

$$\begin{aligned} \boldsymbol{x}(t+1) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_1\boldsymbol{\omega}(t) + \boldsymbol{B}_2\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t), \\ & \boldsymbol{A} \in \mathbb{R}^{n\times n},\ \boldsymbol{B}_1 \in \mathbb{R}^{n\times n_\omega},\ \boldsymbol{B}_2 \in \mathbb{R}^{n\times n_u},\ \boldsymbol{C} \in \mathbb{R}^{n_y\times n}, \end{aligned} \tag{3.35}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^n$, $\boldsymbol{\omega}(t) \in \mathbb{R}^{n_\omega}$, $\boldsymbol{u}(t) \in \mathbb{R}^{n_u}$, $\boldsymbol{y}(t) \in \mathbb{R}^{n_y}$ are respectively the state at time t, the disturbance input at time t, the control input at time t, and the output at time t. Consider the constrained linear optimization problem defined at the current time t as

$$\begin{aligned} \min_{\boldsymbol{u}(t+k),\boldsymbol{u}(t+k+1),\dots,\boldsymbol{u}(t+k+N-1)} & \sum_{k=1}^{N-1} \left(\boldsymbol{q}^{\mathrm{T}}\boldsymbol{y}(t+k) + \boldsymbol{r}^{\mathrm{T}}\boldsymbol{u}(t+k)\right) \\ & + \boldsymbol{s}^{\mathrm{T}}\boldsymbol{y}(t+N) \end{aligned} \tag{3.36}$$

subject to:

recursion (3.35) for $k = 1, \dots, N$ with initial state $\boldsymbol{x}(t)$,
$\boldsymbol{G}\boldsymbol{y}(t+k) + \boldsymbol{g} \leq 0, \quad k = 1, \dots, N,$
$\boldsymbol{M}\boldsymbol{u}(t+k) + \boldsymbol{m} \leq 0, \quad k = 1, \dots, N-1,$

where $\boldsymbol{q} \in \mathbb{R}^{n_y}$, $\boldsymbol{r} \in \mathbb{R}^{n_u}$, and $\boldsymbol{s} \in \mathbb{R}^{n_y}$ are weighting factors, and where $\boldsymbol{G} \in \mathbb{R}^{n_1\times n_y}$, $\boldsymbol{g} \in \mathbb{R}^{n_1}$, $\boldsymbol{M} \in \mathbb{R}^{n_2\times n_u}$, $\boldsymbol{m} \in \mathbb{R}^{n_2}$ define the output and control constraints, respectively. This problem can be written in the standard LP form using the explicit expression of the solution of (3.35), which is given as

$$\boldsymbol{x}(t+k) = \boldsymbol{A}^k\boldsymbol{x}(t) + \sum_{s=0}^{k-1} \boldsymbol{A}^s\Big(\boldsymbol{B}_2\boldsymbol{u}(t+k-1-s) + \boldsymbol{B}_1\boldsymbol{\omega}(t+k-1-s)\Big), \tag{3.37}$$

for all $k \in \mathbb{N}$. Define

$$\boldsymbol{x}_N(t) = \begin{bmatrix} \boldsymbol{x}(t+1) \\ \vdots \\ \boldsymbol{x}(t+N) \end{bmatrix},\ \boldsymbol{y}_N(t) = \begin{bmatrix} \boldsymbol{y}(t+1) \\ \vdots \\ \boldsymbol{y}(t+N) \end{bmatrix},\ \boldsymbol{u}_N(t) = \begin{bmatrix} \boldsymbol{u}(t) \\ \vdots \\ \boldsymbol{u}(t+N-1) \end{bmatrix},$$

$$\boldsymbol{\omega}_N(t) = \begin{bmatrix} \boldsymbol{\omega}(t) \\ \vdots \\ \boldsymbol{\omega}(t+N-1) \end{bmatrix},\ \boldsymbol{A}_N = \begin{bmatrix} \boldsymbol{A} \\ \vdots \\ \boldsymbol{A}^N \end{bmatrix},\ \boldsymbol{q}_N = \begin{bmatrix} \boldsymbol{q} \\ \vdots \\ \boldsymbol{q} \end{bmatrix},\ \boldsymbol{C}_N = \begin{bmatrix} \boldsymbol{C} & & \\ & \ddots & \\ & & \boldsymbol{C} \end{bmatrix},$$

3

$$\boldsymbol{r}_N = \begin{bmatrix} \boldsymbol{r} \\ \vdots \\ \boldsymbol{r} \end{bmatrix}, \ \boldsymbol{G}_N = \begin{bmatrix} \boldsymbol{G} \\ \vdots \\ \boldsymbol{G} \end{bmatrix}, \ \boldsymbol{g}_N = \begin{bmatrix} \boldsymbol{g} \\ \vdots \\ \boldsymbol{g} \end{bmatrix}, \ \boldsymbol{M}_N = \begin{bmatrix} \boldsymbol{M} \\ \vdots \\ \boldsymbol{M} \end{bmatrix}, \ \boldsymbol{m}_N = \begin{bmatrix} \boldsymbol{m} \\ \vdots \\ \boldsymbol{m} \end{bmatrix},$$

$$\boldsymbol{B}_{2,N} = \begin{bmatrix} \boldsymbol{B}_2 & 0 & \cdots & 0 \\ \boldsymbol{A}\boldsymbol{B}_2 & \boldsymbol{B}_2 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ \boldsymbol{A}^{N-1}\boldsymbol{B}_2 & \boldsymbol{A}^{N-2}\boldsymbol{B}_2 & \dots & \boldsymbol{B}_2 \end{bmatrix}, \ \boldsymbol{B}_{1,N} = \begin{bmatrix} \boldsymbol{B}_1 & 0 & \cdots & 0 \\ \boldsymbol{A}\boldsymbol{B}_1 & \boldsymbol{B}_1 & \cdots & 0 \\ \vdots & \cdots & \ddots & 0 \\ \boldsymbol{A}^{N-1}\boldsymbol{B}_1 & \boldsymbol{A}^{N-2}\boldsymbol{B}_1 & \dots & \boldsymbol{B}_1 \end{bmatrix}.$$

Then, using

$$\boldsymbol{y}_N = \boldsymbol{C}_N\Big(\boldsymbol{A}_N\boldsymbol{x}(t) + \boldsymbol{B}_{2,N}\boldsymbol{u}_N(t) + \boldsymbol{B}_{1,N}\boldsymbol{\omega}_N(t)\Big),$$

the optimization problem (3.36) can be rewritten as

$$\begin{aligned} &\min_{\boldsymbol{u}_N} \left[\boldsymbol{q}_N^{\mathrm{T}} \quad \boldsymbol{s}^{\mathrm{T}}\right] \boldsymbol{C}_N\Big(\boldsymbol{A}_N\boldsymbol{x}(t) + \boldsymbol{B}_{1,N}\boldsymbol{\omega}_N(t) + \boldsymbol{B}_{2,N}\boldsymbol{u}_N(t)\Big) \\ &\text{subject to:} \\ &\qquad \boldsymbol{G}_N\boldsymbol{C}_N\boldsymbol{B}_{2,N}\boldsymbol{u}_N + \boldsymbol{G}_N\boldsymbol{C}_N\Big(\boldsymbol{A}_N\boldsymbol{x}(t) + \boldsymbol{B}_{1,N}\boldsymbol{\omega}_N\Big) \le \boldsymbol{g}_N \\ &\qquad \boldsymbol{M}_N\boldsymbol{u}_N \le \boldsymbol{m}_N, \end{aligned}$$

or equivalently as

$$\begin{aligned} &\min_{\boldsymbol{u}_N} \left[\boldsymbol{q}_N^{\mathrm{T}} \quad \boldsymbol{s}^{\mathrm{T}}\right] \boldsymbol{B}_{2,N}\boldsymbol{u}_N(t) \\ &\text{subject to:} \\ &\qquad \boldsymbol{G}_N\boldsymbol{C}_N\boldsymbol{B}_{2,N}\boldsymbol{u}_N + \boldsymbol{G}_N\boldsymbol{C}_N\Big(\boldsymbol{A}_N\boldsymbol{x}(t) + \boldsymbol{B}_{1,N}\boldsymbol{\omega}_N\Big) \le \boldsymbol{g}_N \\ &\qquad \boldsymbol{M}_N\boldsymbol{u}_N \le \boldsymbol{m}_N, \end{aligned}$$

which is in the form of standard linear program in the variable $\boldsymbol{u}_N$.

4

Robustly Positively Invariant Sets for Discrete-time Linear Positive Systems: Application to Tube-based MPC Approach

"Science can amuse and fascinate us all, but it is engineering that changes the world."

Isaac Asimov, 1920 – 1992

For linear discrete-time systems subject to an infinity-norm bounded disturbance, this chapter presents a Linear-Programming-based method for simultaneous calculation of an infinity-norm bounded robustly positively invariant set and a (constrained) state feedback gain that minimizes the L_∞-norm of the output over this set. This result can be applied in tube-based MPC, where the robustly positively invariant set is used to tighten the nominal state and control constraints. In addition, this method is used to derive an infinity-norm bounded terminal constraint set and an infinity-norm-based terminal cost function for (tube-based) MPC approach, guaranteeing recursive feasibility and (robust) asymptotic stability of the closed-loop system. Application of the developed approach is demonstrated in a case study involving baggage handling systems.

This chapter is based on [1].

4.1. Introduction

For linear systems subject to an additive disturbance, tube-based Model Predictive Control (MPC) provides a relatively simple approach to robust MPC design. A key element of this approach is the *tube*, which is the bundle of all trajectories of the uncertain system for a given disturbance set. Characterizing the tube is very closely related to calculating robustly positively invariant and robustly control invariant sets for the error system, which describes the error dynamics between the uncertain system and the nominal system (i.e., without disturbance). The problem of finding robustly positively invariant and robustly control invariant sets for a linear system is, in general, a complex problem. For a linear system subject to linear state and control constraints, the solution is an iterative process involving polyhedral projections and Minkowski differences [2–4], which results in a finite number of iterations only under certain conditions that depend on the system matrices $(\boldsymbol{A}, \boldsymbol{B})$ and the geometry of admissible state, and control, and disturbance sets [5, 6]. This limits practical applicability of tube-based MPC to small-scale systems, as for large-scale system polyhedral projections and Minkowski differences become prohibitively expensive.

In this chapter, we propose a solution for calculating robustly positively invariant and robustly control invariant sets for discrete-time linear positive systems subject to infinity-norm bounded constraints and infinity-norm bounded disturbances. We show that the problem of finding a robustly positively invariant set can be formulated as a linear program (LP), which can be efficiently solved even for large-scale systems. The LP-based approach for finding a robustly positively invariant set is later used in designing a tube-based MPC controller, where a constrained state feedback gain is calculated to not only stabilize the error system, but also to minimize the $\|L\|_\infty$ norm of the output subject to the given disturbance set. Therefore, the tube-based MPC controller ensures the "smallest" deviation from the nominal trajectory in the presence of disturbances for a given control "budget", which determines the maximum control effort that can be generated by the feedback controller. We show that this state feedback controller can be obtained via a linear program as well.

Based on the approach developed for calculating a robustly control invariant set, we also develop a linear program for calculating a terminal constraint set and a terminal cost function for the (tube-based) MPC approach such that recursive feasibility and asymptotic stability of the closed-loop system is theoretically guaranteed.

To illustrate its applicability, we apply the developed methods in a case study involving automated baggage handling systems (BHSs), which have recently received a lot of attention due to growing demand for air travel. The reader is referred to [7] for a complete description of modern BHSs and their basic components. From a high-level perspective, the control problems associated with BHSs revolve around routing, dispatching, and scheduling of high-speed destination coded vehicles (DCVs) that transport the pieces of baggage between loading stations and unloading stations via various routes in the system. DCVs need to be controlled in such a way that pieces of luggage are delivered at their pre-assigned unloading stations within pre-specified time windows. A detailed description of the high-level control problems of BHSs can be found in [7, 8].

Having developed a modeling framework for BHSs based on *link densities*, we design a nominal MPC control strategy based on the base value of the baggage demand profile at loading stations, which is assumed to be a known function of time and, hence, fully predictable. As long as the actual baggage demand at loading stations is the nominal demand, this controller generates a routing policy that is optimal in the sense of *energy cost* and delivery time deviations from the target time window, while satisfying the capacity constraints of the BHS. We then consider structured uncertainty in the baggage demand profile and design a tube-based MPC approach to minimize the effect of uncertainty on the performance of the system, where the feedback gain for the tube-based MPC controller is calculated via the solution proposed in this chapter. In addition, we show how the terminal constraint set and the terminal cost function for the MPC and tube-based MPC problems can be calculated using our proposed approach. The effectiveness of tube-based MPC approach in the face of baggage demand uncertainty is then demonstrated by a simulation study.

The current chapter is different with respect to [8] and [9] in several key aspects: firstly, the solution proposed solution in [8] assumes full knowledge of the baggage demand and its future prediction at the loading stations, and hence is not robust against any possible uncertainty in the baggage demand. In the current chapter, the tube-based MPC approach is meant to deal with partially uncertain baggage demand. Secondly, in both [8] and [9], stability and *recursive feasibility* of the MPC controller are achieved indirectly by choosing a sufficiently long prediction horizon, whereas, here, these problems are directly addressed by introducing an appropriate *terminal state constraint* and a *terminal cost* [10, 11]. Finally, the two-level solution proposed in [9] incorporates an L_2-optimal feedback controller, which is formulated as Linear Matrix Inequalities (LMIs). In the current chapter we take advantage of the intrinsic nonnegative dynamics of our model, which allows for Linear Programming (LP) formulations of the output L_∞-norm optimal feedback controller. Obviously, the LP formulations outweigh the LMI approach in terms of scalability of the solution to large-scale BHSs.

The rest of this chapter is organized as follows. In Section 4.2 the terminology and definitions used throughout the chapter are introduced. In Section 4.3, we present some known results on discrete-time LTI positive systems and extend them to discrete-time systems, and formulate the output L_∞-norm optimization problem for a discrete-time linear time-invariant positive system. The nominal and tube-based MPC problems are then formulated in Section 4.4. The dynamical model of the BHS is developed in Section 4.5. Using the developed model of the BHS, the proposed control approaches are then applied to a simulated case study and the results are presented in Section 4.6. Finally, we conclude this chapter in Section 4.7, pointing out possible directions for future research. Proofs of the propositions and theorems presented in the chapter are provided in Appendix 4.A.

4.2. Notation, terminology and definition

For sets $\mathbb{A}$ and $\mathbb{B}$, $\mathbb{A} \bigoplus \mathbb{B} \coloneqq \{a + b \mid a \in \mathbb{A}, b \in \mathbb{B}\}$ is the Minkowski sum of the two sets. For non-negative integers i and j, $i \leq j$, $\mathbb{N}_{i:j}$ is the finite set $\{i,\ i+1,\ \ldots,\ j\}$ of integers and $\mathbb{N}_{i,\infty}$ is the infinite set $\{i,\ i+1,\ \ldots\}$ of integers. For a vector $\boldsymbol{x} \in \mathbb{R}^n$, x_i denotes its i-th element. In a similar manner for a matrix $\boldsymbol{A} \in \mathbb{R}^{n\times m}$, $A_{i,j}$ denotes the element on the i-th row and the j-th column, and $\boldsymbol{A}_{i,:}$ and $\boldsymbol{A}_{:,j}$ respectively denote the i-th row of the matrix and its j-th column. We denote the all-ones vector in $\mathbb{R}^n_+$, the identity matrix of dimension n, and any zero matrix of suitable dimension, respectively, by 1_n, $\boldsymbol{I}_n$, and 0. For matrices $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times m}$, $\boldsymbol{A} \geq \boldsymbol{B}$ ($\boldsymbol{A} > \boldsymbol{B}$), or equivalently $\boldsymbol{A} - \boldsymbol{B} \in \mathbb{R}^{n\times m}_+$ ($\boldsymbol{A} - \boldsymbol{B} \in \mathbb{R}^{n\times m}_{+,\mathrm{s}}$) means that all entries of $\boldsymbol{Z} := \boldsymbol{A} - \boldsymbol{B}$ are non-negative (positive).

For a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\mathrm{conv}(\boldsymbol{A}) := \{\sum_{i=1}^n x_i \boldsymbol{A}_{:,i} \mid \sum_{i=1}^n x_i = 1,\ x_i \geq 0,\ i \in \mathbb{N}_{1:n}\}$ is the convex set generated by its columns. The infinity-induced -norm of $\boldsymbol{A} \in \mathbb{R}^{m\times n}$ is defined as $\|A\|_{\infty-\mathrm{ind}} := \sup_{\boldsymbol{x}\in\mathbb{R}^n\setminus\{0\}} \frac{\|\boldsymbol{A}\boldsymbol{x}\|_\infty}{\|\boldsymbol{x}\|_\infty}$; it is known [12] that $\|A\|_{\infty-\mathrm{ind}} = \max_{i\in\mathbb{N}_{1:m}}(|\boldsymbol{A}|1_n)_i$, where $|\cdot|$ denotes the element-wise absolute value of its argument. For a single-column matrix $\boldsymbol{a} \in \mathbb{R}^m$, this definition renders into the conventional vector infinity norm $\|\boldsymbol{a}\|_{\infty-\mathrm{ind}} = \|\boldsymbol{a}\|_\infty := \max_{i\in\mathbb{N}_{1:m}}(|\boldsymbol{a}|)_i$. For a matrix $\boldsymbol{A} \in \mathbb{R}^{m\times n}_+$ and a positive scalar γ, the inequality $\|\boldsymbol{A}\|_{\infty-\mathrm{ind}} < \gamma$ can then be equivalently expressed as $\boldsymbol{A}1_n < \gamma 1_m$. For a discrete-time signal of dimension n, defined as the map $\boldsymbol{x} : \mathbb{N} \to \mathbb{R}^n$, its L_∞ norm is defined as $\|\boldsymbol{x}\|_{L_\infty} := \sup_{t\in\mathbb{N}} \|\boldsymbol{x}(t)\|_\infty$.

Define $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{\omega} \in \mathbb{R}^{n_\omega}$, and $\boldsymbol{z} \in \mathbb{R}^{n_z}$ respectively as the state, disturbance, and monitored/controlled output. An autonomous (i.e., without control input) discrete-time linear time-invariant system of the form

$$\begin{aligned} \boldsymbol{x}^+ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{\omega}, \\ \boldsymbol{z} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_{11}\boldsymbol{\omega} \\ &\ \text{with } \boldsymbol{A} \in \mathbb{R}^{n\times n}, \boldsymbol{B}_1 \in \mathbb{R}^{n\times n_u},\ \boldsymbol{C} \in \mathbb{R}^{n_z\times n},\ \boldsymbol{D}_{11} \in \mathbb{R}^{n_z\times n} \end{aligned} \tag{4.1}$$

is called a *positive system* [13–15] if $\boldsymbol{A} \geq 0$, $\boldsymbol{B}_1 \geq 0$, $\boldsymbol{C} \geq 0$, and $\boldsymbol{D}_{11} \geq 0$. For system (4.1) with initial state $\boldsymbol{x}_0 := \boldsymbol{x}(0)$, and disturbance sequence $\boldsymbol{\omega}_k := (\boldsymbol{\omega}(0),\ \boldsymbol{\omega}(1),\ \ldots,\ \boldsymbol{\omega}(k-1),\ \boldsymbol{\omega}(k))$ of length $k+1$, $k \in \mathbb{N}$, the state and output trajectories at time $t \in \mathbb{N}_{0:k}$ are given explicitly by the formula

$$\boldsymbol{x}\big(t; \boldsymbol{x}_0, \boldsymbol{\omega}_k\big) := \boldsymbol{A}^t\boldsymbol{x}_0 + \sum_{s=0}^{t-1} \boldsymbol{A}^s\big(\boldsymbol{B}_1\boldsymbol{\omega}_k(t-s)\big), \tag{4.2a}$$

$$\boldsymbol{z}\big(t; \boldsymbol{x}_0, \boldsymbol{\omega}_k\big) := \boldsymbol{C}\boldsymbol{x}\big(t; \boldsymbol{x}_0, \boldsymbol{\omega}_k\big) + \boldsymbol{D}_{11}\boldsymbol{\omega}_k(t+1). \tag{4.2b}$$

For a positive system with $\boldsymbol{x}_0 \geq 0$, and $\boldsymbol{\omega}_k \in \mathbb{R}^{n_\omega(k+1)}_+$, $k \in \mathbb{N}$, it is obvious that $\boldsymbol{x}\big(t; \boldsymbol{x}_0, \boldsymbol{\omega}_k\big) \geq 0$ and $\boldsymbol{z}\big(t; \boldsymbol{x}_0, \boldsymbol{\omega}_k\big) \geq 0$ for all $t \in \mathbb{N}_{0:k}$.

4.3. Output L_∞-norm-optimal Feedback Control for Positive Systems

4.3.1. Preliminaries

In this section, we focus on a system of the form

$$\begin{aligned} \boldsymbol{x}^+ &= \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{\omega} + \boldsymbol{B}_2\boldsymbol{u}, \\ \boldsymbol{z} &= \boldsymbol{C}\boldsymbol{x} + \boldsymbol{D}_{11}\boldsymbol{\omega} + \boldsymbol{D}_{12}\boldsymbol{u}, \\ & \boldsymbol{A} \in \mathbb{R}^{n\times n},\ \boldsymbol{B}_1 \in \mathbb{R}_+^{n\times n_\omega},\ \boldsymbol{B}_2 \in \mathbb{R}^{n\times n_u},\ \boldsymbol{C} \in \mathbb{R}^{n_z\times n}, \\ & \boldsymbol{D}_{11} \in \mathbb{R}_+^{n_z\times n},\ \boldsymbol{D}_{12} \in \mathbb{R}^{n_z\times n}. \end{aligned} \tag{4.3}$$

Given compact sets $\mathbb{X}_0 \subseteq \mathbb{R}^n$, $\mathbb{W} \subseteq \mathbb{R}^{n_\omega}$, $\mathbb{U} \subseteq \mathbb{R}^{n_u}$ as the set of initial conditions, the disturbance set, and the set of admissible controls with $0 \in \mathbb{X}_0$, $0 \in \mathbb{W}$, and $0 \in \mathbb{U}$, we are interested in the smallest, in the sense of the L_∞-norm of the output, robustly positively invariant set [2] $\mathbb{X} \supseteq \mathbb{X}_0$ achievable with a linear state feedback law of the form $\boldsymbol{\mu} : \boldsymbol{x} \mapsto \boldsymbol{K}\boldsymbol{x}$, $\boldsymbol{u}(t) = \boldsymbol{\mu}(\boldsymbol{x}(t))$ that renders the closed-loop system positive. Hence, a feedback gain $\boldsymbol{K} \in \mathbb{R}^{nu\times n}$ and a set $\mathbb{X} \supseteq \mathbb{X}_0$ are sought such that the closed-loop system

$$\begin{aligned} \boldsymbol{x}^+ &= (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K})\boldsymbol{x} + \boldsymbol{B}_1\boldsymbol{\omega}, \\ \boldsymbol{z} &= (\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K})\boldsymbol{x} + \boldsymbol{D}_{11}\boldsymbol{\omega}, \\ \boldsymbol{u} &= \boldsymbol{K}\boldsymbol{x}, \end{aligned} \tag{4.4}$$

is positive, has the property that $\boldsymbol{x} \in \mathbb{X}$ implies $\boldsymbol{u} \in \mathbb{U}$ and $\boldsymbol{x}^+ \in \mathbb{X}$ for all $\boldsymbol{\omega} \in \mathbb{W}$, and achieves minimum $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$ defined as

$$\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}} := \max_{\boldsymbol{\omega}\in\mathbb{W},\boldsymbol{x}_0\in\mathbb{X}} \|\boldsymbol{z}\|_{L_\infty} = \sup_{k\in\mathbb{N},\, t\in\mathbb{N}_{0:k},\, \boldsymbol{x}_0\in\mathbb{X},\, \boldsymbol{\omega}_k\in\mathbb{W}^{k+1}} \|\boldsymbol{z}(t;\boldsymbol{x}_0,\boldsymbol{\omega}_k)\|_\infty.$$

We consider the case where both the robustly positively invariant state set in question, and the given admissible control and disturbance sets are infinity-norm bounded sets of the form $\mathbb{Y} := \{\boldsymbol{y} \in \mathbb{R}^{n_y} \mid \|\boldsymbol{Y}^{-1}\boldsymbol{y}\|_\infty \le 1,\ \boldsymbol{Y} := \mathrm{diag}(\boldsymbol{y}_M),\ \boldsymbol{y}_M \in \mathbb{R}_{+,s}^{n_y}\}$.

4.3.2. Upper Bound on L_∞-norm of Output

We first present the following result regarding establishing an upper bound on $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$ for the open-loop positive system (4.1) that will be used when discussing the state feedback design problem.

Proposition 4.1 (Bound on $\boldsymbol{z}_{L_\infty,\mathbb{W},\mathbb{X}}$). Consider the positive system (4.1) with a positive scalar $\gamma \in \mathbb{R}_{+,s}$, $\mathbb{W} := \{\boldsymbol{\omega} \in \mathbb{R}^{n_\omega} \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty \le 1,\ \boldsymbol{\Omega} := \mathrm{diag}(\boldsymbol{\omega}_M),\ \boldsymbol{\omega}_M \in \mathbb{R}_{+,s}^{n_\omega}\}$, $\mathbb{X} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \le 1,\ \boldsymbol{P} := \mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathbb{R}_{+,s}^n\}$, and $\mathbb{X}_0 = \mathrm{conv}(\boldsymbol{M}^\mathrm{T})$, $\boldsymbol{M} \in \mathbb{R}^{n\times r}$. Then $\mathbb{X}$ is a robustly positively invariant set containing $\mathbb{X}_0$ for the system with $\boldsymbol{\omega} \in \mathbb{W}$, and $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}}$ of the system is bounded as $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}} < \gamma$ if and only if

$$(\boldsymbol{A}-\boldsymbol{I}_n)\boldsymbol{P}1_n+\boldsymbol{B}_1\boldsymbol{\Omega}1_{n_\omega}<0, \tag{4.5a}$$
$$\boldsymbol{C}\boldsymbol{P}1_n+\boldsymbol{D}_{11}\boldsymbol{\Omega}1_{n_\omega}-\gamma 1_{n_z}<0, \tag{4.5b}$$
$$\boldsymbol{P}\geq \mathrm{diag}\left(\left[\|\boldsymbol{M}(:,1)\|_\infty\ \dots\ \|\boldsymbol{M}(:,n)\|_\infty\right]^{\mathrm{T}}\right). \tag{4.5c}$$

See Appendix 4.A for the proof.

Remark 1. Note that with $\boldsymbol{\Omega}=\boldsymbol{I}_{n_\omega}$ and $\mathbb{X}_0=\{0\}$, inequalities 4.5 of Proposition 4.1 are necessary and sufficient conditions for establishing γ as an upper bound to the L_∞-gain of the system, defined as $\|\boldsymbol{G}\|_\infty \coloneqq \sup_{\boldsymbol{\omega}\in\mathcal{L}_\infty}\frac{\|\boldsymbol{z}\|_{L_\infty}}{\|\boldsymbol{\omega}\|_{L_\infty}}$ [16, 17]. Hence, the search for upper bound to L_∞-gain of the system, can be considered an special case of finding an upper bound to $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$.

Remark 2. For the positive system (4.1), the statements of proposition 4.1 are also valid when the state and disturbance are restricted to their corresponding positive orthants; that is, when $\mathbb{X}$ and $\mathbb{W}$ are respectively replaced by $\mathbb{X}_+\coloneqq\mathbb{X}\cap\mathbb{R}^n_+$ and $\mathbb{W}_+\coloneqq\mathbb{W}\cap\mathbb{R}^{n_\omega}_+$.

4.3.3. Derivation of the Feedback Gain

Now, we will discuss the state feedback design problem. The method presented in this section will be used in Section 4.4.2 for designing a tube-based MPC controller. Given the infinity norm-bounded sets $\mathbb{W}\coloneqq\{\boldsymbol{\omega}\in\mathbb{R}^{n_\omega}\mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty\leq 1,\ \boldsymbol{\Omega}\coloneqq \mathrm{diag}(\boldsymbol{\omega}_M),\ \boldsymbol{\omega}_M\in\mathbb{R}^{n_\omega}_{+,s}\}$, $\mathbb{U}\coloneqq\{\boldsymbol{u}\in\mathbb{R}^{n_u}\mid\|\boldsymbol{U}^{-1}\boldsymbol{u}\|_\infty\leq 1,\ \boldsymbol{U}\coloneqq\mathrm{diag}(\boldsymbol{u}_M),\ \boldsymbol{u}_M\in\mathbb{R}^{n_u}_{+,s}\}$, and the polyhedral set $\mathbb{X}_0=\mathrm{conv}(\boldsymbol{M}^{\mathrm{T}})$, $\boldsymbol{M}\in\mathbb{R}^{n\times r}$, existence and derivation of the feedback gain $\boldsymbol{K}$ and the robustly positively invariant set $\mathbb{X}\supseteq\mathbb{X}_0$ are the subject of the following theorem:

Theorem 4.2. *The feedback gain $\boldsymbol{K}$ renders the closed-loop system 4.4 positive with minimum $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$ over a robustly positively invariant set $\mathbb{X}\coloneqq\{\boldsymbol{x}\in\mathbb{R}^n\mid\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty\leq 1,\ \boldsymbol{P}=\mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p}\in\mathbb{R}^n_{+,s}\}$, $\mathbb{X}\supseteq\mathbb{X}_0$ such that $\boldsymbol{K}\boldsymbol{x}\in\mathbb{U}$ for all $\boldsymbol{x}\in\mathbb{X}$ and all $\boldsymbol{\omega}\in\mathbb{W}$, if and only if there is a solution $\gamma\in\mathbb{R}_{+,s}$, $\boldsymbol{p}\in\mathbb{R}^n_{+,s}$, $\boldsymbol{Y}_1\in\mathbb{R}^{n_u\times n}_+$ and $\boldsymbol{Y}_2\in\mathbb{R}^{n_u\times n}_+$ to the linear program*

$$\inf_{\boldsymbol{p}\in\mathbb{R}^n_{+,s},\boldsymbol{Y}_1,\boldsymbol{Y}_2\in\mathbb{R}^{n_u\times n}_+,\gamma>0}\gamma\,, \tag{4.6a}$$

subject to :

$$\boldsymbol{A}\boldsymbol{P}+\boldsymbol{B}_2(\boldsymbol{Y}_1-\boldsymbol{Y}_2)\geq 0\,,\ \boldsymbol{C}\boldsymbol{P}+\boldsymbol{D}_{12}(\boldsymbol{Y}_1-\boldsymbol{Y}_2)\geq 0\,, \tag{4.6b}$$
$$\begin{bmatrix}(\boldsymbol{A}-\boldsymbol{I}_n)\boldsymbol{P}+\boldsymbol{B}_2(\boldsymbol{Y}_1-\boldsymbol{Y}_2) & \boldsymbol{B}_1\boldsymbol{\Omega}1_{n_\omega}\\ \boldsymbol{C}\boldsymbol{P}+\boldsymbol{D}_{12}(\boldsymbol{Y}_1-\boldsymbol{Y}_2) & \boldsymbol{D}_{11}\boldsymbol{\Omega}1_{n_\omega}-\gamma 1_{n_z}\end{bmatrix}\begin{bmatrix}1_n\\1\end{bmatrix}<0\,, \tag{4.6c}$$
$$\boldsymbol{P}\geq\mathrm{diag}\left(\left[\|\boldsymbol{M}(:,1)\|_\infty\ \dots\ \|\boldsymbol{M}(:,n)\|_\infty\right]^{T}\right), \tag{4.6d}$$
$$\begin{bmatrix}\boldsymbol{Y}_1 & \boldsymbol{Y}_2\end{bmatrix}\boldsymbol{S}\leq\boldsymbol{U}1_{n_u,2^n}\text{, where}\begin{cases}\boldsymbol{S}_{i,j}=1,\ \boldsymbol{S}_{i+n,j}=0 & \boldsymbol{v}_i^j>0\\ \boldsymbol{S}_{i,j}=0,\ \boldsymbol{S}_{i+n,j}=1 & \text{otherwise}\end{cases}, \tag{4.6e}$$

for all $i \in \mathbb{N}_{1:n}$, $j \in \mathbb{N}_{1:2^n}$, *where* $\boldsymbol{v}^j \in \mathbb{R}^n$, $j \in \mathbb{N}_{1:2^n}$ *is the* j*-th vertex of the unit hypercube in* $\mathbb{R}^n$*. The optimal feedback gain* $\boldsymbol{K}^*$ *is retrieved from the optimization variables* $\boldsymbol{P}$*,* $\boldsymbol{Y}_1$*, and* $\boldsymbol{Y}_2$ *using* $\boldsymbol{K}^* = (\boldsymbol{Y}_1 - \boldsymbol{Y}_2)\boldsymbol{P}^{-1}$*. In addition, the closed-loop system under the optimal feedback control* $\boldsymbol{u} \coloneqq \mu(\boldsymbol{x}) = \boldsymbol{K}^*\boldsymbol{x}$ *is globally ultimately bounded in the set* $\mathbb{X}$*.*

the proof can be found in Appendix 4.A.

Note that, as highlighted in Remark 1, with $\boldsymbol{\Omega} = \boldsymbol{I}_n$, $\mathbb{U} = \mathbb{R}^{n_u}$, and $\mathbb{X}_0 = \{0\}$, Theorem 4.2 yields a positive closed-loop system with minimum L_∞-gain. The problem of closed-loop L_∞-gain minimization with control constraints has been discussed in [17, 18], where the state of the system is restricted to the positive orthant. However, that approach does not automatically apply to the case when the state of the system belongs to a infinity-norm bounded set in $\mathbb{R}^n$.

The linear program (4.6) involves $n + 2nn_u + 1$ optimization variables, and $2n_z + 3n + 2^n n_u$ linear inequalities, which may get prohibitively complex for large-scale systems with an increasing number of variables. The number of inequality constraints can be significantly reduced by replacing the $2^n n_u$ inequalities in 4.6e by the conservative approximation $[\boldsymbol{Y}_1\, \boldsymbol{Y}_2]1_{2n} \le \boldsymbol{U}1_{n_u}$, which consists of n_u inequalities. In addition, the number of optimization variables can be reduced by employing $\boldsymbol{Y} \coloneqq \boldsymbol{Y}_1 - \boldsymbol{Y}_2$, $\boldsymbol{Y} \in \mathbb{R}^{n_u \times n}$ and replacing 4.6e by the conservative approximation $\sum_{i=1}^{n} \|\boldsymbol{U}^{-1}\boldsymbol{Y}_{:,i}\|_\infty \le 1$, which is equivalent to $\boldsymbol{Y}[\boldsymbol{v}^1 \;\dots\; \boldsymbol{v}^{2^n}] \le \boldsymbol{U}1_{n_u,2^n}$, where $\boldsymbol{v}^j \in \mathbb{R}^n$, is the j-th vertex of the unit n-dimensional hypercube.

4.3.4. Derivation of the Feedback Gain With Soft Control Constraints

Rather than specifying hard control constraints in the form of $\boldsymbol{u} \in \mathbb{U}$, as discussed in Section 4.3.3, one may be interested in keeping the control effort relatively small, which is applicable when the resulting feedback control only constitutes a part of the final control input to the system, and the control input constraint is handled by a higher-level controller. In addition, in this case, infeasibility of the optimization problem (4.6) due to an overly stringent specification of control constraint is avoided. The optimization problem

$$\inf_{\boldsymbol{p}\in\mathbb{R}^n_{+,s},\boldsymbol{Y}_1,\boldsymbol{Y}_2\in\mathbb{R}^{n_u\times n}_{+},\gamma>0,r\ge 0} \gamma + c_r r\,, \tag{4.7a}$$

subject to :

(4.6b), (4.6c), (4.6d),

$$\begin{bmatrix} \boldsymbol{Y}_1 & \boldsymbol{Y}_2 \\ -\boldsymbol{Y}_1 & -\boldsymbol{Y}_2 \end{bmatrix} \boldsymbol{S} \le r1_{2n_u,2^n}\,, \text{where} \begin{cases} \boldsymbol{S}_{i,j} = 1,\, \boldsymbol{S}_{i+n,j} = 0 & \boldsymbol{v}^j_i > 0 \\ \boldsymbol{S}_{i,j} = 0,\, \boldsymbol{S}_{i+n,j} = 1 & \text{otherwise} \end{cases}, \tag{4.7b}$$

$i \in \mathbb{N}_{1:n}$, $j \in \mathbb{N}_{1:2^n}$,

where $\boldsymbol{v}^j \in \mathbb{R}^n$, $j \in \mathbb{N}_{1:2^n}$ is the j-th vertex of the unit n-dimensional hypercube in $\mathbb{R}^n$, imposes no constraint on the control input and achieves an arbitrary trade-off, determined by the weighting factor $c_r > 0$, between $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$ and $\|\boldsymbol{u}\|_{L_\infty}$. The optimal feedback gain is then given by $\boldsymbol{K}^* = (\boldsymbol{Y}_1 - \boldsymbol{Y}_2)\boldsymbol{P}^{-1}$.

Since $\mathbb{X}$ is robustly positively invariant, it holds that $\sum_{i=1}^{n} \|\boldsymbol{Y}_{1:,i} - \boldsymbol{Y}_{2:,i}\|_\infty \geq \|\boldsymbol{Kx}\|_\infty = \|\boldsymbol{u}\|_{L_\infty}$ for any $\boldsymbol{x} \in \mathbb{X}$, hence, constituting an upper bound to $\|\boldsymbol{u}\|_{L_\infty}$. Using this upper bound with $\boldsymbol{Y} \coloneqq \boldsymbol{Y}_1 - \boldsymbol{Y}_2$, the $2^{n+1}n_u$ inequalities of 4.7b are decreased to $2nn_u$ inequalities in the following linear program, which minimizes a mixed objective function composed of a penalty term for $\|\boldsymbol{z}\|_{L_\infty,\mathbb{W},\mathbb{X}}$ and a penalty term for the upper bound of $\|\boldsymbol{u}\|_{L_\infty}$:

$$\inf_{\boldsymbol{p}\in\mathbb{R}^n_{+,s},\, \boldsymbol{Y}\in\mathbb{R}^{n_u\times n},\, \gamma>0,\, \boldsymbol{r}\in\mathbb{R}^n_+} \gamma + c_r 1_n{}^{\mathrm{T}}\boldsymbol{r}, \tag{4.8a}$$

subject to :

4.6b, 4.6c, 4.6d,

$$\begin{bmatrix} \boldsymbol{Y} \\ -\boldsymbol{Y} \end{bmatrix} \leq 1_{2n_u}\boldsymbol{r}^{\mathrm{T}}. \tag{4.8b}$$

The optimal feedback gain is then given by $\boldsymbol{K}^* = \boldsymbol{Y}\boldsymbol{P}^{-1}$.

4.4. Model Predictive Control

This section explains the concept of model predictive control (MPC), especially within the scope of its application to a BHS described by the dynamic model (4.3). The external input $\boldsymbol{\omega}(t) \coloneqq \overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)$ is assumed to be composed of $\overline{\boldsymbol{\omega}}(t)$ with known values for all t, and a nonnegative additive unmeasurable component $\tilde{\boldsymbol{\omega}}(t)$. Within the context of BHSs, $\overline{\boldsymbol{\omega}}(t)$ corresponds to the baseline baggage demand due to the known flight schedule and $\tilde{\boldsymbol{\omega}}(t)$ corresponds to unplanned excess of demand. The system is endowed with state and control constraints of the form

$$\boldsymbol{x}(t) \in \mathbb{X} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n | 0 \leq \boldsymbol{x} \leq \boldsymbol{x}_{\max}\},\ \forall\, t \in \mathbb{N}, \tag{4.9a}$$

$$\boldsymbol{u}(t) \in \mathbb{U} \coloneqq \{\boldsymbol{u} \in \mathbb{R}^{n_u} | 0 \leq \boldsymbol{u} \leq \boldsymbol{u}_{\max}\},\ \forall\, t \in \mathbb{N}, \tag{4.9b}$$

corresponding to operational constrains such as capacity limits and actuation saturation.

At every time step, based on the current state of the system and future values of the known component of the external input, a constrained finite-horizon optimization problem with a linear cost subject to (4.9) will be solved yielding a sequence of optimal control inputs. According to the receding horizon policy, only the first step of this sequence is applied to the system, and this process is repeated at the next time step [10, 11].

We develop two MPC strategies. One that achieves optimal performance when $\tilde{\boldsymbol{\omega}}(t) = 0$ and one that provides robustness in presence of $\tilde{\boldsymbol{\omega}}(t)$. In both cases, the underlying optimization problem is formulated as an LP program by employing a linear cost function together with the linear constraints (4.9). For both MPC schemes, while optimizing for the performance defined by the cost function, stability of closed-loop system is guaranteed by proper selection of weighting vectors of the cost function.

4.4.1. Baseline MPC

For the baseline MPC design, we consider the system (4.3) with $\boldsymbol{\omega}(t) := \overline{\boldsymbol{\omega}}(t)$, which is a known function of time t.

Baseline MPC Problem Formulation

Let N_h be the prediction horizon. Given the current time step t and the current state of the system $\boldsymbol{x} := \boldsymbol{x}(t)$, the finite-horizon optimization problem that needs to be solved is defined as

$$\mathcal{P}_{N_\mathrm{h}}(t,\boldsymbol{x}) : V^*_{N_\mathrm{h}}(t,\boldsymbol{x}) = \min_{\boldsymbol{u}_{N_\mathrm{h}}}\{V_{N_\mathrm{h}}\big(t,\boldsymbol{x},\boldsymbol{u}_{N_\mathrm{h}}(t)\big) \mid \boldsymbol{u}_{N_\mathrm{h}}(t) \in \mathcal{U}_{N_\mathrm{h}}(t,\boldsymbol{x})\}, \tag{4.10a}$$

where,

$$V_{N_\mathrm{h}}\big(t,\boldsymbol{x},\boldsymbol{u}_{N_\mathrm{h}}(t)\big) = \sum_{k=0}^{N_\mathrm{h}-1}\Big(\boldsymbol{q}_\mathrm{z}{}^\mathrm{T}(t+k)\boldsymbol{z}(t+k) + \boldsymbol{q}_\mathrm{u}{}^\mathrm{T}(t+k)\boldsymbol{u}(t+k)\Big) \tag{4.10b}$$
$$+ V_\mathrm{f}(t+N_\mathrm{h},\boldsymbol{x}(t+N_\mathrm{h})),$$
$$\mathcal{U}_{N_\mathrm{h}}(t,\boldsymbol{x}) = \big\{\boldsymbol{u}_{N_\mathrm{h}} \mid 0 \le \boldsymbol{x}(t+k) \le \boldsymbol{x}_\mathrm{max},\ 0 \le \boldsymbol{u}(t+k) \le \boldsymbol{u}_\mathrm{max}, \tag{4.10c}$$
$$\forall\ k \in \mathbb{N}_{0:N_\mathrm{h}-1},\ \boldsymbol{x}(t+N_\mathrm{h}) \in \mathbb{X}_\mathrm{f}\big\},$$
$$\boldsymbol{x}(t+k) = \boldsymbol{x}\big(t+k;\boldsymbol{x},\boldsymbol{u}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\big),\ k \in \mathbb{N}_{0:N_\mathrm{h}}, \tag{4.10d}$$
$$\boldsymbol{z}(t+k) = \boldsymbol{z}\big(t+k;\boldsymbol{x},\boldsymbol{u}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\big),\ k \in \mathbb{N}_{0:N_\mathrm{h}-1}, \tag{4.10e}$$

with $\boldsymbol{u}_{N_\mathrm{h}}(t) := \big(\boldsymbol{u}(t),\dots,\boldsymbol{u}(t+N_\mathrm{h}-1)\big)$ and $\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t) := (\overline{\boldsymbol{\omega}}(t),\dots,\overline{\boldsymbol{\omega}}(t+N_\mathrm{h}-1))$ respectively being the sequence of to-be-calculated control actions and the sequence of known disturbances over the prediction horizon. In above, the weighting functions for the stage cost $\boldsymbol{q}_\mathrm{z} : \mathbb{N} \to \mathbb{R}^{n_z}_{+,\mathrm{s}}$ and $\boldsymbol{q}_\mathrm{u} : \mathbb{N} \to \mathbb{R}^{n_u}_{+,\mathrm{s}}$, the terminal cost function $V_\mathrm{f} : (\mathbb{N},\mathbb{X}) \to \mathbb{R}_{+,\mathrm{s}}$, and the terminal constraint set $\mathbb{X}_\mathrm{f} \subseteq \mathbb{X} \subseteq \mathbb{R}^n_+$, are design parameters, the choice of which will be discussed in the sequel.

Assume $\boldsymbol{x} \in \mathcal{X}_{N_\mathrm{h}}(t)$, where $\mathcal{X}_{N_\mathrm{h}}(t)$ is the domain of $\mathcal{P}_{N_\mathrm{h}}(t,\cdot)$, which is the set of all states at time step t for which the optimization problem has a solution. The solution to the optimization problem $\mathcal{P}_{N_\mathrm{h}}(t,\boldsymbol{x})$ is then the optimal control sequence $\boldsymbol{u}^*_{N_\mathrm{h}}(t,\boldsymbol{x}) = \Big(\boldsymbol{u}^*\big(t;t,\boldsymbol{x}\big),\dots,\boldsymbol{u}^*\big(t+N_\mathrm{h}-1;t,\boldsymbol{x}\big)\Big)$. Finally, according to the receding horizon policy, the control action to be applied to the system at the current time step is given as the first element of the optimal control sequence; that is

$$\boldsymbol{u}_\mathrm{mpc}(t) := \boldsymbol{u}^*(t;t,\boldsymbol{x}). \tag{4.11}$$

Note that the even though the system described by (4.3) is time-invariant, the optimization problem and the resulting optimal control sequence are time-dependent due time-varying $\overline{\boldsymbol{\omega}}_{N_h}$ and the time-dependent weighting factors.

4.4.2. Tube-based MPC

Here we assume that the external input to the system (4.3) is in the form of $\boldsymbol{\omega}(t) = \overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)$, where, just as before, $\overline{\boldsymbol{\omega}}(t)$ is known and $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W} := \{\boldsymbol{\omega} \ge$

$0 \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty \leq 1,\ \boldsymbol{\Omega} = \mathrm{diag}(\boldsymbol{\omega}_M),\ \boldsymbol{\omega}_M \in \mathbb{R}^{n_\omega}_{+,s}\}$ for all $t \in \mathbb{N}$. The control scheme discussed in this section aims at an optimal performance of the closed-loop system while being robust against perturbations of the external input due to $\tilde{\boldsymbol{\omega}}$. Among others, *tube-based MPC* is an MPC-based approach that allows us to achieve this goal. This approach has the advantage that it can be derived with little modification to the baseline MPC problem 4.10 and it admits a simple implementation. The design of a tube-based MPC controller involves the following steps: i) characterize the *bounding tube* of uncertain system trajectories for all $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W}$, ii) design a stabilizing feedback controller to ensure boundedness of the generated tube, and iii) design an MPC controller, satisfying state and control constraints for all state trajectories in the tube, in order to generate the nominal trajectory. Tube-based MPC is covered in depth in [10, Chapter 3], of which we provide an overview.

4

Bounding Tube of System Trajectories

In tube-based MPC approach, a *bundle* of trajectories containing states of the uncertain system for all $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W}$ is used to represent the state trajectory of the uncertain system. The bundle of state trajectories is called a *tube* and is supported by the nominal trajectory, which is the trajectory of the system driven by a nominal control signal $\overline{\boldsymbol{u}}(\cdot)$ with $\tilde{\boldsymbol{\omega}}(\cdot) = 0$. The *size* of the tube represents deviation from the nominal trajectory for all values of $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W}$. Consider system (4.3) and let the uncertain and nominal systems be respectively expressed by

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}_2\boldsymbol{u}(t) + \boldsymbol{B}_1\overline{\boldsymbol{\omega}}(t) + \boldsymbol{B}_1\tilde{\boldsymbol{\omega}}(t), \tag{4.12}$$

$$\boldsymbol{z}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}_{12}\boldsymbol{u}(t) + \boldsymbol{D}_{11}\overline{\boldsymbol{\omega}}(t) + \boldsymbol{D}_{11}\tilde{\boldsymbol{\omega}}(t) \tag{4.13}$$

$$\overline{\boldsymbol{x}}(t+1) = \boldsymbol{A}\overline{\boldsymbol{x}}(t) + \boldsymbol{B}_2\overline{\boldsymbol{u}}(t) + \boldsymbol{B}_1\overline{\boldsymbol{\omega}}(t), \tag{4.14}$$

$$\overline{\boldsymbol{z}}(t) = \boldsymbol{C}\overline{\boldsymbol{z}}(t) + \boldsymbol{D}_{12}\overline{\boldsymbol{u}}(t) + \boldsymbol{D}_{11}\overline{\boldsymbol{\omega}}(t), \tag{4.15}$$

where $\overline{\boldsymbol{x}}(t)$ and $\overline{\boldsymbol{z}}(t)$ are, respectively, the state and output of the nominal system, and $\boldsymbol{x}(t)$ and $\boldsymbol{z}(t)$ are those of the disturbed (uncertain) system, and $\overline{\boldsymbol{u}}(t)$ is the nominal control signal generated by the MPC controller, and $\boldsymbol{u}(t)$ is the control input to the disturbed system, which is of the form

$$\boldsymbol{u}(t) = \overline{\boldsymbol{u}}(t) + \boldsymbol{K}(\boldsymbol{x}(t) - \overline{\boldsymbol{x}}(t)), \tag{4.16}$$

where the feedback gain $\boldsymbol{K}$ is such that $\boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{K}$ is stable, guaranteeing boundedness of the generated tube. Hence, with $\boldsymbol{x}_\mathrm{e}(t) \coloneqq \boldsymbol{x}(t) - \overline{\boldsymbol{x}}(t)$ and $\boldsymbol{z}_\mathrm{e}(t) \coloneqq \boldsymbol{z}(t) - \overline{\boldsymbol{z}}(t)$,

$$\boldsymbol{x}_\mathrm{e}(t+1) = (\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K})\boldsymbol{x}_\mathrm{e}(t) + \boldsymbol{B}_1\tilde{\boldsymbol{\omega}}(t), \tag{4.17a}$$

$$\boldsymbol{z}_\mathrm{e}(t+1) = (\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K})\boldsymbol{x}_\mathrm{e}(t) + \boldsymbol{D}_{11}\tilde{\boldsymbol{\omega}}(t), \tag{4.17b}$$

describe the error dynamics. Assuming $\boldsymbol{x}(0) = \overline{\boldsymbol{x}}(0)$, it follows from (4.17) that $\boldsymbol{x}_\mathrm{e}(t) \in \mathbb{S}(t)$ for all $t \in \mathbb{N}$, where

$$\mathbb{S}(t) \coloneqq \boldsymbol{B}_1\mathbb{W} \oplus \boldsymbol{A}_e\boldsymbol{B}_1\mathbb{W} \oplus \ldots \oplus\ \boldsymbol{A}_e^{t-1}\boldsymbol{B}_1\mathbb{W} = \sum_{i=0}^{t-1} \boldsymbol{A}_e^i\boldsymbol{B}_1\mathbb{W}, \tag{4.18}$$

with $\boldsymbol{A}_e \coloneqq \boldsymbol{A}+\boldsymbol{B}_2\boldsymbol{K}$. The set $\mathbb{S}(t)$ contains the state trajectories of the error system with $\boldsymbol{x}_\mathrm{e}(0)=0$ for all possible sequences of $(\tilde{\boldsymbol{\omega}}(0),\dots,\tilde{\boldsymbol{\omega}}(t-1))$ and for any $t \in \mathbb{N}$. Hence, for the disturbed system we have $\boldsymbol{x}_\mathrm{e}(t) \in \boldsymbol{X}(t) \coloneqq \{\overline{\boldsymbol{x}}(t)\} \oplus \mathbb{S}(t)$, $\forall t \in \mathbb{N}$.

For the MPC use case, the bundle of all trajectories over the prediction horizon $N_\mathrm{h} \in \mathbb{N}$ starting from the initial state $\boldsymbol{x}(t) = \overline{\boldsymbol{x}}(t) = \overline{\boldsymbol{x}}$ under the nominal control sequence $\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t) = \left(\overline{\boldsymbol{u}}(t),\ \dots,\ \overline{\boldsymbol{u}}(t+N_\mathrm{h}-1)\right)$ and the known external input sequence $\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t) = \left(\overline{\boldsymbol{\omega}}(t),\ \dots,\ \overline{\boldsymbol{\omega}}(t+N_\mathrm{h}-1)\right)$ for all possible values of disturbance sequence $\tilde{\boldsymbol{\omega}}_{N_\mathrm{h}}(t) = \left(\tilde{\boldsymbol{\omega}}(t),\ \dots,\ \tilde{\boldsymbol{\omega}}(t+N_\mathrm{h}-1)\right) \in \mathbb{W}^{N_\mathrm{h}}$, is the tube generated at $\boldsymbol{x}(t)$ by $\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t)$ and $\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)$, defined as

$$\begin{aligned}\boldsymbol{X}_{N_\mathrm{h}}\left(\overline{\boldsymbol{x}},\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\right) &\coloneqq \Big(\boldsymbol{X}\left(0;\overline{\boldsymbol{x}},\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\right),\dots, \\ \boldsymbol{X}\left(N_\mathrm{h};\overline{\boldsymbol{x}},\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\right)\Big) &= \Big(\{\overline{\boldsymbol{x}}\},\{\overline{\boldsymbol{x}}(t+1)\}\oplus\mathbb{S}(1),\dots,\{\overline{\boldsymbol{x}}(t+N_\mathrm{h})\}\oplus\mathbb{S}(N_\mathrm{h})\Big).\end{aligned} \tag{4.19}$$

The tube described by 4.19 is generated assuming $\boldsymbol{x}_\mathrm{e}(0)=0$ (i.e., $\boldsymbol{x}(t)=\overline{\boldsymbol{x}}(t)$). An outer bounding tube taking initial conditions into account can be obtained by using $\mathbb{S}_\infty \coloneqq \lim_{t\to\infty}\mathbb{S}(t)$ instead of $\mathbb{S}(t)$, $t \in \mathbb{N}_{0:N_\mathrm{h}}$, which is the minimal robust positive invariant set for the system (4.17) with $\boldsymbol{A}_e$ stable [10, 19]. However, $\mathbb{S}_\infty$ is often very difficult to calculate requiring one to use an approximation of it. In addition, using $\mathbb{S}_\infty$ one only takes into account initial conditions $\boldsymbol{x}_\mathrm{e}(0) \in \mathbb{S}_\infty$.

Our approach utilizes Proposition 4.1 and the associated state feedback derivation method of Section 4.3.3 for system 4.17 to compute a robustly positively invariant set $\mathbb{S}'$, which includes a pre-specified minimum set $\mathbb{S}_\mathrm{min}$, and a state feedback gain $\boldsymbol{K}$. The state feedback gain $\boldsymbol{K}$ calculated in this manner not only does stabilize the error system, but it also achieves minimal $\|\boldsymbol{z}_\mathrm{e}\|_{L_\infty,\mathbb{W},\mathcal{S}_\mathrm{min}}$ with constrained control effort. An outer tube containing trajectories of the uncertain system for all $\boldsymbol{x}_\mathrm{e}(0) \in \mathbb{S}'$ and all $\tilde{\boldsymbol{\omega}} \in \mathbb{W}$ is constructed as

$$\begin{aligned}\boldsymbol{X}'_{N_\mathrm{h}}\left(\overline{\boldsymbol{x}},\overline{\boldsymbol{u}}_{N_\mathrm{h}}(t),\overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\right) \coloneqq &\Big(\{\overline{\boldsymbol{x}}(t)\}\oplus\mathbb{S}',\{\overline{\boldsymbol{x}}(t+1)\}\oplus\mathbb{S}',\dots, \\ &\{\overline{\boldsymbol{x}}(t+N_\mathrm{h})\}\oplus\mathbb{S}'\Big).\end{aligned} \tag{4.20}$$

Tube-based MPC Problem Formulation

The tube-based MPC problem is then defined as a modified version of the original MPC problem 4.10, where the set $\mathbb{S}'$ is used to tighten the original MPC constraints 4.10c so that the original MPC constraints are satisfied by the tube $\boldsymbol{X}'_{N_\mathrm{h}}(\boldsymbol{x},\overline{\boldsymbol{u}}_{N_\mathrm{h}},\overline{\boldsymbol{\omega}}_{N_\mathrm{h}})$. Let $\mathbb{S}' \coloneqq \{\boldsymbol{e} \mid 0 \le \boldsymbol{e} \le \boldsymbol{p},\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$ and $\boldsymbol{K} \in \mathbb{R}_+^{n_u\times n}$, respectively, be the robustly positively invariant set and the associated state feedback gain obtained using the method of Section 4.3.3 for the error system 4.17 with $\tilde{\boldsymbol{\omega}} \in \mathbb{W}$. At the current state of the nominal system $\overline{\boldsymbol{x}} \coloneqq \overline{\boldsymbol{x}}(t)$, the tube-based MPC problem $\overline{\mathcal{P}}_{N_\mathrm{h}}(t,\overline{\boldsymbol{x}})$ is defined as $\mathcal{P}_{N_\mathrm{h}}(t,\boldsymbol{x})$ with $\boldsymbol{x} = \overline{\boldsymbol{x}}$ and with tightened constraints $0 \le \overline{\boldsymbol{x}}(t+k) \le \boldsymbol{x}_\mathrm{max} - \boldsymbol{p}$ and $0 \le \overline{\boldsymbol{u}}(t+k) \le \boldsymbol{u}_\mathrm{max} - \boldsymbol{K}\boldsymbol{p}$, $k \in \mathbb{N}_{0:N_\mathrm{h}-1}$, $\overline{\boldsymbol{x}}(t+N_\mathrm{h}) \in \overline{\mathbb{X}}_\mathrm{f}$ replacing the original constraints in 4.10, where $\overline{\mathbb{X}}_\mathrm{f}$ is a tightened version of the terminal constraint set $\mathbb{X}_\mathrm{f}$. The choice of $\mathbb{X}_\mathrm{f}$ and $\overline{\mathbb{X}}_\mathrm{f}$ will be discussed in detail in Section 4.4.3.

MPC Problem Formulation With Improved Tube Base

In the tube-based MPC approach, it is also possible to optimize for the base of the tube $\overline{\boldsymbol{x}} := \overline{\boldsymbol{x}}(t)$ to achieve improved performance with respect to the MPC cost function. Let $\overline{\boldsymbol{x}} := \overline{\boldsymbol{x}}(t) \in \mathcal{X}_{N_\mathrm{h}}(t)$ and $\boldsymbol{x} := \boldsymbol{x}(t) \in \{\overline{\boldsymbol{x}}(t)\} \oplus \mathbb{S}'$ respectively be the current state of the nominal system and that of the uncertain system, where $\mathcal{X}_{N_\mathrm{h}}(t)$ is the domain of $\overline{\mathcal{P}}_{N_\mathrm{h}}(t, \overline{\boldsymbol{x}})$. The MPC problem with improved tube base is then defined as

$$\mathcal{P}^*_{N_\mathrm{h}}(t, \boldsymbol{x}) : V^*_{N_\mathrm{h}}(t, \boldsymbol{x}) = \min_{\boldsymbol{v}, \overline{\boldsymbol{u}}_{N_\mathrm{h}}} \{ V_{N_\mathrm{h}}\big(t, \boldsymbol{x}, \boldsymbol{v}, \overline{\boldsymbol{u}}_{N_\mathrm{h}}(t)\big) \mid \boldsymbol{x} - \boldsymbol{p} \le \boldsymbol{v} \le \boldsymbol{x}, \quad \overline{\boldsymbol{u}}_{N_\mathrm{h}}(t) \in \mathcal{U}_{N_\mathrm{h}}(t, \boldsymbol{v}) \}, \tag{4.21a}$$

where,

$$V_{N_\mathrm{h}}\big(t, \boldsymbol{x}, \boldsymbol{v}, \overline{\boldsymbol{u}}_{N_\mathrm{h}}(t)\big) = \sum_{k=0}^{N_\mathrm{h}-1} \Big(\boldsymbol{q}_\mathrm{z}{}^\mathrm{T}(t+k)\overline{\boldsymbol{z}}(t+k) + \boldsymbol{q}_\mathrm{u}{}^\mathrm{T}(t+k)\overline{\boldsymbol{u}}(t+k) \Big) + V_\mathrm{f}\big(t + N_\mathrm{h}, \overline{\boldsymbol{x}}(t + N_\mathrm{h})\big), \tag{4.21b}$$

$$\mathcal{U}_{N_\mathrm{h}}(t, \boldsymbol{v}) = \big\{ \boldsymbol{u}_{N_\mathrm{h}} \mid 0 \le \overline{\boldsymbol{x}}(t+k) \le \boldsymbol{x}_\mathrm{max} - \boldsymbol{p},\ 0 \le \overline{\boldsymbol{u}}(t+k) \le \boldsymbol{u}_\mathrm{max} - \boldsymbol{K}\boldsymbol{p}, \quad \forall\ k \in \mathbb{N}_{0:N_\mathrm{h}-1},\ \overline{\boldsymbol{x}}(t + N_\mathrm{h}) \in \overline{\mathbb{X}}_\mathrm{f} \big\}, \tag{4.21c}$$

$$\overline{\boldsymbol{x}}(t+k) = \boldsymbol{x}\big(t+k; \boldsymbol{v}, \overline{\boldsymbol{u}}_{N_\mathrm{h}}(t), \overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\big),\ k \in \mathbb{N}_{0:N_\mathrm{h}}, \tag{4.21d}$$

$$\overline{\boldsymbol{z}}(t+k) = \boldsymbol{z}\big(t+k; \boldsymbol{v}, \overline{\boldsymbol{u}}_{N_\mathrm{h}}(t), \overline{\boldsymbol{\omega}}_{N_\mathrm{h}}(t)\big),\ k \in \mathbb{N}_{0:N_\mathrm{h}-1}. \tag{4.21e}$$

The solution to this new optimization problem, is the improved tube-base $\overline{\boldsymbol{x}}^*$ and the optimal control sequence $\overline{\boldsymbol{u}}^*_{N_\mathrm{h}}(t, \overline{\boldsymbol{x}}^*) = \Big(\overline{\boldsymbol{u}}^*(t; t, \overline{\boldsymbol{x}}^*), \ldots, \overline{\boldsymbol{u}}^*(t + N_\mathrm{h} - 1; t, \overline{\boldsymbol{x}}^*)\Big)$. The constraint $\boldsymbol{x} - \boldsymbol{p} \le \boldsymbol{v} \le \boldsymbol{x}$ ensures that the first element of the tube 4.20 contains the current state of the uncertain system (i.e., $\boldsymbol{x} \in \overline{\boldsymbol{x}}^* \oplus \mathbb{S}'$). Once again, the MPC control action at the current time step is the first element of the optimal control sequence; that is $\overline{\boldsymbol{u}}_\mathrm{mpc}(t) := \overline{\boldsymbol{u}}^*(t; t, \overline{\boldsymbol{x}}^*)$. The subsequent tube base $\overline{\boldsymbol{x}}^+$ is given by $\overline{\boldsymbol{x}}^+ = \boldsymbol{A}\overline{\boldsymbol{x}}^* + \boldsymbol{B}_1(\tilde{\boldsymbol{\omega}} + \overline{\boldsymbol{\omega}}) + \boldsymbol{B}_2 + \overline{\boldsymbol{u}}_\mathrm{mpc}$.

Integration of Controllers

Given sets $\mathbb{W} := \{\boldsymbol{\omega} \ge 0 \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty \le 1,\ \boldsymbol{\Omega} = \mathrm{diag}(\boldsymbol{\omega}_M),\ \boldsymbol{\omega}_M \in \mathbb{R}^{n_\omega}_{+,s}\}$ and $\mathbb{S}_\mathrm{min} \subset \mathbb{R}^n_+$, the set $\mathbb{S}' := \{\boldsymbol{e} \mid 0 \le \boldsymbol{e} \le \boldsymbol{p},\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$ with $\mathbb{S}_\mathrm{min} \subseteq \mathbb{S}'$, and the associated feedback gain $\boldsymbol{K}$ minimizing $\|\boldsymbol{z}_\mathrm{e}\|_{L_\infty, \mathbb{W}, \mathbb{S}'}$ are calculated offline. At every time step, the optimization problem 4.21 is then solved and the control input $\boldsymbol{u}(t) := \overline{\boldsymbol{u}}_\mathrm{mpc}(t) + \boldsymbol{K}\big(\boldsymbol{x}(t) - \overline{\boldsymbol{x}}^*(t)\big)$ is applied to the (uncertain) system.

4.4.3. Recursive Feasibility and Asymptotic Stability

The design parameters need to be chosen such that *asymptotic stability* of the closed-loop system under the control law (4.11) and *recursive feasibility* [10, Chapter 2] of the optimization problem (4.10) are guaranteed. Asymptotic stability of the nominal (i.e., without disturbance) closed-loop system is achieved by choosing a strictly decreasing terminal cost function $V_{\mathrm{f}}(\boldsymbol{x}) : \boldsymbol{x} \mapsto \boldsymbol{q}_{\mathrm{f}}(\boldsymbol{x})$ over a control invariant terminal constraint set. Recursive feasibility of the controlled system is a property requiring that $\boldsymbol{x}(t) \in \mathcal{X}_{N_{\mathrm{h}}}(t)$ implies $\boldsymbol{x}(t+1) \in \mathcal{X}_{N_{\mathrm{h}}}(t+1)$ for any $t \in \mathbb{N}$. It is well known that recursive feasibility in the presence of an unknown disturbance is achieved if the terminal constraint set $\mathbb{X}_{\mathrm{f}}$ is *robustly control invariant* [2, 10, 20–22]. Calculating a (robustly) control invariant $\mathbb{X}_{\mathrm{f}}$ in general is a non-trivial task. A classical approach, introduced in [3], gives a *maximal robustly controlled invariant set* as the fixed-point solution to a recursion over sets. However, only under certain conditions that depend on the system matrices $(\boldsymbol{A}, \boldsymbol{B})$ and the geometry of admissible state, and control, and disturbance sets [5, 6], the maximal robustly control invariant set can be determined in a finite number of iterations (i.e., in general the fixed-point might not exist). This approach is further adapted in [4] to ensure that the resulting set at every iteration is robust control invariant; hence an arbitrarily precise inner approximation of the maximal robust control set can be calculated. Nonetheless, practical application of such approaches is mostly limited to small-scale system with few states. For example, with polyhedral state and control constraints, each iteration involves operations such as Minkowski differences and polyhedral projections, which become quickly untractable for polyhedra of large dimensions.

4

We propose a linear-programing-based method for simultaneous calculation of the set $\mathbb{X}_{\mathrm{f}}$ and the terminal cost function $V_{\mathrm{f}}(\cdot)$ over $\mathbb{X}_{\mathrm{f}}$. The set $\mathbb{X}_{\mathrm{f}}$ obtained in this manner has the property that it is control invariant when $\tilde{\boldsymbol{\omega}} = 0$ and is robustly control invariant otherwise. In addition $\boldsymbol{V}_{\mathrm{f}}(\cdot)$ is strictly decreasing over $\mathbb{X}_{\mathrm{f}}$ when $\tilde{\boldsymbol{\omega}} = 0$.

Proposition 4.3. Consider system 4.3 with the external input $\boldsymbol{\omega}(t) \coloneqq \overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)$, where $\overline{\boldsymbol{\omega}}(t)$ is known and $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W} \coloneqq \{\boldsymbol{\omega} \in \mathbb{R}^{n_\omega} \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty \leq 1,\ \boldsymbol{\Omega} \coloneqq \mathrm{diag}(\boldsymbol{\omega}_M),$ $\boldsymbol{\omega}_M \in \mathbb{R}^{n_\omega}_{+,s}\}$ for all $t \in \mathbb{N}$. Using the terminal constraint set $\mathbb{X}_{\mathrm{f}} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1,\ \boldsymbol{P} \coloneqq \mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$, the MPC problem 4.21 is recursively feasible if there exists a "feed forward" control input $\overline{\boldsymbol{u}}(t)$, a feedback gain $\boldsymbol{K}(t)$, and a scalar $0 \leq \lambda < 1$ satisfying

$$\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K}(t) \geq 0\,, \tag{4.22a}$$

$$\big(\boldsymbol{A} + \boldsymbol{B}_2\boldsymbol{K}(t) - \lambda \boldsymbol{I}_n\big)\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{\omega}_M < 0\,, \tag{4.22b}$$

$$0 \leq \boldsymbol{K}(t)\boldsymbol{p} \leq \boldsymbol{u}_{\max} - \overline{\boldsymbol{u}}(t)\,, \tag{4.22c}$$

$$\begin{bmatrix}\boldsymbol{B}_2^{\mathrm{T}} & \boldsymbol{D}_{12}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{u}}(t) = -\begin{bmatrix}\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{D}_{11}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{\omega}}(t)\,, \tag{4.22d}$$

for all $t \in \mathbb{N}$.

See Appendix 4.A for the proof.

Proposition 4.4. Suppose conditions of Proposition 4.3 are met for the set $\mathbb{X}_{\mathrm{f}} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1,\ \boldsymbol{P} \coloneqq \mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$ corresponding to the control law $\boldsymbol{u}(t, \boldsymbol{x}) \coloneqq \overline{\boldsymbol{u}}(t) + \boldsymbol{K}(t)\boldsymbol{x}(t)$, and let $r(\cdot) \coloneqq \mathbb{N} \to \mathbb{R}_{+,s}$ be a non-increasing positive bounded function such that $\lim_{t\to\infty} r(t) > 0$. Then, with the terminal constraint set $\mathbb{X}_{\mathrm{f}}$ and the terminal cost function $V_{\mathrm{f}}(t, \boldsymbol{x})\colon (t, \boldsymbol{x}) \mapsto r(t)\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$, the origin is asymptotically stable for the nominal closed-loop system (i.e., with $\tilde{\boldsymbol{\omega}}(t) = 0$ for all

$t \in \mathbb{N}$) under the MPC control law 4.11 if, in addition to 4.22, it holds that

$$\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K}(t) \geq 0\,, \tag{4.23a}$$

$$(\lambda - 1)r(t) \leq -\big(\boldsymbol{q}_{\mathrm{z}}^{\mathrm{T}}(t)(\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K}(t)) + \boldsymbol{q}_{\mathrm{u}}^{\mathrm{T}}(t)\boldsymbol{K}(t)\big)\boldsymbol{p}\,, \tag{4.23b}$$

for all $t \in \mathbb{N}$.

The proof is provided in Appendix 4.A. Proposition 4.3 and Proposition 4.4 do not automatically result in a finite number of inequalities as they require 4.22 and 4.23 to hold for all $t \in \mathbb{N}$. However, with $\overline{\boldsymbol{\omega}}(\cdot)$ being a periodic function of time such that $\overline{\boldsymbol{\omega}}(t+T) = \overline{\boldsymbol{\omega}}(t)$ for some $T \in \mathbb{N}$, the following propositions, as restatements of Propositions 4.3 and 4.4 for the periodic case, result in a finite number of inequalities.

4

Proposition 4.5. Suppose $\boldsymbol{q}_{\mathrm{z}} \colon \mathbb{N} \to \mathbb{R}^{n_z}_{+,s}$, $\boldsymbol{q}_{\mathrm{u}} \colon \mathbb{N} \to \mathbb{R}^{n_u}_{+,s}$, and $\overline{\boldsymbol{\omega}} \colon \mathbb{N} \to \mathbb{R}^{n_\omega}_{+,s}$ are given T-periodic functions of time with $T \in \mathbb{N}$, and $0 \leq \lambda < 1$ is a given scalar. Consider the nominal MPC problem 4.10 for the system 4.3 with terminal constraint set $\mathbb{X}_{\mathrm{f}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1\}$ and the terminal cost function $V_{\mathrm{f}}(\boldsymbol{x}) \colon (t, \boldsymbol{x}) \mapsto r\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$. Under the MPC control law 4.11, the origin is asymptotically stable for the closed-loop system with $\boldsymbol{\omega}(t) := \overline{\boldsymbol{\omega}}(t)$ and the optimization problem 4.10 is recursively feasible with $\boldsymbol{\omega}(t) := \overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)$ if the following set of linear (in)equalities in variables $\boldsymbol{p}$, $\boldsymbol{Y}_0$, …, $\boldsymbol{Y}_{T-1}$, r, $\overline{\boldsymbol{u}}(0)$, …, $\overline{\boldsymbol{u}}(T-1)$ is feasible for all $i \in \mathbb{N}_{0:T-1}$:

$$\begin{aligned}
&\boldsymbol{p} > 0\,, \boldsymbol{P} = \mathrm{diag}(\boldsymbol{p})\,,\\
&\boldsymbol{A}\boldsymbol{P} + \boldsymbol{B}_2\boldsymbol{Y}_i \geq 0\,,\\
&\big((\boldsymbol{A} - \lambda\boldsymbol{I}_n)\boldsymbol{P} + \boldsymbol{B}_2\boldsymbol{Y}_i\big)1_n + \boldsymbol{B}_1\boldsymbol{\omega}_M < 0\,,\\
&0 \leq \boldsymbol{Y}_i 1_n \leq \boldsymbol{u}_{\max} - \overline{\boldsymbol{u}}(i)\,,\\
&\begin{bmatrix}\boldsymbol{B}_2^{\mathrm{T}} & \boldsymbol{D}_{12}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{u}}(i) = -\begin{bmatrix}\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{D}_{11}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{\omega}}(i)\,,\\
&\boldsymbol{C}\boldsymbol{P} + \boldsymbol{D}_{12}\boldsymbol{Y}_i \geq 0\,,\\
&r > 0\,,\\
&(\lambda - 1)r \leq -\boldsymbol{q}_{\mathrm{z}}^{\mathrm{T}}(i)\boldsymbol{C}\boldsymbol{P}1_n - \big(\boldsymbol{q}_{\mathrm{z}}^{\mathrm{T}}(i)\boldsymbol{D}_{12} + \boldsymbol{q}_{\mathrm{u}}^{\mathrm{T}}(i)\big)\boldsymbol{Y}_i 1_n\,.
\end{aligned}$$

Proposition 4.6. Suppose $\overline{\boldsymbol{\omega}} \colon \mathbb{N} \to \mathbb{R}^{n_\omega}_{+,s}$ is a given T-periodic function of time with $T \in \mathbb{N}$, and $\boldsymbol{q}_{\mathrm{z}} \colon \mathbb{N} \to \mathbb{R}^{n_z}_{+,s}$ and $\boldsymbol{q}_{\mathrm{u}} \colon \mathbb{N} \to \mathbb{R}^{n_u}_{+,s}$ are given bounded functions of time, and $0 \leq \lambda < 1$ is a given scalar. Consider the nominal MPC problem 4.10 for the system 4.3 with terminal constraint set $\mathbb{X}_{\mathrm{f}} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1\}$ and the terminal cost function $V_{\mathrm{f}}(\boldsymbol{x}) \colon (t, \boldsymbol{x}) \mapsto r\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$. Under the MPC control law 4.11, the origin is asymptotically stable for the closed-loop with $\boldsymbol{\omega}(t) := \overline{\boldsymbol{\omega}}(t)$ and the optimization problem 4.10 is recursively feasible with $\boldsymbol{\omega}(t) := \overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)$ if the following set of linear (in)equalities in variables $\boldsymbol{p}$, $\boldsymbol{Y}$, r, $\overline{\boldsymbol{u}}(0)$, …, $\overline{\boldsymbol{u}}(T-1)$ is feasible for all $i \in \mathbb{N}_{0:T-1}$:

$$\boldsymbol{p} > 0\,, \boldsymbol{P} = \mathrm{diag}(\boldsymbol{p})\,, \tag{4.24a}$$
$$\boldsymbol{A}\boldsymbol{P} + \boldsymbol{B}_2\boldsymbol{Y} \geq 0\,, \tag{4.24b}$$
$$\big((\boldsymbol{A} - \lambda\boldsymbol{I}_n)\boldsymbol{P} + \boldsymbol{B}_2\boldsymbol{Y}\big)1_n + \boldsymbol{B}_1\boldsymbol{\omega}_M < 0\,, \tag{4.24c}$$
$$0 \leq \boldsymbol{Y}1_n \leq \boldsymbol{u}_{\max} - \overline{\boldsymbol{u}}(i)\,, \tag{4.24d}$$
$$\begin{bmatrix}\boldsymbol{B}_2^{\mathrm{T}} & \boldsymbol{D}_{12}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{u}}(i) = -\begin{bmatrix}\boldsymbol{B}_1^{\mathrm{T}} & \boldsymbol{D}_{11}^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}\overline{\boldsymbol{\omega}}(i)\,, \tag{4.24e}$$
$$\boldsymbol{C}\boldsymbol{P} + \boldsymbol{D}_{12}\boldsymbol{Y} \geq 0\,, \tag{4.24f}$$
$$r > 0\,, \tag{4.24g}$$
$$(\lambda - 1)r \leq -\|\boldsymbol{q}_{\mathrm{z}}\|_{L_\infty}{1_{n_z}}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{P}1_n - \big(\|\boldsymbol{q}_{\mathrm{z}}\|_{L_\infty}{1_{n_z}}^{\mathrm{T}}\boldsymbol{D}_{12} + \|\boldsymbol{q}_{\mathrm{u}}\|_{L_\infty}{1_{n_u}}^{\mathrm{T}}\big)\boldsymbol{Y}1_n\,. \tag{4.24h}$$

4.5. Model Description of BHS

4.5.1. Evolution of Link Density

In this section, we present a modeling framework for BHSs that will be used in combination with the methods developed in Sections 4.3 and 4.4 to illustrate the application of those methods in designing a BHS control system. The DCV-based BHS network is symbolically modeled as a directed graph. This symbolic relation implies that not all components of DCV-based baggage handling system are present in its graph representation. Nonetheless, the most important components of the system for the control point of view, including the loading stations, which are entry points of pieces of baggage to the BHS network), the unloading stations, which are departure points of pieces of baggage from the network, the early baggage storage (EBS), which is an automated storage to temporarily store the early baggage, the central DCV storage (CDS), which is the parking location of empty DCVs, and the network junctions, which connect various parts of the network, are present in this model. Within this modeling framework, loading and unloading stations, network junctions, the CDS and the EBS are represented by links of the graph. The nodes of the graph simply re-distribute the inbound DCV flows from their upstream links over their downstream links.

A loading station $i \in \{1, \dots, N_O\}$ is represented by a pair of two distinct links (o_i, s_i), where o_i is the origin link that transports pieces of baggage to the loading station, and where s_i is the source link that transports empty DCVs to the loading station from the CDS. It is assumed that the pieces of baggage are loaded onto empty DCVs, hence converting empty DCVs to loaded DCVs, at the downstream node of (o_i, s_i). The set of all origin links and all source links are respectively denoted by $O = \{o_1, \dots, o_{N_O}\}$ and $S = \{s_1, \dots, s_{N_O}\}$.

The CDS of the system is represented by a unique external link CDS, where empty DCVs, having left the unloading stations, are stored and from where they are dispatched to the source links.

An unloading station $i \in \{1, \dots, N_D\}$ is represented by a destination link d_i. Loaded DCVs are unloaded and converted to unloaded DCVs at the downstream node of d_i. The resulting empty DCVs are then transported to the CDS. The set of all destination links is denoted by $D = \{d_1, \dots, d_{N_D}\}$.

A network junction $i \in \{1, \dots, N_{\text{net}}\}$ is represented by a network link n_i, which (partially) connects an origin link to a destination link. The set of all network links is $N = \{n_1, \dots, n_{\text{net}}\}$. The early baggage storage (EBS) of the system is represented by the graph link EBS $\in N$, where loaded DCVs can be temporarily stored.

Let $L = O \cup S \cup N \cup D \cup \{\text{CDS}\}$ denote the set of all links of the graph. For a link $d \in D$, let $P_d \subseteq L$ be the set of links that are on some directed path that include link d. In addition, for a link $x \in L$, L_x^- and L_x^+ respectively denote the set of incoming and outgoing links of x. For any $x \in O \cup N$, $d \in D$ and $y \in N \cup D$, $q_{x\to y,d}$ [DCV/s] is the link flow from link x to link y with destination d given by

$$q_{x\to y,d}(t) = \begin{cases} f^{\text{l}}_{x\to y,d}(t), & \text{if } y \in P_d \cap L_x^+ \\ 0, & \text{otherwise}. \end{cases} \tag{4.25}$$

where $f^{\text{l}}_{x\to y,d}$ [DCV/s] is the flow of loaded DCVs from link x to link y with destination d. Note that this definition implies that only loaded DCVs can flow from $x \in O \cup N$ to $y \in N \cup D$ and that the flow of loaded DCVs with destination index d is zero if y is not a downstream link to x or if d cannot be reached via y. For any d_1 and $d_2 \in D$, $q^{\text{out}}_{d_1,d_2}$ [DCV/s] is the rate at which the DCVs destined for d_2 leave destination link d_1, which is defined as

$$q^{\text{out}}_{d_1,d_2}(t) = \begin{cases} f^{\text{l}}_{d_1\to \text{CDS}}(t), & \text{if } d_1 = d_2 \\ 0, & \text{otherwise}. \end{cases} \tag{4.26}$$

where $f^{\text{l}}_{d_1\to \text{CDS}}(t)$ [DCV/s] is the rate at which loaded DCVs in destination link d_1 are unloaded and dispatched to the central DCV storage CDS. This definition implies that only loaded DCVs destined for link d_1 are unloaded in d_1 and then shipped to the central DCV storage. For any $o \in O$ and $d \in D$, the rate ([Bag/s]) at which pieces of baggage destined for d enter origin link o is defined as

$$q^{\text{in}}_{o,d}(t) = \begin{cases} f^{\text{l}}_{o,d}(t), & o \in P_d \\ 0, & \text{otherwise}. \end{cases} \tag{4.27}$$

where $f^{\text{l}}_{o,d}$ [Bag/s] is the destination-indexed baggage demand at the origin link o. For any $s \in S$, the flow [DCV/s] of empty DCVs to the source link s is given as

$$q^{\text{in}}_s(t) = f^{\text{u}}_{\text{CDS}\to s}(t), \tag{4.28}$$

where $f^{\text{u}}_{\text{CDS}\to s}(t)$ is the link-to-link flow [DCV/s] of empty (unloaded) DCVs from the central DCV storage to the source link s. In this modeling framework, as depicted in Fig 4.51, pieces of baggage flow into the origin links, where they are picked up by empty DCVs on the source links. Loaded DCVs then make their way through the network links toward the destination links, possibly after being stored in the EBS. Loaded DCVs are finally unloaded at the destination links, where they are dispatched to the central DCV storage.

Accumulation of loaded DCVs and pieces of baggage on links of the graph is modeled by a destination-index link density [DCV/m], whereas the link density of

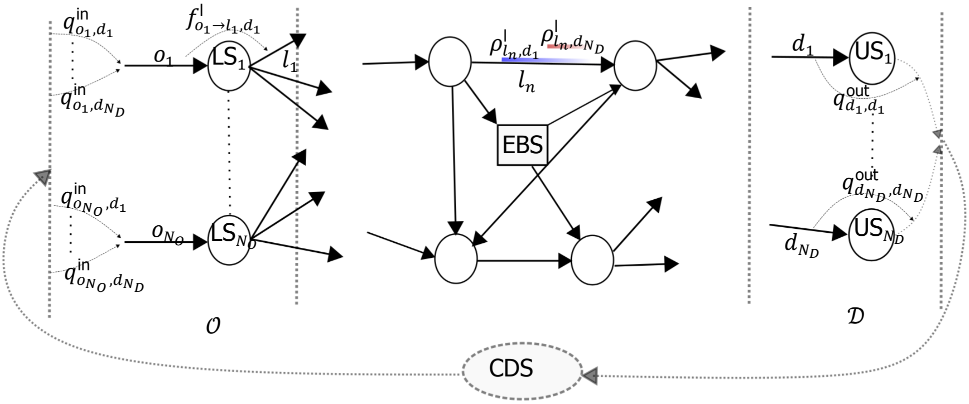


Figure 4.51: Graph representation of BHS with loading stations stations LS_i and their unique upstream links o_i, $i = 1, \ldots, N_O$, and with unloading stations stations US_i and their unique upstream links d_i, $i = 1, \ldots, N_D$, and with EBS.

empty DCVs is non-destination-indexed. We assume that empty DCVs enter the source links from the central DCV storage and leave the graph network at the loading stations, where they are entirely *converted* to loaded DCVs by picking up the luggage accumulated on the origin links. Even though origin links technically carry luggage and loaded DCVs flow in the network links, we do not distinguish between luggage densities on origin links and loaded DCVs densities in the rest of the network, using the same notation for both. It is also assumed that loaded DCVs leave the network at destination links, *producing* empty DCVs. Hence, empty DCVs only exist on the central DCV storage and on the source links and their link densities need not to be modeled in the rest of the network.

For any $o \in O$ and $d \in D$, the number of loaded DCVs on the origin link $o \in O$ with destination $d \in D$ and the corresponding link density are respectively given by

$$x^{\mathrm{l}}_{o,d}(t+1) = x^{\mathrm{l}}_{o,d}(t) + \Delta t\Big(q^{\mathrm{in}}_{o,d}(t) - F^{\mathrm{out}}_{o,d}(t)\Big), \tag{4.29}$$

$$\rho^{\mathrm{l}}_{o,d}(t) = \frac{1}{l_o} x^{\mathrm{l}}_{o,d}(t) \tag{4.30}$$

where Δt is discretization time step and $F^{\mathrm{out}}_{o,d}$ is the total outflow the link given by

$$F^{\mathrm{out}}_{o,d}(t) = \sum_{r \in L^+_o} q_{o \to r,d}(t) \tag{4.31}$$

The number of empty (unloaded) DCVs on any source link $s \in S$ and the associated link density of empty DCVs are respectively given by

$$x^{\mathrm{u}}_s(t+1) = x^{\mathrm{u}}_s(t) + \Delta t\Big(q^{\mathrm{in}}_s(t - t_s) - F^{\mathrm{out}}_s(t)\Big), \tag{4.32}$$

$$\rho^{\mathrm{u}}_s(t) = \frac{1}{l_s} x^{\mathrm{u}}_s(t), \tag{4.33}$$

where the total outflow of the link is given by

$$F_s^{\text{out}}(t) = \sum_{d \in D} F_{o,d}^{\text{out}}(t), \tag{4.34}$$

with o being the corresponding origin link to the source link s, (i.e., the pair (o,s) belong to the same loading station).

For any $n \in N$ and $d \in D$, the number of loaded DCVs on link n with destination d and the corresponding link densities are respectively described by the following equation:

$$x_{n,d}^{\text{l}}(t+1) = x_{n,d}^{\text{l}}(t) + \Delta t\Big(\sum_{r \in L_n^-} q_{r \to n,d}(t - t_n) - \sum_{r \in L_n^+} q_{n \to r,d}(t)\Big), \tag{4.35}$$

$$\rho_{n,d}^{\text{l}}(t) = \frac{1}{l_n} x_{n,d}^{\text{l}}(t), \tag{4.36}$$

For any destination links $d_1 \in D$ and $d_2 \in D$, the number of loaded DCVs at d_1 with destination d_2 and the corresponding density of loaded DCVs are given as:

$$x_{d_1,d_2}^{\text{l}}(t+1) = x_{d_1,d_2}^{\text{l}}(t) + \Delta t\Big(\sum_{r \in L_{d_1}^-} q_{r \to d_1,d_2}(t - t_{d_1}) - q_{d_1,d_2}^{\text{out}}(t)\Big), \tag{4.37}$$

$$\rho_{d_1,d_2}^{\text{l}}(t) = \frac{1}{l_{d_1}} x_{d_1,d_2}^{\text{l}}(t), \tag{4.38}$$

In a similar manner, for the EBS and for any $d \in D$, the number of loaded DCVs with destination d stored in the storage and the corresponding density of loeade DCVs are respectively given as

$$x_{\text{EBS},d}^{\text{l}}(t+1) = x_{\text{EBS},d}^{\text{l}}(t) + \Delta t\Big(\sum_{r \in L_{\text{EBS}}^-} q_{r \to \text{EBS},d}(t - t_{\text{EBS}}) - \sum_{r \in L_{\text{EBS}}^+} q_{\text{EBS} \to r,d}(t)\Big), \tag{4.39}$$

$$\rho_{\text{EBS},d}^{\text{l}}(t) = \frac{1}{l_{\text{ebs}}} x_{\text{EBS},d}^{\text{l}}(t) \tag{4.40}$$

The density of empty DCVs in the central DCV storage is described by the following equations:

$$x_{\text{CDS}}^{\text{u}}(t+1) = x_{\text{CDS}}^{\text{u}}(t) + \Delta t\Big(\sum_{d \in D} q_{d,d}^{\text{out}}(t - t_{\text{CDS}}) - q_s^{\text{in}}(t)\Big), \tag{4.41}$$

$$\rho_{\text{CDS}}^{\text{u}}(t) = \frac{1}{l_{\text{CDS}}} x_{\text{CDS}}^{\text{u}}(t), \tag{4.42}$$

Note that (4.31) and (4.34), and (4.41) and (4.37) jointly guarantee that empty DCVs accumulate only in the central DCV storage and on the source links, and that loaded DCVs only accumulate on the rest of the network.

4.5.2. Inputs, Outputs and Constraints

The manipulated variables of the model are the DCV flows $q^{\mathrm{l}}_{r\to y,d}$ with $r \in O \cup N$, $d \in D$, $o \in O \cap P_d$, $y \in N \cup D$, and $q^{\mathrm{l}}_{d_1,d_2}$ with d_1, $d_2 \in D$, and $q^{\mathrm{in}}_{\mathrm{s}}$ with $s \in S$. Since DCV flows are intrinsically non-negative, we limit the flows to non-negative values in the model. Moreover, we constrain the total outflow capacity of a link as follows:

$$0 \leq \sum_{d} \sum_{y} q_{r\to y,d}(t) \leq q_r^{\max}, \qquad r \in O \cup N, d \in D, y \in N \cup D \tag{4.43}$$

$$0 \leq \sum_{d_2} q_{d_1,d_2}(t) \leq q_{d_1}^{\max}, \qquad d_1 \in D, d_2 \in D \tag{4.44}$$

$$0 \leq \sum_{s} q_s^{\mathrm{in}}(t) \leq q_{\mathrm{CDS}}^{\max}, \qquad s \in S. \tag{4.45}$$

Model outputs are the link densities, which are intrinsically non-negative and bounded. The non-negativity constraint along with capacity constraints are included in the model as below:

$$0 \leq \sum_{d\in D} \rho^{\mathrm{l}}_{o,d}(t) \leq \rho_o^{\mathrm{l},\max}, \qquad o \in O, d \in D, \tag{4.46}$$

$$0 \leq \sum_{d\in D} \rho^{\mathrm{l}}_{n,d}(t) \leq \rho_n^{\mathrm{l},\max}, \qquad n \in N, d \in D \tag{4.47}$$

$$0 \leq \rho^{\mathrm{u}}_{s}(t) \leq \rho_s^{\mathrm{u},\max}, \qquad s \in S \tag{4.48}$$

$$0 \leq \sum_{d_2\in D} \rho^{\mathrm{l}}_{d_1,d_2}(t) \leq \rho_{d_1}^{\mathrm{l},\max}, \qquad d_1 \in D, d_2 \in D \tag{4.49}$$

$$0 \leq \rho^{\mathrm{u}}_{\mathrm{CDS}}(t) \leq \rho_{\mathrm{CDS}}^{\mathrm{u},\max}, \tag{4.50}$$

$$0 \leq \sum_{d\in D} \rho^{\mathrm{l}}_{\mathrm{EBS},d}(t) \leq \rho_{\mathrm{EBS}}^{\mathrm{l},\max}. \tag{4.51}$$

The variable $0 \leq q^{\mathrm{in}}_{o,d}(t) \leq {q^{\mathrm{in}}_{o,d}}^{\max}$, $(o,d) \in (O \times D)$ is the disturbance input, about which *partial* information is availabe in terms of its nominal and maximal values.

4.6. Case Study

The methods developed in this chapter are now deployed on a BHS network with two loading stations and two unloading stations, which is presented in Figure 4.61. A model in the form of 4.3 with $n = 47$ states describing DCV queue lengths, $n_{\mathrm{u}} = 106$ control inputs representing link to link flows, $n_{\mathrm{d}} = 848$ states representing delayed samples of control inputs[1], $n_{\mathrm{w}} = 4$ external inputs representing baggage demand, and $n_{\mathrm{z}} = n$ monitored outputs representing link densities, is developed as

[1]With link length of 64 [m] and DCV travel speed of 8 [m/s], a travel time of 8 samples is obtained for a sample time of $\Delta t = 1$ [s].

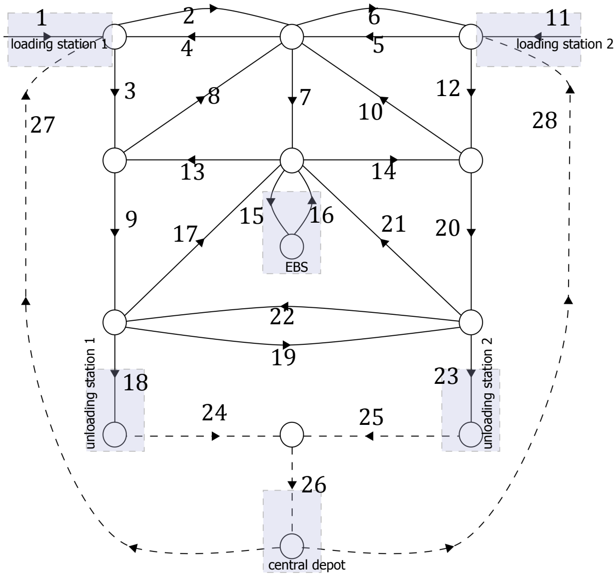


Figure 4.61: Graph representation of BHS with loading stations (links 1 and 11), source links 27 and 28, unloading stations (links 18 and 23), EBS link 15, and the CDS (link 26). Solid lines are associated with bags/loaded DCVs and dashed lines correspond to empty DCVs. Each link is assumed to have a length of 64 meters with fixed DCV travel speed of 8 [m/s].

$$\begin{bmatrix} \boldsymbol{x}(t+1) \\ \hline \boldsymbol{u}(t) \\ \boldsymbol{u}(t-1) \\ \vdots \\ \boldsymbol{u}(t-d+2) \\ \boldsymbol{u}(t-d+1) \end{bmatrix} = \left[\begin{array}{c|ccccc} \boldsymbol{I}_n & & & \Delta t\boldsymbol{A}_{12} & & \\ 0 & 0 & 0 & \dots & & 0 \\ 0 & \boldsymbol{I} & 0 & \ddots & & \vdots \\ \vdots & 0 & \ddots & \ddots & & 0 \\ 0 & \vdots & \ddots & \boldsymbol{I} & & 0 \\ 0 & 0 & \dots & 0 & & 0 \end{array}\right] \begin{bmatrix} \boldsymbol{x}(t) \\ \hline \boldsymbol{u}(t-1) \\ \boldsymbol{u}(t-2) \\ \vdots \\ \boldsymbol{u}(t-d+1) \\ \boldsymbol{u}(t-d) \end{bmatrix} + \begin{bmatrix} \Delta t\boldsymbol{B}_u \\ \hline \boldsymbol{I} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \boldsymbol{u}(t) \tag{4.52a}$$

$$+ \begin{bmatrix} \Delta t\boldsymbol{B}_\omega \\ \hline 0 \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \big(\overline{\boldsymbol{\omega}}(t) + \tilde{\boldsymbol{\omega}}(t)\big)$$

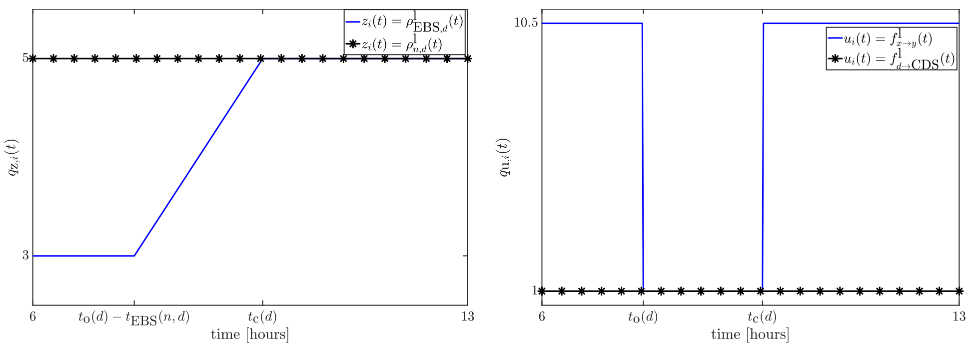
$$\boldsymbol{z}(t) = \left[\boldsymbol{C}_1 \mid 0\right] \begin{bmatrix} \boldsymbol{x}(t) \\ \hline \boldsymbol{u}(t-1) \\ \vdots \\ \boldsymbol{u}(t-d) \end{bmatrix}, \tag{4.52b}$$

where $\Delta t = 1$ [s] is the model sample time and the system matrices $\boldsymbol{A}_{12} \geq 0$, $\boldsymbol{B}_u$, $\boldsymbol{B}_\omega \geq 0$, $\boldsymbol{C}_1 \geq 0$ are appropriately defined based on the modeling framework of Section 4.5.

4.6.1. Configuring the MPC Controller

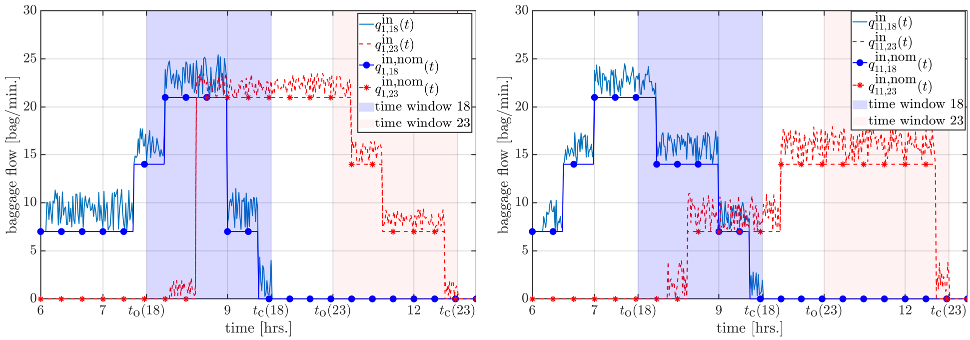
We impose a maximum link density of $\rho^{\mathrm{l}}_{\max} = 0.8$ [DCV/m] on all links, and maximum link-to-link flow of 100 [DCV/min.], which corresponds to $\boldsymbol{x}^{\mathrm{T}}_{\max} = [48\, \mathbf{1}_n{}^{\mathrm{T}}\; 100\, \mathbf{1}_{n_{\mathrm{d}}}{}^{\mathrm{T}}]$ [DCV] and $\boldsymbol{u}_{\max} = 100\, \mathbf{1}_{n_{\mathrm{u}}}$ [DCV/min.] in 4.10c. The MPC controller has a prediction horizon of $N_{\mathrm{h}} = 25$ steps with control sample time $\Delta t = 1$ [s]. The controller employs $J = \boldsymbol{q}_{\mathrm{z}}{}^{\mathrm{T}}(t)\boldsymbol{z}(t) + 12\boldsymbol{q}_{\mathrm{u}}{}^{\mathrm{T}}(t)\boldsymbol{u}(t)$ as the stage cost in the MPC objective function 4.10b and 4.21b, where the weighting function $\boldsymbol{q}_{\mathrm{z}} \coloneqq \mathbb{N} \to \mathbb{R}^{n_{\mathrm{z}}}_{+,s}$ and $\boldsymbol{q}_{\mathrm{u}} \coloneqq \mathbb{N} \to \mathbb{R}^{n_{\mathrm{u}}}_{+,s}$ are depicted in Fig. 4.62. The weighting function $\boldsymbol{q}_{\mathrm{z}}(\cdot)$ is designed such that baggage flow arriving at loading stations is first directed to the EBS and is later sent to a destination link as the corresponding unloading station opens. The control inputs weighting function $\boldsymbol{q}_{\mathrm{u}}(\cdot)$ incentivizes DCVs with destination d, $d \in \{18, 23\}$ to arrive at the corresponding unloading station during its designated time window. In addition, it prevents circular free movements of DCVs in the network by making the loaded DCVs stay in the EBS for as long as possible. The controller assumes the planned baggage profile $\overline{\boldsymbol{\omega}}(\cdot)$ of Fig. 4.63 and unplanned baggage demand $\tilde{\boldsymbol{\omega}}(t) \in \mathbb{W} \coloneqq \{\boldsymbol{\omega} \,|\, 0 \leq \boldsymbol{\omega}^{\mathrm{T}} \leq [4.5\;\; 2.5\;\; 3.5\;\; 4]\}$ for all $t \in \mathbb{N}$. The terminal constraint set $\mathbb{X}_{\mathrm{f}} \coloneqq \{\boldsymbol{x} \geq 0 \,|\, \|\boldsymbol{P}_{\mathrm{f}}^{-1}\boldsymbol{x}\|_\infty \leq 1,\, \boldsymbol{P}_{\mathrm{f}} = \mathrm{diag}(\boldsymbol{p}_{\mathrm{f}})\}$ and the terminal cost function $V_{\mathrm{f}}(\boldsymbol{x}) \coloneqq r\|\boldsymbol{P}_{\mathrm{f}}^{-1}\boldsymbol{x}\|_\infty$ are calculated by solving the linear inequalities 4.24 of Proposition 4.6 for system 4.52 as $\boldsymbol{p}_{\mathrm{f}} = 10\, \mathbf{1}_{n+n_{\mathrm{d}}}$ and $r = 1250$. For tube-based MPC, the solution to the linear program 4.6 of Theorem 4.2 with $\boldsymbol{U} = 10\, \boldsymbol{I}_{n_{\mathrm{u}}}$, $\boldsymbol{M} = \boldsymbol{I}_{n+n_{\mathrm{d}}}$, and $\boldsymbol{Y}_2 = 0$ yields $\boldsymbol{p}$, $\boldsymbol{Y}_1$, and $\gamma = 0.09$. The state feedback controller is then obtained as $\boldsymbol{K} = \boldsymbol{Y}(\mathrm{diag}(\boldsymbol{p}))^{-1}$, which achieves $\|\boldsymbol{z}_{\mathrm{e}}\|_{L_\infty, \mathbb{W}, \mathbb{X}_{\mathrm{e}}} < 0.09$ for the error system 4.17 with bounded control effort $0 \leq \boldsymbol{K}\boldsymbol{x}_{\mathrm{e}} \leq 10\, \mathbf{1}_{n_{\mathrm{u}}}$ for all $\boldsymbol{x}_{\mathrm{e}} \in \mathbb{X}_{\mathrm{e}} \coloneqq \{\boldsymbol{x} \in \mathbb{R}^{n+n_{\mathrm{d}}} \,|\, 0 \leq \boldsymbol{x} \leq \boldsymbol{p}\}$.

The linear programs for determining the terminal cost and constraint set, and calculating the feedback gain $\boldsymbol{K}$, and the ones associated with the MPC optimization problems 4.10 and 4.21 are solved using IBM ILOG CPLEX Optimization Studio (version 12.63) [23] connector for Matlab.



(a) (b)

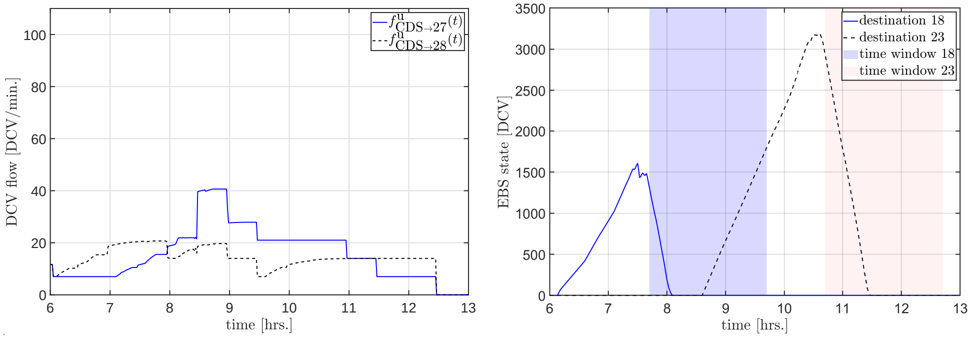
Figure 4.62: Weighting functions used in the MPC controller design; $t_{\mathrm{EBS}}(n,d)$ is the travel time from link n to destination d on the shortest path connecting the two, and $t_{\mathrm{o}}(d)$ and $t_{\mathrm{c}}(d)$, respectively, mark the opening and closing time of destination link d. (a): link density weighting function $q_{\mathrm{z}}(\cdot)$ for links with loaded DCVs. (b): weighting function $q_{\mathrm{u}}(\cdot)$ for link-to-link flows.



(a) Loading station 1.

(b) Loading station 2.

Figure 4.63: Planned and disturbed baggage profile per destination at the loading stations. The opening and closing time of destination $d \in \{18, 23\}$ are marked by $t_o(d)$ and $t_c(d)$, respectively.



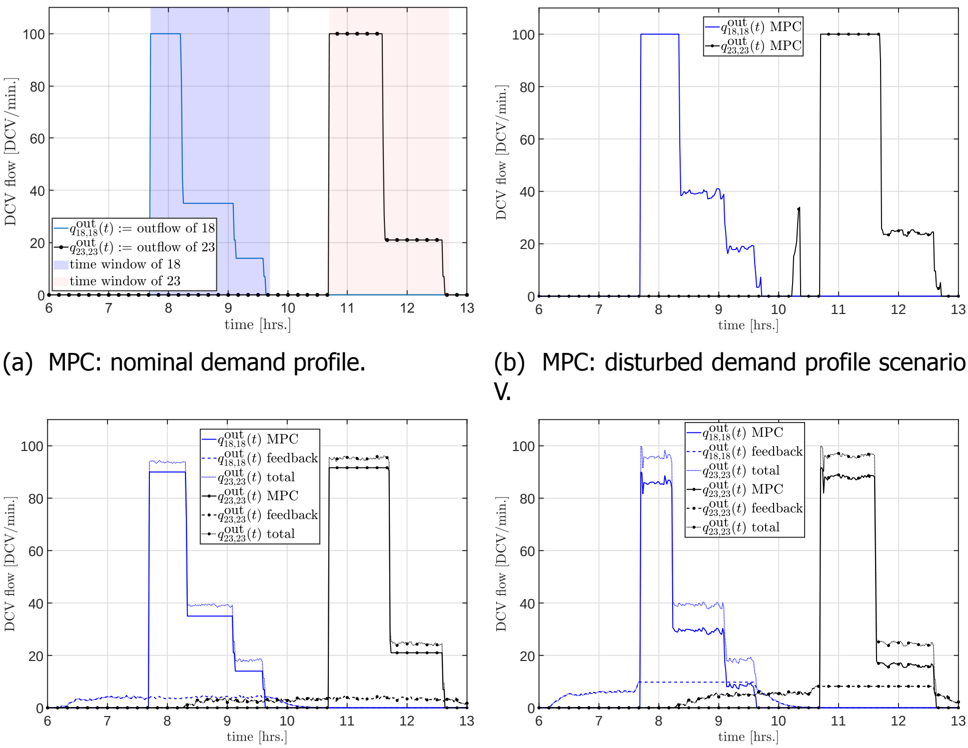
(a) Outflow of DCVs from CDS to the unloading stations.

(b) DCV accumulation in the EBS

Figure 4.64: MPC control under planned baggage demand.

4.6.2. Simulation Results

To compare the performance of MPC, tube-based MPC (TB-MPC), and tube-based MPC with optimized tube base (Optimized TB-MPC) control strategies a series of simulations are conducted based on the morning baggage demand profile of Fig. 4.63, which depicts the nominal baggage demand profile and a disturbed profile. In all simulations, all empty DCVs are initially located at the CDS with no DCV being present on any other link of the BHS network. First, effectiveness of the MPC control strategy under the nominal baggage demand profile is tested. As observed in Fig. 4.64 and Fig. 4.65a, the MPC controller performs as expected in the sense that early baggage demand is directed to EBS, empty DCVs are optimally routed from the CDS to the unloading stations, and the DCV flows arrive at the designated destinations during their respective time window. Next, we consider a series of scenarios that correspond to disturbance sequences $\tilde{\boldsymbol{\omega}}_{t_{\text{sim}}}(i) = (\tilde{\boldsymbol{\omega}}(i;0), \tilde{\boldsymbol{\omega}}(i;1), \ldots, \tilde{\boldsymbol{\omega}}(i;t_{\text{sim}}))$, $i \in \mathbb{N}_{1:7}$, where $\tilde{\boldsymbol{\omega}}_{t_{\text{sim}}}(i) \in \mathbb{W}^{t_{\text{sim}}}$ for all $i \in \mathbb{N}_{1:7}$



(a) MPC: nominal demand profile.

(b) MPC: disturbed demand profile scenario V.

(c) TB-MPC: disturbed demand profile scenario V.

(d) Optim. TB-MPC: disturbed demand profile scenario V.

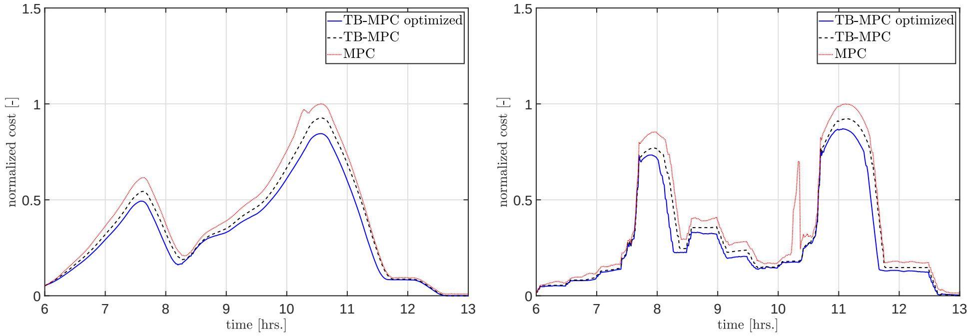
Figure 4.65: MPC control under (disturbed) baggage demand. Flow of DCVs leaving the network at the unloading stations.

and t_{sim} is the simulation time. A qualitative comparison between the three control strategies is provided in Fig. 4.65 for the sample scenario V. For each scenario, performance of the MPC controller is compared against those of the TB-MPC and Optimized TB-MPC controllers, based on the following two criteria: i) the total relative baggage demand that misses the closing time of its designated destination of the time window defined as

$$\mathrm{loss}(i) = \frac{\sum_{d\in\{18,23\}} \sum_{t=t_{\mathrm{c}}(d)}^{t_{\mathrm{sim}}} q_{d,d}^{\mathrm{out}}(i;t)}{\sum_{t=0}^{t_{\mathrm{sim}}} 1_{n_{\mathrm{w}}}{}^{\mathrm{T}}\big(\tilde{\boldsymbol{\omega}}(i;t) + \overline{\boldsymbol{\omega}}(t)\big)}, \ i \in \mathbb{N}_{1:7},$$

and ii) the total closed loop cost defined as

$$J_{\mathrm{cl}}(i) = \sum_{t=0}^{t_{\mathrm{sim}}} \boldsymbol{q}_{\mathrm{z}}{}^{\mathrm{T}}(t)\boldsymbol{z}(i;t) + 12\boldsymbol{q}_{\mathrm{u}}{}^{\mathrm{T}}(t)\boldsymbol{u}(i;t), \ i \in \mathbb{N}_{1:7}.$$



(a) Optimal value of predicted cost over the prediction horizon. (b) Current closed-loop cost.

Figure 4.66: Comparison of MPC, TB-MPC and optimized TB-MPC approaches in terms of optimal prediction cost and current cost for the closed-loop system.

The results are summarized in Table 4.61, where we have also listed the relative total baggage demand for the i-th scenario, defined as $\sum_{t=0}^{t_{\mathrm{sim}}} 1_{n_{\mathrm{w}}}{}^{\mathrm{T}} \tilde{\boldsymbol{\omega}}(i;t) / \sum_{t=0}^{t_{\mathrm{sim}}} 1_{n_{\mathrm{w}}}{}^{\mathrm{T}} \overline{\boldsymbol{\omega}}(t)$. The total closed loop cost $J_{\mathrm{cl}}(i)$ of TB-MPC and TB-MPC Optimized approaches are normalized by that of the MPC approach for all $i \in \mathbb{N}_{1:7}$.

It it observed from Table 4.61 that as we deviate from the nominal scenario, the MPC control strategy performs the worst compared to the tube-based variants in terms of the loss measure. In addition, the difference between closed-loop performance of the tube-based methods and the MPC approach increases in significance for larger deviations from the nominal scenario. It is also evident that optimizing the tube base enhances the performance of the TB-MPC, but the difference is not as significant. The difference between the three methods can also be seen in Fig. 4.66, where the optimal prediction cost at the current state $V_{N_{\mathrm{h}}}\big(t, \boldsymbol{x}(t), \boldsymbol{u}^*_{N_{\mathrm{h}}}(t)\big)$ and the current closed-loop cost $J_{\mathrm{cl}}(t) = \boldsymbol{q}_{\mathrm{z}}{}^{\mathrm{T}}(t)\boldsymbol{z}(t) + 12\boldsymbol{q}_{\mathrm{u}}{}^{\mathrm{T}}(t)\boldsymbol{u}(t)$ are depicted.

4.7. Conclusions

For linear discrete time systems subject to an infinity-norm bounded additive disturbance $\boldsymbol{\omega} \in \mathbb{W}$, we have shown that the simultaneous problem of finding a robustly positively invariant set $\mathbb{X}$ and a (constrained) state feedback gain $\boldsymbol{K}$ that minimizes the $\|L\|_\infty$ norm of the output over this set for all $\boldsymbol{\omega} \in \mathbb{W}$ can be formulated as a linear program when $\boldsymbol{K}$ renders the closed-loop system positive. This solution is then leveraged in a tube-based MPC approach, where the feedback gain $\boldsymbol{K}$ renders the error system positive and the set $\mathbb{X}$ is used to characterize a bounding tube for trajectories of the uncertain system and to tighten the nominal MPC state and control input constraints. The feedback gain calculated by this approach has also the advantage that it ensures minimal deviation between the nominal and uncertain trajectories for a given control "budget" in terms of the maximal effort allowed by the feedback controller.

Table 4.61: Overview of simulation results for MPC, TB-MPC, and TB-MPC Optimized control strategies under various disturbance scenarios.

Scenario	Norm. demand [-]	Performance criteria	Control Strategy		
			MPC	TB-MPC	Optim. TB-MPC
Nominal	1.00	loss [%]	0	-	-
		norm. closed-loop cost [-]	1.00	-	-
Scenario I	1.13	loss [%]	0.13	0.06	0.00
		norm. closed-loop cost [-]	1.00	0.88	0.79
Scenario II	1.14	loss [%]	0.32	0.16	0.15
		norm. closed-loop cost [-]	1.00	0.87	0.77
Scenario III	1.16	loss [%]	0.15	0.10	0.00
		norm. closed-loop cost [-]	1.00	0.83	0.77
Scenario IV	1.19	loss [%]	0.62	0.29	0.26
		norm. closed-loop cost [-]	1.00	0.80	0.75
Scenario V	1.22	loss [%]	0.69	0.30	0.28
		norm. closed-loop cost [-]	1.00	0.75	0.72
Scenario VI	1.25	loss [%]	3.03	0.56	0.50
		norm. closed-loop cost [-]	1.00	0.78	0.70
Scenario VII	1.27	loss [%]	5.81	1.02	0.96
		norm. closed-loop cost [-]	1.00	0.69	0.61

The proposed solution has also been used for calculating an infinity-norm bounded terminal constraint set and an infinity-norm based terminal cost function for the MPC and tube-based MPC approaches via linear programs to ensure recursive feasibility and (robust) asymptotic stability of the closed-loop system.

It has been shown via a case study that the proposed approach can be used to efficiently design (tube-based) MPC approaches for large-scale linear systems as all key components of our approach (i.e., calculating the feedback gain, the terminal constraint set and the terminal cost) and the MPC optimization problem are formulated as linear programs.

Finally, we would like to emphasize that the methods developed in Section 4.3 do not require the open-loop system to be positive; hence they are applicable to the wider class of linear systems.

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4

Appendix

4.A. Proofs

In this section we provide proofs of the propositions and theorems presented in the chapter. First, we present and prove the following proposition, which can be regarded as an extension of the Theorem 1.3.12 of [15] to system 4.1, that will be consequently used in the proof of Proposition 4.1.

Proposition 4.7. The following statements are equivalent for a positive system (4.1) with a set of states $\mathbb{X} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1,\ \boldsymbol{P} := \mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$, a disturbance set $\mathbb{W} := \{\boldsymbol{\omega} \in \mathbb{R}^{n_\omega} \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_\infty \leq 1,\ \boldsymbol{\Omega} := \mathrm{diag}(\boldsymbol{\omega}_M),\ \boldsymbol{\omega}_M \in \mathbb{R}^{n_\omega}_{+,s}\}$, and a positive scalar $\lambda \leq 1$.

i) $\boldsymbol{Ap} + \boldsymbol{B}_\omega\boldsymbol{\omega}_M < \lambda\boldsymbol{p}$.

ii) $\mathbb{X}^+ \subset \lambda\mathbb{X}$, where $\mathbb{X}^+ := \{\boldsymbol{Ax} + \boldsymbol{B}_\omega\boldsymbol{\omega} \mid \boldsymbol{\omega} \in \mathbb{W},\ \boldsymbol{x} \in \mathbb{X}\}$.

iii) The function $V : \boldsymbol{x} \mapsto \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$ is a Lyapunov function for the system over the set $\overline{\mathbb{X}} := \mathbb{R}^n - \mathrm{int}(\mathbb{X})$, which is strictly decreasing along system trajectories in $\overline{\mathbb{X}}$; hence, the system state is globally ultimately bounded [2] in $\mathbb{X}$.

Proof. (ii $\implies$ i): Since $\boldsymbol{A} \geq 0$ and $\boldsymbol{B}_1 \geq 0$, it holds that

$$x^+_{\max} := \max_{\boldsymbol{x}\in\mathbb{X},\ \boldsymbol{\omega}\in\mathbb{W}} \boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{\omega} = \boldsymbol{Ap} + \boldsymbol{B}_1\boldsymbol{\omega}_M$$
$$x^+_{\min} := \min_{\boldsymbol{x}\in\mathbb{X},\ \boldsymbol{\omega}\in\mathbb{W}} \boldsymbol{Ax} + \boldsymbol{B}_1\boldsymbol{\omega} = -(\boldsymbol{Ap} + \boldsymbol{B}_1\boldsymbol{\omega}_M).$$

Since, by assumption, $\boldsymbol{x}^+ \in \mathrm{int}(\lambda\mathbb{X})$ for all $\boldsymbol{x} \in \mathbb{X}$ and $\boldsymbol{\omega} \in \mathbb{W}$, it must then hold that

$$-\lambda\boldsymbol{p} < x^+_{\max} < \lambda\boldsymbol{p} \tag{4.53a}$$
$$-\lambda\boldsymbol{p} < x^+_{\min} < \lambda\boldsymbol{p}, \tag{4.53b}$$

which is equivalent to $\boldsymbol{Ap} + \boldsymbol{B}_\omega\boldsymbol{\omega}_M < \lambda\boldsymbol{p}$. Conversely, $\boldsymbol{Ap} + \boldsymbol{B}_\omega\boldsymbol{\omega}_M < \lambda\boldsymbol{p}$ implies 4.53, which, in turn, implies that $\boldsymbol{x}^+ \in \mathrm{int}(\lambda\mathbb{X})$ for all $\boldsymbol{x} \in \mathbb{X}$ and $\boldsymbol{\omega} \in \mathbb{W}$.

(i $\implies$ iii): assuming (i) holds, for any scalar $c \geq 1$ it follows that $\boldsymbol{A}c\boldsymbol{p} + \boldsymbol{B}_1\boldsymbol{\omega}_M < \lambda c\boldsymbol{p}$ and that $\boldsymbol{x}^+ \in \mathrm{int}(\lambda c\mathbb{X})$ for all $\boldsymbol{x} \in c\mathbb{X}$, and all $\boldsymbol{\omega} \in \mathbb{W}$. Define $\mathbb{X}(c) := \{\boldsymbol{x} \in \mathbb{R} \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty = c\}$ as the boundary of $c\mathbb{X}$. For any $\boldsymbol{x} \in \mathbb{X}(c) \subseteq c\mathbb{X}$ it then holds that $\|\boldsymbol{P}^{-1}\boldsymbol{x}^+\|_\infty < c\lambda$ for all $\boldsymbol{\omega} \in \mathbb{W}$. Hence, $\|\boldsymbol{P}^{-1}\boldsymbol{x}^+\|_\infty - \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty < (\lambda - 1)c = (\lambda - 1)\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$ for all $\boldsymbol{\omega} \in \mathbb{W}$. The proof is complete by noting that $c \geq 1$ is chosen arbitrarily; so $\boldsymbol{x} \in \overline{\mathbb{X}}$ implies that there exists a $c \geq 1$ such that $\boldsymbol{x} \in \mathbb{X}(c)$.

Conversely, $\|P^{-1}x^+\|_\infty < \lambda\|P^{-1}x\|_\infty$ for some[2] $x \in \mathbb{X}(1)$ and a positive $\lambda \leq 1$, implies that $-\lambda p < x^+ < \lambda p$ or, equivalently, $x^+ \in \mathrm{int}(\lambda\mathbb{X})$ for all $\omega \in \mathbb{W}$, and, consequently, $Ap + B_\omega \omega_M < \lambda p$. □

Proof of Proposition 4.1. For the positive system 4.1, equivalence of 4.5a to the set $\mathbb{X}$ being robustly positively invariant is due to statement (i) of Proposition 4.7 with $\lambda = 1$. Note that the set $\overline{\mathbb{X}}_0 := \left\{x \in \mathbb{R}^n \;\middle|\; \left\|\mathrm{diag}\left(\left[\|M(:,1)\|_\infty \; \dots \; \|M(:,n)\|_\infty\right]^{\mathrm{T}}\right)^{-1} x\right\|_\infty < 1\right\}$ is the smallest norm-infinity bounded set containing $\mathbb{X}_0$. Thus, 4.5c is equivalent to having $\mathbb{X} := \{x \mid \|P^{-1}x\|_\infty \leq 1\}$ include $\mathbb{X}_0$. Finally, note that for $\overline{\omega}_k := (\omega_M, \dots, \omega_M) \in \mathbb{W}^{k+1}$, it follows that

$$
\begin{aligned}
&z(t; p, \overline{\omega}_k) - Cp - D_{11}\omega_M = \qquad\qquad (4.54)\\
&\begin{cases}
0 & t = 0,\ k \in \mathbb{N}\\
C\left(\sum_{s=0}^{t-1} A^s\right)(A - I)p + C\left(\sum_{s=0}^{t-1} A^s\right)B_1\omega_M & t \in \mathbb{N}_{0:k},\ k \in \mathbb{N}\\
\qquad < -C\left(\sum_{s=0}^{t-1} A^s\right)B_1\omega_M + C\left(\sum_{s=0}^{t-1} A^s\right)B_1\omega_M = 0. &
\end{cases}
\end{aligned}
$$

Hence, it holds that

$$
-Cp - D_{11}\omega_M \leq z(t; -p, -\overline{\omega}_k) \leq z(t; x_0, \omega_k) \leq z(t; p, \overline{\omega}_k) \leq Cp + D_{11}\omega_M
$$

for any $x_0 \in \mathbb{X}$, $k \in \mathbb{N}$, $\omega_k \in \mathbb{W}^{k+1}$, and for all $t \in \mathbb{N}_{0:k}$, with the equality holding for $t = 0$, $x_0 = p$ ($x_0 = -p$), and $\omega_k = \overline{\omega}_k$ ($\omega_k = -\overline{\omega}_k$). It then follows from the definition of $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}}$ that $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}} = \|Cp + D_{11}\omega_M\|_\infty$. Hence, $Cp + D_{11}\omega_M < 1_{n_z}\gamma$ is equivalent to $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}} < \gamma$. □

Proof of Theorem 4.2. The proof is constructed by applying Propositions 4.7 and 4.1 to the closed-loop system 4.4. Letting $K = (Y_1 - Y_2)P^{-1}$, constraint 4.6b is equivalent to the closed-loop system 4.4 being positive. Constraints 4.6c and 4.6d are then equivalent to 4.5 of Proposition 4.1, expressed in the closed-loop system matrices. For the closed-loop system, it then follows from Proposition 4.1 that γ is an upper bound to $\|z\|_{L_\infty,\mathbb{W},\mathbb{X}}$. Equation 4.6e guarantees that the control input constraint $\|U^{-1}Kx\|_\infty \leq 1$ is satisfied by all vertices of $\mathbb{X}$ and, thus, by any $x \in \mathbb{X}$. Finally, the closed-loop system is globally ultimately bounded in $\mathbb{X}$ due to the last statement of Proposition 4.7. □

Proof of Proposition 4.3. Define $u(t, x) := \overline{u}(t) + K(t)x(t)$, which satisfies the control constraint $0 \leq u(t) \leq u_{\max}$ for all $t \in \mathbb{N}$. The resulting closed-loop system is

$$
\begin{aligned}
x(t+1) &= (A + B_2K(t))x(t) + B_1\tilde{\omega}(t),\\
z(t) &= (C + D_{12}K(t))x(t) + D_{11}\tilde{\omega}(t),\\
\Delta u(t) &:= u(t, x) - \overline{u}(t) = K(t)x,
\end{aligned}
$$

[2]Note that assuming the Lyapunov forward difference holds for $x \in \mathbb{X}(1)$ does not cause any loss of generality as for any $y \in \mathbb{X}(c)$, with c being an arbitrary positive scalar, $c^{-1}y \in \mathbb{X}(1)$.

4

for all $t \in \mathbb{N}$. Since 4.22a renders the closed-loop system positive for all $t \in \mathbb{N}$, it follows from 4.22b and Proposition 4.7 that the set $\mathbb{X}_\mathrm{f} := \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty \leq 1,\ \boldsymbol{P} := \mathrm{diag}(\boldsymbol{p}),\ \boldsymbol{p} \in \mathbb{R}^n_{+,s}\}$ is robustly positively invariant for the closed-loop system. Therefore, it holds for system 4.3 that for all $\boldsymbol{x}(t) \in \mathbb{X}_\mathrm{f}$, there exists a feasible control input in the form of $\boldsymbol{u}(t, \boldsymbol{x}) := \overline{\boldsymbol{u}}(t) + \boldsymbol{K}(t)\boldsymbol{x}(t)$ such that $\boldsymbol{x}(t+1) \in \mathbb{X}_\mathrm{f}$ for all $\tilde{\boldsymbol{\omega}} \in \mathbb{W}$. □

Proof of Proposition 4.4. First note that for any $\boldsymbol{x}(t) \in \mathbb{X}_\mathrm{f}$, we have $\|\boldsymbol{P}^{-1}\boldsymbol{x}^+\|_\infty - \|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty < (\lambda - 1)\|\boldsymbol{P}^{-1}\boldsymbol{x}\|_\infty$ and $\|\boldsymbol{P}^{-1}\boldsymbol{x}\|\boldsymbol{p} \geq \boldsymbol{x}$. Assuming 4.23 holds, it then follows for any $\boldsymbol{x}(t) \in \mathbb{X}_\mathrm{f}$ that

$$
\begin{aligned}
\boldsymbol{V}_\mathrm{f}(t+1, \boldsymbol{x}(t+1)) - \boldsymbol{V}_\mathrm{f}(t, \boldsymbol{x}(t)) &= r(t+1)\|\boldsymbol{P}^{-1}\boldsymbol{x}(t+1)\|_\infty - r(t)\|\boldsymbol{P}^{-1}\boldsymbol{x}(t)\|_\infty < \\
&(\lambda - 1)r(t)\|\boldsymbol{P}^{-1}\boldsymbol{x}(t)\|_\infty \leq \|\boldsymbol{P}^{-1}\boldsymbol{x}(t)\|_\infty\big(\boldsymbol{q}_\mathrm{z}^\mathrm{T}(t)(\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K}(t)) + \boldsymbol{q}_\mathrm{u}^\mathrm{T}(t)\boldsymbol{K}(t)\big)\boldsymbol{p} \\
&\leq -\big(\boldsymbol{q}_\mathrm{z}^\mathrm{T}(t)(\boldsymbol{C} + \boldsymbol{D}_{12}\boldsymbol{K}(t)) + \boldsymbol{q}_\mathrm{u}^\mathrm{T}(t)\boldsymbol{K}(t)\big)\boldsymbol{x}(t) = -\big(\boldsymbol{q}_\mathrm{z}^\mathrm{T}(t)\boldsymbol{z}(t) + \boldsymbol{q}_\mathrm{u}^\mathrm{T}(t)\Delta\boldsymbol{u}(t)\big).
\end{aligned}
$$

Hence, the time-varying version of "basic stability assumption" [10, Assumption 2.23 (a)] is satsified. In addition the time-varying version of basic stability assumption [10, Assumption 2.33 (b)] is automatically fulfilled as there exists a $c_1 > 0$ such that $\boldsymbol{q}_\mathrm{z}^\mathrm{T}(t)\boldsymbol{z} + \boldsymbol{q}_\mathrm{u}^\mathrm{T}(t)\Delta\boldsymbol{u} \geq c_1 1_{n_z+n_u}{}^\mathrm{T}[\boldsymbol{z}^\mathrm{T}\ \Delta\boldsymbol{u}^\mathrm{T}]^\mathrm{T}$ with any positive bounded functions $\boldsymbol{q}_\mathrm{z}(\cdot)\colon \mathbb{N} \to \mathbb{R}^{n_z}_{+,s}$ and $\boldsymbol{q}_\mathrm{u}(\cdot)\colon \mathbb{N} \to \mathbb{R}^{n_u}_{+,s}$, and $V_\mathrm{f}(t, \boldsymbol{x}) \leq c_2\|\boldsymbol{x}\|_\infty$ with some scalar $c_2 > 0$ for all $t \in \mathbb{N}$ and for any $\boldsymbol{x} \in \mathbb{X}_\mathrm{f}$. Therefore, due to [10, Theorem 2.39], the origin is asymptotically stable in $\mathcal{X}_{N_\mathrm{h}}(t)$ at each $t \in \mathbb{N}$ for the system $\boldsymbol{x}^+ = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}_2\big(\boldsymbol{K}(t) + \overline{\boldsymbol{u}}(t)\big) + \boldsymbol{B}_1\overline{\boldsymbol{\omega}}(t)$. □

5

Linear Positive Systems May Have a Reachable Subset from the Origin That is Either Polyhedral or Nonpolyhedral

”The beauty of mathematics only shows itself to more patient followers.”

Maryam Mirzakhani, 1977 – 2017

Positive systems with positive inputs and positive outputs are used in several branches of engineering, biochemistry, and economics. Both control theory and system theory require the concept of reachability of a time-invariant discrete-time linear positive system. The subset of the state set that is reachable from the origin is therefore of interest. The reachable subset is in general a cone in the positive vector space of the positive real numbers. It is established in this chapter that the reachable subset can be either a polyhedral or a nonpolyhedral cone. For a single-input case, a characterization is provided of when the infinite-time and the finite-time reachable subset are polyhedral. An example is provided for which the reachable subset is nonpolyhedral. Finally, for the case of polyhedral reachable subset(s), a method is provided to verify if a target set can be reached from the origin using positive inputs.

The current chapter is based on [1].

5.1. Introduction

5.1.1. Motivation and Scope

In this chapter, the focus is on the reachable subset from the origin of a single-input time-invariant discrete-time linear positive system. It will be proven that such a reachable subset can be either a polyhedral or a nonpolyhedral cone. A characterization is provided of when this reachable subset is polyhedral.

A positive system may arise in many areas of science and of engineering, such as econometrics [2], bio-chemical reactors [3], compartmental systems [4, 5], and transportation system [6, 7],to name a few. The variables in such systems represent growth rates, concentration levels, mass accumulation, flows, etc. Obviously, variables of this nature can only assume values that are zero or strictly positive.

For problems of control and system theory with positive systems, a solid body of concepts, theorems, and algorithms has been developed. Of particular interest is the theory of linear positive systems [8], which is based on the theory of positive matrices and their geometric equivalent, polyhedral cones, [9–12].

While the theory of linear positive systems overlaps with the theory of linear systems, there are distinct differences between the two. Therefore, several concepts of linear systems cannot be directly generalized to linear positive systems without reformulation. One such property is the notion of reachability and controllability of a linear positive system.

The motivation of the investigation of reachability and controllability of a linear positive system is in 1) their use in control theory as an equivalent condition for the existence of a control law for particular control objectives; and 2) in the theory of realization and of system identification. In a positive system, as it arises in the research areas mentioned above, one may want to know whether from a specified initial state a particular terminal state can be reached by application of a positive input to the system. The state to be reached can be a set of concentrations of chemical substances in bio-reactor or a concentration in a compartment which e.g. a model of tissue in a human being. More generally, one may want to characterize all states of a linear positive system that can be reached from the zero initial state using positive inputs, which is also the object of interest for realization theory of linear positive systems. The choice for the reachable subset from the origin is essential for realization theory. Observability of a linear positive system is then of interest only for states in the reachable set. A characterization of that view of observability does currently not exist in the literature. The condition formulated in the paper [13] is too strong because it is based on the assumption that the reachable set from the origin is the entire positive vector space $\mathbb{R}_+^n$. Therefore, characterizing all states of a linear positive system that can be reached from the zero initial state using positive inputs is the problem to be investigated in this chapter. More details on the problem formulation may be found in Section 5.3.

5.1.2. Previous Work

Below the vector space of tuples of the positive real numbers will be referred to as the *positive vector space*; it is formally defined in Section 5.2.

Controllability and reachability of a discrete-time linear positive system has been widely studied and there is a considerable literature. This literature is briefly surveyed below. In most of the literature it is emphasized that the characterization of controllability of a discrete-time linear positive system takes a very different form than that of its counterpart for discrete-time linear systems [14–16]. In addition, while reachability of a linear system may be achieved in a number of steps equal to the state-space dimension, [17], for discrete-time linear positive systems this does not hold. For a linear positive system the number of steps required to reach a certain point in the positive orthant can be larger than the dimension of the system, as noted in [15], where this is illustrated using the model of a pharmacokinetic system.

The concept of reachability used in the literature of discrete-time linear positive systems is whether every state of the positive vector space can be reached from the origin either in finite time or in infinite time. The result is then a characterization of this considered concept of reachability. Publications that are based on that approach include [14, 18–21].

Reachability of a discrete-time linear positive system is characterized using a graph-theoretic approach, and canonical reachable or canonical controllable forms are derived as well in [14, 21]. The authors of [20] have established a link between positive state controllability and positive input controllability of a related system, which is then used to obtain a controllability criterion. A survey of results on controllability and reachability of positive systems is provided in [22, 23]. Controllability results for special classes of 1D and 2D positive systems are provided in [24].

It is worth mentioning that the constrained reachability problem for a discrete-time linear system in the presence of disturbance with respect to a target tube or a target set has been widely discussed in the literature [25–28]. Among others [26] investigates this problem by constructing a sequence of target sets. The reachability problem is then transformed into a certain inclusion check on the last target set of this sequence. The authors of [26] also provide an approximate bounding ellipsoid algorithm to calculate the sequence of target sets and the associated input sequence. In [25], constrained reachability with respect to a target set is studied as a special case of reachability with respect to a target tube, and the authors provide an algorithm to construct the sequence of modified target sets when these sets are known to be polyhedral. In the above-mentioned literature, checking reachability or controllability of a target set requires one to directly or indirectly construct certain modified target sets in an iterative manner. In addition, it is not known in advance whether a target set can be reached in finite time.

5.1.3. Contribution of This Work

The contribution of this chapter to control and system theory is described next. Attention is restricted to a time-invariant discrete-time linear positive system. The problem for a continuous-time linear positive system is different. The results are mostly for a single-input system. The object of interest is the reachable subset from the origin state in either finite time or in infinite time. The problem is to characterize this reachable subset, in particular to determine whether the reachable subset is either a polyhedral cone or a nonpolyhedral cone. This problem is of interest to both control theory and to realization theory.

The problem considered in this chapter differs from the reachability or controllability problems treated in the literature. In the literature, the problem whether any state of the positive vector space can be reached by use of a positive input from the zero initial state has been investigated and a corresponding characterization of this concept has been provided. In this chapter the focus is on the characterization of the reachability subset which will often be a strict subset of the positive vector space. Moreover, it will be investigated whether the reachable subset is a polyhedral cone or a nonpolyhedral cone. In the existing literature the reachable subset has to equal the positive vector space which is a polyhedral cone. Surprisingly, as presented in this chapter, there exists an example of a linear positive system of which the reachable subset from the origin is a nonpolyhedral cone in the positive vector space. A consequence of this is that the reachable subset has to be investigated for the following cases: for a prespecified finite time, for an arbitrary finite time, and for infinite time. It will also be shown that the reachable subset can in general not be determined in a number of steps that equals the dimension of the state set but that the number of steps can be strictly larger than the dimension of the state set.

The specific contributions of the chapter are then as follows. A characterization of when the infinite-time reachable subset is a polyhedral cone, is provided in Theorem 5.13. A related result regarding the geometry of reachable sets for discrete-time linear systems (not necessarily positive) with positive control inputs can be found in [29]. A characterization of when the finite-time reachable subset from the origin is a polyhedral cone, is provided in Theorem 5.15. An example of linear positive system for which the reachable subset is nonpolyhedral is provided in Example 4. Results for the problem of when the reachable set contains a particular cone of terminal states are summarized in Proposition 5.18 and in Proposition 5.19.

The structure of the chapter is described next. Section 5.2 presents necessary background knowledge on positive matrices and positive systems. It also reports key terminology of controllability and reachability and links this to linear positive systems while highlighting existing view of the characterization of controllability and reachability of linear positive systems in the literature. Section 5.3 presents the approach of this chapter and the problem formulation. The characterization of the infinite-time reachable set as a polyhedral cone is provided in Section 5.4. The characterization of the finite-time reachable set as a polyhedral cone is provided in Section 5.5. Numerical verifiable conditions for the polyhedrality of the reachset in terms of the spectrum of the system matrix are provided also in those sections. Section 5.6 provides results on how to determine reachability for a specified control objective in the form of a subset of the positive vector space of a linear positive system.

5.2. Preliminaries

5.2.1. Positive Real Numbers, Positive Matrices, and Cones

The reader is assumed to be familiar with the integers, the real numbers, and vector spaces. Denote the set of the integers by $\mathbb{Z}$, the set of strictly-positive integers by $\mathbb{Z}_+ = \{1, 2, \ldots\}$, and the set of the natural numbers by $\mathbb{N} = \{0, 1, 2, \ldots\}$. For $n \in \mathbb{Z}_+$ denote $\mathbb{Z}_n = \{1, 2, \ldots, n\}$.

The *real numbers* are denoted by $\mathbb{R}$, the set of the *positive real numbers* or the *positive numbers* by $\mathbb{R}_+ = [0, \infty)$, and the set of the *strictly-positive real numbers* by $\mathbb{R}_{s+} = (0, \infty) \in \mathbb{R}_+$. The term *positive real numbers* is preferred by the authors over the term *nonnegative real numbers* which occurs in the literature. The term *positive real numbers* is used in the book [30, p. 19].

Define the *positive vector space* of tuples of the positive real numbers as the tuple $(\mathbb{R}_+, \mathbb{R}_+^n)$ with the algebraic operations described next. The set of the positive real numbers is closed with respect to addition and to multiplication. There does not exist an additive inverse while in the subset $(0, \infty)$ there always exists a multiplicative inverse. The set of positive vectors $\mathbb{R}_+^n$ is closed with respect to addition but there does not exist an additive inverse in this set. The vector of all-ones in $\mathbb{R}^n$ is denoted by $\mathbb{1}_n$. When used without a subscript $\mathbb{1}$ is a vector of appropriate dimension of which all elements are equal one.

For an integer $m \in \mathbb{Z}_+$ and a set of positive vectors $\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_m, \boldsymbol{a}_i \in \mathbb{R}_+^n$ define in the positive vector space the set

$$\text{conv}([\boldsymbol{a}_1 \ \ldots \ \boldsymbol{a}_m]) = \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \boldsymbol{x} = \sum_{i=1}^{m} \lambda_i \boldsymbol{a}_i, \ \lambda_i \geq 0, \ i = 1, \ldots, m, \ \sum_{i=1}^{m} \lambda_i = 1\} \quad (5.1)$$

as the *convex polytope* generated by $\boldsymbol{a}_i$, $i = 1, \ldots, m$.

Define in the vector space of the real numbers $\mathbb{R}^n$ the *open ball* with center $x \in \mathbb{R}^n$ and with radius $r \in (0, \infty)$ as the set

$$B(\boldsymbol{x}, r) = \{\boldsymbol{y} \in \mathbb{R}^n \mid \|\boldsymbol{y} - \boldsymbol{x}\|_2 < r\}. \quad (5.2)$$

The norm on $\mathbb{R}^n$ is the Euclidean norm, $\|\boldsymbol{x}\|_2 = (\sum_{i=1}^{n} x_i^2)^{1/2}$. This norm is also used on $\mathbb{R}_+^n$. An open ball in the positive vector space $\mathbb{R}_+^n$ is defined in a similar manner with $\boldsymbol{y} \in \mathbb{R}$ replaced by $\boldsymbol{y} \in \mathbb{R}_+^n$ in (5.2).

A *positive matrix* $\boldsymbol{A}$ of size $n \times m$ for $n, m \in \mathbb{Z}_+$ is a matrix of which each element $A_{i,j} = A_{ij}$ belongs to the positive real numbers $\mathbb{R}_+$. The set of such matrices is denoted by $\mathbb{R}_+^{n \times m}$.

The geometric view point of positive vectors is formulated in terms of rays and of cones as defined next. A *ray* is a half line $Y \subset \mathbb{R}_+^n$ for $n \in \mathbb{Z}_+$ described by a direction vector $\boldsymbol{x} \in \mathbb{R}_+^n \backslash \{0\}$ such that for all $c \in \mathbb{R}_+$, Y contains all elements of the form $c \cdot \boldsymbol{x} \in Y$. Equivalently,

$$\exists \ n \in \mathbb{Z}_+, \ \exists \ \boldsymbol{x} \in \mathbb{R}_+^n \backslash \{0\}, \quad C(x) = \{c \cdot \boldsymbol{x} \in \mathbb{R}_+^n \mid \forall \ c \in \mathbb{R}_+\}.$$

Below $c \cdot \boldsymbol{x}$ will be denoted by $c \ \boldsymbol{x}$.

A *cone* is a nonempty subset $C \subseteq \mathbb{R}_+^n$ such that (1) if $\boldsymbol{x} \in C$ and $c \in \mathbb{R}_+$ then $c\,\boldsymbol{x} \in C$; and (2) if $\boldsymbol{x},\ \boldsymbol{y} \in C$ then $\boldsymbol{x}+\boldsymbol{y} \in C$. It follows that $0 \in C$ for any cone C. By definition, a cone always includes the zero element of the positive vector space. That zero element is called the *apex* of the cone. Cones with an apex not at zero of the positive vector space are not used in this chapter.

A cone C is called a *polyhedral cone* if there exists an integer $m \in \mathbb{Z}_+$ and a set of positive vectors $a_1,\ a_2,\ \ldots,\ a_m \in C \subseteq \mathbb{R}_+^n$ such that, for any $\boldsymbol{x} \in C$ there exists positive real numbers $y_i \in \mathbb{R}_+$ for $i = 1, \ldots, m$, $\boldsymbol{x} = \sum_{i=1}^m\ y_i\ \boldsymbol{a}_i$. Equivalently, C is a polyhedral cone if

$$
\begin{aligned}
& \exists\ m \in \mathbb{Z}_+,\ \exists\ \boldsymbol{a}_1,\ \ldots, \boldsymbol{a}_m \in C, \text{ such that,} \\
C \ &= \ \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \exists\ \boldsymbol{y} \in \mathbb{R}_+^m \text{ such that } \boldsymbol{x} = \boldsymbol{A}\boldsymbol{y}\}, \text{ where,} \\
\boldsymbol{A} \ &= \ \begin{bmatrix} \boldsymbol{a}_1 & \boldsymbol{a}_2 & \ldots & \boldsymbol{a}_m \end{bmatrix} \in \mathbb{R}_+^{n\times m},\quad \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}.
\end{aligned}
$$

In the representation used above, the cone will also be denoted by

$$
C = \mathrm{cone}([\boldsymbol{a}_1\ \ldots\ \boldsymbol{a}_m]) = \mathrm{cone}(\boldsymbol{A}) \tag{5.3}
$$

for the positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times m}$ with the understanding that the cone is generated by the columns of the matrix $\boldsymbol{A}$. Moreover, with little abuse of the notation for $\boldsymbol{A} \in \mathbb{R}_+^{n\times m}$ and $\boldsymbol{X} \in \mathbb{R}_+^{n\times p}$, the cone generated by stacking up the $m+p$ columns of the matrices $\boldsymbol{A}$ and $\boldsymbol{X}$ will be denoted by $C = \mathrm{cone}([\boldsymbol{A}\ \boldsymbol{X}])$.

A cone is called a *nonpolyhedral cone* if it is not polyhedral. This implies that there does not exist a finite number $m \in \mathbb{Z}_+$ as in the above definition. The term *round cone* could also be used in this case. An example of a round cone is the well known ice cream cone which may be found in [9, Ex. 1.2.2].

An example of a polyhedral cone is given by

$$
\begin{aligned}
C \ &= \ \{\boldsymbol{x} \in \mathbb{R}_+^4 \mid \exists\ \boldsymbol{y} \in \mathbb{R}_+^4 \text{ such that } \boldsymbol{x} = \boldsymbol{A}\boldsymbol{y}\}, \\
\text{with } \boldsymbol{A} \ &= \ \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}.
\end{aligned}
$$

A *boundary ray* of a cone C is a ray of the cone that lies on the boundary of the cone. A ray lies on the boundary of a cone if for every $\epsilon \in (0,1)$ sufficiently small and for every element $\boldsymbol{x}$ of the ray, the ball $B(\boldsymbol{x},\epsilon)$ includes an element outside the cone.

It is called an *extreme (boundary) ray* of the cone if it cannot be written as the strict convex combination of two different rays. Thus $\boldsymbol{x} \in C$ is an extreme ray if there do not exist vectors $\boldsymbol{y},\ \boldsymbol{z} \in C$ that are boundary rays and a scalar $c \in (0,1)$ such that $\boldsymbol{x} = c\ \boldsymbol{y} + (1-c)\ \boldsymbol{z}$. In the above example, each of the columns of the matrix $\boldsymbol{A}$ is an extremal ray of cone $\mathrm{cone}(\boldsymbol{A})$.

More technical concepts and results regarding positive matrices may be found in Appendix 5.A because these are well known and not a contribution of this chapter.

The reader may find additional information on positive real numbers, positive matrices, and cones in the books [9, 31, 32].

5.2.2. Linear Positive Systems

Definition 5.1. Define a *discrete-time linear positive system* with *system matrix* $\boldsymbol{A}$ and *input matrix* $\boldsymbol{B}$ by the representation

$$\begin{aligned} \boldsymbol{x}(t+1) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t),\ t_0 \in \mathbb{N},\ \boldsymbol{x}(\boldsymbol{t}_0) = \boldsymbol{x}_0, \\ & \boldsymbol{A} \in \mathbb{R}_+^{n\times n},\ \boldsymbol{B} \in \mathbb{R}_+^{n\times m},\ t \in T = \{t_0,\ t_0+1,\ t_0+2, \dots\}, \\ & \boldsymbol{x}_0 \in \mathbb{R}_+^n,\ \boldsymbol{u} : T \to \mathbb{R}_+^m,\ \boldsymbol{x} : T \to \mathbb{R}_+^n. \end{aligned} \tag{5.4}$$

An explicit expression for the state function of a discrete-time linear positive system is well known and provided by the formula

$$\boldsymbol{x}(t) = \boldsymbol{A}^{t-t_0}\boldsymbol{x}_0 + \sum_{s=t_0}^{t-t_0-1} \boldsymbol{A}^s\ \boldsymbol{B}\boldsymbol{u}(t-1-s),\ \forall\ t \in T, \tag{5.5}$$

$$(t_0, \boldsymbol{x}_0) \overset{\boldsymbol{u}(t_0:t-1)}{\mapsto} (t, \boldsymbol{x}(t)),\ \text{where} \tag{5.6}$$

$$\boldsymbol{u}(t_0 : t-1) = (\boldsymbol{u}(t_0),\ \boldsymbol{u}(t_0+1),\ \dots, \boldsymbol{u}(t-1)).$$

For a *time-invariant discrete-time linear positive system* we may assume $t_0 = 0$ in Definition 5.1 and in the explicit solution (5.5) as the time axis can be shifted to the zero time without affecting the trajectories.

Definition 5.1 requires that the mathematical objects of the definition exist. An alternative definition, which may be found in the literature, defines a linear positive system as a linear system with as state space $X = \mathbb{R}^n$ and requires that for any initial state $\boldsymbol{x}_0 \in \mathbb{R}_+^n$ and any positive input function $\boldsymbol{u} : T \to \mathbb{R}_+^m$, the resulting state function $\boldsymbol{x}$ is such that for all $t \in T$, $\boldsymbol{x}(t) \in \mathbb{R}_+^n$. It can then be proven that this alternative definition leads to the condition that the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are positive matrices. Thus the alternative definition leads back to the form of Definition 5.1.

Books on positive systems or books with chapters on positive systems include [8, 16, 24].

5.2.3. Terminology of Controllability and Reachability

The literature of control and system theory is not standardized in regard to the terms controllability and reachability. The authors have chosen to use in this chapter the terms as introduced by R.E. Kalman in Chapter 2 of the book [33, Def. 2.13, Def. 2.14, p. 32]. Almost the same definitions may be found in [34, Def. 3.1.1]. Related papers of Kalman on controllability are [35, 36].

Consider the discrete-time linear system with the representation (5.4) and the corresponding solution (5.5). Associate with this system the *initial tuple* $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ consisting of the initial time t_0 and the initial state $\boldsymbol{x}_0$ where t_0 will often be taken to be zero, $t_0 = 0$, and the *terminal tuple* $(t_1, \boldsymbol{x}_1)$ consisting of the terminal time t_1 and the terminal state x_1 where $t_1 \in T$ and $\boldsymbol{x}_1 = \boldsymbol{x}(t_1)$. The solution displayed above is then denoted as the *transition*

$$(t_0, \boldsymbol{x}_0) \stackrel{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}(t_1)).$$

In system theory one often distinguishes between reachability and controllability: for reachability one considers an initial tuple consisting of an initial time and an initial state as fixed and one has to determine which tuples of a terminal time and a terminal state can be reached by the use of a positive input; for controllability one considers a terminal time and terminal state as fixed and one has to determine from which tuples of an initial time and an initial state one can reach the selected terminal state at the terminal time by the use of a positive input.

In case of a time-invariant system the concepts of reachability and of controllability do not depend on the initial time because the time axis can be shifted to the zero time without affecting the trajectories.

Definition 5.2. Consider a linear positive system as defined in Definition 5.1.

(a) Fix an initial tuple $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$. The terminal tuple $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$ is called *reachable* from the initial tuple (i.e., can be *reached* from the initial tuple), if there exists a positive input $\boldsymbol{u} : \{t_0, t_0 + 1, \ldots, t_1 - 1\} \to \mathbb{R}_+^m$ such that the transition $(t_0, \boldsymbol{x}_0) \stackrel{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}(t_1)) = (t_1, \boldsymbol{x}_1)$ exists for this system. (R.E. Kalman states this for $\boldsymbol{x}_0 = 0$.) The terminal tuple is called *reachable from the origin* if it is reachable from the initial tuple $(t_0, 0) \in T \times \mathbb{R}_+^n$.

Define the *reachable set* from $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ as

$$\text{Reachset}(t_0, \boldsymbol{x}_0) = \left\{ \begin{array}{l} \boldsymbol{x}_1 \in \mathbb{R}_+^n | \; \exists \; t_1 \in T, \; \exists \; \boldsymbol{u} : \{t_0, \ldots, t_1 - 1\} \to \mathbb{R}_+^m, \\ \text{such that } (t_0, \boldsymbol{x}_0) \stackrel{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, x_1) \end{array} \right\}.$$

(b) Fix a terminal tuple $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$. The initial tuple $(t_0, \boldsymbol{x}_0) \in T \times \mathbb{R}_+^n$ is called *controllable* to the terminal tuple (i.e., can be *controlled* to the terminal tuple) if there exists an input $\boldsymbol{u} : \{t_0, t_0 + 1, \ldots, t_1 - 1\} \to \mathbb{R}_+^m$ such that the transition $(t_0, \boldsymbol{x}_0) \stackrel{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, \boldsymbol{x}_1)$ exists for this system. (R.E. Kalman requires that the terminal state $\boldsymbol{x}_1 = 0$.) The initial tuple is called *controllable to the origin* if it is controllable to the terminal tuple $(t_1, 0) \in T \times \mathbb{R}_+^n$.

Define the *controllable set* to the terminal tuple $(t_1, \boldsymbol{x}_1) \in T \times \mathbb{R}_+^n$ as

$$\text{Conset}(t_1, \boldsymbol{x}_1) = \left\{ \begin{array}{l} \boldsymbol{x}_0 \in \mathbb{R}_+^n | \; \exists \; t_0 \in T, \; \exists \; \boldsymbol{u} : \{t_0, \ldots, t_1 - 1\} \to \mathbb{R}_+^m, \\ \text{such that } (t_0, \boldsymbol{x}_0) \stackrel{\boldsymbol{u}(t_0:t_1-1)}{\mapsto} (t_1, x_1) \end{array} \right\}.$$

For linear systems, not necessarily a linear positive system, the following result holds.

Lemma 5.3. [34, Lemma 3.1.5] Consider a time-invariant discrete-time linear system (not necessarily a linear positive system). The system is a reachable system on the interval $\{t_0, \ldots, t_1\}$, if and only if it is reachable from the origin on the same interval.

The above result does not hold for linear positive systems as the following example shows.

Example 1. Consider the time-invariant linear positive system

$$\boldsymbol{x}(t+1) = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{x}(t) + \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \boldsymbol{u}(t), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0.$$

Then the reachable set from the origin is the full positive vector space $\mathbb{R}_+^2$. If $\boldsymbol{x}_0 = (1,1)^T$ then the reachable set from that initial state equals

$$X(\boldsymbol{x}_0) = \{\boldsymbol{x} \in \mathbb{R}_+^2 | \ x_1 \geq 0.5 x_{0,1} = 0.5, \ x_2 \geq x_{0,2} = 1\}.$$

Hence the state $\boldsymbol{x}_1 = (0.4, 0.4)^T$ can never be reached from $\boldsymbol{x}_0$ using positive inputs. Thus reachability from the origin and from an arbitary initial state of the positive vector space are different concepts for linear positive systems.

From the above example it is clear that the reachable set from the origin and the reachable set from an arbitrary initial state are different objects. In this chapter attention is restricted to the reachable set from the origin.

5.2.4. Existing Results on Reachability and Controllability of Linear Positive Systems

The existing view of the characterization of controllability and reachability as known in the literature, is discussed below. In most papers of the literature, the characterization of controllability or of reachability of a linear positive system is based on the following definition.

Definition 5.4. [16, p. 74, Def. 7]. A linear positive system is said to be completely reachable if all states $\boldsymbol{x} \geq 0$ are reachable in finite time from the origin, that is, if $X_{\mathrm{r}} = \mathbb{R}_+^n$, where X_{r} denotes the cone of all reachable states in finite time using a positive input.

The underlying idea behind Definition 5.4 probably originates from making an analogy to reachability of linear systems. This definition is based on the assumption that the state space equals $X = \mathbb{R}^n$. Note that in Definition 5.4 $X_{\mathrm{r}} \subseteq \mathbb{R}_+^n$ by definition, hence the equality $X_{\mathrm{r}} = \mathbb{R}_+^n$ holds if in addition $\mathbb{R}_+^n \subseteq X_{\mathrm{r}}$. The following theorem states a necessary and sufficient condition for reachability with respect to Definition 5.4 for the single-input case.

Theorem 5.5. *[16, Th. 27]. A discrete-time linear positive system with a single-input is completely reachable if it is possible to reorder its state variablesi in such a way that the input $\boldsymbol{u}$ directly influences only $\boldsymbol{x}_1$, and $\boldsymbol{x}_i$ directly influences $\boldsymbol{x}_{i+1}$ for $i = 1, 2, \ldots, n-1$.*

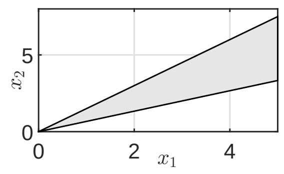

Figure 5.21: Example 2. The shaded area, associated with K, represents the region of interest for which controllability needs to be checked.

Additional results may be found in [16, Ch. 8]. The criterion for complete reachability of a linear positive system with multiple inputs based on Definition 5.4 is more involved, but it is required that the controllability matrix of the corresponding linear system, $[\boldsymbol{B}\ \boldsymbol{AB}\ \dots\ \boldsymbol{A}^k\boldsymbol{B}]$, includes a monomial submatrix of dimension n, for some $k \in \mathbb{N}_+$ [14, 15, 19, 21, 22]. Such conditions are often too strong to be satisfied by most practical linear positive systems.

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For several examples of linear positive systems, complete reachability as of Definition 5.4 is not required. For example in economic systems, one would be interested to know whether a certain growth rate can be achieved, which corresponds to checking whether a certain extremal ray of a cone inside the positive vector space is reachable. In bio-chemical reactors, it may be of interest to know whether a set of desired mass concentrations can be reached by applying a particular input (for example, a flow of materials).

An example follows that illustrates the concept of reachability stated above.

Example 2. Consider the discrete-time time-invariant linear positive system

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}u(t),\ \boldsymbol{x}(0) = \boldsymbol{x}_0,$$

with

$$\boldsymbol{A} = \begin{bmatrix} 4 & 4 \\ 11 & 2 \end{bmatrix},\ \boldsymbol{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix},\ \boldsymbol{x}_0 = 0.$$

It is of interest to determine whether the states in the cone $K \subset \mathbb{R}^2_+$, defined by (5.7) and illustrated by Figure 5.21, can be reached in finite time:

$$K : \begin{cases} 3x_1 - 2x_2 \geq 0, \\ 3x_2 - 2x_1 \geq 0, \\ x_1 \geq 0,\ x_2 \geq 0. \end{cases} \tag{5.7}$$

Since $K \subset \mathbb{R}^2_+$, in order to answer this question using the classical approach, one needs to check the reachability of $\mathbb{R}^2_+$, which is very conservative considering the fact that K occupies only a small portion of $\mathbb{R}^2_+$. It can be verified that

$$\begin{bmatrix} \boldsymbol{b} & \boldsymbol{Ab} & \dots & \boldsymbol{A}^k\boldsymbol{b} \end{bmatrix} = \begin{bmatrix} 2 & 12 & \cdots \\ 1 & 24 & \cdots \end{bmatrix}$$

does not include a monomial submatrix of dimension 2 for any $k \in \mathbb{N}_+$. Therefore, the conditions of Theorem 5.5 do not hold and we cannot deduce anything about the reachability of K. Nevertheless, invoking Theorem 5.15 and using the results of Section 5.6, it turns out that K is reachable from the origin in a finite number of steps.

5.3. Approach of This Chapter

The chapter changes the focus of reachability of a linear positive system. In the classical literature the system is reachable from the origin if the reach set from the origin equals the entire positive vector space $\mathbb{R}_+^n$.

In this chapter, the approach is to determine the reachable set from the origin, in either finite time or in infinite time, as defined below. The reachable set is then the main object of study. In this chapter, there is no requirement that the reachable set from the origin equals the positive vector space $\mathbb{R}_+^n$.

In the late 1960s and the 1970s the geometric view point gained momentum in control and system theory. This viewpoint was developed by W.M. Wonham, [37], for time-invariant linear control systems using the concept of a linear subspace of a vector space. The geometric approach to control of nonlinear control systems was described in the book [38]. Later this led to the development of control theory in differential-geometric structures, [39, 40], and in algebraic-geometric structures such as rings [41].

In the geometric approach to control systems the main concept is the reachable set from the origin. In the context of observability, it is the kernel of the output map, but that will not be treated in this chapter. For linear positive systems, the main geometric concept is a cone in the positive vector space $\mathbb{R}_+^n$. This geometric object allows the use of abstract algebra for theory and algorithms. Therefore, in this chapter the geometric approach to linear positive systems is used.

Based on this new view point, the system theoretic problem under study is: Characterize the reachable set from the origin of a linear positive system. The reachable set from the origin is by definition a cone in the positive vector space. A question is then: Is the reachable set from the origin a polyhedral cone or a nonpolyhedral cone?

Remark 3. The above formulation has been for decades the approach to reachability in system theory. The reachable set from the origin is defined as stated above. The reachable set in general may be a strict subset of the ambient space in which it is situated. The reader may want to look at the definitions of the reachable subset for discrete-time polynomial systems, [41], for continuous-time polynomial systems, [42], rational systems, [43], and infinite-dimensional linear systems, [44].

5.3.1. Concepts

The reachable set and its role in the problem of reachability and of controllability of linear positive systems have been already discussed in the literature [14, 15, 19, 21, 22]. Below the concept inspired by [15] is used. Recall that only reachability from the origin, the zero initial state, is considered and that the system is restricted to have an input with only one component. Recall the formula of the state transition of a time-invariant discrete-time linear positive system as

$$\begin{aligned} \boldsymbol{x}(t) &= \sum_{s=0}^{t-1} \boldsymbol{A}^s\, \boldsymbol{b}\, u(t-1-s) \\ &= \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \dots & \boldsymbol{A}^{t-1}\boldsymbol{b} \end{bmatrix} \begin{bmatrix} \boldsymbol{u}(t-1) \\ \boldsymbol{u}(t-2) \\ \vdots \\ \boldsymbol{u}(0) \end{bmatrix} \end{aligned} \tag{5.8}$$

with $\text{conmat}_k(\boldsymbol{A}, \boldsymbol{b}) = \begin{bmatrix} \boldsymbol{b} & \boldsymbol{A}\boldsymbol{b} & \dots & \boldsymbol{A}^{k-1}\boldsymbol{b} \end{bmatrix}$ being the *controllability matrix of index* k.

It is useful to have notation for the infinite reachable set and to contrast that with the finite reachable set, which is the purpose of the following definition.

Definition 5.6. Consider a single-input time-invariant discrete-time linear positive system with representation

$$\boldsymbol{x}(t+1) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{b}\boldsymbol{u}(t),\ \boldsymbol{x}(0) = 0. \tag{5.9}$$

Define the following subsets of the state space: the k*-step reachable subset from the origin*, the *finite-time reachable subset from the origin*, and the *infinite-time reachable subset from the origin*, respectively as the sets,

$$\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}) = \left\{ \boldsymbol{x} \in \mathbb{R}_+^n |\ \exists \boldsymbol{u} : \mathbb{N}^{k-1} \to \mathbb{R}_+,\ (0,0) \overset{\boldsymbol{u}}{\mapsto} (k, \boldsymbol{x}) \right\},\ \forall\ k \in \mathbb{Z}_+, \tag{5.10}$$

$$\text{Reachset}_f(\boldsymbol{A}, \boldsymbol{b}) = \cup_{k=0}^{\infty} \text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}), \tag{5.11}$$

$$\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = \overline{\text{Reachset}_f(\boldsymbol{A}, \boldsymbol{b})}. \tag{5.12}$$

Here, the notation $\overline{S}$ denotes the closure of the set S with respect to the Euclidean topology.

The reachable subsets defined above are subsets of the state set. To simplify the terminology, in the remainder of the chapter these sets are referred to as the *reachable set from the origin* or as the *reachable set*, without the use of the term *subset*.

Once a reachable set has been defined, there is no need for the concept of complete reachability.

Proposition 5.7*.* The k-step reachable subset, the finite-time reachable subset, and the infinite-time reachable subset of Definition 5.6, each from the zero initial state, equal respectively the expressions

$$\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big(\text{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big), \tag{5.13}$$

$$\text{Reachset}_f(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \boldsymbol{A}^2\boldsymbol{b}\ \dots]\big), \tag{5.14}$$

$$\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = \overline{\text{Reachset}_f(\boldsymbol{A}, \boldsymbol{b})},\ \text{where} \tag{5.15}$$

$$\text{conmat}_k(\boldsymbol{A}, \boldsymbol{b}) = [\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \boldsymbol{A}^2\boldsymbol{b}\ \dots\ \boldsymbol{A}^{k-1}\boldsymbol{b}] \tag{5.16}$$

Proof. The proof is skipped as it can be derived in a straightforward manner. The reader is referred to [22, 45] for similar proofs. The proof could also be deduced from the corresponding definition in [15]. □

5.3.2. Problem Formulation

Having characterized the infinite-time and the finite-time reachable sets from the origin, the main questions of this chapter are discussed next.

Problem 5.8. For a single-input time-invariant linear positive system, the problems to be addressed in this chapter are:

(a) Is the finite-time reachable set from the origin $\text{Reachset}_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$ a polyhedral cone or a nonpolyhedral cone?

(b) Is the infinite-time reachable set from the origin $\text{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$ a polyhedral cone or a nonpolyhedral cone?

(c) If the control objective is specified as a cone in the positive vector space or as a subset of that space, is that control objective subset then contained in the reachable set from the origin?

Note that the k-time reachable set is by definition always a polyhedral set.

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5.4. When Is the Infinite-Time Reachable Set a Polyhedral Set?

In this section, we investigate the polyhedrality of $\text{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$, and characterize this in terms of a necessary and sufficient conditions on the system matrix $\boldsymbol{A}$.

The reader is expected to have knowledge of concepts and of results of positive linear algebra as summarized in Appendix 5.A. The notations used below may be found in Appendix 5.A.

As summarized in Appendix 5.A, a positive matrix which is nonzero and of dimension $n \geq 2$ is either irreducible or can be fully reduced. The analysis of the matrix $\boldsymbol{A}^k$ for $k \in \mathbb{Z}_+$ or for its limit, $\lim_{k\to\infty} \boldsymbol{A}^k$, can then be carried out (1) for irreducible positive matrices and, (2) for fully reduced matrices. Below the case of an irreducible system matrix $\boldsymbol{A}$ is carried out. The case of a fully reduced positive matrix is then relatively simple based on the results for the irreducible case [9].

For the remainder of this section, the reader should keep in mind the restriction to an irreducible positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$.

Proposition 5.9. Consider the linear positive system given in (5.4). Assume that $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ is irreducible with cyclicity index $1 \leq h \leq n$ and $\boldsymbol{b} \in \mathbb{R}_+^n$. Then, the infinite-time reachable set from the origin, $\text{Reachset}_{\infty}(\boldsymbol{A}, \boldsymbol{b})$, is polyhedral if and only if there exists a $k^* \in \mathbb{Z}_+$ such that

$$\boldsymbol{A}^{k^*}\boldsymbol{b} \in \text{cone}\left([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \ \boldsymbol{A}_{\text{f},0}b\ \ldots\ \boldsymbol{A}_{\text{f},h-1}\boldsymbol{b}]\right), \tag{5.17}$$

where matrices $\boldsymbol{A}_{\text{f},i}$ are introduced in Definition 5.24.

Proof. The result is almost obvious by geometric considerations except for the presence of the set of vectors $\{\boldsymbol{A}_{\mathrm{f},0}b, \ldots, \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}\}$.

Sufficiency: We will show that

$$C = \mathrm{cone}\left([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}]\right)$$

is $\boldsymbol{A}$-invariant. Let $\boldsymbol{x} = \sum_{i=0}^{k^*-1} c_i \boldsymbol{A}^i\boldsymbol{b} + \sum_{i=0}^{h-1} c_{\mathrm{f},i}\boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}$ for arbitrary positive coefficients $\boldsymbol{c} \in \mathbb{R}_+^{k^*}$ and $\boldsymbol{c}_{\mathrm{f}} \in \mathbb{R}_+^h$. We then have

$$\boldsymbol{A}\boldsymbol{x} = \sum_{i=0}^{k^*-1} c_i \boldsymbol{A}^{i+1}\boldsymbol{b} + \sum_{i=0}^{h-1} c_{\mathrm{f},i}\boldsymbol{A}\boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}. \tag{5.18}$$

Using (5.17), and noting that (see Definition 5.24)

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}_{\mathrm{f},i} &= \boldsymbol{A}_{\mathrm{f},i+1}, \quad i = 0, \ldots, h-2 \\ \boldsymbol{A}\boldsymbol{A}_{\mathrm{f},h-1} &= \left(\rho(\boldsymbol{A})\right)^h \boldsymbol{A}_{\mathrm{f},0}, \end{aligned} \tag{5.19}$$

(5.18) can be expressed as $\boldsymbol{A}\boldsymbol{x} = \sum_{i=0}^{k^*-1} c'_i \boldsymbol{A}^i\boldsymbol{b} + \sum_{i=0}^{h-1} c'_{\mathrm{f},i}\boldsymbol{A}_{\mathrm{f},i}\boldsymbol{b}$ for some $\boldsymbol{c}' \in \mathbb{R}_+^{k^*}$ and some $\boldsymbol{c}'_{\mathrm{f},i} \in \mathbb{R}_+^h$. This proves that $\boldsymbol{A}\boldsymbol{x} \in C$ for any $\boldsymbol{x} \in C$. Hence, the system trajectory (5.8) remains in C and $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = C$ is polyhedral.

Necessity: Let $\boldsymbol{x}_\infty = \lim_{k\to\infty} \dfrac{\boldsymbol{A}^k\boldsymbol{b}}{\left(\rho(\boldsymbol{A})\right)^k}$. Note that $\boldsymbol{x}_\infty$ is characterized by the set of h vectors $\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}, \ldots, \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}$ [15, Th. 2](also see proof of Lemma 5.25.) In fact, Lemma 5.25 states that $\boldsymbol{x}_\infty \in \mathrm{cone}([\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. By the definition of $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ as the closure of $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$, and by the above explanation of $\boldsymbol{x}_\infty$, the extremal rays of the polyhedral
$\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ belong to the sequence $\{\boldsymbol{A}^k\boldsymbol{b} \in \mathbb{R}_+^n,\ k \in \mathbb{N}\}$ or are extremal rays of the cone, $\mathrm{cone}([\boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. Again, by the assumption that $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral, there exists a finite $k^* \in \mathbb{Z}_+$ such that
$\boldsymbol{A}^{k^*}\boldsymbol{b} \in \mathrm{cone}([\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}])$. □

It is clear that if (5.17) is established for an integer $k^* \in \mathbb{Z}_+$, it will hold for any $k \geq k^*$. The smallest integer $k^* \in \mathbb{Z}_+$ satisfying (5.17) is called the *vertex number* and denoted by k_{vert}^∞ of the reachable set $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$. Following the steps of the proof of Proposition 5.9, we can put forward the following corollary.

Corollary 5.10. Given $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ irreducible with cyclicity index $h \in \{1, \ldots, n\}$ and $\boldsymbol{b} \in \mathbb{R}_+^n$, the following statements are equivalent:

(a) $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral.

(b) There exists an integer $k_{\mathrm{vert}}^\infty \in \mathbb{Z}_+$ such that $\mathrm{cone}\left([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^{k-1}\boldsymbol{b}\ \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}]\right)$ is $\boldsymbol{A}$-invariant for $k \geq k_{\mathrm{vert}}^\infty$.

(c) There exists an integer $k_{\mathrm{vert}}^\infty \in \mathbb{Z}_+$ such that for all $k \geq k_{\mathrm{vert}}^\infty$, the matrix equation

$$\boldsymbol{A}\boldsymbol{M} = \boldsymbol{M}\boldsymbol{X}, \text{ has a solution } \boldsymbol{X} \in \mathbb{R}_+^{(k+h)\times(k+h)}, \text{ where}$$
$$\boldsymbol{M} = \left[\ \boldsymbol{b}\quad \boldsymbol{A}\boldsymbol{b}\quad \ldots\quad \boldsymbol{A}^{k-1}\boldsymbol{b}\quad \boldsymbol{A}_{\mathrm{f},0}\boldsymbol{b}\quad \ldots\quad \boldsymbol{A}_{\mathrm{f},h-1}\boldsymbol{b}\ \right].$$

Definition 5.11. A square positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ is said to have a *positive recursion* if the following holds:

$$\exists m \in \mathbb{N},\ \exists c_i \in \mathbb{R}_+ \text{ for } i = 0, \dots, m-1 \text{ such that} \tag{5.20}$$

$$\boldsymbol{A}^m = \sum_{i=0}^{m-1} c_i \boldsymbol{A}^i,$$

or, equivalently, if

$$g(\lambda) = \lambda^m - \sum_{i=0}^{m-1} c_i \lambda^i = 0, \quad \forall \lambda \in \operatorname{spec}(\boldsymbol{A}).$$

In terms of the characteristic polynomial of $\boldsymbol{A}$, $p_{\boldsymbol{A}}$, the existence of a positive recursion implies that $g = p_{\boldsymbol{A}}\, Q$, where Q is a polynomial of degree q with $0 \le q \le m$. It is then immediate that

$$m = n + q \ge n. \tag{5.21}$$

Before presenting our main results on polyhedrality of reachable subsets, we report a key theorem ([46, Th. 5]). In the following $\mathcal{Q}$ denotes the set of all real polynomials of the form $c_n x^n - \sum_{i=0}^{n-1} c_i x^i$, where $n \ge 1$, $c_n > 0$, and $c_i \ge 0$ for all i.

Theorem 5.12. *[46, Th. 5] Let $\{a_1, \dots, a_k\}$ be given complex numbers, and let $P(x)$ be the polynomial $x^k - a_1 x^{k-l} - \cdots - a_k$. Then conditions (A), (B) and (C) below are equivalent:*

(A) Any infinite sequence $(u_n)_{n\ge 0}$ of complex numbers which satisfies the recursion $u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \cdots + a_k u_n$ for $n \ge 0$, also satisfies a recursion with positive coefficients.

(B) The polynomial $P(x)$ divides a polynomial in $\mathcal{Q}$.

(C) In case the polynomial $P(x)$ has a positive root r, then all conditions (1)-(4) below are satisfied:

- *(C1) $r \ge |\alpha|$ for any root α of $P(x)$.*
- *(C2) if $\alpha = r$ for some root α of $P(x)$, then α/r is a root of unity.*
- *(C3) all roots $P(x)$ with absolute value r are simple.*
- *(C4) if $P(r) = P(r\epsilon) = 0$, where $\epsilon^k = 1$ with $k \ge 1$ minimal, then $P(x)$ has no roots of the form $s\omega$ where $0 < s < r$ and $\omega^k = 1$.*

We are now in the position to state a characterization of Proposition 5.9 in terms of spec($\boldsymbol{A}$), hence, providing numerically verifiable conditions as to when (5.17) holds.

Theorem 5.13 (Polyhedrality of Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$)**.** *Given an irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ and $\boldsymbol{b} \in \mathbb{R}_+^n$, the following statements are equivalent:*

(a) *The infinite-time reachable subset is polyhedral, hence there exists an integer* $k^* \in \mathbb{Z}_+$ *such that*

$$\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big([\text{conmat}_{k^*}(\boldsymbol{A}, \boldsymbol{b}) \;\; \boldsymbol{A}_{f,0}\boldsymbol{b} \;\; \dots \;\; \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\big).$$

Denote the lowest integer for which the above equality holds by $k_{vert}^\infty \in \mathbb{Z}_+$.

(b) *The matrix* $\boldsymbol{A}_2$ *defined in Definition 5.22, satisfies a positive recursion.*

(c) *If there exists a positive* $\lambda_r \in \text{spec}(\boldsymbol{A}_2)$, *then the following conditions all hold:*

(c1) $\lambda_r = \rho(\boldsymbol{A}_2)$.

(c2) *For any* $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$, $\lambda = \rho(\boldsymbol{A}_2)\exp(\phi_\lambda 2\pi i)$, *where* $\phi_\lambda \in \mathbb{Q}$ *is a rational number.*

(c3) $\sigma^\rho(\boldsymbol{A}_2)$, *defined in Definition 5.22, includes only simple eigenvalues.*

(c4) *Given* $M \in \mathbb{Z}_+$ *by Lemma 5.23, no* $\lambda^- \in \sigma^-(\boldsymbol{A}_2)$ *has a polar angle which is an integer multiple of* $2\pi/Mh$.

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Note that the condition (a) of Theorem 5.13 involves the determination of the integer k^*, which is in principle a test with an infinite number of steps. Similarly, condition (b) is a test with an infinite number of steps. However, condition (c) of the theorem is a finite test though it requires the exact eigenvalues.

Proof. (a)⇒(b)⇒(c): Since $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral, according to Corollary 5.10, there is a sufficiently large $k \geq n - h$ such that the equation

$$\boldsymbol{A}[\boldsymbol{b} \; \boldsymbol{A}\boldsymbol{b} \; \dots \; \boldsymbol{A}^{k-1}\boldsymbol{b} \; \boldsymbol{A}_{f,0} \; \dots \; \boldsymbol{A}_{f,h-1}] = [\boldsymbol{b} \; \boldsymbol{A}\boldsymbol{b} \; \dots \; \boldsymbol{A}^{k-1}\boldsymbol{b} \; \boldsymbol{A}_{f,0}\boldsymbol{b} \; \dots \; \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\boldsymbol{X}$$

has a solution $\boldsymbol{X} \geq 0$. It can be easily verified using (5.17)-(5.19) that

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1 & 0 \\ \boldsymbol{X}_3 & \boldsymbol{X}_2 \end{bmatrix}, \; \boldsymbol{X}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & & 0 & \alpha_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{k-1} \end{bmatrix}, \tag{5.22}$$

$$\boldsymbol{X}_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \rho^h(\boldsymbol{A}) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \; \boldsymbol{X}_3 = \begin{bmatrix} 0 & 0 & \cdots & 0 & \beta_0 \\ 0 & 0 & \cdots & 0 & \beta_1 \\ 0 & 0 & & 0 & \beta_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 0 & \beta_{h-1} \end{bmatrix}. \tag{5.23}$$

constitutes a solution, where $\boldsymbol{X}_1 \in \mathbb{R}_+^{k\times k}$, $\boldsymbol{X}_2 \in \mathbb{R}_+^{h\times h}$, and $\boldsymbol{X}_3 \in \mathbb{R}_+^{h\times k}$. Let $p_{\boldsymbol{X}_1}(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{X}_1)$ and $p_{\boldsymbol{X}_2}(\lambda) = \det(\lambda\boldsymbol{I} - \boldsymbol{X}_2)$. Since by assumption, $k \geq n - h$ and $\text{rank}\big(\text{conmat}_n(\boldsymbol{A}, \boldsymbol{b})\big) = n$, due to [47, Lemma 3.10], $p_{\boldsymbol{A}}(\lambda)$ divides $p_{\boldsymbol{X}}(\lambda) = p_{\boldsymbol{X}_1}(\lambda)$ $p_{\boldsymbol{X}_2}(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))(\lambda^k - \alpha_{k-1}\lambda^{k-1} - \dots - \alpha_0)$. Since $\boldsymbol{A}$ is irreducible with cyclicity index h, $p_{\boldsymbol{A}}(\lambda)$ can be expressed as $p_{\boldsymbol{A}}(\lambda) = p_{\boldsymbol{A}_1}(\lambda)p_{\boldsymbol{A}_2}(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))p_{\boldsymbol{A}_2}(\lambda)$.

Therefore, $p_{\boldsymbol{A}_2}(\lambda)$ divides $p_{\boldsymbol{X}_2}(\lambda)$, which, due to statements (A) and (B) of Theorem 5.12, proves that $\boldsymbol{A}_2$ has a positive recursion of the form $\boldsymbol{A}_2^{k^*} - \gamma_{k^*-1}\boldsymbol{A}_2^{k^*-1} - \cdots - \gamma_0 \boldsymbol{I} = 0$ for some $n-h \le k^* \le k$ and for some $\boldsymbol{\gamma} \in \mathbb{R}_+^{k^*}$. Assume $\boldsymbol{A}_2$ has a positive eigenvalue. Since $\boldsymbol{A}_2$ satisfies a positive recursion, the statements (C1-C4) in (C) of Theorem 5.12 hold for $p_{\boldsymbol{A}_2}(\lambda)$. It is straightforward to check that this implies that (c1)-(c4) holds[1].

(c)⇒(b)⇒(a): Assume $\boldsymbol{A}_2$ has a positive eigenvalue. We need to prove that statements (c1)-(c4) imply a positive recursion for $\boldsymbol{A}_2$ of the form $\boldsymbol{A}_2^{k^*} - \alpha_{k^*-1}\boldsymbol{A}_2^{k^*-1} - \cdots - \alpha_0 \boldsymbol{I} = 0$, for $k^* \ge n-h$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^{k^*}$, and that, in turn, implies polyhedrality of the infinite-time reachable subset.
First we show that the statements (c1)-(c4) imply the statements (C1)-(C4) of Theorem 5.12. The statement $\lambda_r \in \sigma^\rho(\boldsymbol{A}_2)$ implies (C1) of Theorem 5.12. The requirement of all $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$ having a rational polar phase implies (C2). The requirement of all $\lambda \in \sigma^\rho(\boldsymbol{A}_2)$ being simple implies (C3), and (C4) is implied from $\sigma^-(\boldsymbol{A}_2)$ including no eigenvalue with polar phase $2\pi m/Mh$ for any $m \in \mathbb{Z}$ [9, Theorem 2.2.20]. Next, invoking the equivalence between (C) and (B) of Theorem 5.12 for $p_{\boldsymbol{A}_2}(\lambda)$, one can observe that there is a polynomial $Q(\lambda)$ of positive degree such that

$$g(\lambda) = p_{\boldsymbol{A}_2}(\lambda)Q(\lambda) = \lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_0 = 0, \tag{5.24}$$

for $k^* \ge n-h$ and $\boldsymbol{\alpha} \in \mathbb{R}_+^{k^*}$. It follows from (5.20) that $\boldsymbol{A}_2$ has a positive recursion, which results in (b).
Given (b), there exists a polynomial $g(\lambda)$ of degree $k^* \ge n-h$ satisfying (5.24), from which one concludes that $p_{\boldsymbol{A}}(\lambda) = p_{\boldsymbol{A}_1}(\lambda)p_{\boldsymbol{A}_2}(\lambda)$ divides $h(\lambda) = p_{\boldsymbol{A}_1}(\lambda)g(\lambda) = (\lambda^h - \rho^h(\boldsymbol{A}))(\lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_0)$. Now consider the equation $\boldsymbol{AM} = \boldsymbol{MX}$ with $\boldsymbol{M} = [\boldsymbol{b}\ \boldsymbol{Ab}\ \ldots\ \boldsymbol{A}^{k^*-1}\boldsymbol{b}\ \boldsymbol{A}_{f,0}\boldsymbol{b}\ \ldots\ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]$, where $\boldsymbol{X} \in \mathbb{R}^{(n+k^*)\times(n+k^*)}$ is an unknown matrix. Since $\text{conmat}_{k^*}(\boldsymbol{A}, \boldsymbol{b})$ is full rank by assumption and $k^* \ge n-h$, $\boldsymbol{M}$ is of full rank as well. Then, it is known from [47, Lemma 10] that $p_{\boldsymbol{A}}(\lambda)$ divides $p_{\boldsymbol{X}}(\lambda)$. Hence, we can choose $\boldsymbol{X}$ such that $p_{\boldsymbol{X}}(\lambda) = h(\lambda)$. A possible choice of $\boldsymbol{X}$, having substituted k^* for k, is then given by (5.22)-(5.23). It is clear from (5.22)-(5.23) that $\boldsymbol{X}$ admits a positive solution. Based on Corollary 5.10, this implies that $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral. □

Remark 4. For a polyhedral $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ the following can be observed:

(a) Due to (5.21) and from the second part of the proof of Theorem 5.13 the vertex number of $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$, k_{vert}^∞, is at least $n-h$, which implies that $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ has at least n generators. It has exactly n generators (i.e., it is simplicial) if and only if the characteristic polynomial $p_{\boldsymbol{A}_2}$ of $\boldsymbol{A}_2$ has non-positive coefficients.

(b) In the view of Lemma 5.25, $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ can be expressed as $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big([\boldsymbol{b}\ \boldsymbol{Ab}\ \ldots\ \boldsymbol{A}^{k-1}\boldsymbol{b}\ \boldsymbol{v}_{f,0}\ \ldots\ \boldsymbol{v}_{f,h-1}]\big)$, where $\boldsymbol{v}_{f,0}, \ldots, \boldsymbol{v}_{f,h-1}$ are the h distinct positive eigenvectors of $\boldsymbol{A}^h$ associated with the eigenvalue $\rho^h(\boldsymbol{A})$.

[1] Condition $\lambda_r \in \sigma^\rho(\boldsymbol{A}_2)$ follows from (C1) of Theorem 5.12, and conditions (c2) and (c3) are, respectively, a direct result of (C2) and (C3). Finally, (c4) is implied from (C4) using Lemma 5.23.

Example 3 (polyhedral Reachset∞$(\boldsymbol{A}, \boldsymbol{b})$). Consider discrete-time linear time-invariant positive system of (5.4) with system matrices

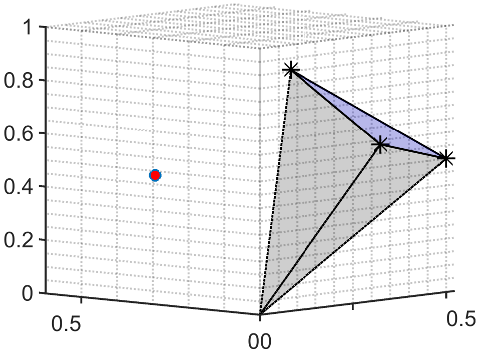
$$\boldsymbol{A} = \begin{bmatrix} 0.9727 & 0 & 0.0263 \\ 0.0388 & 0.1273 & 0.2156 \\ 0 & 3.4497 & 0 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

where $\boldsymbol{A}$ is primitive, i.e., is irreducible with cyclicity index $h = 1$. We have spec$(\boldsymbol{A}) = \{1, 0.9, -0.8\}$. We can assume $\boldsymbol{A}_1 = 1$, and $\boldsymbol{A}_2 =$ diag$(0.9, -0.8)$. Using Theorem 5.13, it is immediate that conditions (c1) and (c2) hold as $\lambda = 0.9$ is a simple eigenvalue of $\boldsymbol{A}_2$, which equals the spectral radius of $\boldsymbol{A}_2$. Condition (c1) holds as well since the polar angle of $\lambda = -0.8$ is not an integer multiple of the polar angle of $\lambda = 0.9$. Hence, it can be concluded that the infinite-time reachable subset Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral. We can also conclude that $\boldsymbol{A}_2$ has a positive recursion, which is readily verified as $p_{\boldsymbol{A}_2}(\lambda) = \lambda^2 - 0.1\lambda - 0.72$. Example 5.41 illustrates the growth of Reachset$_k(\boldsymbol{A}, \boldsymbol{b})$. It can be observed that Reachset$_\mathrm{f}(\boldsymbol{A}, \boldsymbol{b})$ is not polyhedral since the cone keeps growing for increasing values of k. Its closure is, however, polyhedral as shown in Figure 5.41d.

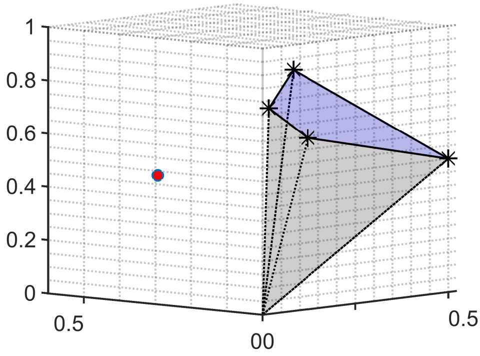
Example 4 (non-polyhedral Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$). Consider the time-invariant discrete-time linear positive system of (5.4) with system matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0.5 \\ 0 & 0.4 & 1 \end{bmatrix}, \ \boldsymbol{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

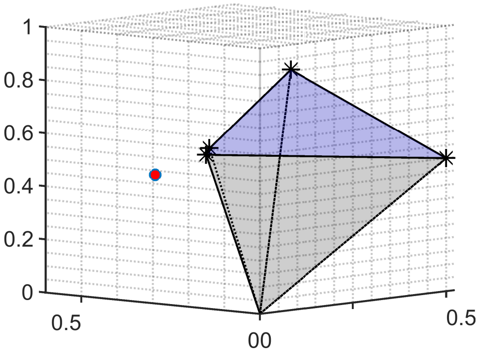
where $\boldsymbol{A}$ has cyclicity index $h = 1$ with spec$(\boldsymbol{A}) = \{-1.05, 0.7116, 1.3383\}$. One can assume $\boldsymbol{A}_1 = 1.3383$ and $\boldsymbol{A}_2 =$ diag$(-1.05, 0.7116)$. It is immediate that condition (c1) of Theorem 5.13 is not satisfied as $0.7116 \neq \rho(\boldsymbol{A}_2)$. Therefore, Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$ is not polyhedral. This is illustrated by Figure 5.42d, from which it is clear that Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$ is approaching a round cone as introduced in Section 5.2.



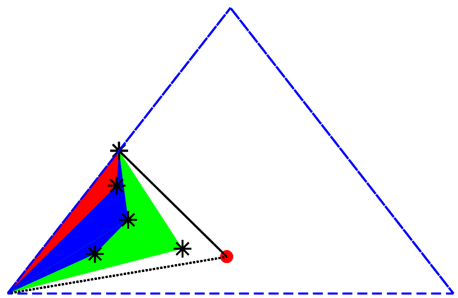
(a) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 3$



(b) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 8$



(c) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 19$



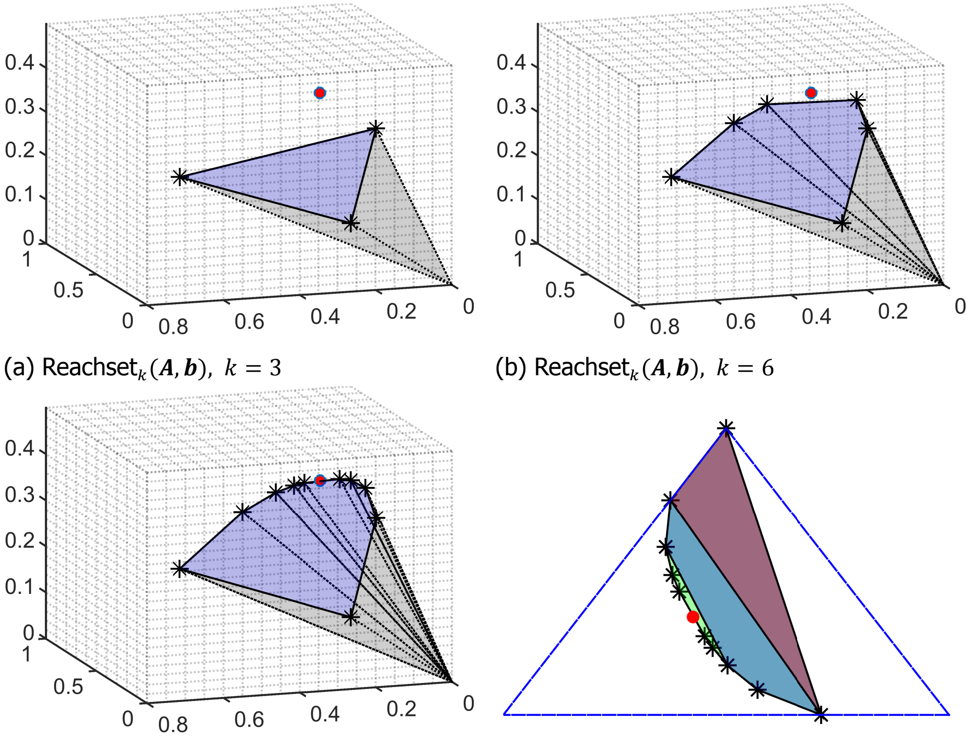
(d) $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, k=3 (red), 8 (blue and red), 19 (green, blue and red) and $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ (the triangle with the red vertex)

Figure 5.41: (a), (b), (c): The growth of the reachability cone $\mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$ of Example 3 for different values of k, where generators of the cone are marked by asterisks, and the Frobenius eigenvector is marked by a red dot. (d): The growth of the reachable cone mapped on the 3-dimensional simplex $S = \{\boldsymbol{x} \in \mathbb{R}^3_+ | \mathbb{1}^{\mathrm{T}}\boldsymbol{x} = 1\}$.

5.5. When Is the Finite-Time Reachable Subsets a Polyhedral Set?

The polyhedrality of the finite-time reachability set from the origin, $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$, will be proven to be a special case of polyhedrality of $\mathrm{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ but with stricter requirements.

In this section we investigate the polyhedrality of the finite-time reachable set from the origin, $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$. Consider a linear positive system with an irreducible system matrix $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$. with the cyclicity index $h \in \{1, \dots, n\}$. It follows from Proposition 5.9 that the finite-time reachable set from the origin $\mathrm{Reachset}_{\mathrm{f}}(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral if and only if there exists a positive integer $k^* \in \mathbb{Z}_+$



(a) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 3$

(b) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 6$

(c) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 10$

(d) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b}),\ k = 3$ (red region), $k = 6$ (red and blue regions), $k = 10$ (red, blue and green regions). $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$ approaches a "round cone".

Figure 5.42: (a), (b), (c): The growth of the reachability cone $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$ of Example 4 for different values of k, where generators of the cone are marked by asterisks, and the Frobenius eigenvector is marked by a red dot. (d): The growth of the reachable cone mapped on the 3-dimensional simplex $S = \{\boldsymbol{x} \in \mathbb{R}^3_+ | \mathbb{1}^\mathrm{T}\boldsymbol{x} = 1\}$.

such that

$$\text{Reachset}_{k^*+1}(\boldsymbol{A}, \boldsymbol{b}) \subseteq \text{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b}), \tag{5.25}$$

$$\Leftrightarrow \quad \boldsymbol{A}^{k^*}\boldsymbol{b} \in \text{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b}). \tag{5.26}$$

The smallest k^* for which (5.26) holds is referred to as the *vertex number*, k_{vert}, of $\text{Reachset}_\text{f}(\boldsymbol{A}, \boldsymbol{b})$. Note that (5.26) also implies that

$$\text{cone}\big([\boldsymbol{A}_{f,0}\boldsymbol{b} \ \ldots \ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]\big) \subseteq \text{Reachset}_{k_{\text{vert}}}(\boldsymbol{A}, \boldsymbol{b}), \tag{5.27}$$

which is clearly a restriction on (5.17).

Corollary 5.14. For an irreducible $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$ with cyclicity index $1 \le h \le n$ and for $b \in \mathbb{R}_+$, equivalence of the following statements follows directly from the above argument:

(a) $\mathrm{Reachset_f}(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral.

(b) There exists an integer $k_{\mathrm{vert}} \in \mathbb{Z}_+$ such that $\mathrm{cone}\big([\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \dots\ \boldsymbol{A}^k\boldsymbol{b}]\big)$ is $\boldsymbol{A}$-invariant for any $k \geq k_{\mathrm{vert}}$.

(c) There exists an integer $k_{\mathrm{vert}} \in \mathbb{Z}_+$ such that for the matrix equation

$$\boldsymbol{A}[\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \dots\ \boldsymbol{A}^{k-1}\boldsymbol{b}] = [\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \dots\ \boldsymbol{A}^{k-1}\boldsymbol{b}]\boldsymbol{X},$$

there exists a solution $\boldsymbol{X} \in \mathbb{R}_+^{(k)\times(k)}$, with $k \geq k_{\mathrm{vert}}$.

(d) Based on (5.27) and Lemma 5.25, there exists an integer $k_{\mathrm{vert}} \in \mathbb{Z}_+$ such that for any $k \geq k_{\mathrm{vert}}$, $\mathrm{cone}\big([\boldsymbol{v}_{f,0}\ \dots\ \boldsymbol{v}_{f,h-1}]\big) \subseteq \mathrm{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$.

The following theorem provides necessary and sufficient conditions on $\mathrm{spec}(\boldsymbol{A})$ for polyhedrality of $\mathrm{Reachset_f}(\boldsymbol{A}, \boldsymbol{b})$. These conditions turn out to be a conservative version of those of Theorem 5.13.

Theorem 5.15 (Polyhedrality of $\mathrm{Reachset_f}(\boldsymbol{A}, \boldsymbol{b})$)**.** *Let $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ be irreducible with index of cyclicity $h \in \{1, \dots, n\}$ and $\boldsymbol{b} \in \mathbb{R}_+^n$. Then the following statements are equivalent:*

(a) The finite-time controllable subset is polyhedral and hence there exists an integer $k^* \in \mathbb{Z}_+$, $k^* \geq k_{\mathrm{vert}}$, such that $\mathrm{Reachset_f}(\boldsymbol{A}, \boldsymbol{b}) = \mathrm{Reachset}_{k^*}(\boldsymbol{A}, \boldsymbol{b})$.

(b) $\boldsymbol{A}$ has a positive recursion.

(c) The matrix $\boldsymbol{A}_2$, defined in Definition 5.22, does not have any positive eigenvalue.

Proof. (a) ⇒ (b) ⇒ (c): Based on Corollary 5.14 with $k \geq n$ we obtain

$$\boldsymbol{A}\big(\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big) = \big(\mathrm{conmat}_k(\boldsymbol{A}, \boldsymbol{b})\big)\boldsymbol{X},$$

where $\boldsymbol{X} \in \mathbb{R}_+^{k\times k}$ is given by

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_0 \\ 1 & 0 & \cdots & 0 & \alpha_1 \\ 0 & 1 & & 0 & \alpha_2 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & \alpha_{k-1} \end{bmatrix}.$$

Since, by assumption, $\mathrm{conmat}_n(\boldsymbol{A}, \boldsymbol{b})$ is of full rank and $k \geq n$, there exists [47, Lemma 3.10] a polynomial $Q(\lambda)$ of positive degree such that $p_{\boldsymbol{A}}(\lambda)Q(\lambda) = p_{\boldsymbol{X}}(\lambda) = \lambda^k - \alpha_{k-1}\lambda^{k-1} - \dots - \alpha_1\lambda - \alpha_0$, which, in the view of Definition 5.11, proves that $\boldsymbol{A}$ has a positive recursion. Noting that (b) is equivalent to condition (B) of Theorem 5.12 ([46, Th. 5]), all conditions (C1)-(C4) are then fulfilled. In particular, (C4) holds as conditions (C1)-(C3) are already satisfied for a positive irreducible matrix due to the Perron-Frobenius theorem [9, Th. 2.1.4, 2.2.20]. Condition (C4) requires that no eigenvalue $\lambda^- \in \sigma^-(\boldsymbol{A})$ has a polar angle of $2\pi k/h$ for $k = 0, \dots, h-1$. Since $\mathrm{spec}(\boldsymbol{A})$ is invariant under a polar rotation of $2\pi m/h$ for any $m \in \mathbb{Z}$, no $\lambda^- \in \sigma^-(\boldsymbol{A})$ is then positive. Noting that for an irreducible matrix, $\left(\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\}\right) \cap \mathbb{R}_{s+} = \emptyset$ and that spec$(\boldsymbol{A}_2) = (\sigma^-(\boldsymbol{A}) \cup \sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\})$, one concludes that $\boldsymbol{A}_2$ has no positive eigenvalue.

(c) ⇒ (b) ⇒ (a): Given (c), we have spec$(\boldsymbol{A}_2) \cap \mathbb{R}_{s+} = \emptyset$. For an irreducible matrix it holds that $\left(\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\}\right) \cap \mathbb{R}_{s+} = \emptyset$. Since spec$(\boldsymbol{A}_2) = \sigma^-(\boldsymbol{A}) \cup (\sigma^\rho(\boldsymbol{A}) \setminus \{\rho(\boldsymbol{A})\})$, it follows that $\sigma^-(\boldsymbol{A}) \cap \mathbb{R}_{s+} = \emptyset$, from which it can be immediately concluded that $\nexists \lambda \in \sigma^-(\boldsymbol{A})$, $\lambda = |\lambda| \exp(i2\pi m/h)$ for any $m \in \mathbb{Z}$. Hence, we establised that (C4) of Theorem 5.12 ([46, Th. 5]) holds for $p_{\boldsymbol{A}}(\lambda)$. Moreover, statements (C1)-(C3) hold as well for $p_{\boldsymbol{A}}$ as $\boldsymbol{A}$ is irreducible. Therefore, due to (B) of Theorem 5.12, there exists a polynomial Q of positive degree, such that $p_{\boldsymbol{A}}(\lambda) Q(\lambda) = \lambda^{k^*} - \alpha_{k^*-1}\lambda^{k^*-1} - \cdots - \alpha_1 \lambda - \alpha_0$, where $k^* \geq n$ and $\alpha_i \geq 0$, $i = 0, 1, \ldots, k^* - 1$. This proves that $\boldsymbol{A}$ has a positive recursion based on Definition 5.11. Then, (a) immediately follows as $\boldsymbol{A}^{k^*}\boldsymbol{b} = \sum_{i=0}^{k^*-1} \alpha_i \boldsymbol{A}^i \boldsymbol{b}$. □

Remark 5. Note that since $\deg\left(Q(\lambda)\right) \geq 0$, k_{vert} of Reachset$_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$ is at least n, and it equals n if and only if $p_{\boldsymbol{A}}(\lambda) = \lambda^n - \alpha_{n-1}\lambda^{n-1} - \cdots - \alpha_1 \lambda - \alpha_0$ with $\alpha_i \geq 0$, $i = 0, \ldots, n-1$. Hence Reachset$_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$ is a simplicial cone (i.e., has n generators) if and only if the characteristic polynomial of $\boldsymbol{A}$ has non-positive coefficients. One such matrix is a cyclic matrix with cyclicity index $h = n$ as $p_{\boldsymbol{A}}(\lambda) = \lambda^n - \rho^n(\boldsymbol{A})$.

5

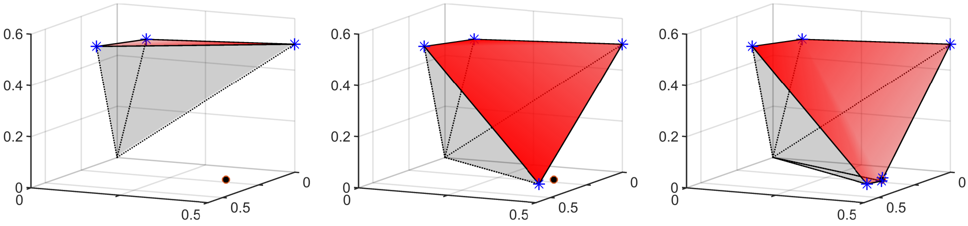
Comparing Theorem 5.13 to Theorem 5.15 reveals that the latter is a restricted version of the former. For example, condition (b) of Theorem 5.13 requires a part of $\boldsymbol{A}$ (i.e., $\boldsymbol{A}_2$) to have a positive recursion while that of Theorem 5.15 requires the entire $\boldsymbol{A}$ to have a positive recursion.

Example 5 (polyhedral Reachset$_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$). Consider the time-invariant discrete-time linear positive system of (5.4) with system matrices

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1.6333 & 1.1049 & 0 \\ 23.5667 & 6.0944 & 0 & 0 \\ 0 & 0 & 1.1225 & 1.0672 \\ 0 & 1.6611 & 0 & 0.7830 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

where $\boldsymbol{A}$ is irreducible with cyclicity index $h = 1$. It can be verified that spec$(\boldsymbol{A}) = \{10, -4, 1 + 1i, 1 - 1i\}$. One can recognize that no eigenvalue of $\boldsymbol{A}_2 = \text{diag}(-4, 1 + i, 1 - i)$ is positive. Therefore, condition (c) of Theorem 5.15 holds and it follows that $\boldsymbol{A}$ has a positive recursion. In fact, it can be verified that in this case it holds that $\boldsymbol{A}^6 = 166.7569\boldsymbol{I}_4 + 16.1434\boldsymbol{A} + 39.7036\boldsymbol{A}^4 + 6.0262\boldsymbol{A}^5$, where $\boldsymbol{I}_4$ denotes the identity matrix of dimension 4×4. In addition, we can conclude that Reachset$_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$ is polyhedral with $k_{\text{vert}} = 6$. This is illustrated by Figure 5.51, where it is observed that Reachset$_k(\boldsymbol{A}, \boldsymbol{b})$ stops growing for $k \geq 6$, i.e., Reachset$_k(\boldsymbol{A}, \boldsymbol{b})$ = Reachset$_6(\boldsymbol{A}, \boldsymbol{b})$ for any $k \geq 6$. One can also notice from Figure 5.51c that $C_{\lim} \subset$ Reachset$_{k_{\text{vert}}}(\boldsymbol{A}, \boldsymbol{b})$, with $C_{\lim}$ introduced in Definition 5.24. Note that in this particular example, since $h = 1$, we have $C_{\lim}$ = cone$(\boldsymbol{A}_{f,0}\boldsymbol{b}) = \{c\boldsymbol{v}_f | c \in \mathbb{R}_{s+}\}$, where $\boldsymbol{v}_f$ is the Frobenius eigenvector of $\boldsymbol{A}^h$.

Remark 6 (Concluding Remark on Theorems 5.13 and 5.15). Theorems 5.13 and 5.15 emphasize the equivalence between the three statements; but this does not



(a) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 3$ (b) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 4$ (c) $\text{Reachset}_k(\boldsymbol{A}, \boldsymbol{b})$, $k = 6$

Figure 5.51: Example 5: growth of the reachability cone mapped on the 3-dimensional simplex $S = \{\boldsymbol{x} \in \mathbb{R}^3_+ | \mathbb{1}^{\mathrm{T}}\boldsymbol{x} = 1\}$; the generators of the cone and the Frobenius eigenvector are, respectively, marked by asterisks and a dot.

imply that all cases are directly verifiable. In fact, it is very difficult to verify statement (b) directly especially since k^{∞}_{vert} and k_{vert} are not known a priori. In practice, statement (a) is practically what one is interested in, and (c) provides numerically verifiable conditions. Statement (b) serves the dual purpose of facilitating the proof and providing insight into otherwise-very-abstract statement (a) and statement (c) by relating them to the matrix having a (partial) positive recursion. Moreover, the characterization (b) will be useful for a different algebraic characterization which is to be developed.

5.5.1. Special Case

So far it has been assumed that $\text{rank}(\text{conmat}_n(\boldsymbol{A}, \boldsymbol{b})) = n$. Based on this assumption, the polyhedrality of the finite-time reachable set only depends on the spectrum of $\boldsymbol{A}$. In addition, $k_{\text{vert}} \geq n$ for $\text{Reachset}_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$. We now point out that in the absence of such an assumption, $\text{Reachset}_{\text{f}}(\boldsymbol{A}, \boldsymbol{b})$ can depend on the structure of $\boldsymbol{b}$ and that the vertex number can be less than n. In particular, it will be shown that $k_{\text{vert}} = h$ if $\boldsymbol{b} \in \mathbb{R}^n_+$ is of a particular structure.

Theorem 5.16. *Let $\boldsymbol{A} \in \mathbb{R}^{n\times n}_+$ be irreducible with cyclicity index h with $0 \leq h \leq n-1$. Then,* $\text{Reachset}_{\text{f}}(\boldsymbol{A}, \boldsymbol{b}) = \text{cone}\big(\text{conmat}_h(\boldsymbol{A}, \boldsymbol{b})\big)$ *if $\boldsymbol{b} \in$* $\text{cone}\big([\boldsymbol{v}_{f,0} \ \cdots \ \boldsymbol{v}_{f,h-1}]\big)$*, where $\boldsymbol{v}_{f,i}$, $i = 0, \dots, h-1$ are the h positive eigenvectors of $\boldsymbol{A}^h$.*

Proof. Assume $\boldsymbol{b} = \sum_{i=0}^{h-1} c_i \boldsymbol{v}_{f,i}$ for some $\boldsymbol{c} \in \mathbb{R}^h_+$. Then, since

$$\boldsymbol{A}^h \boldsymbol{b} = \sum_{i=0}^{h-1} c_i \rho^h(\boldsymbol{A}) \boldsymbol{v}_{f,i} = \rho^h(\boldsymbol{A}) \boldsymbol{b},$$

it is immediate to see that $\boldsymbol{A}\big(\text{conmat}_h(\boldsymbol{A}, \boldsymbol{b})\big) = \big(\text{conmat}_h(\boldsymbol{A}, \boldsymbol{b})\big)\boldsymbol{X}$ has a positive solution

$$\boldsymbol{X} = \begin{bmatrix} 0 & 0 & \cdots & 0 & \rho^h(\boldsymbol{A}) \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix},$$

which, in the view of Corollary 5.14, completes the proof. □

For $\boldsymbol{A}$ primitive (i.e., $h = 1$), this results in the obvious case of Reachset$_\text{f}(\boldsymbol{A}, \boldsymbol{b})$ being a ray along the Frobenius eigenvector $\boldsymbol{v}_f$ of $\boldsymbol{A}$ when $\boldsymbol{b} = c\boldsymbol{v}_f$ for any $c \geq 0$.

5.6. Does the Reachable Set Contain a Pre-specified Set?

A direct consequence of polyhedrality of infinite- or finite-time reachable subset discussed in Section 5.4 and Section 5.5 is that it enables us to determine whether a given subset of the positive vector space is reachable from the origin. Given a cone $C_\text{obj} \subseteq \mathbb{R}_+^n$ of control objectives or a subset of $\mathbb{R}_+^n$, the problem considered here is to investigate whether C_obj is contained in Reachset$_\text{f}(\boldsymbol{A}, \boldsymbol{b})$ or in Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b})$. Of particular interest is when $C_\text{obj} \subset \mathbb{R}_+^n$ is a polyhedral cone or a polytope. Note that if the control objective cone C_obj is not polyhedral then once can outer approximate it by a polyhedral cone $C_\text{out} \subseteq \mathbb{R}_+^n$ such that $C_\text{obj} \subset C_\text{out}$.

Here, it is assumed that the reachability cone or its closure is polyhedral and that its corresponding vertex number or an upper bound of it is known. Note that the authors are not aware of any method to directly compute an upper bound for k_vert or for k_vert^∞. Nonetheless, such an upper bound could be imposed by length of the control sequence that can be practically applied. Let $N \in \mathbb{Z}_+$ denote an upper bound to k_vert^∞, or, where applicable, an upper bound to k_vert. Hence Reachset$_\infty(\boldsymbol{A}, \boldsymbol{b}) =$ cone$\left([\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b} \ \boldsymbol{v}_{f,0} \ \ldots \ \boldsymbol{v}_{f,h-1}]\right)$ and/or Reachset$_\text{f}(\boldsymbol{A}, \boldsymbol{b}) =$ cone$\left([\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}\boldsymbol{b}]\right)$.

Proposition 5.17. Let $C_\text{obj} =$ cone$([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m])$ or $C_\text{obj} =$ conv$([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m])$, where $\boldsymbol{p}_i \in \mathbb{R}_+^n$, $i = 1, \ldots, m$. Then

(a) C_obj is reachable in finite time if and only if

$$\forall\ \boldsymbol{p} \in \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m\},\quad \boldsymbol{p} \in \text{Reachset}_\text{f}(\boldsymbol{A}, \boldsymbol{b}).$$

(b) C_obj is reachable in infinite time (to be called *almost reachable*) if and only if

$$\begin{aligned} &\forall \boldsymbol{p} \in \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m\},\ \boldsymbol{p} \in \text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b}),\ \text{and} \\ &\exists\ \boldsymbol{p}' \in \{\boldsymbol{p}_1, \ldots, \boldsymbol{p}_m\} \text{ such that } \boldsymbol{p}' \notin \text{Reachset}_\text{f}(\boldsymbol{A}, \boldsymbol{b}). \end{aligned}$$

Proof. The proof is obvious from Definition 5.14 and considering the fact that a cone can be expressed as a positive combination of its generators. □

It is obvious from Proposition 5.17, that checking for reachability from the origin involves checking the following condition for each $i \in \{1, \dots, m\}$:

$$\exists \boldsymbol{x}_i \in \{\boldsymbol{z} | \boldsymbol{M}\boldsymbol{z} = \boldsymbol{p}_i, \boldsymbol{z} \in \mathbb{R}_+^N\}, \tag{5.28}$$

where $\boldsymbol{M} \in \mathbb{R}_+^{n \times N}$. Depending on the problem being investigated, either $\boldsymbol{M} = [\boldsymbol{b} \ \dots \ \boldsymbol{A}^{N-1}\boldsymbol{b} \ \boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}]$ or $\boldsymbol{M} = [\boldsymbol{b} \ \dots \ \boldsymbol{A}^{N-1}\boldsymbol{b}]$.

In general, since $N \geq n$ (see Remark 4 and Remark 5), (5.28) defines an underdetermined system of equations. It is known that the positive solution of (5.28) is not unique in general [48, 49], and that uniqueness is guaranteed when the solution is sufficiently sparse [48]. The author of [50] characterizes necessary and sufficient conditions on the polytope $P = \text{conv}(\boldsymbol{M})$ for uniqueness of the solution, and he proves that a unique solution exists if and only if P is k-neighborly [2]. In [49, 53], an equivalent condition is presented in terms of the null space of $\boldsymbol{M}$. In this regard, this problem relates to the *sparse measurement* problem, where the aim is to reconstruct a positive sparse vector from lower-dimensional linear measurements [54]. The results in this field do not directly apply here as the necessary sparsity condition is usually not met. In addition, we are not interested in finding the sparsest solution of (5.28), which is normally an NP-hard problem [48].

Consider for $n \in \mathbb{Z}_+$ the positive matrix $A \in \mathbb{R}_+^{n \times n}$. Let $N \in \mathbb{Z}_+$ with $N > n$ be an upper bound of k_{vert} or an upper bound of k_{vert}^{∞}. Denote by $C(N,n)$ size of the set of all n-subsets of $\mathbb{Z}_N = \{1, \dots, N\}$. Let the index set $\mathcal{I}_j$ be an n-subset (i.e., $|\mathcal{I}_j| = n$) of $\mathbb{Z}_N$ for $j = 1, 2, \dots, C(N,n)$ such that $\cup_{j=1}^{C(N,n)} \mathcal{I}_j = \mathbb{Z}_N$ and $\mathcal{I}_j \neq \mathcal{I}_k$, $j, k = 1, 2, \dots, C(N,n)$, $j \neq k$.

Let $\boldsymbol{I}_{\mathcal{I}_j}$ denote the matrix with n columns, where the columns are chosen from the columns of $\boldsymbol{I}_N$ (i.e., the identity matrix of dimension N) according to the index set $\mathcal{I}_j$ and let $C_{\text{obj}} = \text{cone}([\boldsymbol{p}_1 \ \dots \ \boldsymbol{p}_m])$.

Proposition 5.18. Consider the above defined objects. Then, for any $i \in \{1, \dots, m\}$, equation (5.28) has a solution $\boldsymbol{x}_i$ if and only if,

$$\boldsymbol{X}^i = \left\{ \boldsymbol{x}_j^i | \boldsymbol{x}_j^i = \boldsymbol{I}_{\mathcal{I}_j}(\boldsymbol{M}\boldsymbol{I}_{\mathcal{I}_j})^{-1}\boldsymbol{p}_i, \ \boldsymbol{x}_j^i \in \mathbb{R}_+^N, \ j = 1, \dots, C(N,n) \right\}, \tag{5.29}$$

is a non-empty set.

Proof. From our assumption we have $\boldsymbol{p}_i \in \text{cone}(\boldsymbol{M})$. Since $N > n$, due to Carathéodory theorem [55], $\boldsymbol{p}_i$ also lies in at least one simplicial cone generated by n columns of $\boldsymbol{M}$. Let $\mathcal{J}^i \subset \{1, \dots, N\}$ with $|\mathcal{J}^i| = n$ be an index set composed of the indices of the columns generating this simplicial cone, and let $\boldsymbol{M}_{\mathcal{J}^i}$ denote the columns of $\boldsymbol{M}$ corresponding to $\mathcal{J}^i$. We can then write $\boldsymbol{p}_i \in \text{cone}(\boldsymbol{M}_{\mathcal{J}^i})$, which can be expressed as $\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i}\boldsymbol{z}^i = \boldsymbol{p}_i$ having a solution $\boldsymbol{z}^i \in \mathbb{R}_+^n$. Since $\boldsymbol{M}$ has full row rank and $\boldsymbol{I}_{\mathcal{J}^i}$ is of full column rank, one obtains $\boldsymbol{z}^i = (\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i})^{-1}\boldsymbol{p}_i$. Finally, we obtain a solution $\boldsymbol{x}_j^i \in \mathbb{R}_+^N$, where $\boldsymbol{x}_j^i = \boldsymbol{I}_{\mathcal{J}^i}\boldsymbol{z}^i = \boldsymbol{I}_{\mathcal{J}^i}(\boldsymbol{M}\boldsymbol{I}_{\mathcal{J}^i})^{-1}\boldsymbol{p}_i$.

The converse is proved in a straightforward manner by noticing that every $\boldsymbol{z} \in \boldsymbol{X}^i$ satisfies (5.28). $\square$

[2] A k-neighborly polytope is a convex polytope in which every set of k or fewer vertices forms a face [51, 52].

Remark 7. Let $\boldsymbol{X}^i = \left\{\boldsymbol{x}_1^i, \ldots, \boldsymbol{x}_{q_i}^i\right\}$ for some $q_i \in \mathbb{Z}_+$. It is then clear from the proof of Proposition 5.18 that the set of solutions of (5.28) is the convex hull of $\boldsymbol{X}^i$, i.e., we have for (5.28) that $\boldsymbol{x}_i \in \mathrm{conv}(\boldsymbol{X}^i)$.

Note that even though Proposition 5.18 provides a method to determine whether $C_{\text{obj}} \subseteq \mathrm{cone}(\boldsymbol{M})$ by checking inclusion of C_{obj} in any simplicial subcone of $\mathrm{cone}(\boldsymbol{M})$, the computational complexity of this method can be prohibitive as the check must be conducted for all $C(N, n)$ simplicial subcones in the worst case. A more practical approach is then presented by the following proposition.

Proposition 5.19. Let

$$\begin{aligned}
\boldsymbol{M}_{\text{f}} &= [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}b],\\
\boldsymbol{M}_{\infty} &= [\boldsymbol{b} \ \ldots \ \boldsymbol{A}^{N-1}b \ \boldsymbol{v}_{\text{f},0} \ \ldots \ \boldsymbol{v}_{\text{f},h-1}],\\
C_{\text{obj}} &= \mathrm{cone}([\boldsymbol{p}_1 \ \ldots \ \boldsymbol{p}_m]).
\end{aligned}$$

Define the following optimization problem for each $i \in \{1, \ldots, m\}$:

$$\begin{aligned}
&\min_{\boldsymbol{x}_i} \boldsymbol{x}_i^{\mathrm{T}} \mathbb{1} \\
&\text{subject to } \boldsymbol{M}\boldsymbol{x}_i = \boldsymbol{p}_i, \quad \text{and } \boldsymbol{x}_i \geq 0.
\end{aligned} \tag{5.30}$$

We then have the following:

(a) The optimization problem (5.30) with $\boldsymbol{M} = \boldsymbol{M}_\infty$ has an optimal solution $\boldsymbol{x}_i^* \in \mathbb{R}_+^N$ if and only if (5.28) has a solution with $\boldsymbol{M} = \boldsymbol{M}_\infty$.

(b) The optimization problem (5.30) with $\boldsymbol{M} = \boldsymbol{M}_{\text{f}}$ has an optimal solution $\boldsymbol{x}_i^* \in \mathbb{R}_+^N$ if and only if (5.28) has a solution with $\boldsymbol{M} = \boldsymbol{M}_{\text{f}}$.

Proof. If (5.28) has a solution, the set $\boldsymbol{X}^i$ in (5.29) is non-empty. As mentioned in Remark 7, the feasible set of (5.30) is $\mathrm{conv}(\boldsymbol{X}^i)$. Therefore, the convex optimization problem with linear penalty function converges to the minimum 1-norm solution in the feasible set. The converse is obvious. □

Example 6. We conclude this section with an example illustrating the application of Proposition 5.19. Consider the system matrices of Example 5. Let C_{obj} be the polytope given by

$$C_{\text{obj}} = \left\{\boldsymbol{p} \in \mathbb{R}_+^4 \middle| \boldsymbol{p} = \sum_{i=1}^{4} \lambda_i \boldsymbol{p}_i, \lambda_i \geq 0, \sum_{i=1}^{4} \lambda_i = 1\right\},$$

where

$$\begin{aligned}
\boldsymbol{p}_1 &= [1, \ 3, \ 1, \ 1]^{\mathrm{T}}, \boldsymbol{p}_2 = [1, \ 3, \ 4, \ 3]^{\mathrm{T}},\\
\boldsymbol{p}_3 &= [1, \ 2, \ 2, \ 1]^{\mathrm{T}}, \boldsymbol{p}_4 = [1, \ 1, \ 2, \ 1]^{\mathrm{T}}.
\end{aligned}$$

We will now check whether the system initially at rest can be steered to any point in C_{obj} in finite time. From Example 5, it is known that $k_{\text{vert}} = 6$. Thus taking $\boldsymbol{M} =$

$[\boldsymbol{b}\ \boldsymbol{A}\boldsymbol{b}\ \ldots\ \boldsymbol{A}^5\boldsymbol{b}]$, we solve the linear programming problem (5.30) using the Dual-Simplex algorithm implemented in the Matlab Optimization Toolbox. The optimal solutions are obtained as

$$\begin{aligned}
\boldsymbol{x}_1^* &= [0.1209,\ 0.3735,\ 0,\ 0.0078,\ 0,\ 0.0001]^\mathsf{T},\\
\boldsymbol{x}_2^* &= [2.3460,\ 0.6165,\ 0.0876,\ 0,\ 0.0003,\ 0]^\mathsf{T},\\
\boldsymbol{x}_3^* &= [0.2989,\ 0.6982,\ 0.0473,\ 0,\ 0.0003,\ 0]^\mathsf{T},\\
\boldsymbol{x}_4^* &= [0.2517,\ 0.7798,\ 0.0071,\ 0,\ 0.0003, 0]^\mathsf{T}.
\end{aligned}$$

Hence, the vertices of C_{obj} can be reached from the origin in a finite number of steps using positive inputs, which are determined by the solution vectors $\boldsymbol{x}_i^*$. Moreover, since $k_{\text{vert}} = 6$, every vertex of C_{obj} can be reached in at most 6 steps from the origin. Since C_{obj} is the convex hull of its vertices, we can conclude that any point $\boldsymbol{p} = \sum_{i=1}^4 \lambda_i \boldsymbol{p}_i \in C_{\text{obj}}$ can be reached from the origin in at most 6 steps using the input sequence $\boldsymbol{u}^* = \sum_{i=1}^4 \lambda_i \boldsymbol{x}_i^*$.

5.7. Conclusions and Future Work

The main contribution of the chapter is the result that the reachable set from the origin of a linear positive system can be either a polyhedral cone or a nonpolyhedral cone depending on the system matrices. Among other applications, this has direct consequences for the realization problem, where the choice for the reachable subset from the origin is essential as observability of a linear positive system is then of interest only for states in the reachable set.

For a single-input case, necessary and sufficient conditions for polyhedrality of the reachable set from the origin and its closure are provided. These conditions are expressed in terms of characteristics of eigenvalues of the system. Finally, the chapter presents a method to determine for a positive linear system whether a given target set in the positive orthant can be reached from the origin.

There are several technical issues to be studied. Is it possible to determine in a finite number of steps for a positive matrix whether there exists a positive recursion for it?

In this chapter, we have focused on the single input case, where $\boldsymbol{b} \in \mathbb{R}_+^n$. The problem of characterizing the reachability set from the origin for the multi-input case is an interesting problem because the results developed here are not directly applicable. The main issue, as noted in [47], is that the direct sum of two non-polyhedral cones may still result in a polyhedral cone. Therefore, one cannot apply the results of this chapter to a set of systems $(\boldsymbol{A}, \boldsymbol{b}_i)$ separately, with $\boldsymbol{b}_i$ being a column of $\boldsymbol{B}$.

Finally, it is also of interest to investigate the geometry of the reachable set when the controllability matrix is not of full rank. As far as the authors of this chapter know, this is still an open issue.

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Appendix

5.A. Positive Matrices

The reader finds in this appendix a summary of the theory of positive matrices including concepts and decompositions as far as is necessary for the understanding of this chapter. This theory is well known and therefore not stated in the body of the chapter.

Decompositions of Positive Matrices

As is well known in the theory of positive matrices, such matrices can be either reducible or irreducible as defined next. See the books [9, 56] for the definitions.

Definition 5.20. Consider a positive matrix $\boldsymbol{A} \in \mathbb{R}_{+}^{n \times n}$ for $n \in \mathbb{Z}_{+}$. Call this matrix *reducible* if

$$\begin{aligned}
& \exists\ \boldsymbol{P} \in \mathbb{R}_{+}^{n \times n}, \text{ a permutation matrix}, \\
& \exists\ n_1, n_2 \in \mathbb{Z}_{+},\ \ \exists\ A_{11} \in \mathbb{R}^{n_1 \times n_1},\ A_{12} \in \mathbb{R}^{n_1 \times n_2},\ A_{22} \in \mathbb{R}^{n_2 \times n_2}, \\
& \text{such that } n = n_1 + n_2 \text{ and} \\
A \ &= \ P \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} P^T. \qquad (5.31)
\end{aligned}$$

Call the matrix A *irreducible* if (1) $A \neq 0$ and (2) A is not reducible.

Call the matrix A *fully reduced* if either $n = 1$ or there exists a transformation by a permutation matrix P so that PAP^T has a decomposition in upper-block-diagonal form with only irreducible submatrices on the block-diagonal. Thus the lower-block-diagonal matrices are all zero. The particular form of a fully-reduced positive matrix is thus

$$A \ = \ P \begin{pmatrix}
A_{11} & A_{12} & A_{13} & \dots & A_{1,n-1} & A_{1,n} \\
0 & A_{22} & A_{23} & \dots & A_{2,n-1} & A_{2,n} \\
0 & 0 & A_{33} & \dots & A_{3,n-1} & A_{3,n} \\
\vdots & \vdots & & \ddots & \vdots & \vdots \\
0 & 0 & & \dots & A_{n-1,n-1} & A_{n-1,n} \\
0 & 0 & & \dots & 0 & A_{n,n}
\end{pmatrix} P^T, \qquad (5.32)$$

where $P \in \mathbb{R}_{+}^{n \times n}$ is a permutation matrix and the matrices on the block-diagonal of (5.32) are all irreducible positive matrices.

Decompositions of Positive Matrices Based on Eigenvalues

Recall that for a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ the *spectrum* is defined as the set of its eigenvalues and the *spectral radius* is defined as $\rho(\boldsymbol{A}) = \max_{\lambda \in \text{spec}(\boldsymbol{A})} |\lambda|$. It follows from [9, Th. 1.3.2] that every positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ has at least one eigenvalue which equals its spectral radius.

Definition 5.21. [9, Def. 2.2.26]. Define for an integer $n \in \mathbb{Z}_+$ and an irreducible positive matrix $A \in \mathbb{R}_+^{n\times n}$, the *index of cyclicity* of A as the number $h \in \mathbb{Z}_+$ such that h equals the maximum number of distinct eigenvalues of A which are in modulus equal to the spectral radius $\rho(A)$. In mathematical notation:

$$h \quad = \quad \max\{k \in \mathbb{Z}_n | \ \forall \ i \in \mathbb{Z}_k, \ |\lambda_i(A)| = \rho(A)\}. \tag{5.33}$$

It follows from the comment above the previous definition that $h \geq 1$. If $h \geq 2$ then one says that the matrix A is *cyclic of index* h.

Definition 5.22. Consider an integer $n \in \mathbb{Z}_+$ and an irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$. Partition the set of eigenvalues into the following two subsets: $\sigma^\rho(\boldsymbol{A})$, which is the spectrum of $\boldsymbol{A}$ on the circle centered at origin with radius $\rho(\boldsymbol{A})$, and $\sigma^-(\boldsymbol{A})$, which is the spectrum of $\boldsymbol{A}$ strictly inside the disc centered at origin with radius $\rho(\boldsymbol{A})$. Hence,

$$\begin{aligned} \sigma^\rho(\boldsymbol{A}) &= \{\lambda \in \mathrm{spec}(\boldsymbol{A}) | \ |\lambda(\boldsymbol{A})| = \rho(\boldsymbol{A})\}, \\ \sigma^-(\boldsymbol{A}) &= \{\lambda \in \mathrm{spec}(\boldsymbol{A}) | \ |\lambda(\boldsymbol{A})| < \rho(\boldsymbol{A})\} \end{aligned} \tag{5.34}$$

with

$$\begin{aligned} \mathrm{spec}(\boldsymbol{A}) &= \sigma^\rho(\boldsymbol{A}) \cup \sigma^-(\boldsymbol{A}), \ \sigma^\rho(\boldsymbol{A}) \cap \sigma^-(\boldsymbol{A}) = \emptyset; \\ & \quad n_1 = |\sigma^\rho(\boldsymbol{A})| = h, \ n_2 = |\sigma^-(\boldsymbol{A})| = n - n_1 = n - h. \end{aligned}$$

In addition, there exists a nonsingular matrix $\boldsymbol{S} \in \mathbb{R}^{n\times n}$ such that the matrix $\boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S}$ is block-diagonal with,

$$\begin{aligned} \boldsymbol{S}^{-1}\boldsymbol{A}\boldsymbol{S} &= \text{block-diag}(\boldsymbol{A}_1, \boldsymbol{A}_2), \\ \boldsymbol{A}_1 &\in \mathbb{R}^{n_1\times n_1}, \ \boldsymbol{A}_2 \in \mathbb{R}^{n_2\times n_2}, \ \mathrm{spec}(\boldsymbol{A}_1) = \sigma^\rho(\boldsymbol{A}), \ \mathrm{spec}(\boldsymbol{A}_2) = \sigma^-(\boldsymbol{A}). \end{aligned} \tag{5.35}$$

Finally, define the sets $\sigma^\rho(\boldsymbol{A}_2)$ and $\sigma^-(\boldsymbol{A}_2)$ in a similar manner with $\boldsymbol{A}$ being replaced by $\boldsymbol{A}_2$ in (5.34) and define the set $\sigma^0(\boldsymbol{A}) \subseteq \sigma^\rho(\boldsymbol{A}_2)$ as the set of all eigenvalues of $\boldsymbol{A}_2$ whose whose polar angle is a rational multiple of 2π.

The notation $|\sigma^\rho(A)|$ denotes the number of elements of the indicated set. That the decomposition (5.35) is indeed a partition follows from Perron-Frobenius theorem [9, Th. 2.1.4, 2.2.20] and from the concept of spectral radius as the maximal value of the absolute values of all eigenvalues. In general, the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ depend on $\boldsymbol{S}$. However, the relations (5.35) hold for any such $\boldsymbol{S}$. When the matrices $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are used in the body of the chapter, then these are characterized by their spectra. Also note that in contrary to $\sigma^\rho(\boldsymbol{A})$, $\sigma^\rho(\boldsymbol{A}_2)$ can be empty set.

Next, we present the following lemma about the existence of a subset of eigenvalues that are among the $(M\ h)$-th root of unity for some $M \in \mathbb{Z}_+$. This lemma is used in the sequel for deriving the conditions on $\mathrm{spec}(\boldsymbol{A})$ for $\boldsymbol{A}$ to have a positive recursion.

Lemma 5.23. Consider the objects of Definition 5.22.

Then, there exists a minimal integer $M \in \mathbb{Z}_+$ such that

$$\sigma_0(\boldsymbol{A}) \subseteq \left\{\lambda \in \text{spec}(\boldsymbol{A}_2) \middle| \lambda = \rho(\boldsymbol{A}_2)\exp(\frac{2\pi k}{Mh}i), k = 0, \ldots, Mh-1\right\}, \tag{5.36}$$

or, equivalently, there exists a minimal integer $M \in \mathbb{Z}_+$ such that the eigenvalues of $\boldsymbol{A}_2/\rho(\boldsymbol{A}_2)$ with unit modulus whose arguments are a rational multiple of 2π are among the $(M\ h)$-th roots of unity.

Proof of Lemma 5.23. Let δ^0 be a set of $n_{\delta^0} \in \mathbb{Z}_+$ members of σ^0 with the property that the difference between the polar angle of no two members of δ^0 is an integer multiple of $2\pi/h$, or formally we define $\delta^0 = \{\lambda_1, \ldots, \lambda_{n_{\delta^0}} \in \sigma^0 | \text{arg}(\lambda_i) - \text{arg}(\lambda_j) \neq 2z\pi/h, i \neq j, z \in \mathbb{Z}\}$. For $\lambda_j \in \delta^0$, $j = 1, \ldots, n_{\delta^0}$, let $\text{arg}(\lambda_j) = \frac{2\pi p_j}{q_j}$. Define the sets $\sigma_j^0 \subset \sigma^0$ for $j = 1, \ldots, n_{\delta^0}$ as

$$\sigma_j^0 = \left\{\lambda \in \text{spec}(\boldsymbol{A}_2) \middle| \lambda = \rho(\boldsymbol{A}_2)\exp\left((k/h + p_j/q_j)2\pi i\right),\ k = 0, \ldots, h-1\right\},$$

or equivalently using the notation $s_{j,k} \equiv kq_j + hp_j (\text{mod } hq_j)$,

$$\sigma_j^0 = \left\{\lambda \in \text{spec}(\boldsymbol{A}_2) \middle| \lambda = \rho(\boldsymbol{A}_2)\exp\left(\frac{s_{j,k}}{hq_j}2\pi i\right),\ k = 0, \ldots, h-1\right\}.$$

It is clear that $\sigma_1^0, \ldots, \sigma_{n_{\delta^0}}^0$ are mutually disjoint. In addition, since the eigenvalues of $\boldsymbol{A}$ are invariant under polar rotation of $2k\pi/h$ for any $k \in \mathbb{Z}$, we have $\sigma^0 = \cup_{j=1}^{n_{\delta^0}} \sigma_j^0$. Noting that $0 \leq s_{j,k} \leq hq_j - 1$ for $k = 0, \ldots, h-1$ and for $j = 1, \ldots, n_{\delta^0}$, one observes that σ_0 has the form proposed in (5.36) by choosing $M = \text{lcm}(q_1, \ldots, q_{n_{\delta^0}})$. □

It follows from [9, Th. 2.2.20] that if the matrix $A \in \mathbb{R}_+^{n\times n}$ is irreducible and if A is of index of cyclicity $h \geq 2$ then there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ and matrices $\{A_{i,i+1} \in \mathbb{R}_+^{n_i \times n_{i+1}},\ i = 0, 1, \ldots, h-1 \ (\text{mod } h)\}$ such that,

$$\sum_{i=0}^{h-1} n_i = n,$$

$$\boldsymbol{A} = \boldsymbol{P}\begin{pmatrix} 0 & \boldsymbol{A}_{1,2} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \boldsymbol{A}_{2,3} & \ldots & 0 & 0 \\ \vdots & & 0 & \ddots & \boldsymbol{A}_{h-2,h-1} & 0 \\ 0 & 0 & 0 & \ldots & 0 & \boldsymbol{A}_{h-1,h} \\ \boldsymbol{A}_{h,1} & 0 & 0 & \ldots & 0 & 0 \end{pmatrix}\boldsymbol{P}^T, \tag{5.37}$$

with square diagonal blocks.

One then says that the positive matrix $\boldsymbol{A}$ is *cogredient* to the block matrix of equation (5.37); [9, Def. 2.1.2].

The irreducible positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n\times n}$ is called *primitive* if its trace is strictly positive; see [9, Def. 2.1.8, Cor. 2.2.28].

It follows from the proof of [9, Th. 2.2.30] that if the matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ is irreducible and if A is of index of cyclicity $h \geq 2$ then there exists a permutation matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ such that

$$\boldsymbol{A}^h = \boldsymbol{P} \begin{pmatrix} \boldsymbol{C}_{1,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \boldsymbol{C}_{2,2} & 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{C}_{h-1,h-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boldsymbol{C}_{h,h} \end{pmatrix} \boldsymbol{P}^T,$$

$\forall\ i \in \mathbb{Z}_h,\ \boldsymbol{C}_{i,i} \in \mathbb{R}_+^{n_i \times n_i}$ are primitive matrices with $\rho(\boldsymbol{C}_{i,i}) = \rho^h(\boldsymbol{A})$;

$$\sum_{i=1}^{h} n_i = n.$$

Sources for the above theory are not only [9] but also the book [31, Ch. 3].

Limits of Powers of Positive Matrices

It follows from Theorem [9, Th. 2.4.1] that for a primitive irreducible matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$, the following limit exists:

$$\lim_{k \to \infty} \left(\frac{\boldsymbol{A}}{\rho(\boldsymbol{A})} \right)^k \in \mathbb{R}_+^{n \times n}.$$

Next the above results can be combined. Consider an irreducible matrix $A \in \mathbb{R}_+^{n \times n}$. Assume that the index of cyclity of A is such that $h \geq 2$. It then follows from the above that A^h is cogredient to a block diagonal matrix with on the diagonal primitive irreducible matrices. From the above existence of the limit then follows that,

$$\boldsymbol{A}^h = \boldsymbol{P} \begin{pmatrix} \boldsymbol{C}_{1,1} & 0 & 0 & \dots & 0 & 0 \\ 0 & \boldsymbol{C}_{2,2} & 0 & \dots & 0 & 0 \\ \vdots & \dots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \dots & \boldsymbol{C}_{h-1,h-1} & 0 \\ 0 & 0 & 0 & \dots & 0 & \boldsymbol{C}_{h,h} \end{pmatrix} \boldsymbol{P}^T,$$

$$\begin{aligned} \lim_{k \to \infty} \left(\frac{\boldsymbol{A}^h}{\rho^h(\boldsymbol{A})} \right)^k &= \boldsymbol{P} \text{ block-diag}\Big(\lim_{k \to \infty} (\boldsymbol{C}_{1,1}/\rho^k(\boldsymbol{C}_{1,1}), \dots, \lim_{k \to \infty} (\boldsymbol{C}_{h,h}/\rho^k(\boldsymbol{C}_{h,h})\Big) \boldsymbol{P}^T \\ &= \boldsymbol{P} \text{ block-diag}(\boldsymbol{C}_{\infty,1,1}, \dots, \boldsymbol{C}_{\infty,h,h}) \boldsymbol{P}^T \in \mathbb{R}_+^{n \times n}. \end{aligned}$$

Next, we introduce the following lemma characterizing the limit behavior of $\text{conmat}_k(\boldsymbol{A}, \boldsymbol{b})$ as $k \to \infty$, which is used for characterizing the infinite-time reachable subset $\text{Reachset}_\infty(\boldsymbol{A}, \boldsymbol{b})$.

Definition 5.24. Let the positive matrix $\boldsymbol{A} \in \mathbb{R}_+^{n \times n}$ be irreducible with index of cyclicity h with $1 \leq h \leq n$ and let $\boldsymbol{b} \in \mathbb{R}_+^m$. Define the matrices and the *limit cone* according to

$$\forall\ i \in \{0,\dots,h-1\},\quad \boldsymbol{A}_{f,i} = \lim_{k\to\infty}\left(\left(\frac{\boldsymbol{A}}{\rho(\boldsymbol{A})}\right)^{h}\right)^{k}\boldsymbol{A}^{i},$$

$$\mathcal{C}_{\lim} = \mathrm{cone}([\boldsymbol{A}_{f,0}\boldsymbol{b}\ \dots\ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]).$$

$$\forall\ i \in \{0,\dots,h-1\},\quad \boldsymbol{A}_{f,i} = \lim_{k\to\infty}\left(\left(\frac{\boldsymbol{A}}{\rho(\boldsymbol{A})}\right)^{h}\right)^{k}\boldsymbol{A}^{i},$$

$$\mathcal{C}_{\lim} = \mathrm{cone}([\boldsymbol{A}_{f,0}\boldsymbol{b}\ \dots\ \boldsymbol{A}_{f,h-1}\boldsymbol{b}]),$$

Define for $i = 0,\dots,h-1$ the positive eigen vectors $\boldsymbol{v}_{f,i} \in \mathbb{R}_{+}^{n}$ of the h distinct eigenvalues of the matrix $\boldsymbol{A}^h$ associated with the Perron root of $\rho^h(\boldsymbol{A})$; thus,

$$\boldsymbol{A}^{h}\ v_{f,i} = \rho^{h}(\boldsymbol{A})\ v_{f,i}.$$

Lemma 5.25. Consider the objects of Definition 5.24. Then the limit cone satisfies

$$\mathcal{C}_{\lim} \subseteq \mathrm{cone}([\boldsymbol{v}_{f,0}\ \dots\ \boldsymbol{v}_{f,h-1}]).$$

Proof of Lemma 5.25. Since $\boldsymbol{A}$ is irreducible, there exists a monomial matrix $\boldsymbol{S} \in \mathbb{R}_{+}^{n\times n}$ [9] such that

$$\hat{\boldsymbol{A}} = \boldsymbol{S}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{S} = \begin{bmatrix} 0_{n_1} & A_1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0_{n_{h-1}} & A_{h-1} \\ A_h & 0 & \dots & \dots & 0_{n_h} \end{bmatrix},\ \hat{\boldsymbol{b}} = \boldsymbol{S}^{\mathrm{T}} b$$

where $0_{n_i} \in \mathbb{R}^{n_i\times n_i}$, $i \in \mathbb{N}$ are square blocks with $\displaystyle\sum_{i=1}^{h} n_i = n$, and where A_i has no zero rows or columns with $L_1 = \displaystyle\prod_{i=1}^{h} A_i$ being an irreducible matrix. Then we have $\hat{\boldsymbol{A}}^{h} = \mathrm{diag}(L_1,\dots,L_h)$, where $L_k = \displaystyle\prod_{i=k}^{h} A_i \prod_{j=1}^{\mathrm{mod}(h+k-1,h)} A_j$ is a primitive matrix of dimension $n_k \times n_k$ with Perron root $\rho^h(\boldsymbol{A})$. Define the matrix $\hat{\boldsymbol{A}}_{f,i} = \lim_{p\to\infty}\dfrac{\hat{\boldsymbol{A}}^{ph}}{\rho^{ph}}\hat{\boldsymbol{A}}^{i}$ for $i = 0,\dots,h-1$. Since L_i, $i = 1,\dots,h$ is primitive, it follows from [9] that

$$\hat{\boldsymbol{A}}_{f,0} = \begin{bmatrix} \boldsymbol{x}_1^1 & \dots & \boldsymbol{x}_1^{n_1} & 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \boldsymbol{x}_2^1 & \dots & \boldsymbol{x}_2^{n_2} & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \boldsymbol{x}_h^1 & \dots & \boldsymbol{x}_h^{n_h} \end{bmatrix},$$

where $\boldsymbol{x}_i^k = c_i^k \boldsymbol{x}_i$ with c_i^k, $k = 1, \dots, n_i$, being some positive scalars and with $\boldsymbol{x}_i \in \mathbb{R}_{S+}^{n_i \times n_i}$ being the Frobenius eigenvector of L_i. Note that due to the block structure of $\hat{\boldsymbol{A}}$, $\hat{\boldsymbol{A}}_{f,i}$ retains the same structure as $\hat{\boldsymbol{A}}_{f,0}$ up to a scaled permutation of its columns for $i = 1, \dots, h-1$. Hence, we have $\hat{\boldsymbol{A}}_{f,i}\hat{\boldsymbol{b}} \in \text{cone}(\boldsymbol{C})$, where

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{x}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{x}_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & \boldsymbol{x}_h \end{bmatrix}.$$

In the original coordinates, we have $\boldsymbol{A}_{f,i}\boldsymbol{b} \in \text{cone}(\boldsymbol{SC})$. Clearly, since the columns of $\boldsymbol{C}$ are the positive eigenvectors of $\hat{\boldsymbol{A}}^h$ and since $\boldsymbol{S}$ is monomial, we have $\boldsymbol{SC} = [\boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}]$, where $\boldsymbol{v}_{f,i} \in \mathbb{R}_+^{n \times n}$ is the $(i+1)$-th positive eigenvector of $\boldsymbol{A}^h$ for $i = 0, \dots, h-1$. This proves that $\text{cone}([\boldsymbol{A}_{f,0}b \ \dots \ \boldsymbol{A}_{f,h-1}b]) \subseteq \text{cone}([\boldsymbol{v}_{f,0} \ \dots \ \boldsymbol{v}_{f,h-1}])$. $\square$

5

5.B. Note on Theorem 5.12

After the publication of our work [1], upon which the current chapter is based, it was brought to the authors' attention that the formulation of Theorem 5.12 [46, Th. 5] (specifically statement C4) is incorrect. Theorem 5.12 concerns the characterization of polynomials admitting a nonnegative recursion that is used in the proofs of Theorems 5.13 and 5.15 of the current chapter. A counter example to Theorem 5.12 is provided by L. Farina and L. Benvenuti in [57], where they also state the correct formulation of the given theorem. The correct formulation of the theorem is reported below for convenience.

Theorem 5.26. *[57, Th. 1] Let $\{a_1, \dots, a_k\}$ be given complex numbers, and let $P(x)$ be the polynomial $x^k - a_1 x^{k-l} - \dots - a_k$. Then conditions (A), (B) and (C) below are equivalent:*

(A) Any infinite sequence $(u_n)_{n \geq 0}$ of complex numbers which satisfies the recursion $u_{n+k} = a_1 u_{n+k-1} + a_2 u_{n+k-2} + \dots + a_k u_n$ for $n \geq 0$, also satisfies a recursion with positive coefficients.

(B) The polynomial $P(x)$ divides a polynomial in Q.

(C) In case the polynomial $P(x)$ has a positive root r, then all conditions (1)-(4) below are satisfied:

(C1) $r \geq |\alpha|$ for any root α of $P(x)$.

(C2) if $\alpha = r$ for some root α of $P(x)$, then α/r is a root of unity.

(C3) all roots $P(x)$ with absolute value r are simple.

(C4) if d is the minimal integer such that $\epsilon^d = 1$ for all roots of unity ϵ which satisfy $P(r\epsilon) = 0$, then $P(x)$ has no roots of the form $s\omega$ where $0 < s < r$ and $\omega^d = 1$.

The difference between the two formulations of statement C4 is that in Theorem 5.12, due to M. Roitman and Z. Rubinstein [46], statement C4 can hold with different values of integer k for different groups of roots with absolute value r, while in Theorem 5.26, due to L. Benvenuti and L. Farina [57], statement C4 must hold with the same value of the integer k (m in their own words) for all roots with absolute value r. It is proved in [57] that the latter formulation of C4 must be used to conclude equivalence between statements A, B, and C of the theorem, which we need for Theorems 5.13 and 5.15.

We investigated whether this change has any consequences for Theorems 5.13 and 5.15, and concluded that conditions laid out by these theorems are not affected by using Theorem 5.26 as the basis for our proofs. Note that the proof of Theorem 5.15 is based on C4 of Theorem 5.12 with $k = h$. In this case k cannot take any other value than h as $\boldsymbol{A}$ has exactly h eigenvalues of the form

$$\rho(\boldsymbol{A}) \exp\left(\frac{2\pi m}{h} i\right), \ m = 0, \dots, h-1.$$

In the case of Theorem 5.13, C4 of Theorem 5.12 must hold with the smallest integer M (see Lemma 5.23) such that

$$\sigma^{\rho}(\boldsymbol{A}_2) \subseteq \left\{\lambda \middle| \ \lambda = \rho(\boldsymbol{A}_2) \exp\left(\frac{2\pi k}{Mh} i\right), k = 0, \dots, Mh-1\right\},$$

where $\sigma^{\rho}(\boldsymbol{A}_2)$ is the set of *all* eigenvalues of $\boldsymbol{A}_2$ with modulus $\rho(\boldsymbol{A}_2)$. Therefore, our proof is based on a *single* value for k ($k = M$) for *all* eigenvalues of $\boldsymbol{A}_2$ with modulus $\rho(\boldsymbol{A}_2)$.

6

Conclusions and Further Research

"I would rather have questions that can't be answered than answers that can't be questioned."

Richard Phillips Feynman, 1918 – 1988

6.1. Concluding Remarks

This thesis has addressed a range of topics on model-based control of baggage handling systems and control of discrete-time linear positive systems, where the results derived for control of positive systems have been subsequently used to develop a scalable tube-based model predictive control (MPC) scheme for large-scale baggage handling systems (BHSs). In short, the thesis has made contributions along the following main directions:

1) Development of a flow-based modeling framework for large-scale logistic systems such as BHSs, and scalable MPC-based control strategies for optimal operation of such systems subject to disturbances.

2) Computation of a robustly positively invariant hypercube for discrete-time linear positive systems subject to additive constraints together with computation of the state feedback controller that minimizes the output L_∞ norm of the output over the hypercube.

3) Characterization of reachable subsets for discrete-time linear positive systems, and formulation of the reachability problem of a polyhedral subset of the positive orthant.

We will now elaborate on the contributions and the results.

6.1.1. Robust Model-based Control of Baggage Handling Systems

A modeling and control design framework for large-scale BHSs has been proposed that is based on approximating movements of pieces of baggage by continuous flows, rather than describing their individual movements. This was necessary to arrive at models suitable for developing model-based control methods discussed in the thesis. The developed models were then used within the MPC framework as MPC enables one to incorporate (partial) knowledge of the disturbance (i.e., the baggage demand), capacity and flow constraints, and a measure of performance in an integrated control design approach.

We have shown that the original nonlinear programming (NLP)-based MPC formulation, which is not scalable to large instances of BHSs or to longer prediction horizons, can be expressed as a linear programming (LP)-based MPC problem or a iterative linear programming (ILP)-based MPC problem, which leads to tractable optimization problems for large BHSs. Especially ILP-based MPC has been shown to achieve accurate predictions in relation to NLP-based MPC, hence, making it possible to use longer prediction horizons. Therefore, this approach enables one to fully benefit from knowledge of the future demand profiles (due to the use of large prediction horizons), while scaling to large BHSs. The performance of the developed MPC schemes has been illustrated by a simulation-based case study. It has been shown that, with fully predictable baggage demand at the loading stations, the ILP-based MPC approach outperforms the current state-of-the-art heuristics-based approach at an affordable computational cost. In addition, the LP-based MPC approach also outperforms the current state-of-the-art approach for short prediction horizons while offering the cheapest solution in terms of the required computational resources.

The MPC solution has then been extended to the cases where only a partial knowledge of the future baggage demand is known. The uncertainty in the baggage demand has been modeled by expressing the baggage demand as the sum of a nominal fully predictable component and a relatively smaller zero-mean random component. To provide robustness against the additive uncertainty, a tube-based control scheme has been employed, which is a low-complexity robust MPC approach consisting of a top-level MPC controller for the nominal system and a bottom-level state feedback controller for the error system. The error system describes the difference between the uncertain system (i.e., the actual system) and the nominal system. We have investigated two different approaches, as the subject of two separate chapters of this thesis, to the design of the state feedback controller along with the appropriately modified version of the nominal constraints so that the nominal MPC problem is still feasible in presence of the error dynamics. Such modifications commonly entail restricting the nominal MPC state and control constraints by the minimal robust invariant set and its image on the control subspace, the computation of which is very difficult in practice for large-scale systems. To make our approach applicable to large-scale systems, we have proposed a method to calculate the smallest robustly positively invariant ellipsoids and hypercubes that include the minimal robust invariant set, and we use these sets to tighten the nominal MPC constraints.

For the first approach, a feedback controller minimizing the induced L_2 gain of the closed-loop error system from the disturbance input to the error output has been developed. The problem of searching for the state feedback gain as well as the associated robustly positively invariant ellipsoid has been formulated as an optimization problem with a linear objective function subject to linear matrix inequalities (LMIs). Furthermore, to impose state and control constraints on the error system, another formulation of this problem has been developed that allows for specification of control and state constraints via suitable additional LMIs.

To develop the second approach, we have first developed some theoretical background for linear positive systems subject to additive disturbance, and we have shown that, for positive systems, the existence of a robustly positively invariant set is characterized by a set of linear inequalities. This result is then exploited to design the state feedback controller for the error system such that the resulting closed-loop system is positive and optimal in the sense of the L_∞ norm of the disturbance-driven output. We have formulated the joint problem of searching for such feedback gain and the corresponding robustly positively invariant hypercube as a linear program. We have also extended the solution to include state and control constraints for the closed-loop error system, and a minimum size for the desired robustly positively invariant hypercube.

Systems of BHS nature are often subject to capacity constraints, which are represented by upper and lower bound on the states and the control inputs. The conservatism introduced by ellipsoidal approximation of such constraints, as needed in the first approach, may lead to an infeasible set of LMIs for large-scale systems. Further to this point, linear programs can generally be solved more efficiently than linear semi-definite optimization problems subject to LMIs. Hence, in contrast to the first approach, which is only suitable for small-scale problems, the latter approach to feedback design scales well for large-scale systems. In addition, the second approach allows us to use the L_∞ norm of the output as the criterion for feedback gain design, which provides a better measure of the worst-case performance of the system than the L_2-gain of the system for systems that are subject to additive disturbances characterized by their minimum and maximum values. Finally, the effectiveness of the latter state feedback design approach has been illustrated in a large-scale BHS case study, where a tube-based MPC controller is designed to deal with structured uncertainty of the baggage demand. In this case study, we have also shown how recursive feasibility and asymptotic stability of the nominal system can be guaranteed by the appropriate choice of the terminal cost function and the terminal constraint set.

Models and methods developed in Chapters 2- 4 can in principle be applied to vehicle-based sorting systems, where the feed sequence can be controlled and there is sufficient buffer capacity. However, their applicability to heavily loaded conveyor-based sorting systems may be limited since such systems cannot be heavily buffered and there is usually no control over the feed sequence.

6.1.2. Reachability Problem for Discrete-time Linear Positive Systems

For discrete-time linear positive systems, we have shown that the reachable subset from the origin using nonnegative controls and its closure can be either a polyhedral cone (i.e., generated by a finite number of extremal rays) or a nonpolyhedral cone depending on system matrices. Whether the reachable subset or its closure is polyhedral is closely linked with the system matrix $\boldsymbol{A}$, or an appropriately decomposed version of it, admitting a positive recursion; that is there exist a positive integer m and non-negative real coefficients c_i, $i = 0, \ldots, m-1$ such that $\boldsymbol{A}^m = \sum_{i=0}^{m-1} c_i \boldsymbol{A}^i$. We have derived necessary and sufficient conditions in terms of the spectrum of the system matrix $\boldsymbol{A}$ for polyhedrality of the reachable subset and its closure.

Polyhedrality of the reachable subset or its closure helps to define an alternative formulation for reachability of discrete-time linear positive systems, the one which concerns whether a subset of the positive orthant can be reached from the origin using a finite sequence of nonnegative controls or can be arbitrarily closely approached from the origin using a sufficiently long nonnegative control sequence. This definition of the reachability problem is more useful from an application point of view, and is in contrast to the classical definition, which is concerned with the reachability of the entire positive orthant. This is owed to the fact that in practice one is often interested in reachability of a certain "target" subset of the positive orthant rather than the entire positive orthant.

For cases where the reachable subset or its closure is a polyhedral cone, we have presented a method to determine whether a given target subset of the positive orthant can be reached from the origin. In case the reachable subset is polyhedral, the developed method can also be used to calculate the finite nonnegative control sequence driving the system from the origin to any point in the target set. The applicability of our proposed approach is illustrated by several examples.

6.2. Recommendations for Further Research

6.2.1. Extensions of Models and Methods Developed for BHSs

The models developed in Chapter 2 for BHSs can in principle be extended to cover other applications of a similar nature such as logistic systems and material handling systems. In this regard it is worthwhile to explore whether alternative ways of modeling density-dependent flow travel times through the network, for example by including the effect of network congestion on travel times, yield models of higher accuracy. Furthermore, in addition to robustness against variations in the demand profile, which was the subject of Chapters 3 and 4, fault tolerance and robustness against model parameters should be integrated in the model-based control design process so that the closed-loop system is robust against model mismatch. Among others, model mismatch can arise from broken components or capacity drop due to closure of a part of the network. Finally, this work has considered a centralized control design with full information of the system. For large-scale systems such as BHSs the centralized controller can be both unreliable, as the control system has a single point of failure, and computationally expensive. Hence, one is motivated to investigate distributed or decentralized robust MPC solutions for such systems.

In this research work, we have addressed high-level model-based control design of large-scale logistic systems. In practice, the decisions made by the high-level controller in terms of the link-to-link transfer rate of material need to be translated to the number of carts, boxes, DCVs, etc. that need to be dispatched from one link to another in a given period of time. The effect of the conversion of the flow variables, generated by the high-level controller, to the discrete actions of lower level controllers should be taken into account when assessing the performance of the control system. To this end, one must develop a corresponding quantity-based *plant* model to asses the performance of the MPC controller designed for the flow-based model. This amounts to employing an elaborate plant model and a simplified control model. In this way, the effect of *realization of actuation* (i.e., translating flows to movements of DCVs) on the performance of the control system can be studied.

The models developed for BHSs in this thesis, and possibly their extensions, can be used in feasibility studies of new BHSs. During the pre-design process and as a part of cost and benefit analysis, extensive simulations based on BHS models can be used to determine maximum system throughput for a given capacity, expected time delays due to baggage transfer, robustness of the system in the event of faults, etc. In addition, for existing BHSs, such model-based simulations can help optimize the system operation along user-defined Key Performance Indicators (KPIs) to maximize utilization of the existing infrastructure as efficiently as possible. The use of model-based simulations as an instrumental part of planing new BHSs or for upgrading existing ones can ultimately lead to sustainable BHSs that are not only efficiently designed, but are also well predictable in terms of maintenance, future costs, and expected profits.

In this thesis, we have focused on modern automated BHSs with respect to modeling, control design, and performance analysis. In doing so, we have omitted the human factor, which has a significant impact on the overall quality of baggage handling service. In fact, the bottleneck in baggage handling is often in the manual handling part. Think of baggage not being supplied in time or removed in timely manner at the unloading stations. Delays made in manual processes can easily propagate and adversely impact the overall quality of the baggage handling service. A very interesting research avenue is, therefore, to analyze the baggage handling process in big airports not in terms of technical capabilities of automated BHSs, nor the control algorithms, but in terms of the human factors. In this regard, one can develop (stochastic) models to determine the effect of human-induced delays on the performance of the BHSs, and include those models in overall control design for BHSs such that the designed control strategy is ultimately sufficiently robust against human-induced delays.

6.2.2. Extensions of the Control Theory of Linear Positive Systems

The work in this thesis on polyhedrality of finite-time and infinite-time reachable subsets for linear positive systems only concernedd single-input systems with irreducible system matrix $\boldsymbol{A}$. To establish a full picture of the reachable subsets, extension of these results to the multiple-input case needs to be investigated. Further generalization of such results should allow to consider systems with a reducible system matrix. In addition, in absence of any structure for the input vector $\boldsymbol{b}$, the conditions derived in Chapter 5 depend only on the spectrum of the system matrix $\boldsymbol{A}$. It is worthwhile to investigate whether an assumed structure for $\boldsymbol{B}$ would lead to different conditions for polyhedrality of the reachable subsets. Finally, to obtain a complete picture of the controllability problem when a subset of the positive orthant is considered as the target set, it is necessary to consider the case where the initial state is not the origin, but an arbitrary point in the positive orthant, and to derive the conditions under which the finite-time controllable subsets are polyhedral.

In Chapter 5, we have addressed the problem of determining whether the reachable subset is a polyhedral cone or a nonpolyhedral cone. In addition to control theory, this problem is of interest to the theory of realization and of system identification. The choice for the reachable subset from the origin is essential for realization theory as observability of a linear positive system is then of interest only for states in the reachable set. A characterization of that view of observability does currently not exist in the literature. The condition formulated in [1] is too strong because it is based on the assumption that the reachable set from the origin is the entire positive vector space $\mathbb{R}^n_+$. Developing necessary and sufficient conditions for the described view of observability (i.e., observability of only a subset of the positive orthant) will then pave the way towards a better understanding of the realization problem for linear positive systems.

6.2.3. Other Application Areas of Control Theory of Linear Positive Systems

Recursive feasibility of the optimization problem and asymptotic stability of the closed-loop system are key properties of any MPC design. Although these properties can be essentially achieved by choosing a sufficiently large prediction horizon, enforcing them explicitly by choosing an appropriate terminal constraint set and a terminal cost function is essential when the prediction horizon cannot be chosen arbitrarily large due to computational concerns. The common solution approach of calculating a (robustly) positively invariant subspace for linear systems requires expensive iterative calculation, which quickly becomes prohibitive for systems with a large number of states and inputs. The results of Chapter 4 on calculating (robustly) positively invariant sets for linear positive systems and the associated state feedback control law defined over those sets can be further developed to calculate appropriate terminal constraint set and a terminal cost function for MPC of linear systems. This approach may be of particular interest in the case of parametric MPC, where the control sequence over a prediction horizon N is parameterized as $\boldsymbol{u}_N(t) = \big(\boldsymbol{K}(t)\boldsymbol{x}(t),\ \dots,\ \boldsymbol{K}(t+N-1)\boldsymbol{x}(t+N-1)\big)$, state and control constraints are

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expressed by ∞-norm bounded sets, and the MPC objective function is of the form $V_N(\boldsymbol{x}) = \sum_{i\in\mathbb{N}_{0:N-1}} \|\boldsymbol{P}^{-1}\boldsymbol{x}(i)\|_\infty + \|\boldsymbol{U}^{-1}\boldsymbol{u}(i)\|_\infty + \|\boldsymbol{Q}^{-1}\boldsymbol{x}(N)\|_\infty$, with $\boldsymbol{P} > 0$, $\boldsymbol{U} > 0$, and $\boldsymbol{Q} > 0$ being diagonal matrices. The methods developed in Chapter 4 may then be extended to find a (robustly) positively invariant hypercube and the corresponding Lyapunov function that can be used as the terminal constraint set and the terminal cost function to guarantee the recursive feasibility of the MPC problem and asymptotic stability of the closed-loop system. This approach is expected to yield a linear programming formulation for both calculating the (robustly) invariant hypercube and calculating the optimal control sequence. In addition, developing similar results as those of Chapter 4 for the case where the disturbance is characterized by a 1-norm bounded set of the form $\mathbb{W} = \{\boldsymbol{\omega} \mid \|\boldsymbol{\Omega}^{-1}\boldsymbol{\omega}\|_1 \leq \boldsymbol{\omega}_{\max}\}$, with a diagonal $\boldsymbol{\Omega} > 0$, should be also considered. In this case, one should investigate whether the problem of searching for a feedback controller that minimizes the L_1-norm of the disturbance driven output and the associated robustly positively invariant set can be expressed as a linear program. In addition, the geometry of such set and the associated Lyapunov function defined over such set is of interest. In a similar manner to what we discussed above, such results could also be useful for the MPC application, where the aim is to design a receding horizon finite horizon controller minimizing a suitably defined L_1-norm based cost function subject to L_1-norm based constraints.

Application of methods developed based on the theory of positive dynamical systems, such as those developed in Chapter 4, is not restricted to the classical examples of positive systems such BHSs, logistic systems, economic systems, bioreactors, pharmacokinetic systems, etc. In fact, the result of Chapter 4 on control synthesis and other similar results for control of linear positive systems only require the closed-loop system to be positive, and neither the control input, the disturbance input, nor the initial state is required to be nonnegative. Hence, other major application fields such as motion control, robotics, process control, etc., where the open-loop system is not necessarily positive, can also benefit from the lower-complexity control design solutions and analysis methods developed for linear positive systems as long as the feedback control is designed in such a manner that the closed-loop system positive. Exploring the relevance and application of methods developed for linear positive systems to a broader application domain is therefore a topic worth more attention.

Many of the well-known control design problems take a simpler form for linear positive systems. For example, it has been shown in [2–4] that the KYP lemma for linear positive systems can be expressed in terms of a diagonal matrix variable rather than a general symmetric matrix, which drastically simplifies the H_∞-design problem. Similarly, in [5–8], it has been shown that stability analysis of an interconnection of systems based on L_1-gain and L_∞-gain criteria and the corresponding controller synthesis problems can be expressed as linear programs. In addition, we have shown in Chapter 4 that the joint problem of computing the smallest robustly invariant hypercube containing the minimal robust invariant subset and computing the corresponding state feedback controller minimizing the L_∞-norm of the disturbance driven output over the said hypercube can be expressed as a linear problem.

While the mentioned works make contributions to analysis of feedback systems and controller synthesis for linear positive systems, potential application of the theory nonnegative matrices and positive dynamical systems in other domains of control theory such as optimal and receding horizon observer design problems, (distributed) parameter estimation problems, and joint state and parameter estimation problems has been mostly unexplored. This is an interesting research area, which could yield promising results as such problems may also benefit from simpler solution forms due to the properties of linear positive systems.

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Summary

Large-scale baggage handling systems, or large-scale logistic system, for that matter, pose interesting challenges to model-based control design. These challenges concern computational complexity, scalability, and robustness of the proposed solutions. This thesis tackles these issues in a collection of papers organized in two overlapping parts. The first part concerns modeling and Model Predictive Control (MPC) design of large-scale baggage handling systems (BHSs), where a modeling framework for BHSs is proposed that is subsequently used to develop an MPC scheme for control of large-scale BHSs. The MPC controller optimizes for the timely arrival of pieces of baggage at their destination within the BHS network under capacity constraints while minimizing the overall cost of transporting pieces of baggage. Several formulations for the resulting constrained optimization problem are proposed, and they are compared with each other in terms of closed-loop performance and computational complexity. It is shown, via simulation studies, that the proposed solutions can outperform a heuristics-based approach commonly used for control of BHSs while scaling well to larger BHS network instances.

In its second part, the thesis focuses on robustness of control design in the face of a partially known disturbance input (i.e., input baggage demand), and especially on developing a scalable tube-based MPC scheme. For this purpose, considering the BHS model essentially as a linear positive system, a linear-programming-based approach is proposed for the joint calculation of a robustly positively invariant subset and a constrained state feedback controller that minimizes the disturbance-driven L_∞ norm of the output over this set. A tube-based MPC control scheme is finally developed by coupling the state feedback controller with a nominal MPC controller, guaranteeing recursive feasibility and asymptotic stability. It is shown via simulation studies that the proposed tube-based approach is effective against unpredictable disturbances. In addition, since the design of both the nominal MPC controller and the state feedback controller involves only linear programs, the proposed tube-based approach scales well to BHS networks of larger size.

Linear positive systems are of interest in several branches of engineering, logistics, biochemistry, and economics. As a spin-off topic and inspired by the applications of the theory of linear positive systems to modeling and control design of systems in the mentioned domains, the third part of the thesis focuses on the reachability analysis of discrete-time linear positive systems. More specifically, we revisit the problem of characterizing the subset of the state space that is reachable from the origin for discrete-time linear positive systems. This problem is of interest in topics such as optimal control of linear positive systems and realization theory of linear positive systems. It is established in this thesis that the reachable subset can be either a polyhedral or a nonpolyhedral cone. For the single-input case, a characterization is provided of when the infinite-time and the finite-time reachable

subsets are polyhedral. Finally, for the case of polyhedral reachable subsets, a method, based on solving a set of linear equations, is provided to verify whether a target set can be reached from the origin using positive inputs.

List of Publications

7. Y. Zeinaly, B. De Schutter, *Robustly Positively Invariant Sets for Discrete-time Linear Positive Systems: Application to a Tube-based MPC Approach*, To be submitted to IEEE Transactions on Control Systems Technology, 2022.

6. Y. Zeinaly, J. H. van Schuppen, B. De Schutter, *Linear Positive Systems May Have a Reachable Subset from the Origin That is Either Polyhedral or Nonpolyhedral*, SIAM Journal on Matrix Analysis and Applications **41**, 1 (2020), pp. 279–307.

5. Y. Zeinaly, B. De Schutter, H. Hellendoorn, *An Integrated Model Predictive Scheme for Baggage Handling Systems: Routing, Line Balancing, and Empty-Cart Management*, IEEE Transactions on Control Systems Technology **23**, 4 (2015), pp. 1536–1545.

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3. Y. Zeinaly, B. De Schutter, H. Hellendoorn, *An MPC Scheme For Routing Problem In Baggage Handling Systems: A Linear Programming Approach*, in 10th IEEE International Conference On Networking, Sensing and Control (ICNSC), (2013), pp. 786–791.

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Curriculum Vitæ

Yashar Zeinaly was born in April 1985 in Urmia, Iran. In August 2007, he obtained his B.Sc. degree in electrical engineering with a specialization in control engineering from Khajeh Nasir Toosi University of Technology, Tehran. Iran. In 2008, he was admitted to the master program Systems, Control & Mechatronics at Chalmers University of Technology, Gothenburg, Sweden. He was a visiting researcher at the University of Auckland, New Zealand, working on his master thesis on model predictive direct torque control of medium voltage drives, which was a project supported by ABB Switzerland. In March 2011, he received the best presentation award at the 2011 Applied Power Electronics Conference (APEC), held in Fort Worth, Texas, for the work he conducted for his master thesis.

He obtained his M.Sc. degree in December 2010, and subsequently joined Delft Center of Systems and Control as a PhD candidate under the supervision of Prof. Bart De Schutter, working on the topic "Model-based Control of Large-scale Logistic System". During his first year of PhD he successfully fulfilled the requirements to be awarded the Dutch Institute of Systems and Control (DISC) certificate from the DISC graduate school. The focus of his PhD was on robust scalable model-based control solutions for large-scale baggage handling system (BHSs), where he used tailor-made methods from the control theory of linear positive systems to develop a low-complexity robust model predictive control (MPC) approaches for applications such as large-scale BHSs. In addition, he worked with Prof. Jan van Schuppen of the Mathematics Department of the Delft University of Technology on characterizing the reachable subset of discrete-time linear positive systems.

He is currently a research engineer within the thermotechnology division of Robert Bosch GmbH. His research interest are modeling and low-complexity model-based control of networked smart energy systems, control theory of linear positive systems and its application in scalable model-based control design of complex systems.

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