CIEM5110-2: FEM, lecture 1.2

Derivation of finite element equations for elastostatics

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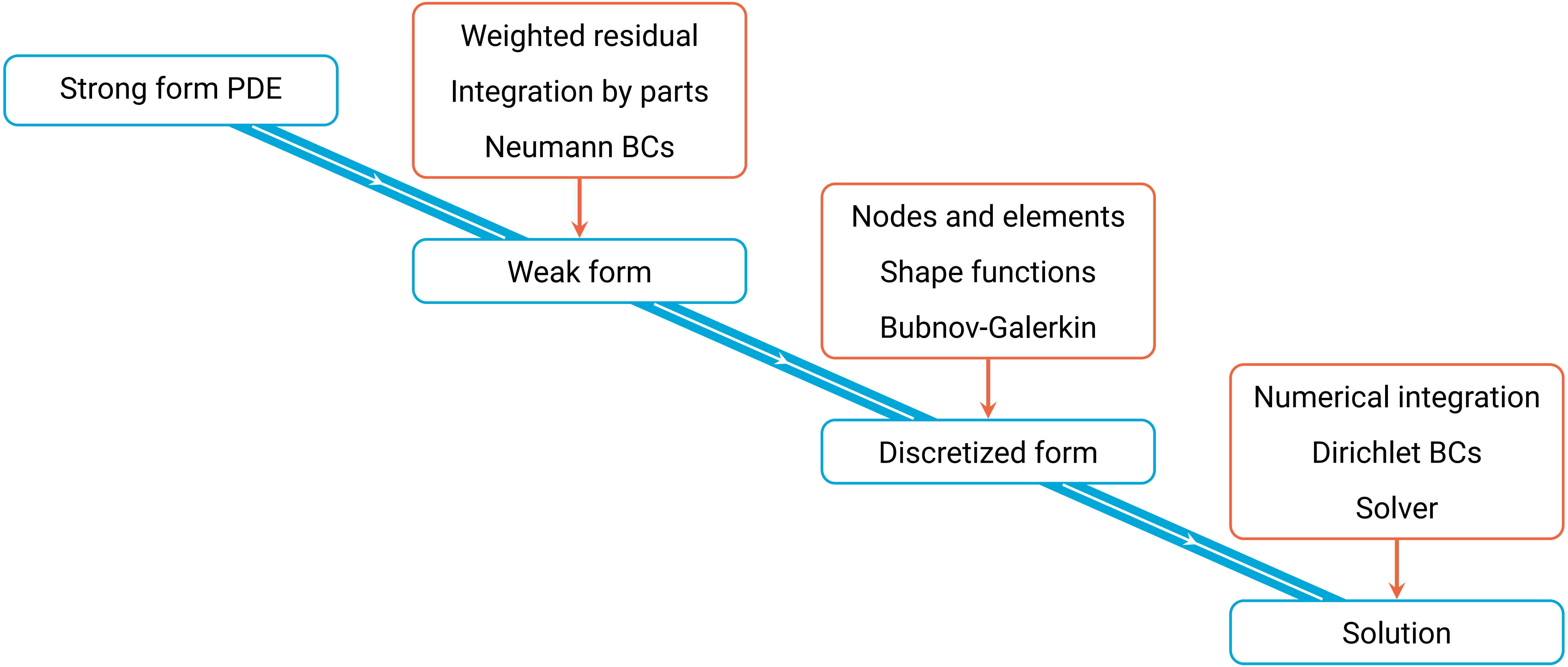
Deriving the finite element method



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Discussion

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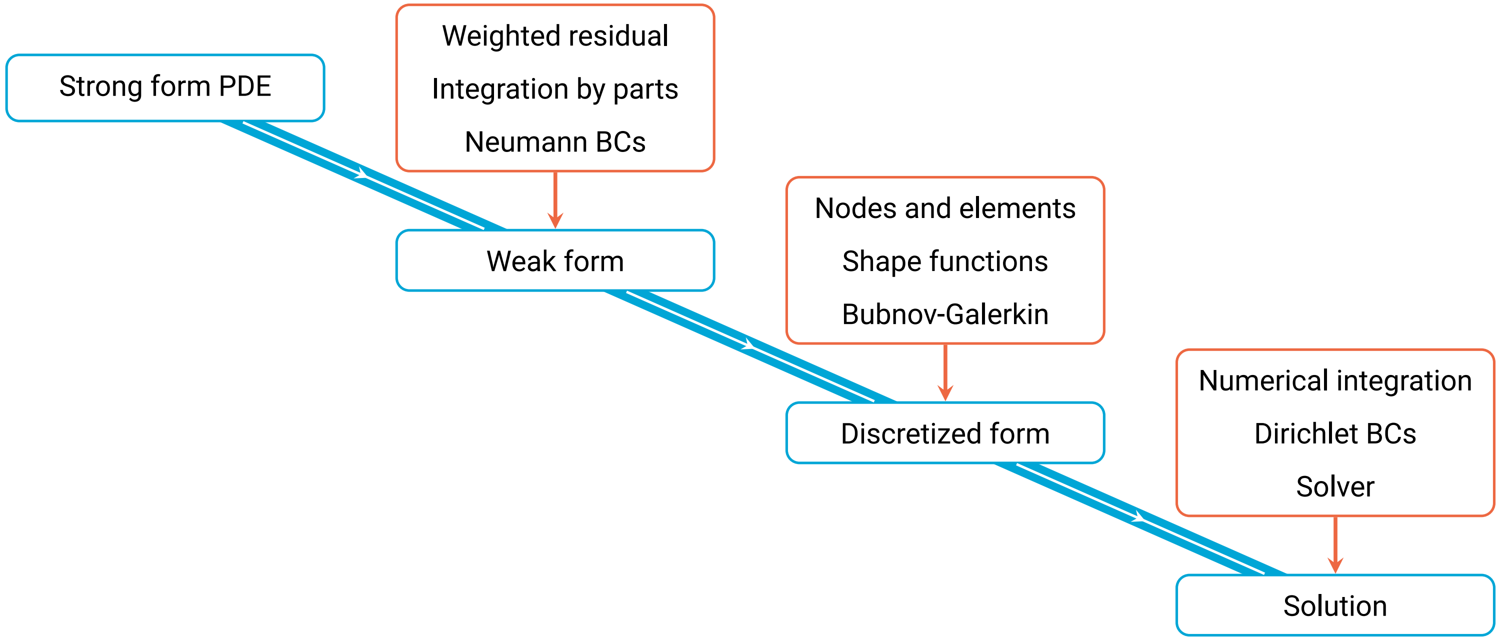
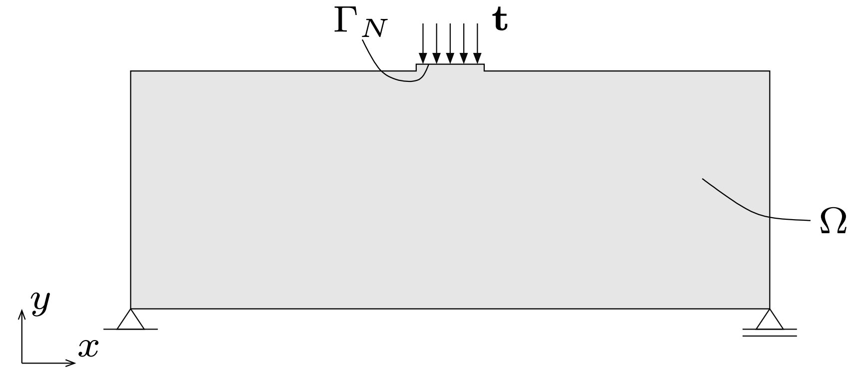

Strong form PDE
Weighted residual
Integration by parts
Neumann BCs
Weak form
Nodes and elements
Shape functions
Bubnov-Galerkin
Discretized form
Numerical integration
Dirichlet BCs
Solver
Solution

The basic ingredients for elastostatics

Equilibrium relation	$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$ $\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = -b_x$ etc
Constitutive relation	$\boldsymbol{\sigma} = \boldsymbol{\mathcal{D}} : \boldsymbol{\varepsilon}$ $\sigma_{xx} = \frac{E}{(1+\nu)(1-2\nu)}\left((1-\nu)\varepsilon_{xx} + \nu\varepsilon_{yy} + \nu\varepsilon_{zz}\right)$ etc
Kinematic relation	$\boldsymbol{\varepsilon} = \nabla^s \mathbf{u}$ $\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$ etc
Geometry and BCs	

Starting point is the equilibrium equation

The partial differential equation (PDE) we want to solve

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}$$

where

- $\boldsymbol{\sigma}$ and $\mathbf{b}$ are a function of coordinates $\mathbf{x}$
- this equation must hold at every point $\mathbf{x}$

Starting point is the equilibrium equation

The partial differential equation (PDE) we want to solve

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \qquad \text{(or} \quad \nabla \cdot \boldsymbol{\mathcal{D}} : \nabla_{\mathrm{s}} \mathbf{u} + \mathbf{b} = \mathbf{0}\text{)}$$

where

- $\boldsymbol{\sigma}$ and $\mathbf{b}$ are a function of coordinates $\mathbf{x}$
- this equation must hold at every point $\mathbf{x}$
- in 1D this is equivalent to the Poisson equation
- in 2D or 3D, the unknown field $\mathbf{u}$ is a vector, different from the Poisson equation

We rewrite the equation in weighted residual form

Premultiply with $\mathbf{w}$ and integrate over domain:

$$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \quad \Rightarrow \quad \int_{\Omega} \mathbf{w} \cdot (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b}) \, \mathrm{d}\Omega = 0, \quad \forall \, \mathbf{w}$$

where

- $\mathbf{w}(\mathbf{x})$ is a (yet unspecified) weight function
- Ω is the domain over which we solve the equation
- just like $\mathbf{u}$, the weight function $\mathbf{w}$ is a vector field
- if this holds for all possible $\mathbf{w}$ (i.e. $\forall \, \mathbf{w}$), the two expressions are equivalent

Now we apply divergence theorem (or Gauss' theorem)

To get rid of second order derivative of $\mathbf{u}$ that is hidden in the term $\nabla \cdot \boldsymbol{\sigma}$:

$$\int_\Omega \mathbf{w} \cdot (\nabla \cdot \boldsymbol{\sigma} + \mathbf{b})\, \mathrm{d}\Omega = 0 \quad \Rightarrow \quad -\int_\Omega \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\sigma}\, \mathrm{d}\Omega + \int_\Omega \mathbf{w} \cdot \mathbf{b}\, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w} \cdot \mathbf{t}\, \mathrm{d}\Gamma = 0$$

where

- Γ_N is the boundary along which an external traction is applied
- $\mathbf{t}$ is that applied traction
- $\nabla^{\mathrm{s}}\mathbf{w} = \frac{1}{2}\left(\nabla\mathbf{w} + (\nabla\mathbf{w})^T\right)$
- actually divergence theorem gives $\nabla\mathbf{w}$ instead of $\nabla^{\mathrm{s}}\mathbf{w}$
- the symmetric gradient can be used because of symmetry of $\boldsymbol{\sigma}$

Substitution of constitutive and kinematic relation (linear elasticity)

Substitution of $\boldsymbol{\sigma} = \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u}$ gives

$$-\int_\Omega \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\sigma}\,\mathrm{d}\Omega + \int_\Omega \mathbf{w}\cdot\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}\cdot\mathbf{t}\,\mathrm{d}\Gamma = 0, \quad \forall\mathbf{w}$$
$$\Rightarrow \quad -\int_\Omega \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u}\,\mathrm{d}\Omega + \int_\Omega \mathbf{w}\cdot\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}\cdot\mathbf{t}\,\mathrm{d}\Gamma = 0, \quad \forall\mathbf{w}$$

where

- $\mathbf{u}(\mathbf{x})$ is the displacement field
- the following strain definition is used $\boldsymbol{\varepsilon} = \nabla^{\mathrm{s}}\mathbf{u}$ or $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$
- and the following constitutive relation $\boldsymbol{\sigma} = \boldsymbol{\mathcal{D}} : \boldsymbol{\varepsilon}$
- where $\boldsymbol{\mathcal{D}}$ is a fourth order tensor

This is our weak form, before any discretization

Find the displacement field $\mathbf{u} \in \mathcal{S}$ that satisfies

$$-\int_{\Omega} \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u} \,\mathrm{d}\Omega + \int_{\Omega} \mathbf{w} \cdot \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w} \cdot \mathbf{t} \,\mathrm{d}\Gamma = 0, \quad \forall \mathbf{w} \in \mathcal{V}$$

where

- $\mathbf{u}(\mathbf{x})$ is the displacement field
- $\mathcal{S}$ is the set of functions to which $\mathbf{u}$ must belong
- $\mathbf{w}(\mathbf{x})$ is the weight function
- $\mathcal{V}$ is the set of functions to which $\mathbf{w}$ must belong
- we have ignored displacement boundary conditions

The same could be obtained directly from virtual work or energy minimization
However, the mathematical procedure presented here also works for other PDEs

Now we approximate both $\mathbf{u}$ and $\mathbf{w}$

Find the displacement field $\mathbf{u}^h \in \mathcal{S}^h$ that satisfies

$$-\int_\Omega \nabla^{\mathrm{s}}\mathbf{w}^h : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u}^h \,\mathrm{d}\Omega + \int_\Omega \mathbf{w}^h \cdot \mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}^h \cdot \mathbf{t}\,\mathrm{d}\Gamma = 0, \quad \forall \mathbf{w}^h \in \mathcal{V}^h$$

where

- the infinite set $\mathcal{S}$ has been reduced to finite set $\mathcal{S}^h$
- the infinite set $\mathcal{V}$ has been reduced to finite set $\mathcal{V}^h$

This abstract operation limits the number of degrees of freedom (in $\mathcal{S}^h$)
and the number of equations (in $\mathcal{V}^h$)

We introduce discretization for $\mathbf{u}^h$ and $\mathbf{w}^h$

Find the displacement field $\mathbf{u}^h \in \mathcal{S}^h$ that satisfies

$$-\int_\Omega \nabla^{\mathrm{s}}\mathbf{w}^h : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u}^h \,\mathrm{d}\Omega + \int_\Omega \mathbf{w}^h \cdot \mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}^h \cdot \mathbf{t}\,\mathrm{d}\Gamma = 0, \quad \forall \mathbf{w}^h \in \mathcal{V}^h$$

given that

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^{n} N_i(\mathbf{x})\mathbf{a}_i, \qquad \mathbf{w}^h(\mathbf{x}) = \sum_{i=1}^{n} N_i(\mathbf{x})\mathbf{c}_i,$$

where

- N_i is the shape function associated with node i
- $\mathbf{a}_i$ is the nodal displacement vector of node i
- $\mathbf{c}_i$ is like an amplitude of the weight function (it will drop out)

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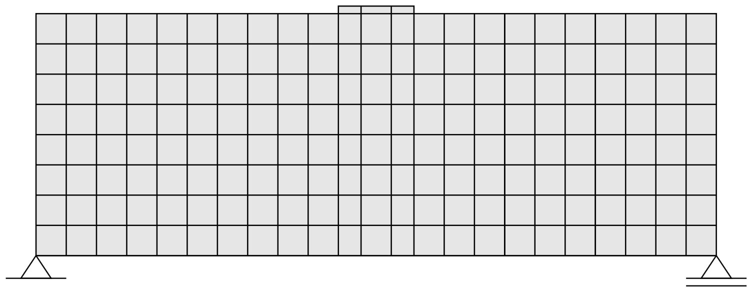
Here the mesh is introduced, shape functions are defined with nodes and elements



We prepare to rewrite the discretized weak form in matrix form

We will substitute

$$\mathbf{u}^h = \mathbf{N}\mathbf{a}, \qquad \mathbf{w}^h = \mathbf{N}\mathbf{c}$$

with (in 2D)

$$\mathbf{N} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \cdots & N_n & 0 \\ 0 & N_1 & 0 & N_2 & \cdots & 0 & N_n \end{bmatrix} \qquad \mathbf{a} = \begin{Bmatrix} a_{1x} \\ a_{1y} \\ a_{2x} \\ a_{2y} \\ \vdots \\ a_{nx} \\ a_{ny} \end{Bmatrix}$$

and

$$\nabla^s \mathbf{u}^h \cong \boldsymbol{\varepsilon}^h = \mathbf{B}\mathbf{a}, \qquad \nabla^s \mathbf{w}^h \cong \mathbf{B}\mathbf{c}$$

with

$$\mathbf{B} = \begin{bmatrix} N_{1,x} & 0 & N_{2,x} & 0 & \cdots & N_{n,x} & 0 \\ 0 & N_{1,y} & 0 & N_{2,y} & \cdots & 0 & N_{n,y} \\ N_{1,y} & N_{1,x} & N_{2,y} & N_{2,x} & \cdots & N_{n,y} & N_{n,x} \end{bmatrix}$$

Here $\mathbf{a}$ and $\mathbf{c}$ are vectors containing values for all nodes

$\nabla^s \mathbf{u}^h$ is a 2$^{\text{nd}}$ order **tensor**, $\boldsymbol{\varepsilon}^h = \mathbf{B}\mathbf{a}$ the engineering strain **vector**

After substitution of these quantities we rewrite the weak form

Find the displacement field $\mathbf{u}^h \in \mathcal{S}^h$ that satisfies

$$\int_\Omega \nabla^{\mathrm{s}} \mathbf{w}^h : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}} \mathbf{u}^h \,\mathrm{d}\Omega = \int_\Omega \mathbf{w}^h \cdot \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}^h \cdot \mathbf{t} \,\mathrm{d}\Gamma, \quad \forall \mathbf{w}^h \in \mathcal{V}^h$$

$$\Downarrow$$

Find the nodal displacements $\mathbf{a}$ that satisfy

$$\int_\Omega (\mathbf{Bc})^T \mathbf{DBa} \,\mathrm{d}\Omega = \int_\Omega (\mathbf{Nc})^T \mathbf{b} \,\mathrm{d}\Omega + \int_{\Gamma_N} (\mathbf{Nc})^T \mathbf{t} \,\mathrm{d}\Gamma, \quad \forall \mathbf{c}$$

where

- we started using Voigt notation: $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are now vectors
- $\mathbf{D}$ is the material stiffness matrix $\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon}$
- we needed tensor notation for writing the strong form and for divergence theorem
- now the engineering notation becomes more convenient

Amplitudes $\mathbf{a}$ and $\mathbf{c}$ can be taken out of the integrals

Find the nodal displacements $\mathbf{a}$ that satisfy

$$\int_\Omega (\mathbf{Bc})^T \mathbf{DBa} \, \mathrm{d}\Omega = \int_\Omega (\mathbf{Nc})^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_N} (\mathbf{Nc})^T \mathbf{t} \, \mathrm{d}\Gamma, \quad \forall \mathbf{c}$$

$$\Downarrow$$

Find the nodal displacements $\mathbf{a}$ that satisfy

$$\mathbf{c}^T \int_\Omega \mathbf{B}^T \mathbf{DB} \, \mathrm{d}\Omega \mathbf{a} = \mathbf{c}^T \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \mathbf{c}^T \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma, \quad \forall \mathbf{c}$$

$\mathbf{c}$ cancels out and we can write this as a system of equations

Finally, the system of equations looks like

Find the nodal displacements $\mathbf{a}$ that satisfy

$$\mathbf{c}^T \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega \mathbf{a} = \mathbf{c}^T \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \mathbf{c}^T \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma, \quad \forall \mathbf{c}$$

$$\Downarrow$$

Find $\mathbf{a}$ such that:

$$\mathbf{K}\mathbf{a} = \mathbf{f}$$

with

$$\mathbf{K} = \int_\Omega \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega, \qquad \mathbf{f} = \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

where

- evaluation of integrals needs to be worked out
- we still need to account for displacement boundary conditions

We evaluate $\mathbf{K}$ and $\mathbf{f}$ element by element

For the stiffness matrix:

$$\mathbf{K} = \int_{\Omega} \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega = \sum_{ie=1}^{ne} \int_{\Omega^e} \mathbf{B}^T \mathbf{D} \mathbf{B} \, \mathrm{d}\Omega = \mathop{\mathsf{A}}_{ie=1}^{ne} \int_{\Omega^e} \mathbf{B}_e^T \mathbf{D} \mathbf{B}_e \, \mathrm{d}\Omega$$

where

- Ω_e is the domain of the element
- $\mathbf{B}_e$ contains only the part of $\mathbf{B}$ that is nonzero in the element
- A takes care of putting the element matrix in the right global position

Similarly, for the force vector:

$$\mathbf{f} = \mathop{\mathsf{A}}_{ie=1}^{ne} \left(\int_{\Omega_e} \mathbf{N}_e^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_{N,e}} \mathbf{N}_e^T \mathbf{t} \, \mathrm{d}\Gamma \right)$$

where

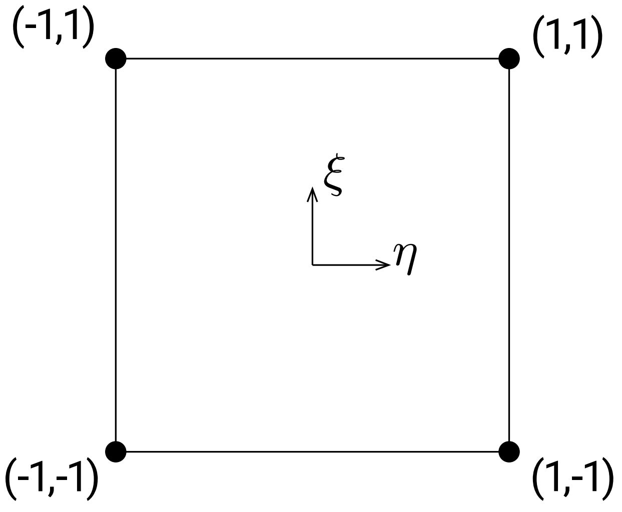
- $\Gamma_{N,e}$ is the part of the element boundary where external tractions are applied
- $\Gamma_{N,e}$ is empty for most elements

Isoparametric mapping is used for integration over elements

A mapping from local to global coordinates is introduced:

$$x(\xi,\eta) = N_i(\xi,\eta)x_i, \qquad y = N(\xi,\eta)_i y_i$$

where ξ and η are the natural coordinates of the reference element



For a 2D quadrilateral element integration is then performed as:

$$\int_{\Omega^e} \mathbf{B}^T\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega = \int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B}j\,\mathrm{d}\xi\,\mathrm{d}\eta$$

where j is the determinant of the Jacobian matrix $\mathbf{J}$:

$$\mathbf{J} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix} = \begin{bmatrix} N_{i,\xi}x_i & N_{i,\xi}y_i \\ N_{i,\eta}x_i & N_{i,\eta}y_i \end{bmatrix}$$

The Jacobian matrix is also used to evaluate shape function gradients in $\mathbf{B}$:

$$\begin{Bmatrix} N_{i,x} \\ N_{i,y} \end{Bmatrix} = \mathbf{J}^{-1} \begin{Bmatrix} N_{i,\xi} \\ N_{i,\eta} \end{Bmatrix}$$

The integrals are evaluated with numerical integration

A finite set of integration point is selected:

$$\int_{-1}^{1}\int_{-1}^{1} \mathbf{B}^T\mathbf{D}\mathbf{B}j\,\mathrm{d}\xi\,\mathrm{d}\eta \approx \sum_{ip=1}^{np} \mathbf{B}^T\mathbf{D}\mathbf{B}jw_{ip}$$

where

- $\mathbf{B}$ and j are evaluated at the integration point (ξ_{ip}, η_{ip})
- w_{ip} is the integration point weight
- jw_{ip} is a measure for the area associated with the integration point

Similarly, for the force vector:

$$\int_{-1}^{1}\int_{-1}^{1} \mathbf{N}^T\mathbf{b}j\,\mathrm{d}\eta\,\mathrm{d}\xi \approx \sum_{ip=1}^{np} \mathbf{N}^T\mathbf{b}jw_{ip}$$

With all this, the PDE is approximated as a system of equations

Strong form		Weak form		Discretized form
$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$	$\Longrightarrow$	$-\int_\Omega \nabla^{\mathrm{s}}\mathbf{w} : \boldsymbol{\mathcal{D}} : \nabla^{\mathrm{s}}\mathbf{u}\,\mathrm{d}\Omega + \int_\Omega \mathbf{w}\cdot\mathbf{b}\,\mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w}\cdot\mathbf{t}\,\mathrm{d}\Gamma = \mathbf{0}$	$\Longrightarrow$	$\mathbf{K}\mathbf{a} = \mathbf{f}$

where

- $\mathbf{a}$ is the vector with nodal displacements, $\mathbf{K}$ the stiffness matrix and $\mathbf{f}$ the force vector

$$\mathbf{K} = \int \mathbf{B}^T\mathbf{D}\mathbf{B}\,\mathrm{d}\Omega, \qquad \mathbf{f} = \int \mathbf{N}^T\mathbf{b}\,\mathrm{d}\Omega + \int \mathbf{N}^T\mathbf{t}\,\mathrm{d}\Gamma$$

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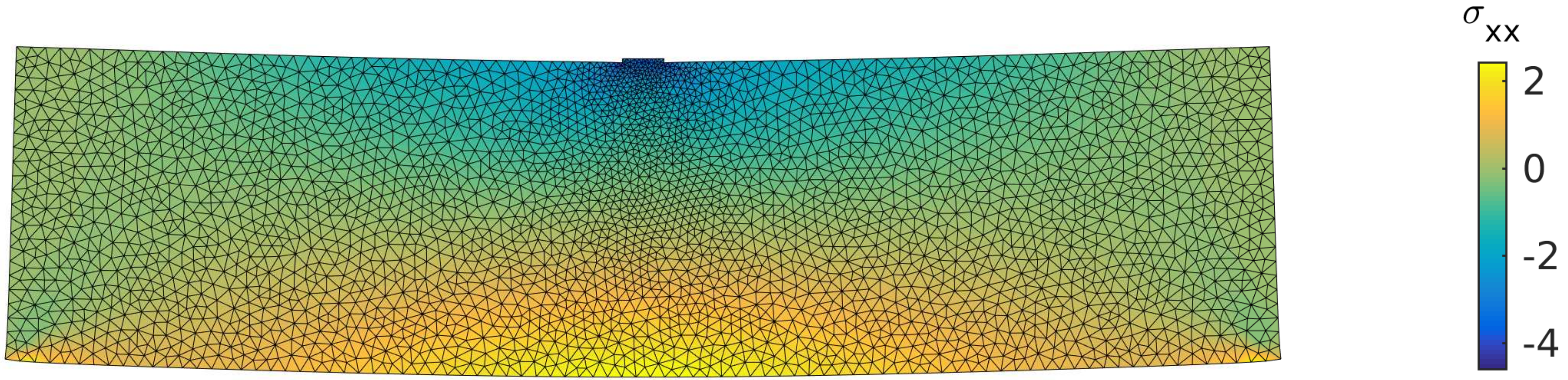
$\nabla \cdot \boldsymbol{\sigma} + \mathbf{b} = 0$	$\Longrightarrow$	$-\int_\Omega \nabla^s \mathbf{w} : \boldsymbol{\mathcal{D}} : \nabla^s \mathbf{u} \, \mathrm{d}\Omega + \int_\Omega \mathbf{w} \cdot \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{w} \cdot \mathbf{t} \, \mathrm{d}\Gamma = \mathbf{0}$	$\Longrightarrow$	$\mathbf{K}\mathbf{a} = \mathbf{f}$

where

- $\mathbf{a}$ is the vector with nodal displacements, $\mathbf{K}$ the stiffness matrix and $\mathbf{f}$ the force vector
- displacement BCs are treated by eliminating prescribed values for $\mathbf{a}$
- once $\mathbf{a}$ is known, stress and strain fields can be computed



Discussion

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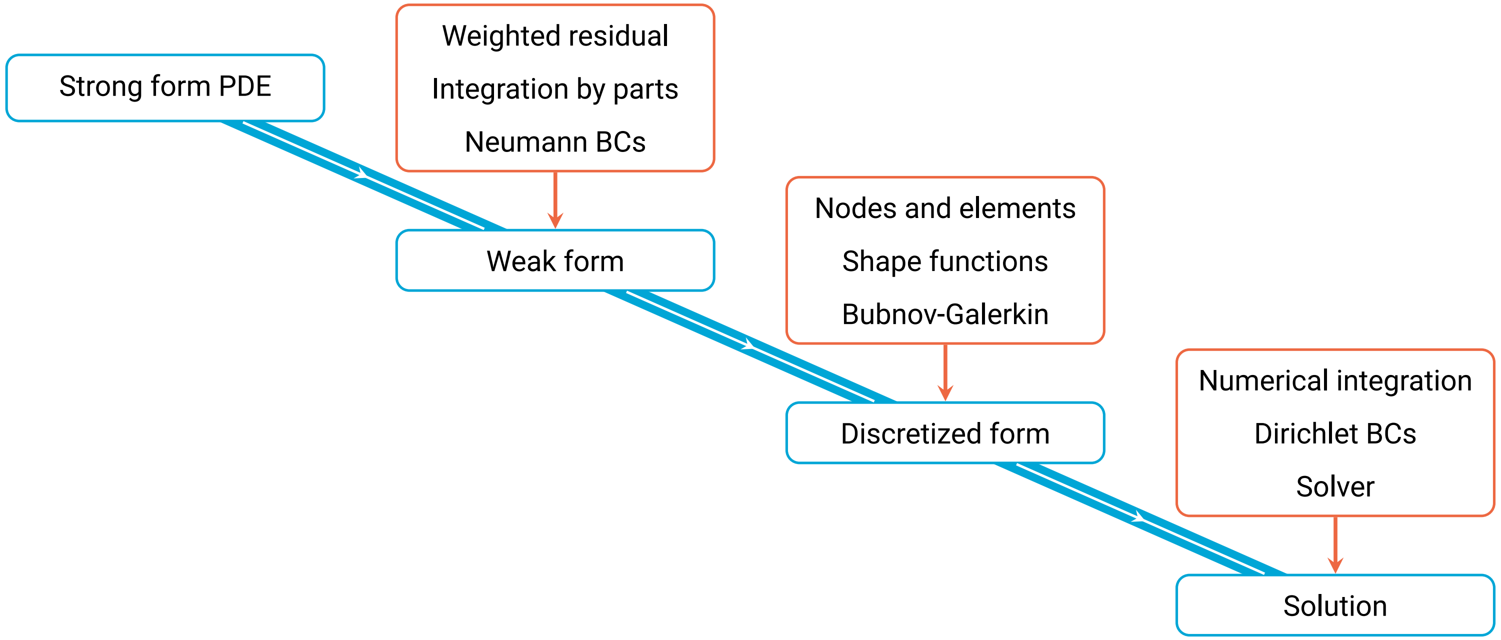

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