CIEM1110-1: FEM, lecture 3.3

Introduction to nonlinear material models

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Agenda for today

1. Nonlinear material models — solver interface
2. An overview of different strategies for modeling nonlinear materials

Recap – nonlinear FEM

Discretized form:

$$\int_\Omega \mathbf{B}^T \boldsymbol{\sigma}\,\mathrm{d}\Omega = \int_\Omega \mathbf{N}^T \mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t}\,\mathrm{d}\Gamma$$

Recap — nonlinear FEM

Discretized form:

$$\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Geometric nonlinearity:

$$\mathbf{B} = \mathbf{B}\left(\mathbf{a}\right)$$

Recap – nonlinear FEM

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Geometric nonlinearity:

$$\mathbf{B} = \mathbf{B}\left(\mathbf{a}\right)$$

Material nonlinearity:

$$\frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \mathbf{D} = \mathbf{D}\left(\mathbf{a}\right)$$

Load control with a nonlinear material

```
Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$
Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps do
    Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
    Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$
    Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
    Evaluate first residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
    repeat
        Solve linear system of equations: $\mathbf{K}^{n+1}\Delta\mathbf{a} = \mathbf{r}$
        Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta\mathbf{a}$
        Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
        Evaluate residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
    until $|\mathbf{r}| <$ tolerance
    $n = n + 1$
end while
```

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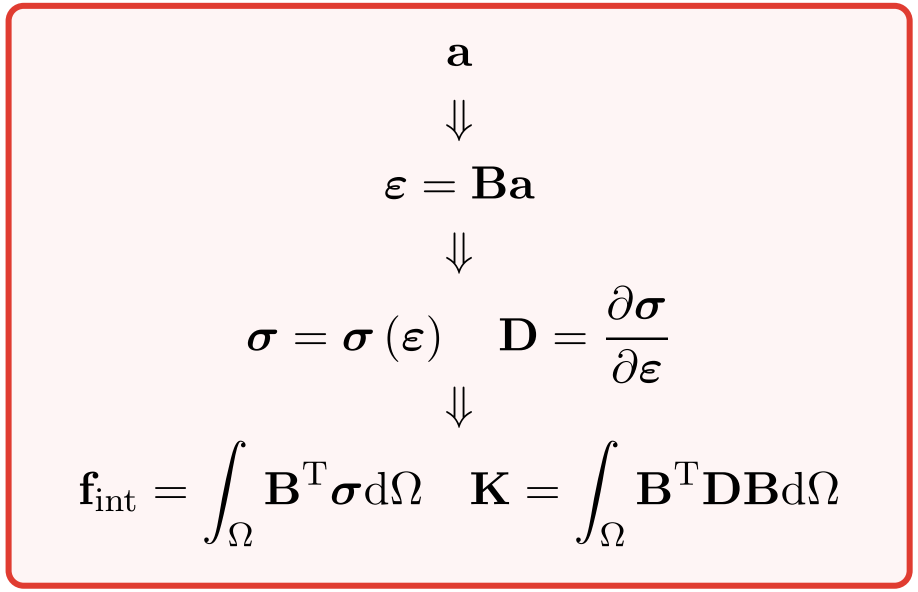
```



$$\mathbf{a}$$

$$\Downarrow$$

$$\boldsymbol{\varepsilon} = \mathbf{B}\mathbf{a}$$

$$\Downarrow$$

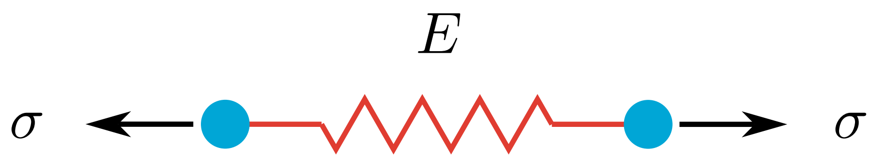
$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) \quad \mathbf{D} = \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}}$$

$$\Downarrow$$

$$\mathbf{f}_{\text{int}} = \int_\Omega \mathbf{B}^{\text{T}}\boldsymbol{\sigma}\,\mathrm{d}\Omega \quad \mathbf{K} = \int_\Omega \mathbf{B}^{\text{T}}\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega$$

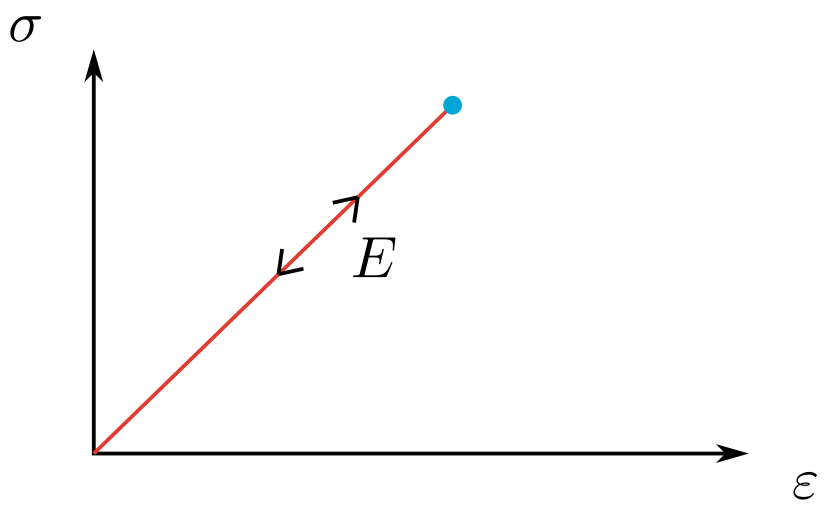
Linear elasticity (recap)

Constant stiffness, full reversibility



In Voigt notation in 3D:

$$\mathbf{D} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}$$



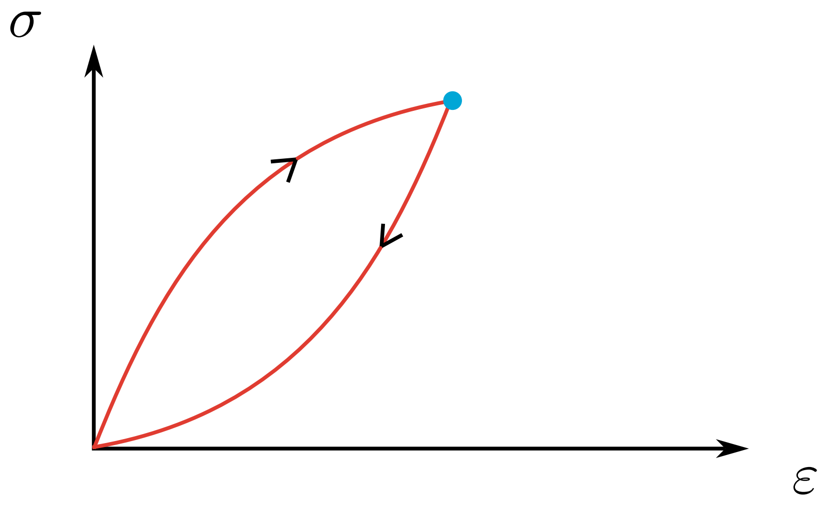
$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$$

Viscoelasticity

Stiffness is time-dependent

Fully reversible response, but stiffer if loaded faster

An integral in time appears, how to handle it?



$$\boldsymbol{\sigma} = \mathbf{D}_{\infty}\boldsymbol{\varepsilon} + \int_{0}^{t} \mathbf{D}_{\mathrm{ve}}\left(t-\tilde{t}\right)\frac{\partial\boldsymbol{\varepsilon}\left(\tilde{t}\right)}{\partial\tilde{t}}\mathrm{d}\tilde{t}$$

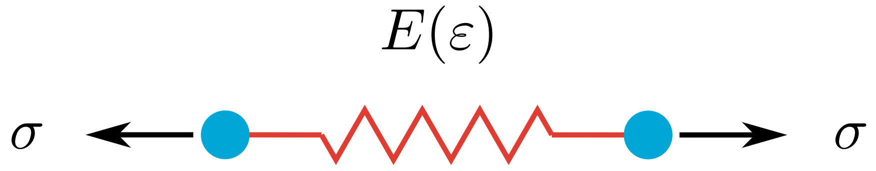
Hyperelasticity

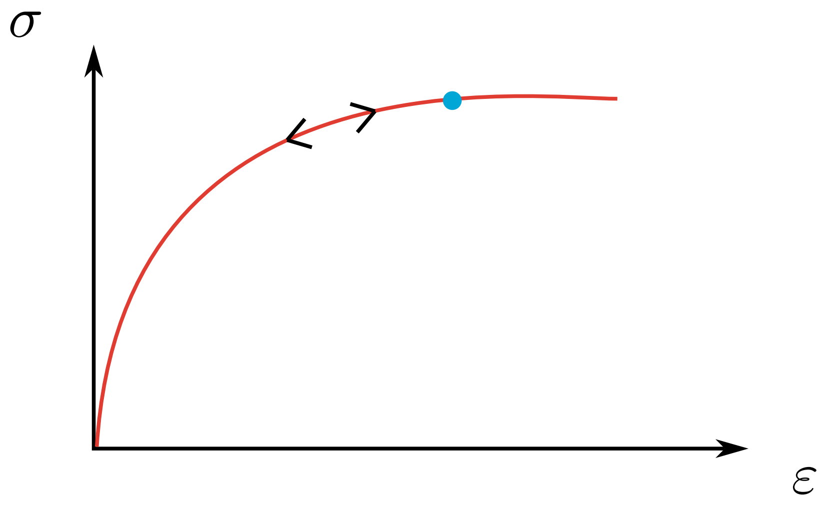
Analogous to a nonlinear spring

Popular for modeling large strains

Still fully reversible

Usually derived from a single scalar potential W



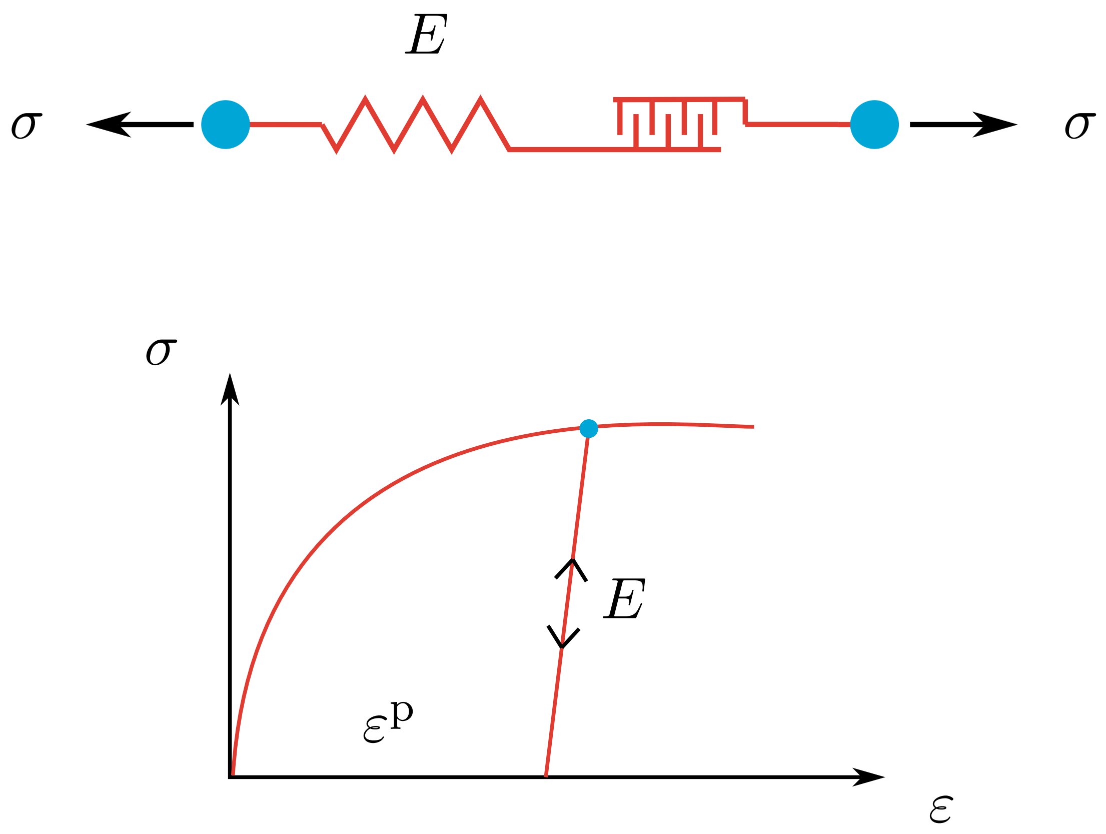


$$W(\boldsymbol{\varepsilon}) \quad \Rightarrow \quad \boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} \quad \Rightarrow \quad \frac{\partial \boldsymbol{\sigma}}{\partial \boldsymbol{\varepsilon}} = \frac{\partial^2 W}{\partial \boldsymbol{\varepsilon}^2}$$

Plasticity

Deformations are split:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$$



$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right)$$

Plasticity

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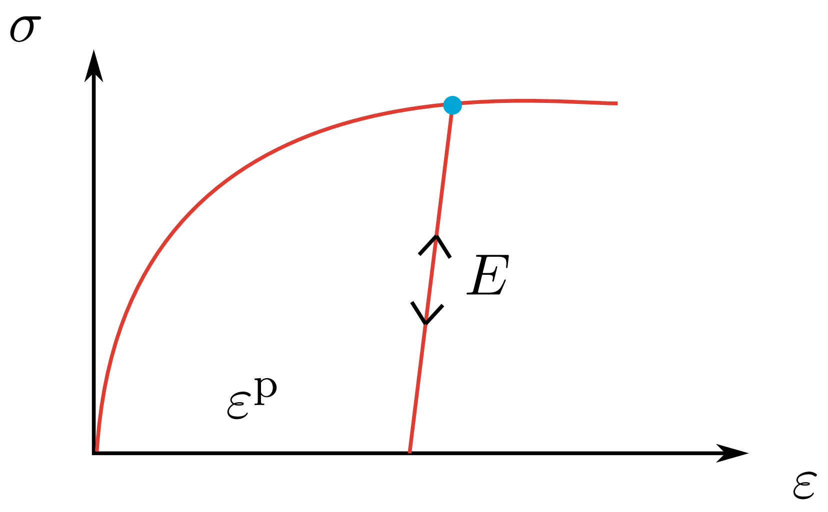
A yield surface defines the plasticity threshold:

$$f\left(\boldsymbol{\sigma}\right) = \tilde{\boldsymbol{\sigma}} - \sigma_y\left(\kappa\right)$$



$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right)$$

Plasticity

Deformations are split:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathrm{e}} + \boldsymbol{\varepsilon}^{\mathrm{p}}$$

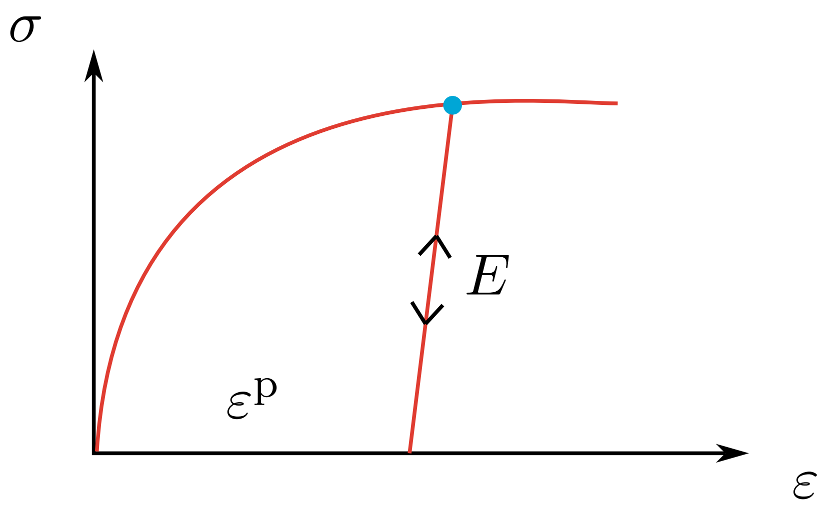
A yield surface defines the plasticity threshold:

$$f\left(\boldsymbol{\sigma}\right) = \tilde{\boldsymbol{\sigma}} - \sigma_y\left(\kappa\right)$$

Plastic flow occurs when the yield surface is pushed:

$$f = 0, \dot{f} = 0 \quad \Rightarrow \quad \dot{\boldsymbol{\varepsilon}}^{\mathrm{p}} = \dot{\gamma}\mathbf{m}$$

We now introduce history variables κ and $\boldsymbol{\varepsilon}^{\mathrm{p}}$

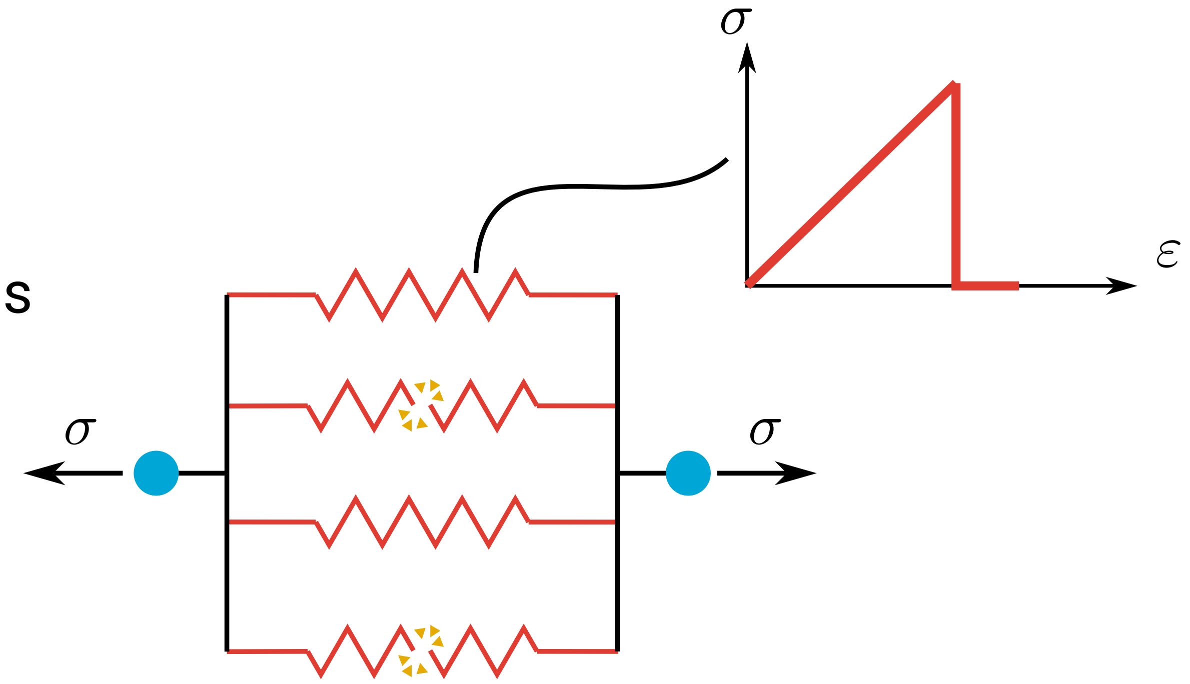


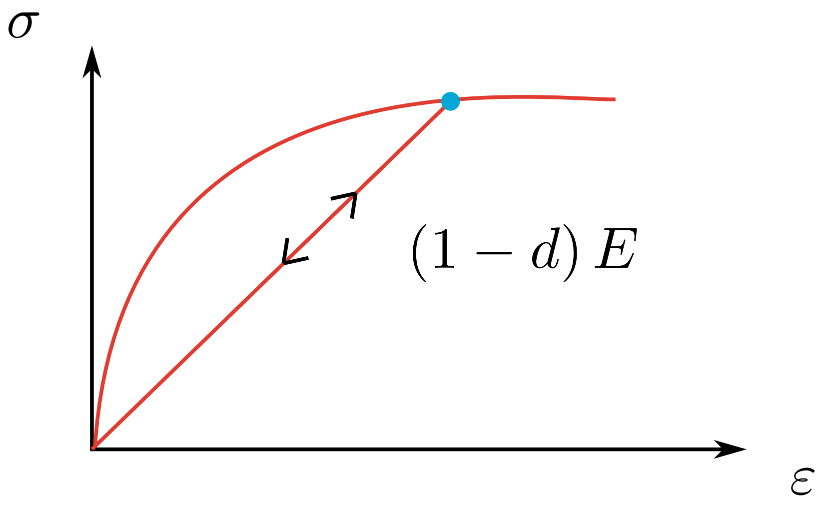
$$\boldsymbol{\sigma} = \mathbf{D}\left(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^{\mathrm{p}}\right)$$

Damage

Loss of load-carrying area modeled as loss of stiffness

$$A = (1 - d)\, A_0$$





$$\boldsymbol{\sigma} = (1 - d)\, \mathbf{D}\boldsymbol{\varepsilon}$$

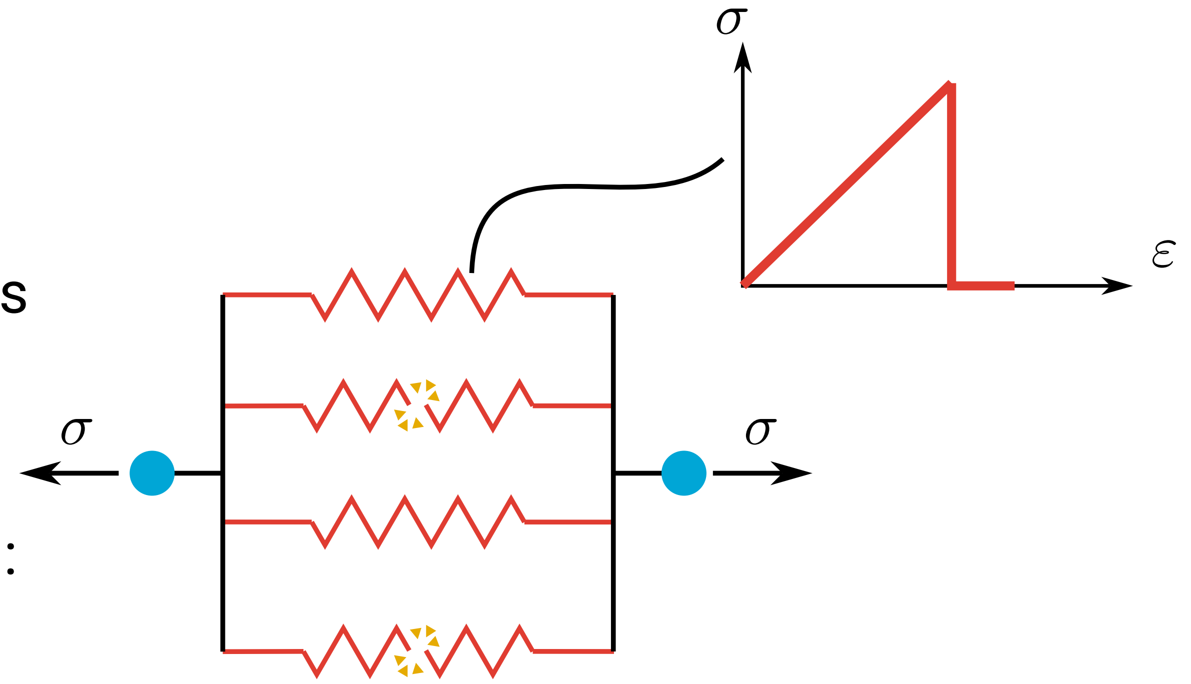
Damage

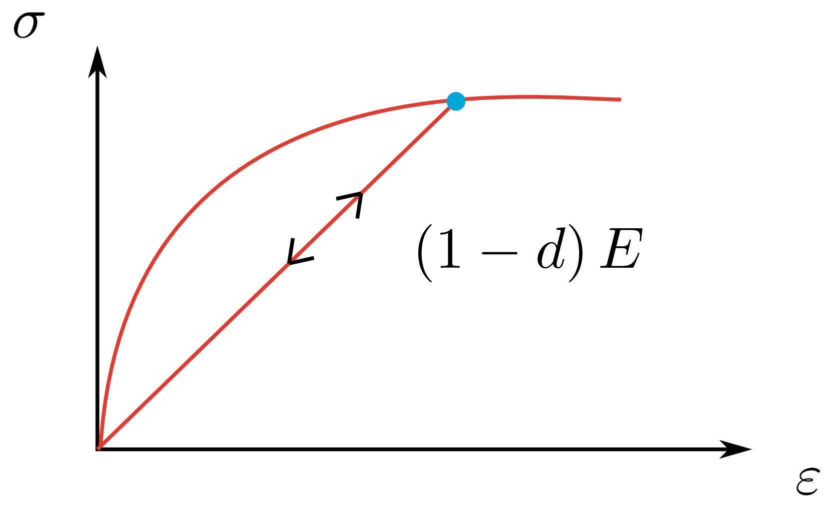
Loss of load-carrying area modeled as loss of stiffness

$$A = (1 - d)\, A_0$$

The damage d evolves according to a loading function:

$$f\left(\tilde{\boldsymbol{\varepsilon}}, \kappa\right) = \tilde{\boldsymbol{\varepsilon}} - \kappa, \quad f \leq 0, \dot{\kappa} \geq 0, f\dot{\kappa} = 0$$





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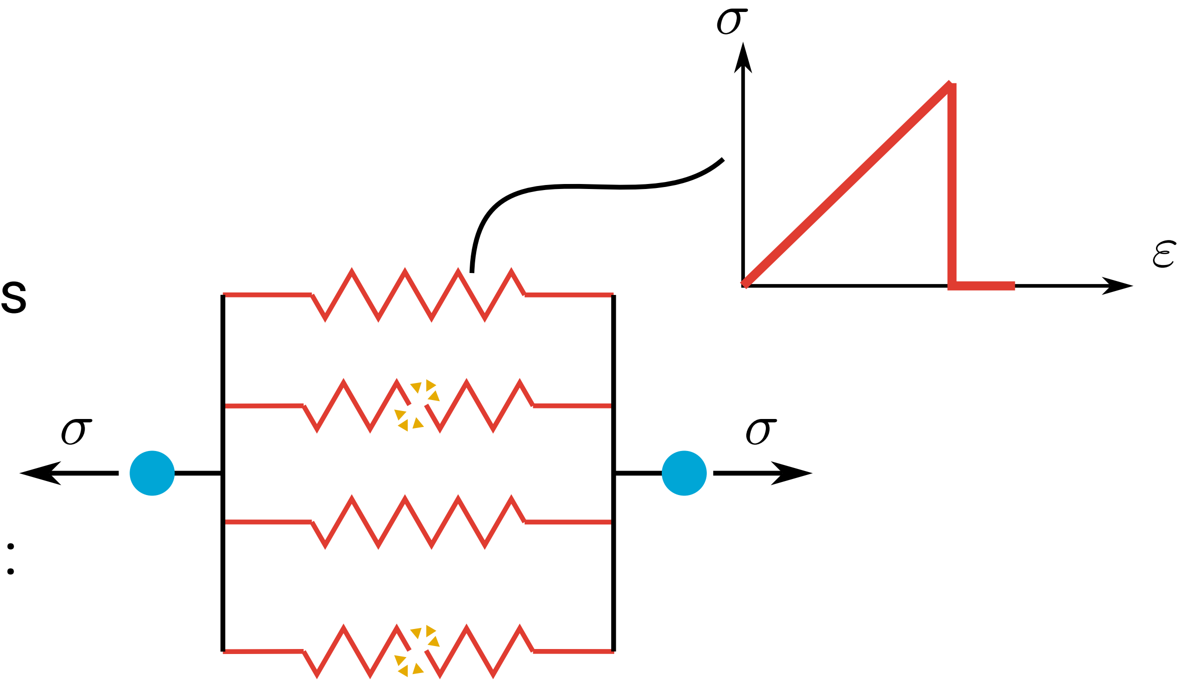
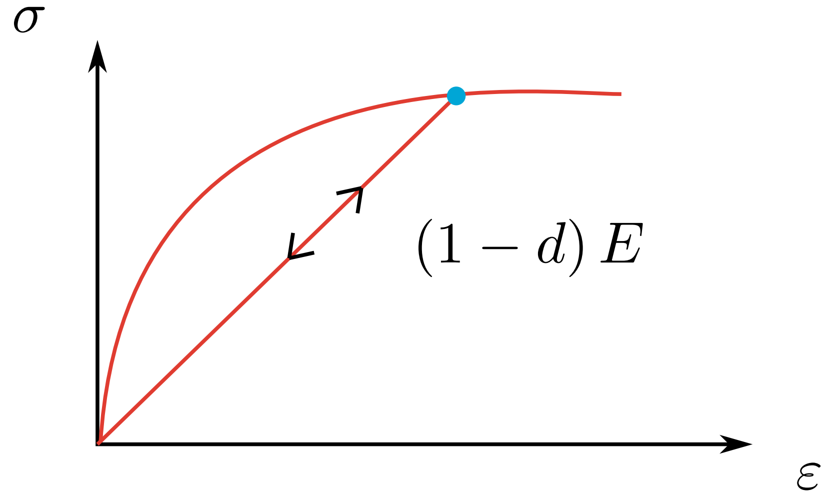
$$f\left(\tilde{\boldsymbol{\varepsilon}}, \kappa\right) = \tilde{\boldsymbol{\varepsilon}} - \kappa, \quad f \leq 0, \dot{\kappa} \geq 0, f\dot{\kappa} = 0$$

and an evolution equation:

$$d = d\left(\kappa\right)$$





$$\boldsymbol{\sigma} = (1 - d)\, \mathbf{D}\boldsymbol{\varepsilon}$$

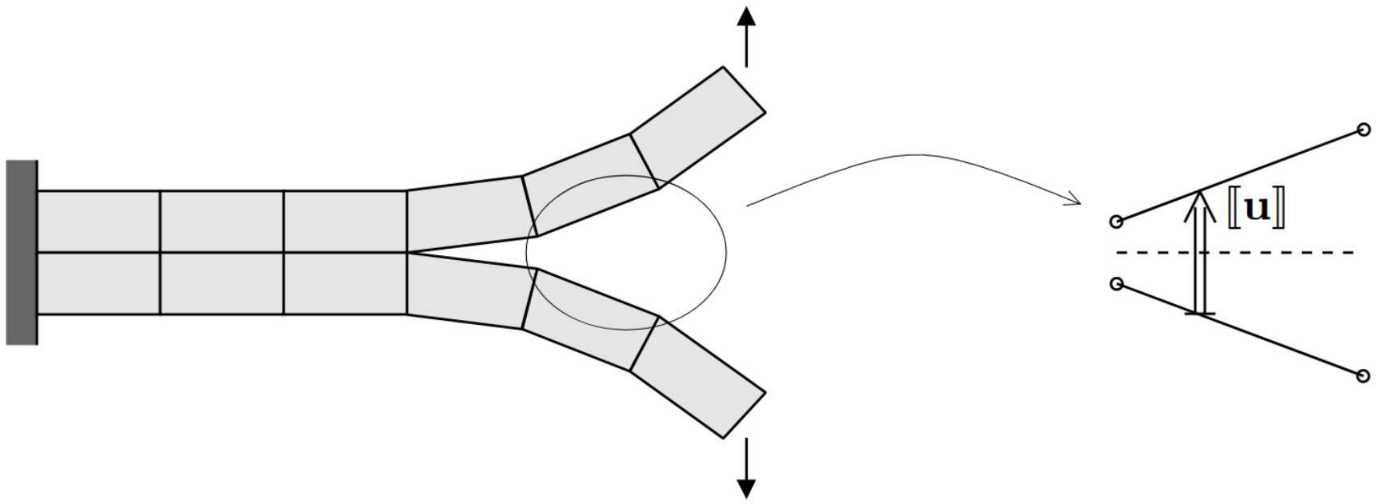
Discontinuous damage

Model an actual displacement discontinuity

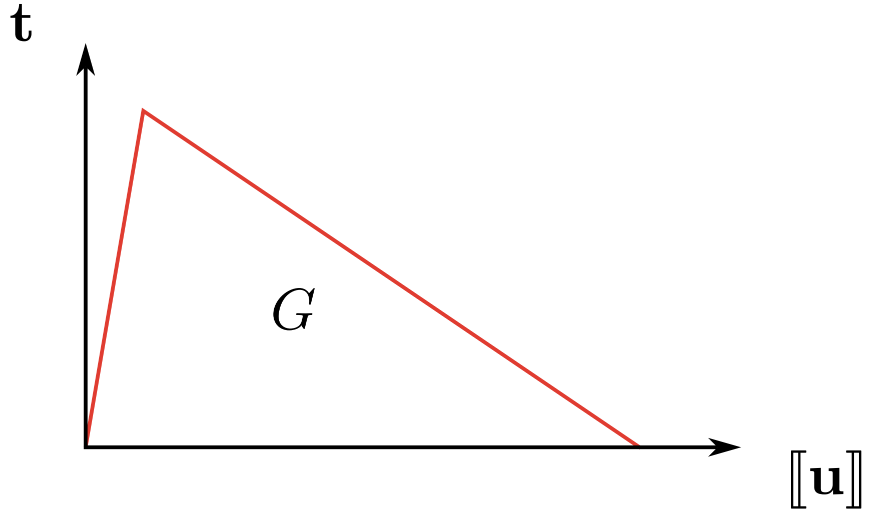
Traction-separation through a cohesive zone law

Special interface elements are needed

Similar to damage, but explicit link to energy dissipation



$$[\![\mathbf{u}]\!] = \mathbf{u}^{\text{top}} - \mathbf{u}^{\text{bot}} \quad \Rightarrow \quad \begin{bmatrix}\mathbf{N} & -\mathbf{N}\end{bmatrix}\begin{bmatrix}\mathbf{a}^{\text{top}} \\ \mathbf{a}^{\text{bot}}\end{bmatrix} \quad \Rightarrow \quad [\![\mathbf{u}]\!] = \mathbf{N}_\Gamma \mathbf{a}_\Gamma$$



$$\mathbf{t} = \mathcal{T}\left([\![\mathbf{u}]\!]\right) \quad \Rightarrow \quad \mathbf{f}_{\text{int}}^{\Gamma} = \int_{\Gamma} \mathbf{N}_{\Gamma}^{\mathrm{T}} \mathbf{t} \, \mathrm{d}\Gamma$$

Displacement control with a history-dependent material

```
Require: Nonlinear relation $\mathbf{f}_{\mathrm{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\mathrm{int}}}{\partial \mathbf{a}}$
Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^{n}$
Update material model: $\boldsymbol{\sigma}^{n+1}, \mathbf{D}^{n+1}, \boldsymbol{\alpha}_{\mathrm{new}} = \mathcal{M}\left(\boldsymbol{\varepsilon}^{n+1}, \boldsymbol{\alpha}_{\mathrm{old}}\right)$
Compute internal force and stiffness: $\mathbf{f}_{\mathrm{int}}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^{n+1} \mathrm{d}\Omega$, $\mathbf{K}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D}^{n+1} \mathbf{B} \mathrm{d}\Omega$
Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = \bar{\mathbf{a}}^{n+1} - \bar{\mathbf{a}}^{n}$
Evaluate first residual: $\mathbf{r} = -\mathbf{f}_{\mathrm{int},f}^{n+1}$
repeat
    Solve linear system of equations: $\mathbf{K}^{n+1} \Delta \mathbf{a} = \mathbf{r}$
    Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta \mathbf{a}$
    Update material model: $\boldsymbol{\sigma}^{n+1}, \mathbf{D}^{n+1}, \boldsymbol{\alpha}_{\mathrm{new}} = \mathcal{M}\left(\boldsymbol{\varepsilon}^{n+1}, \boldsymbol{\alpha}_{\mathrm{old}}\right)$
    Compute internal force and stiffness: $\mathbf{f}_{\mathrm{int}}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \boldsymbol{\sigma}^{n+1} \mathrm{d}\Omega$, $\mathbf{K}^{n+1} = \int_{\Omega} \mathbf{B}^{\mathrm{T}} \mathbf{D}^{n+1} \mathbf{B} \mathrm{d}\Omega$
    Evaluate residual: $\mathbf{r} = -\mathbf{f}_{\mathrm{int},f}^{n+1}$
    Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = 0$
until $|\mathbf{r}| <$ tolerance
Commit material history: $\boldsymbol{\alpha}_{\mathrm{old}} = \boldsymbol{\alpha}_{\mathrm{new}}$
```