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Computer Science > Computational Complexity

arXiv:1706.06708 (cs)
[Submitted on 21 Jun 2017 (v1), last revised 27 Apr 2018 (version, v2)]

Solving the Rubik's Cube Optimally is NP-complete

Erik D. Demaine, Sarah Eisenstat, Mikhail Rudoy

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In this paper, we prove that optimally solving an $n \times n \times n$ Rubik's Cube is NP-complete, reducing from the Hamiltonian Cycle problem on square grid graphs. This improves the previous result that optimally solving an $n \times n \times 1$ Rubik's Cube with missing stickers is NP-complete. We prove this result first for a simpler case of the Rubik's Square and then proceed with a similar but more complicated proof for the Rubik's Cube case.

Comments: 35 pages, 8 figures
Subjects: Computational Complexity (cs.CC); Computational Complexity



←

Cube Solver

Click ▶ to start solve

Reset

⋮

←

Verlinde² Symposium

ERIK

HERMAN

TWINS



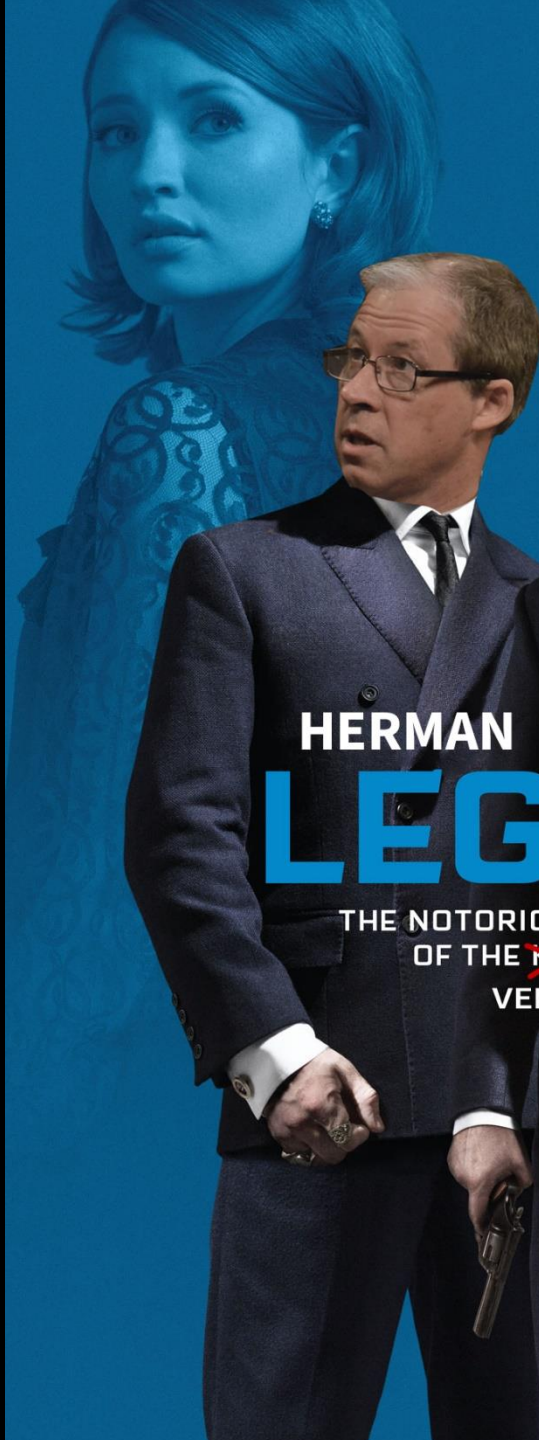
Only their
mother can
tell them
apart.

AN
IVAN
REITMAN
FILM

TWINS KELLY CROFTON, PHILIP WAIKOP, RONNIE DARTLETT, ANDER WILLIAMS, DAVIDO.

Verlinde² Symposium

FROM THE ACADEMY AWARD WINNING SCREENWRITER OF
AMSTERDAM ~~THE~~ **CONFIDENTIAL**
AND THE PRODUCERS OF
THE THEORY OF EVERYTHING



HERMAN

ERIK

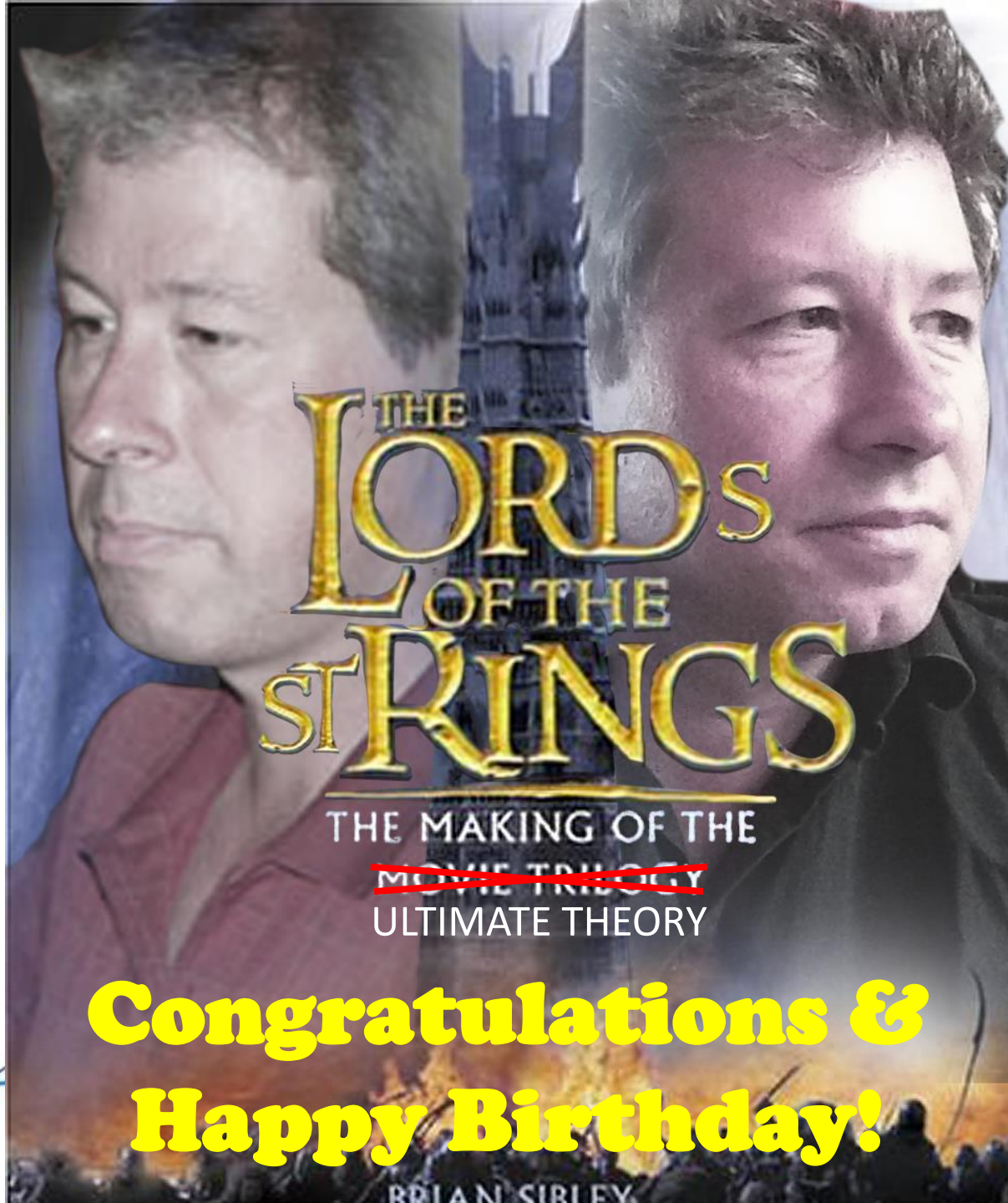
LEGEND

THE NOTORIOUS TRUE STORY
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VERLINDE

THE LORD'S OF THE STRINGS

THE MAKING OF THE
~~MOVIE TRILOGY~~
ULTIMATE THEORY

BRIAN SIBLEY



The image features a central, glowing CPU chip with intricate circuitry extending outwards. The background is a dense, blue-toned digital space filled with floating numbers and data streams, creating a sense of depth and complexity. The overall aesthetic is futuristic and technological.

Complexity = Anything

Alexandre Belin, RCM, Shan-Ming Ruan, Gabor Sarosi & Antony Speranza
[arXiv:2111.02429; arXiv:2208.0xxxx]



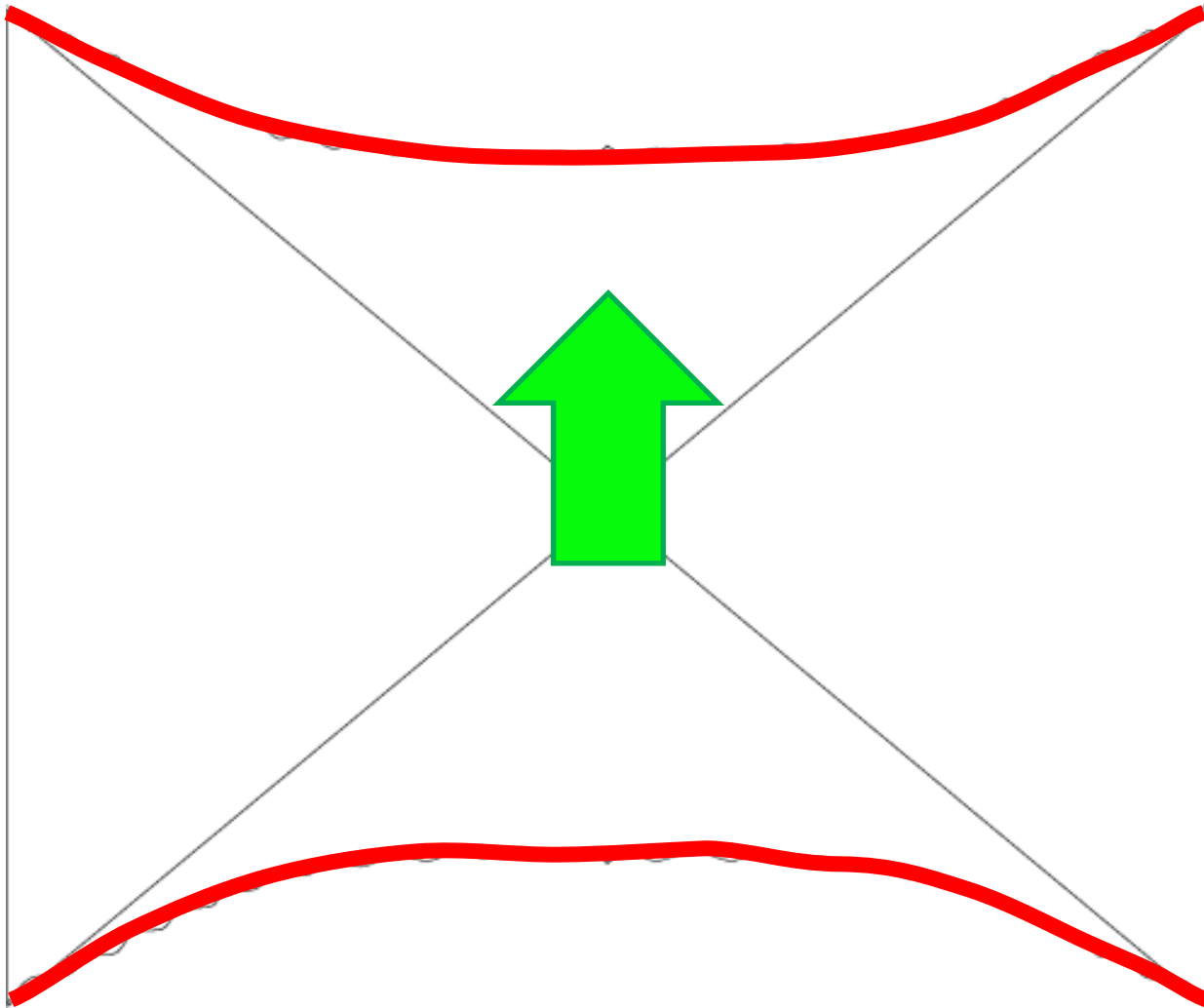
**COMPLEXITY=ANYTHING
&
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EVERYWHERE
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天馬行空

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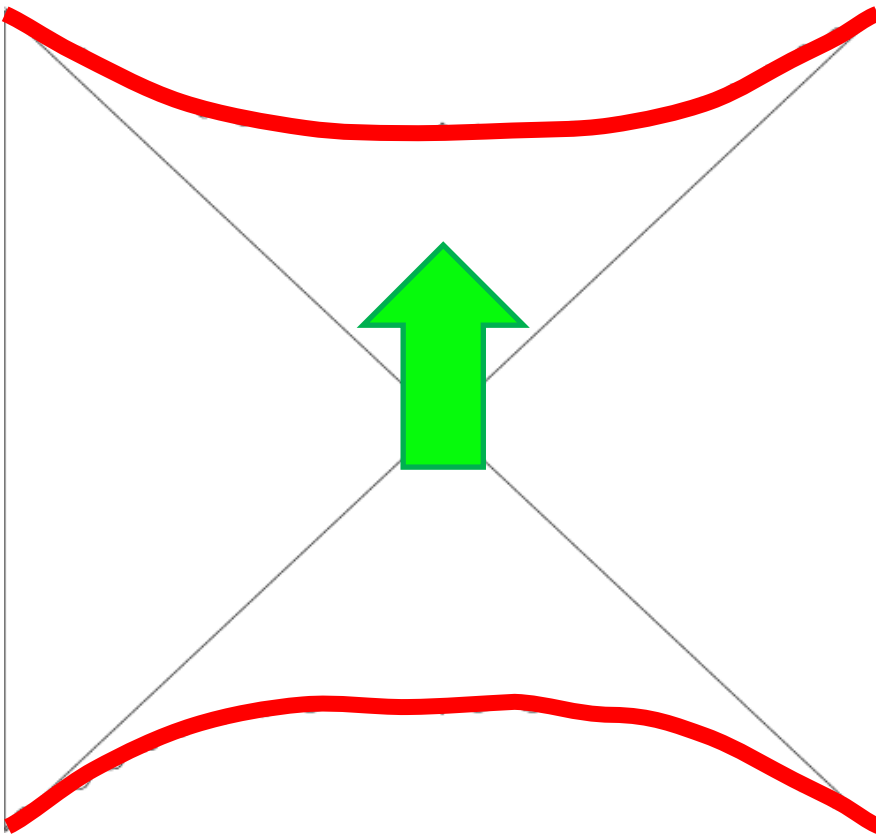
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- “to understand the rich geometric structures that exist behind the horizon and which are predicted by general relativity.”



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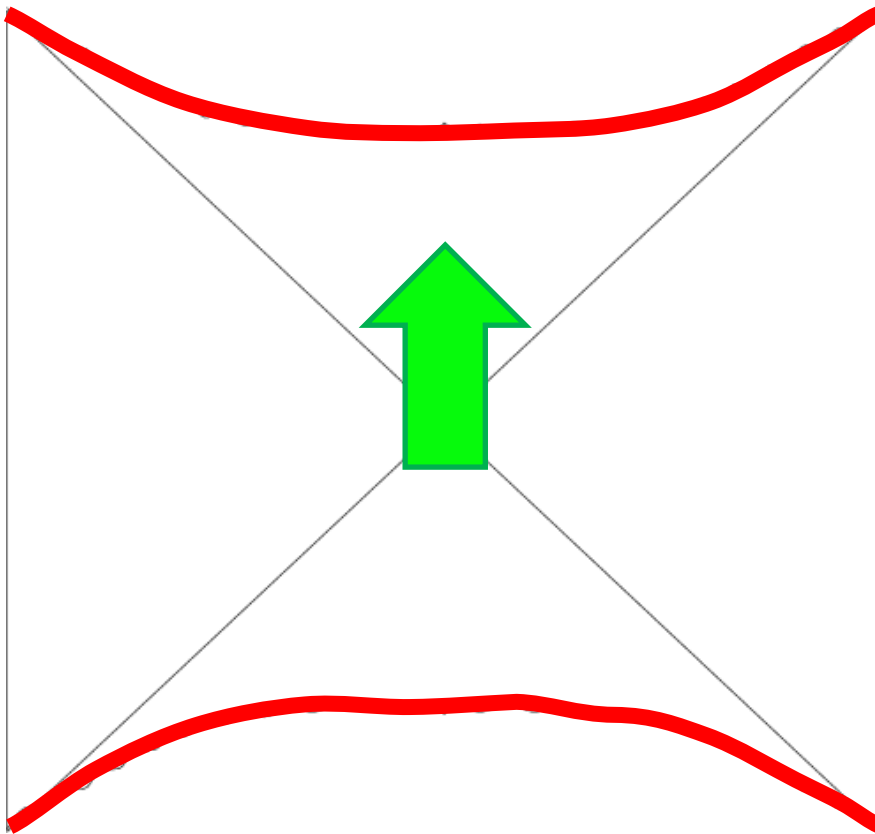


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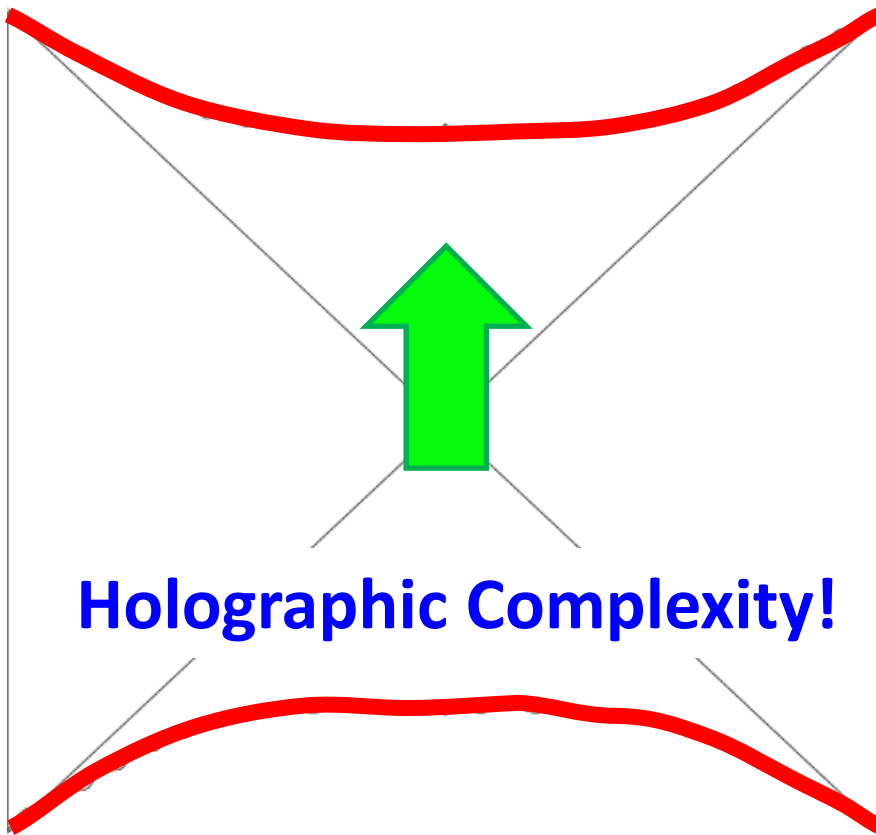
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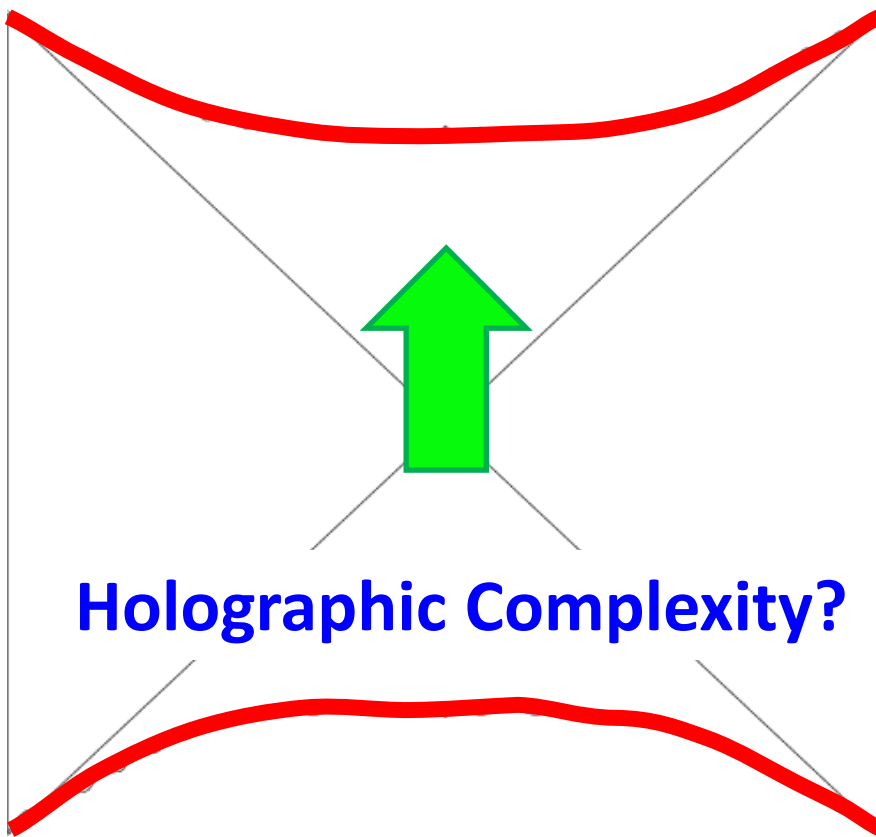
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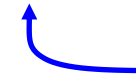
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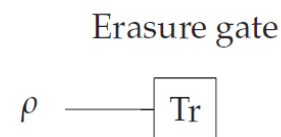
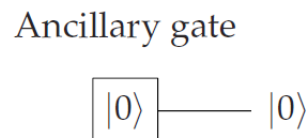
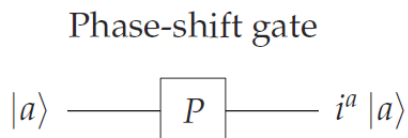
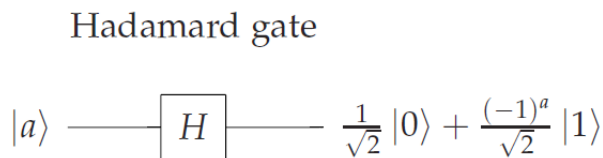
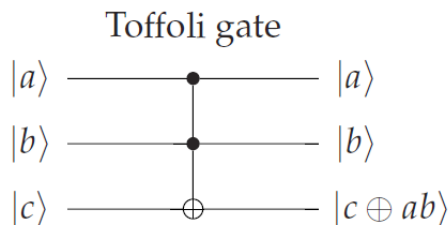
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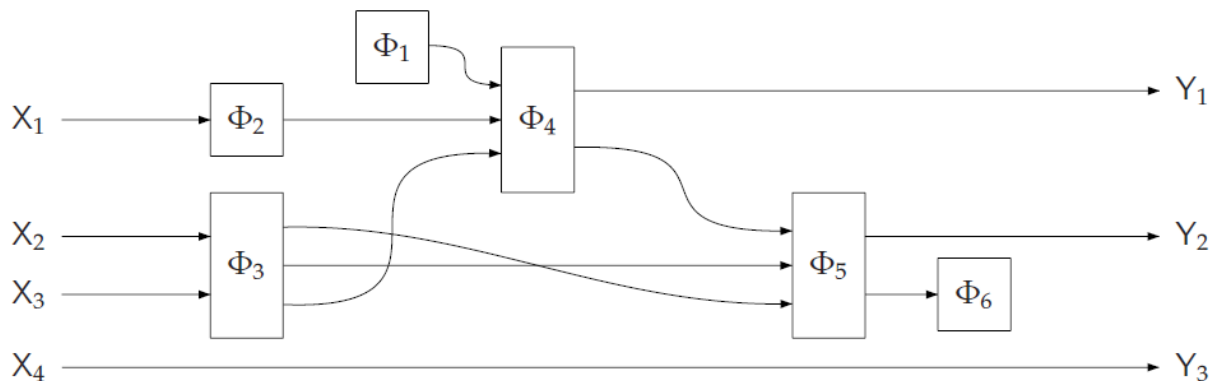
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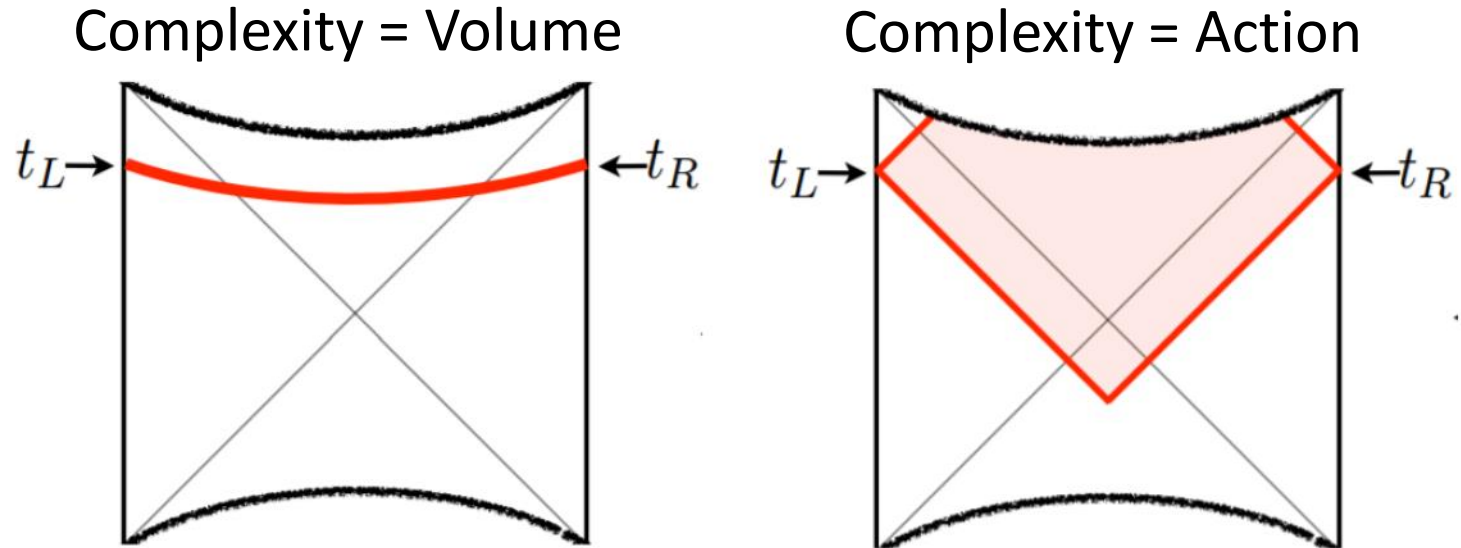
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- does the answer depend on the choices?? **YES!!** go beyond questions like 'poly vs exp'

Holographic Complexity: A Tale of Two Dualities

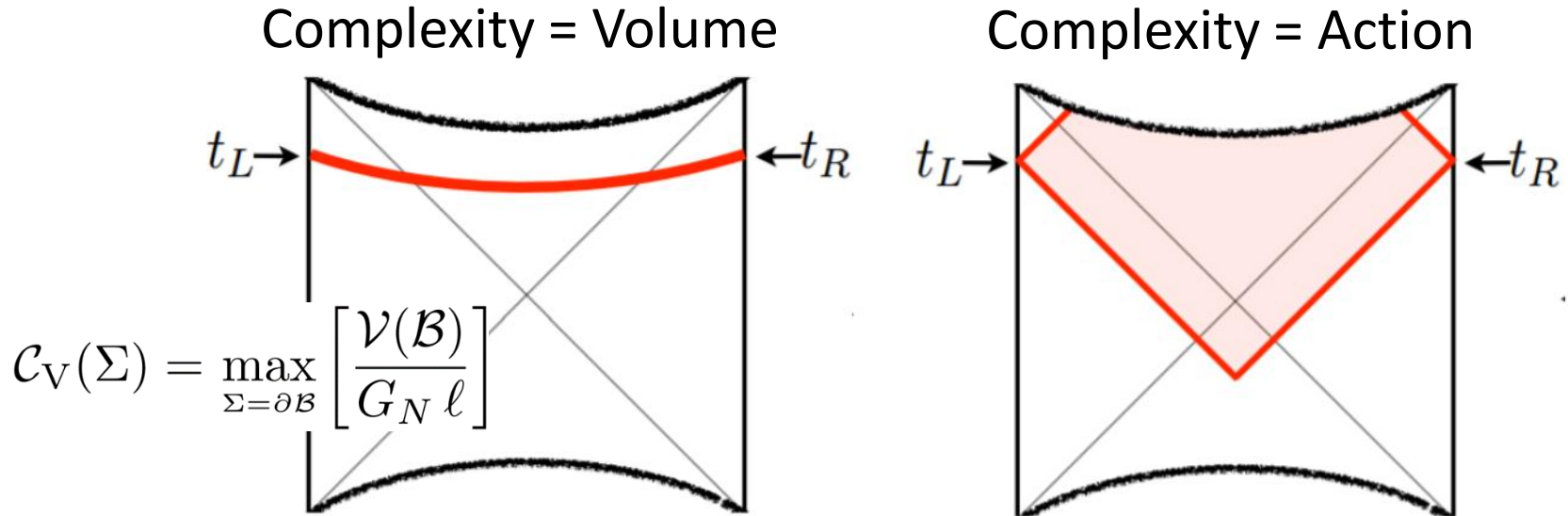
- [complexity=volume](#): evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)
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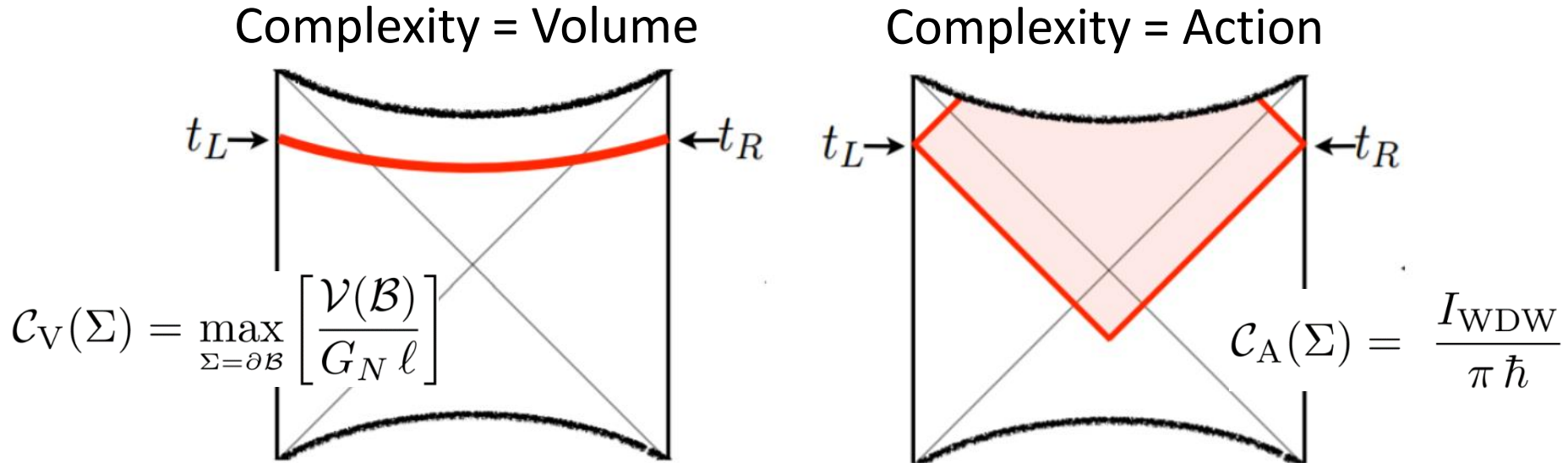
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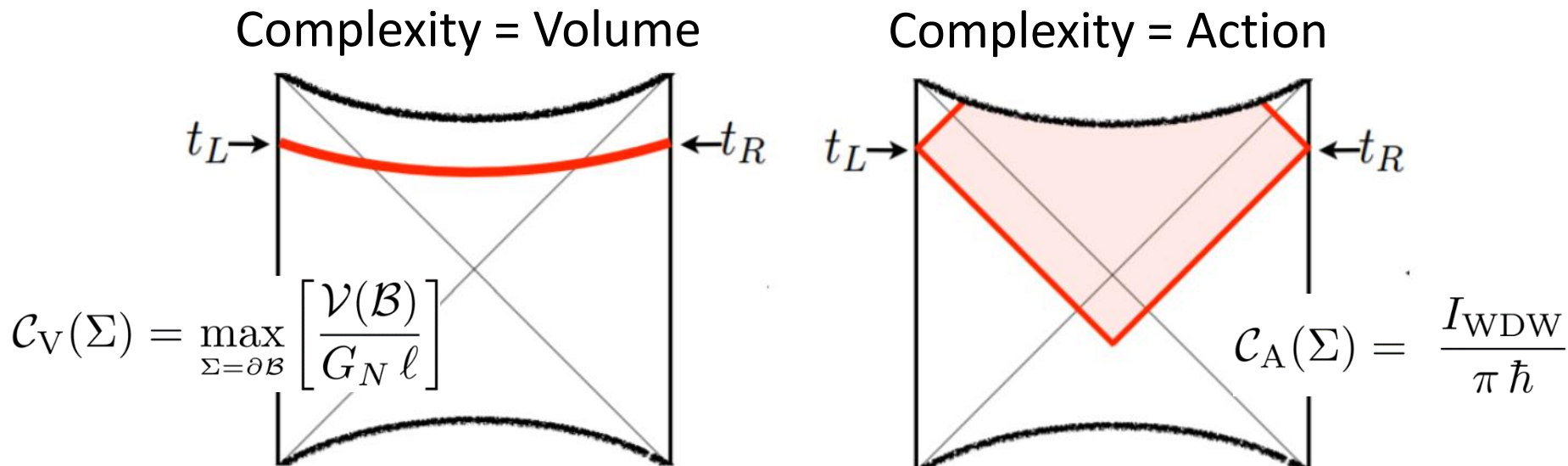
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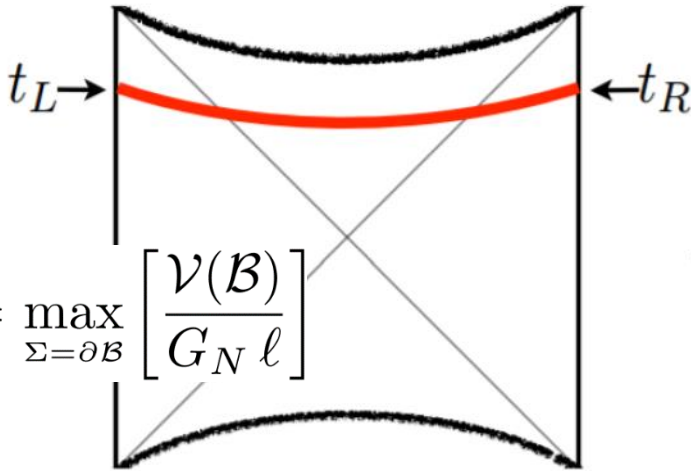
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- both of these gravitational “observables” probe the black hole interior (at arbitrarily late times on boundary)

Holographic Complexity: A Tale of Two ^vDualities

or more

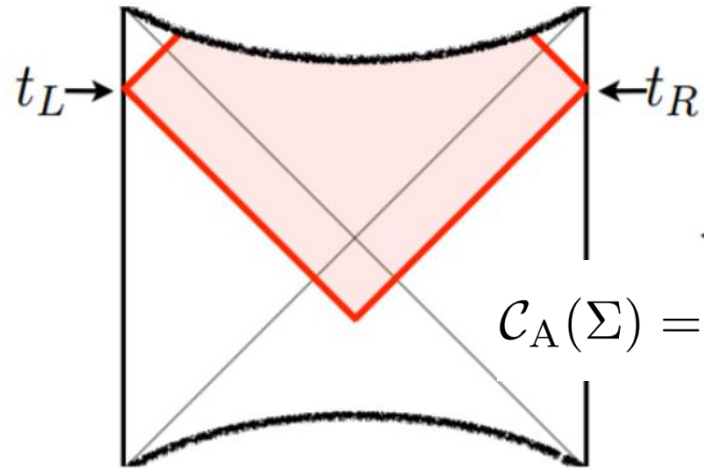
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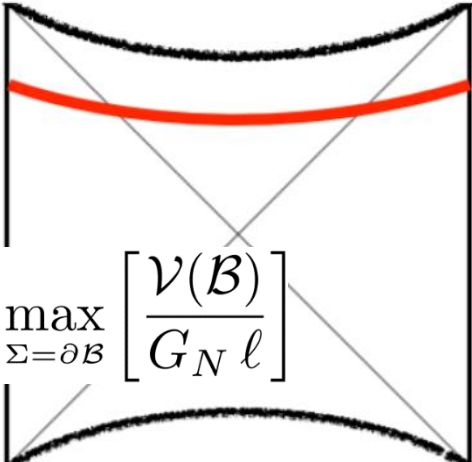
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- complexity=volume2.0: evaluate spacetime volume of WDW patch

$$\mathcal{C}'_V(\Sigma) = \frac{V_{\text{WDW}}}{G_N \ell^2} \quad (\text{Couch, Fischler \& Nguyen})$$

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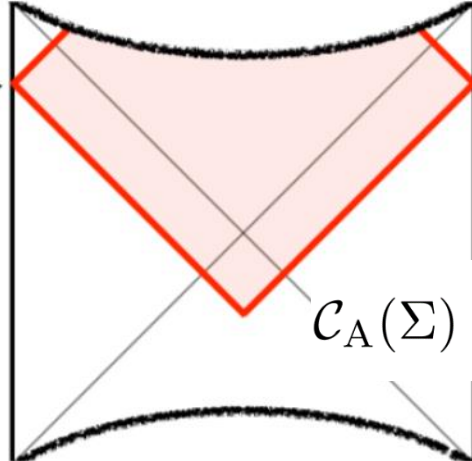
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The diagram shows a rectangular region with curved top and bottom boundaries. A red curve is drawn across the region, representing a volume-minimizing surface. The left and right boundaries are labeled t_L and t_R respectively. The formula for complexity is given as:

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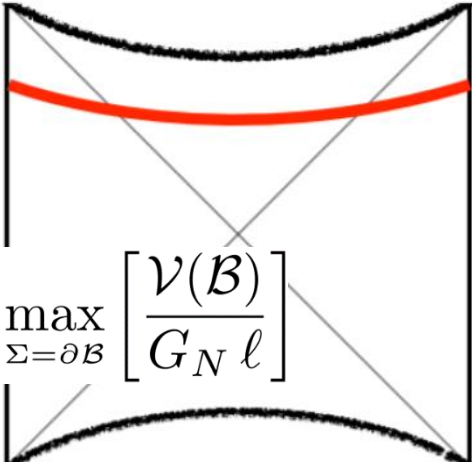
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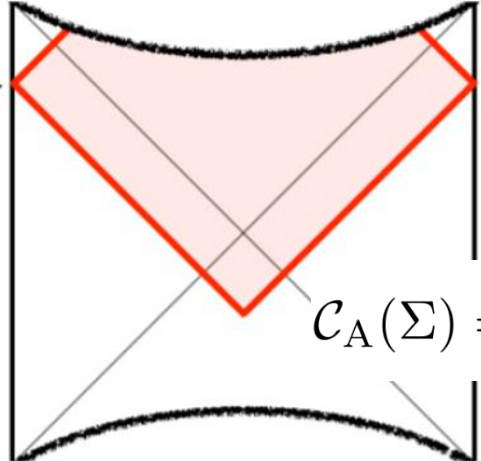
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- connection of complexity=volume to AdS/MERA

- linear growth (at late times)

(d = boundary dimension)

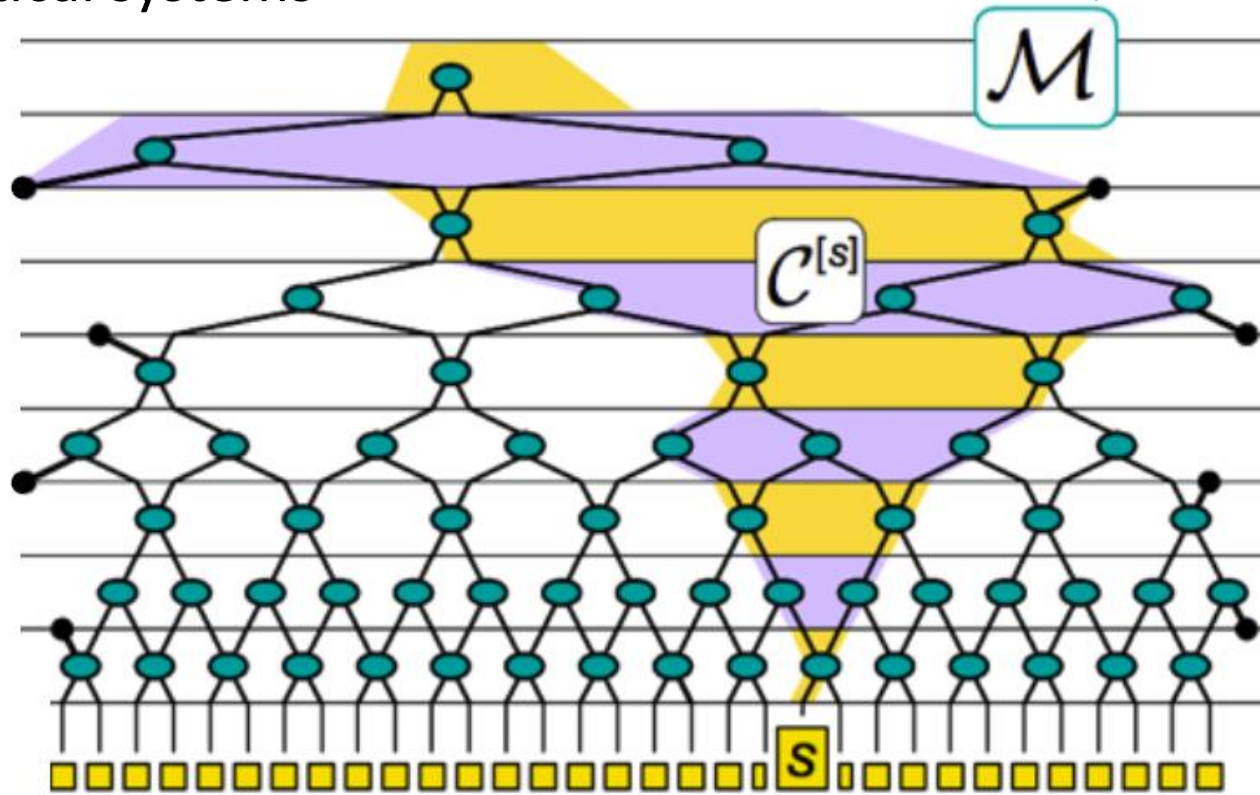
$$\left. \frac{d\mathcal{C}_V}{dt} \right|_{t \rightarrow \infty} = \frac{8\pi}{d-1} M \quad (\text{planar})$$

$$\left. \frac{d\mathcal{C}_A}{dt} \right|_{t \rightarrow \infty} = \frac{2M}{\pi}$$

- “switchback” effect (information scrambling)

AdS/MERA:

- MERA (Multi-scale Entanglement Renormalization Ansatz) provides efficient tensor network representation of ground-state wave-function in $d=2$ critical systems (Vidal; Vidal & Evenbly)



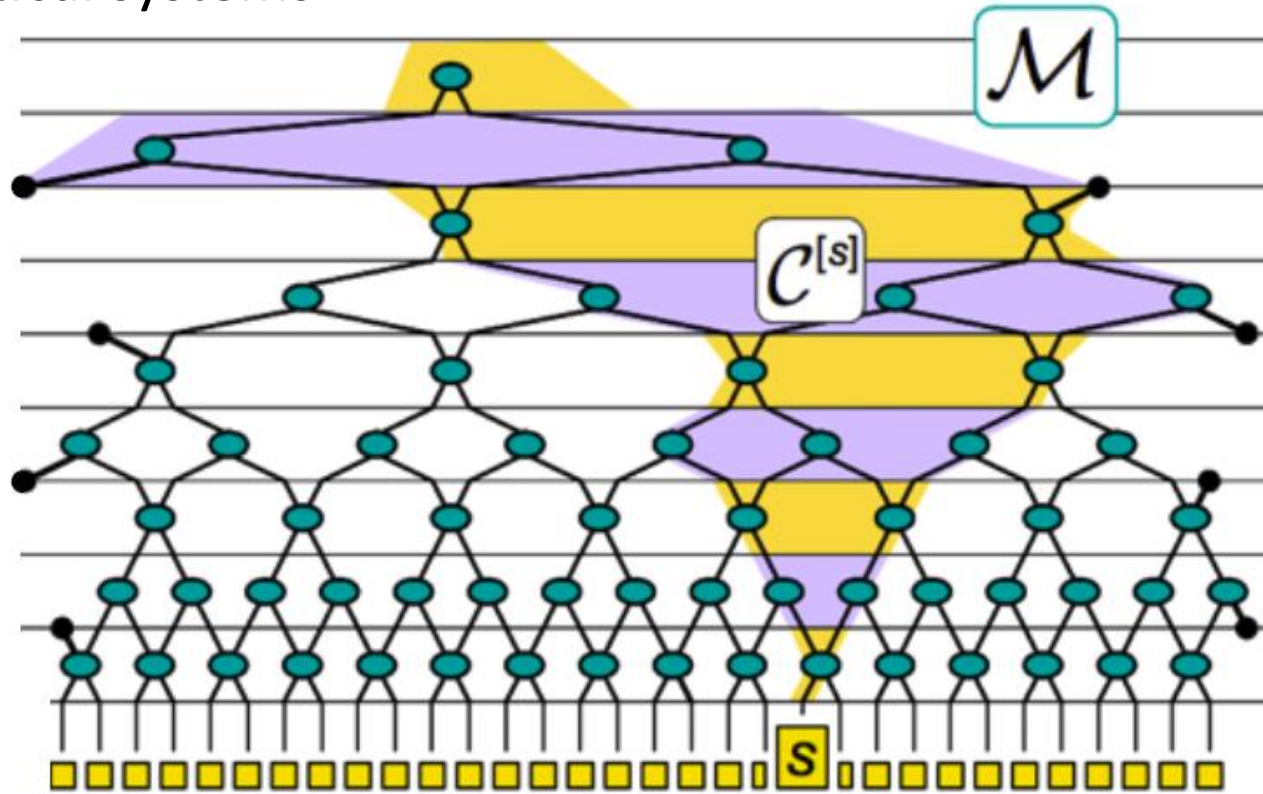
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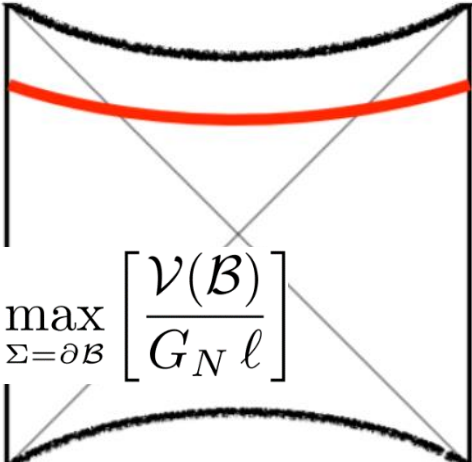
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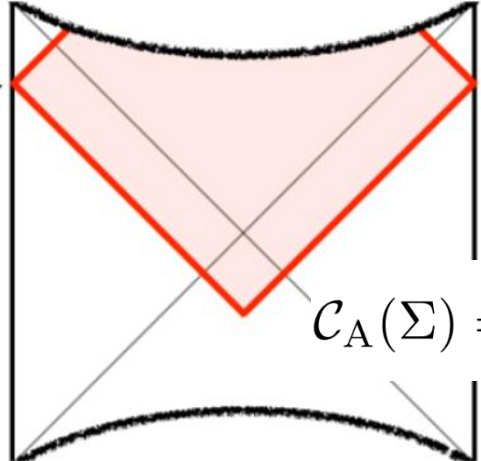
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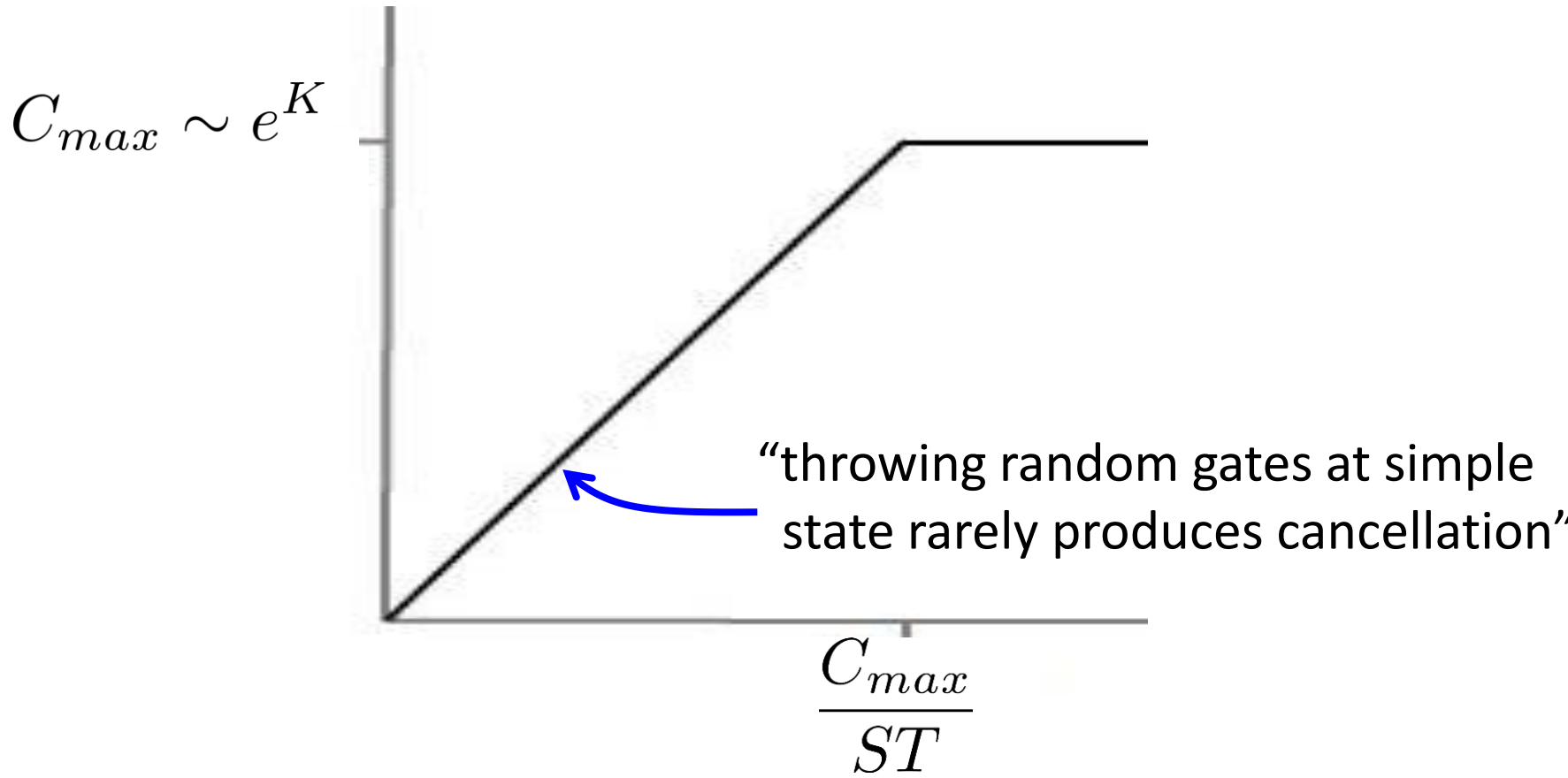
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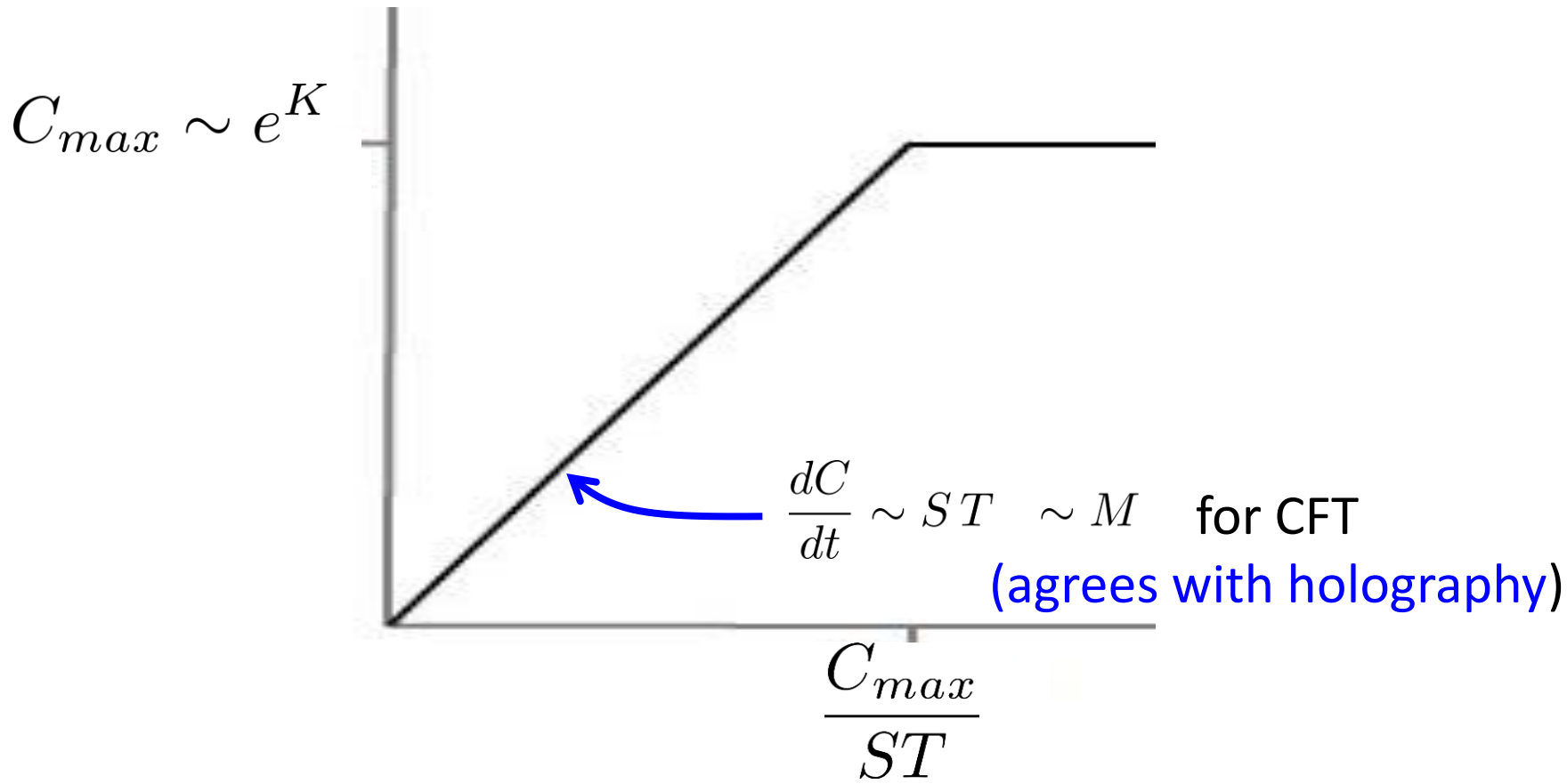
Linear Growth (at late times):

- large number of degrees of freedom + Hamiltonian is chaotic
 → for very long time, growth is linear in time
- rate proportional # degrees of freedom K → thermal entropy S ;
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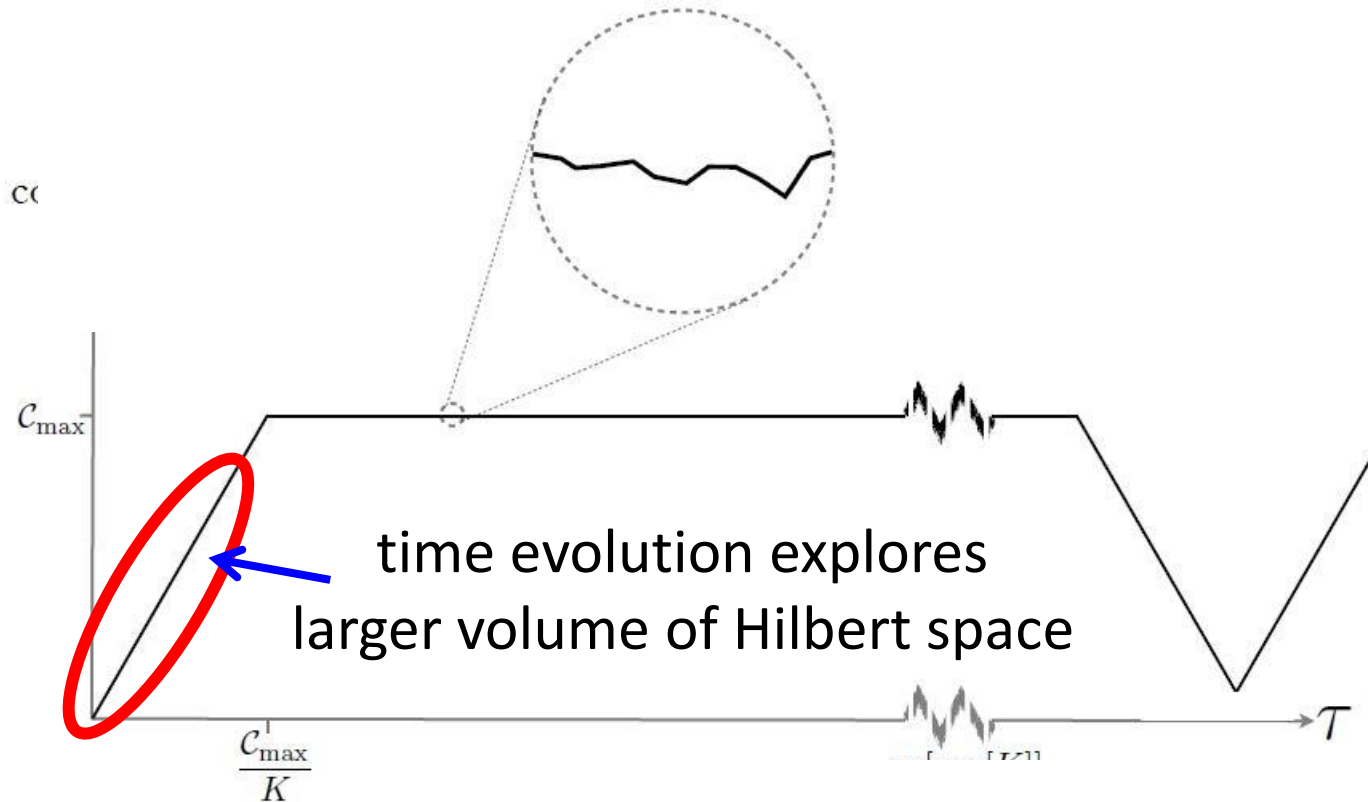
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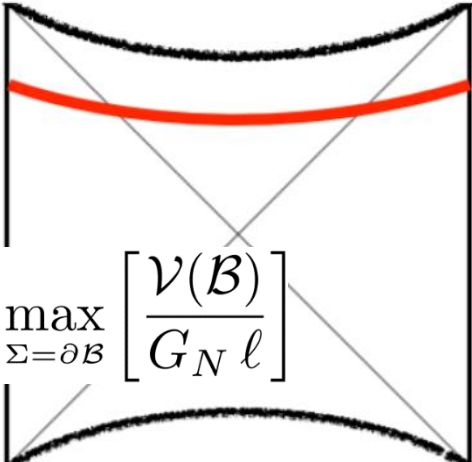
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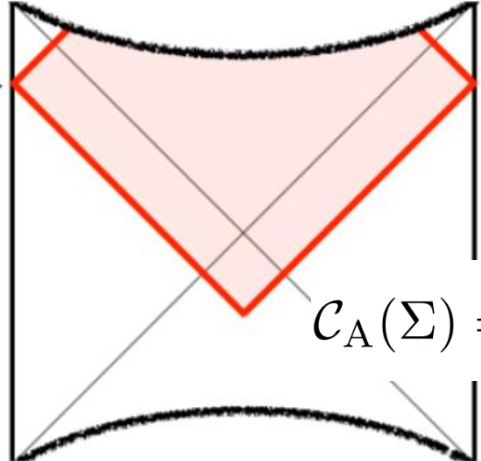
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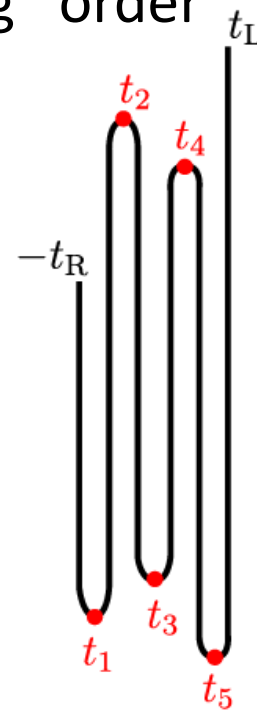
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“Switchback effect”:

- consider perturbed thermofield double state:

$$|\Psi(t_L, t_R)\rangle = e^{-iH_L t_L - iH_R t_R} W_L(t_n) \dots W_L(t_1) |\psi_{\text{TFD}}(0)\rangle$$

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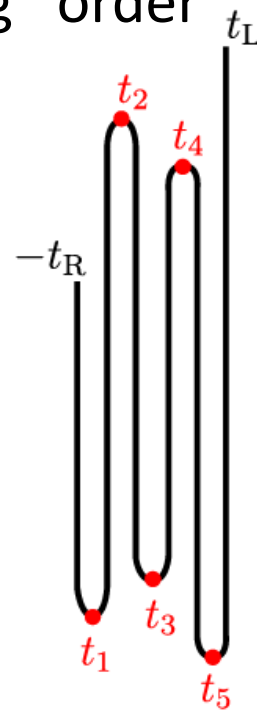
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- complexity $\propto |t_R - t_1| + |t_2 - t_1| + \dots$

$$+ |t_L - t_n| - 2n t_*$$



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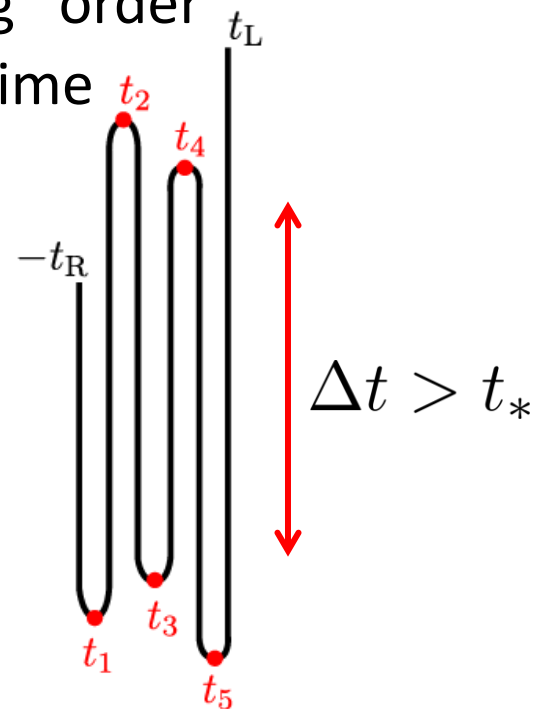
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- complexity $\propto t_f - 2n t_*$ total “folded” time

of folds

scrambling time:

$$t_* = \frac{1}{2\pi T} \log(N^2)$$



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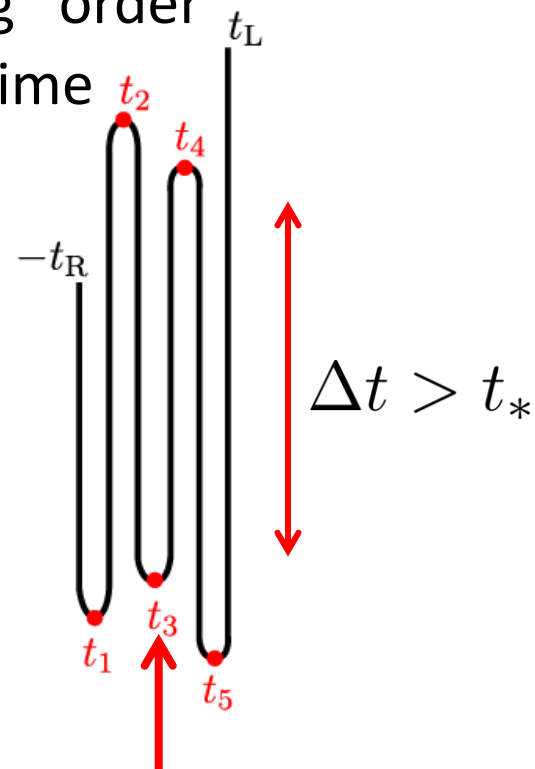
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- complexity $\propto t_f - 2n t_*$ total “folded” time

of folds

scrambling time:

$$t_* = \frac{1}{2\pi T} \log(N^2)$$



$$\dots e^{-iH t_4} W_L(0) e^{iH(t_4 - t_3)} W_L(0) e^{iH(t_3 - t_2)} W_L(0) e^{iH t_2} \dots$$

“Switchback effect”:

- consider perturbed thermofield double state:

$$|\Psi(t_L, t_R)\rangle = e^{-iH_L t_L - iH_R t_R} W_L(t_n) \dots W_L(t_1) |\psi_{\text{TFD}}(0)\rangle$$

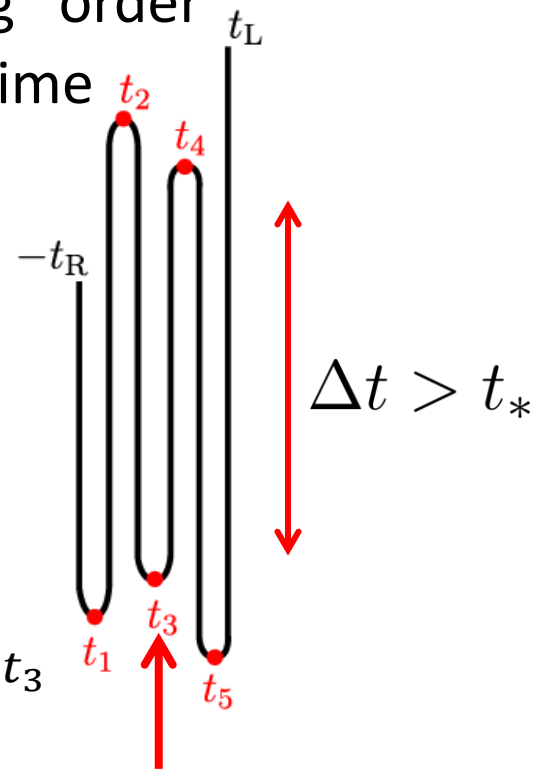
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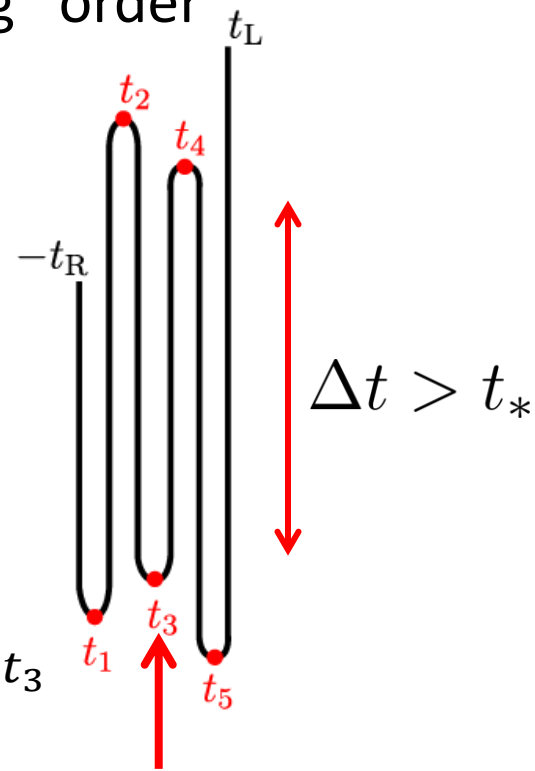
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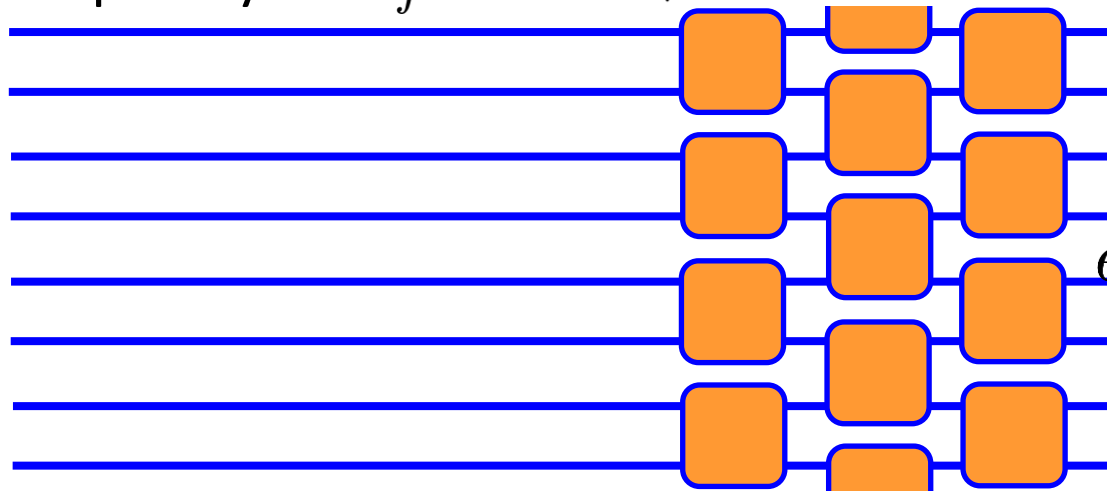
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e^{iHt_3}

without $W_L(0)$, e^{-iHt_3} cancels completely with e^{iHt_3}

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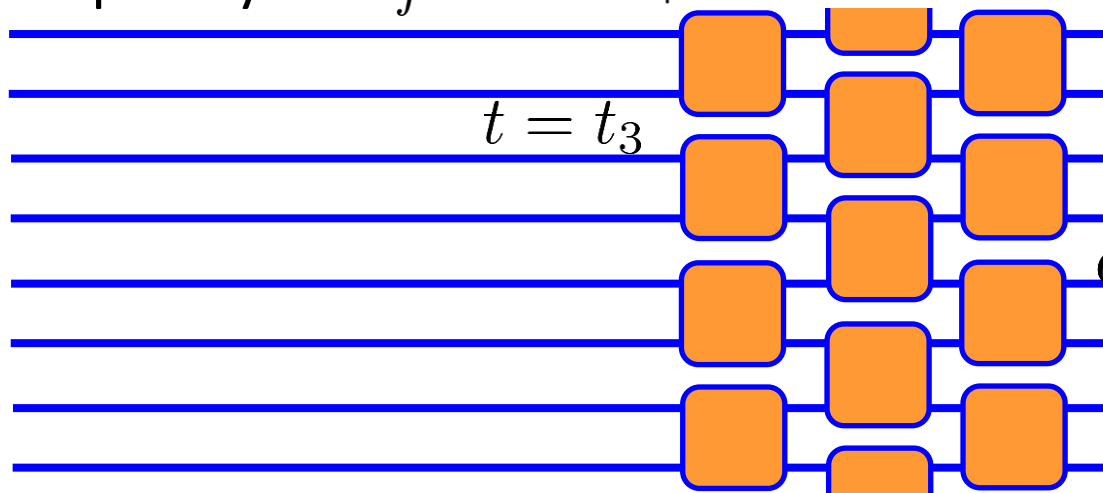
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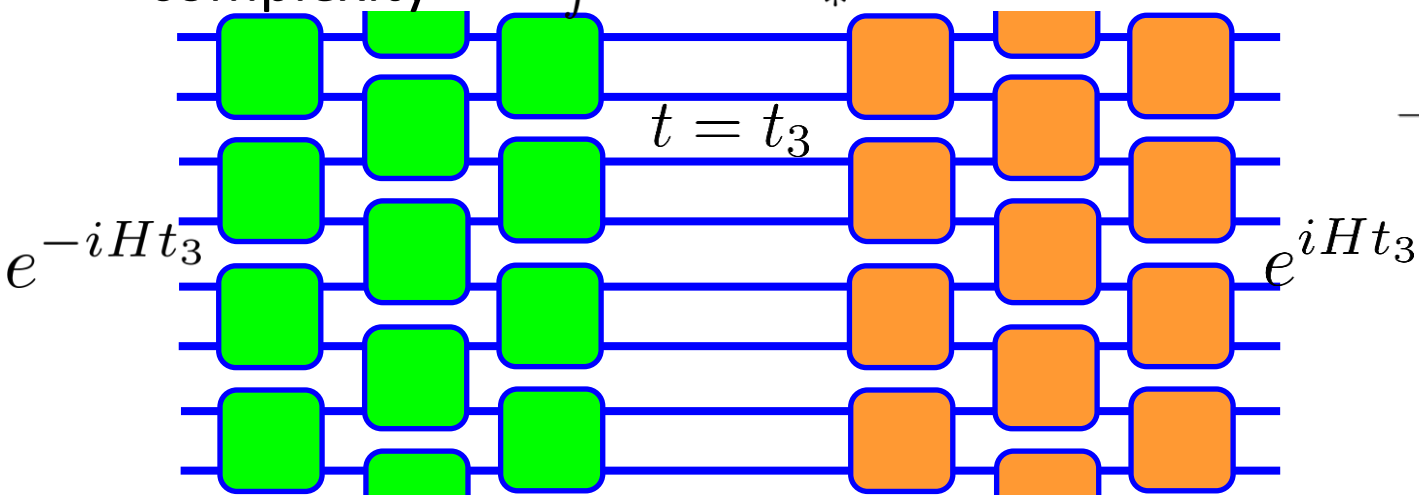
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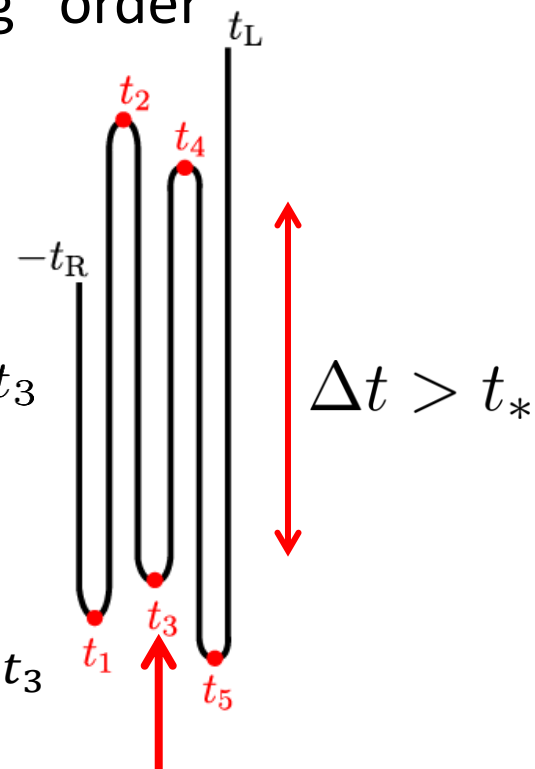
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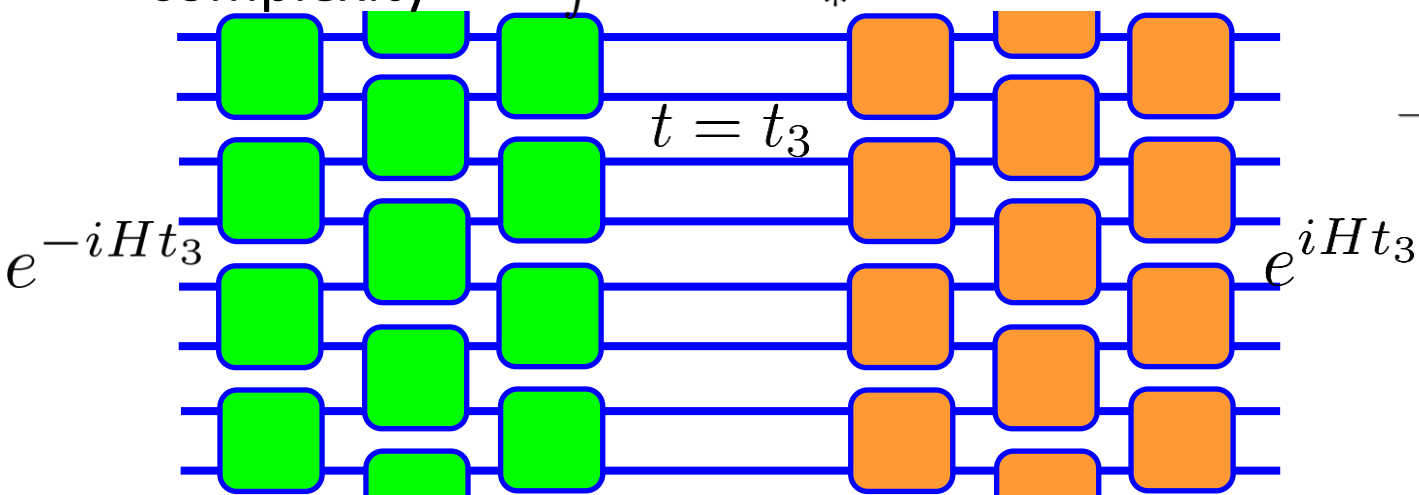
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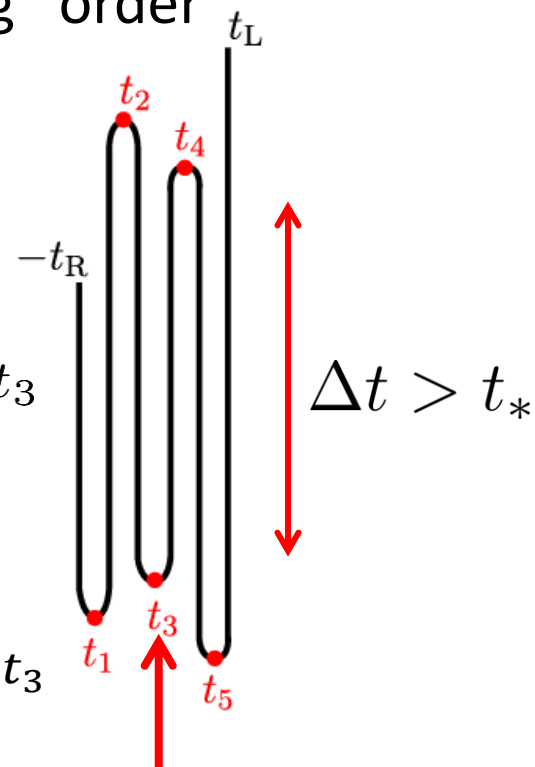
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$$\dots e^{-iHt_4} W_L(0) e^{iHt_4} \dots e^{-iHt_2} W_L(0) e^{iHt_2} \dots$$

$$\text{Green Gate} = (\text{Orange Gate})^{-1}$$



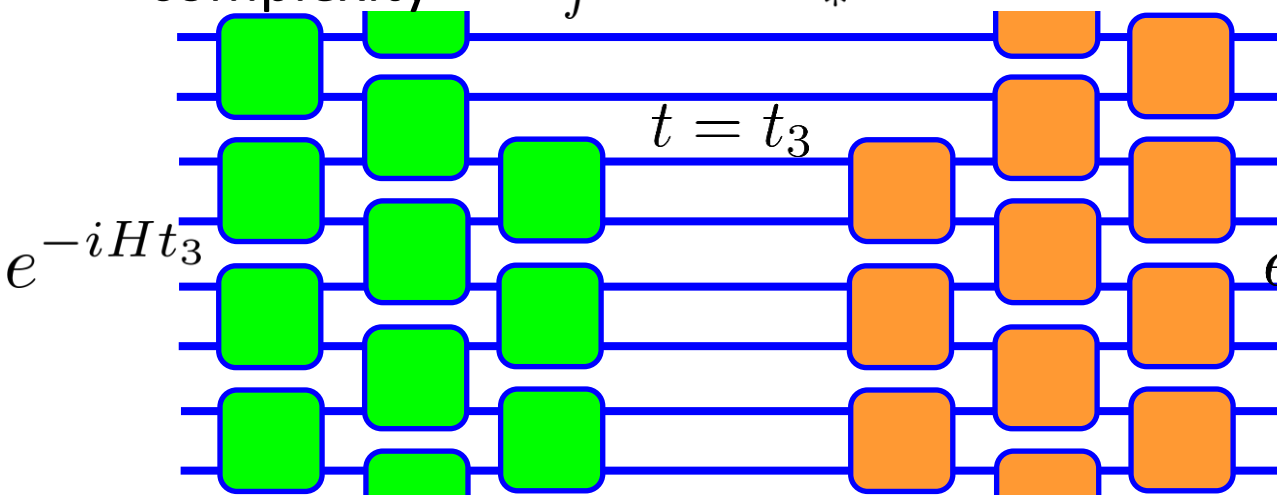
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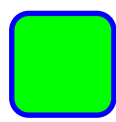
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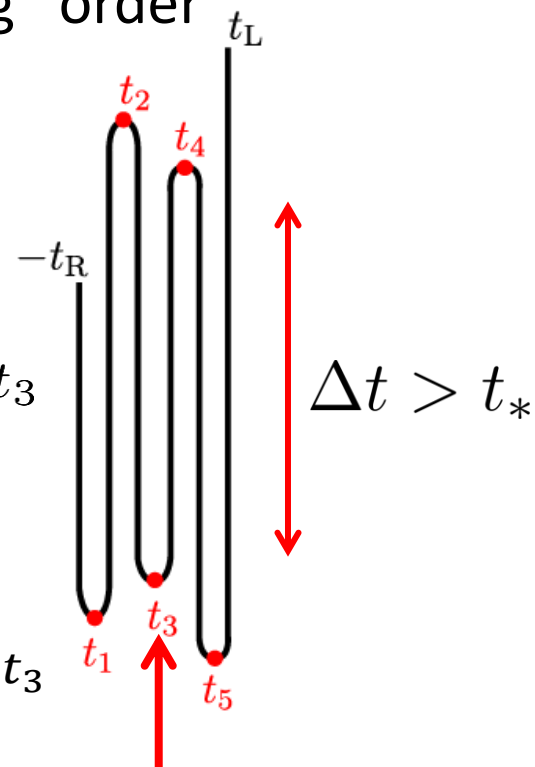


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$$\dots e^{-iHt_4} W_L(0) e^{-iHt_3} \dots e^{-iHt_2} W_L(0) e^{iHt_2} \dots$$



$$= (\text{orange square})^{-1}$$



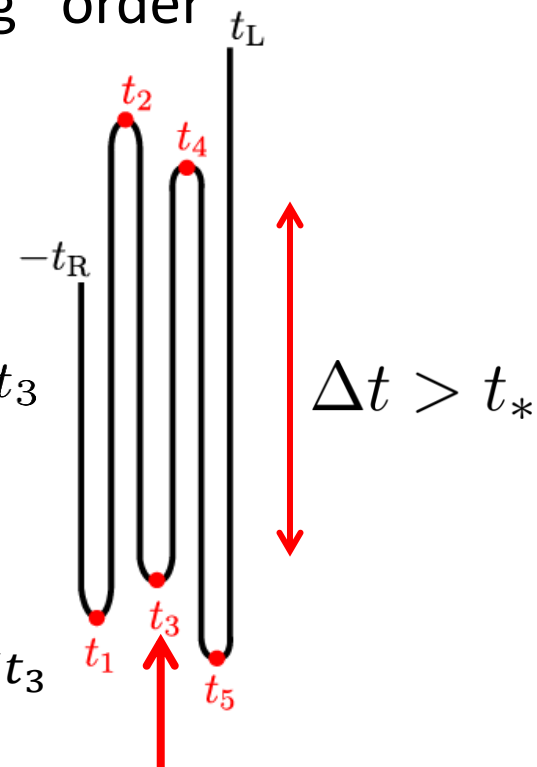
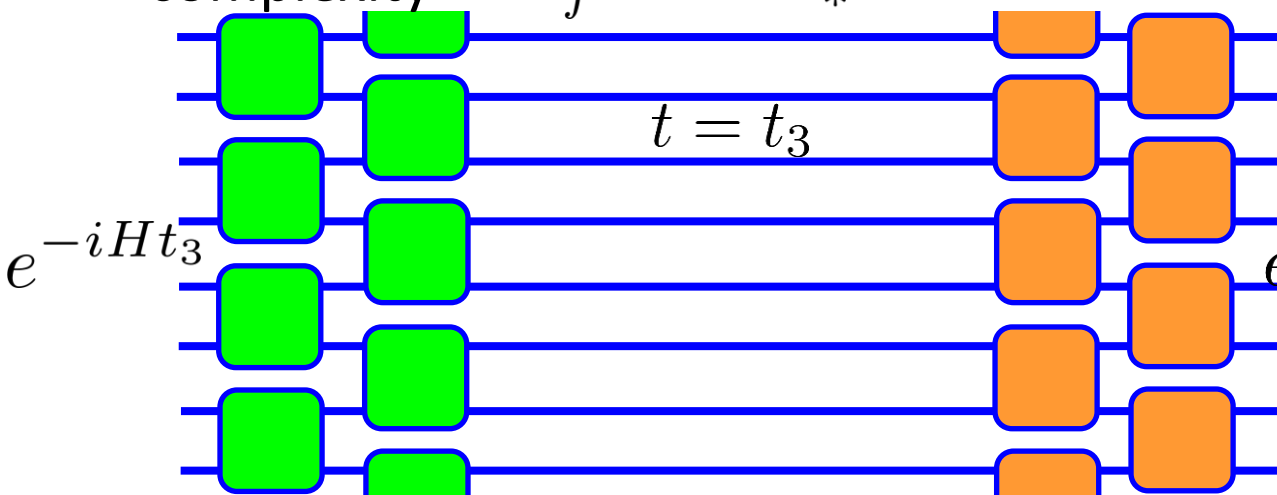
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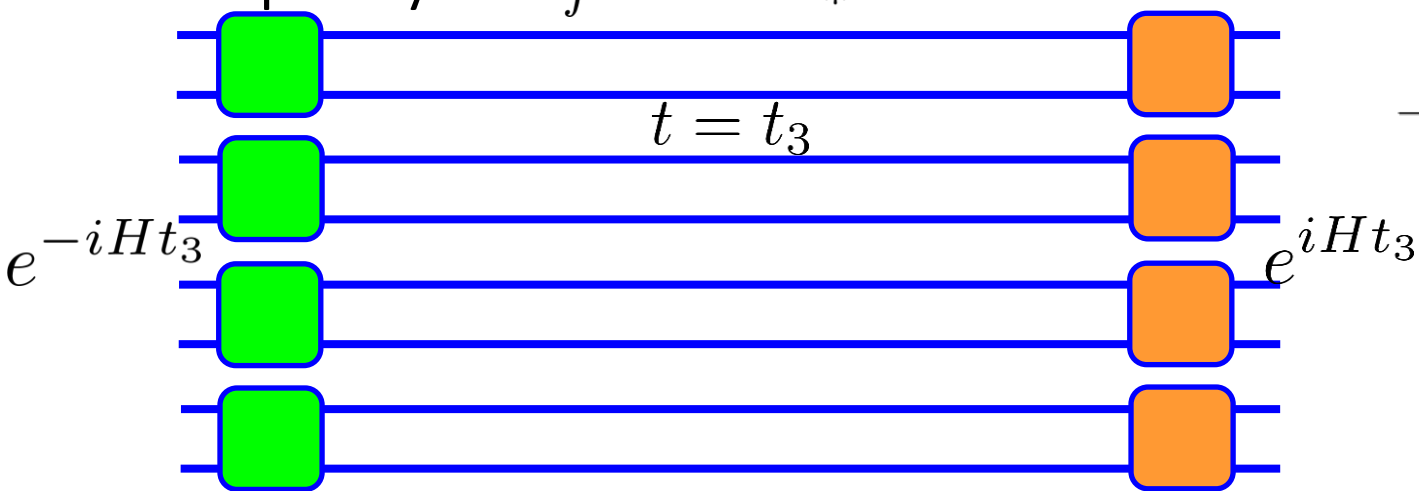
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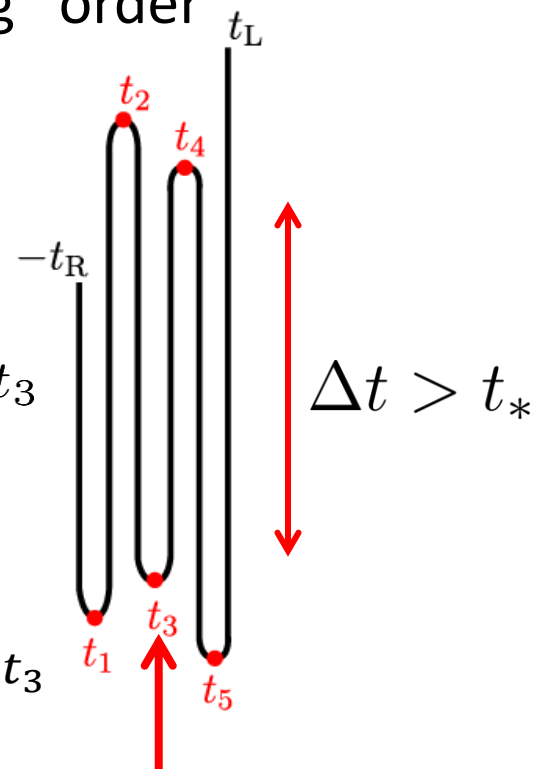
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without $W_L(0)$, e^{-iHt_3} cancels completely with e^{iHt_3}



$$\dots e^{-iHt_4} W_L(0) e^{-iHt_3} \left(\text{Green Square} = \left(\text{Orange Square} \right)^{-1} \right) e^{-iHt_2} W_L(0) e^{iHt_2} \dots$$

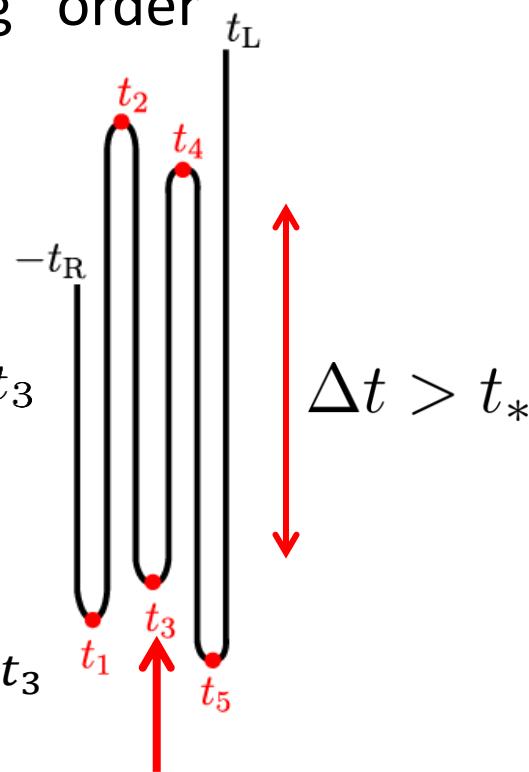
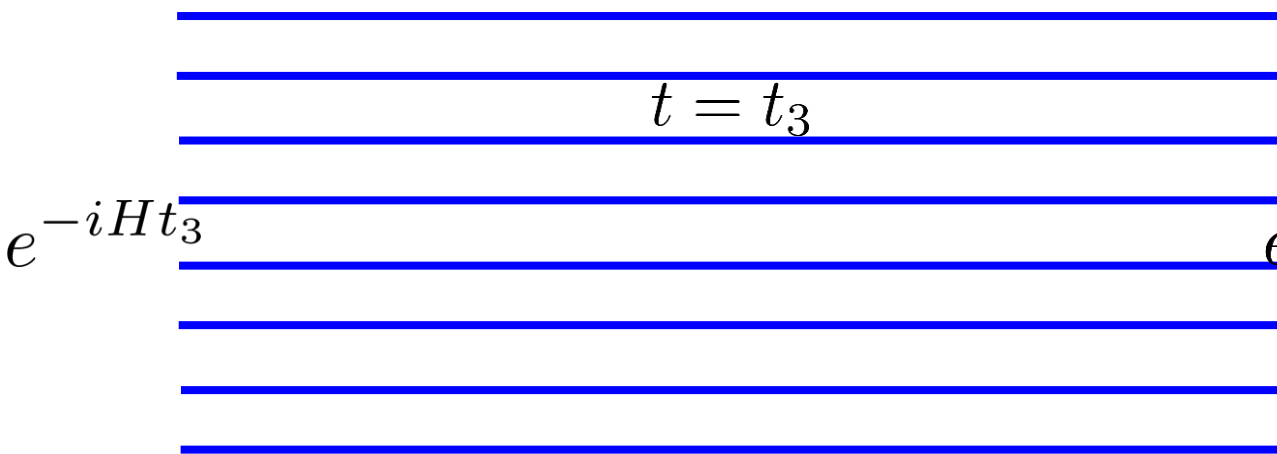
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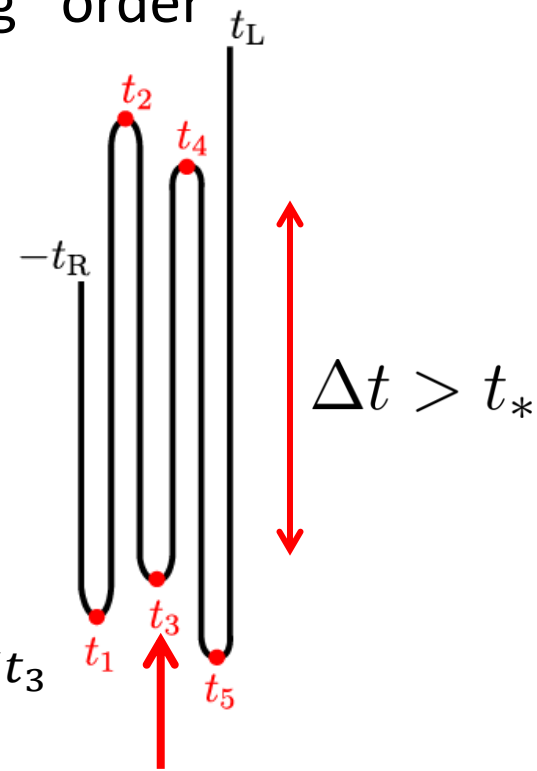
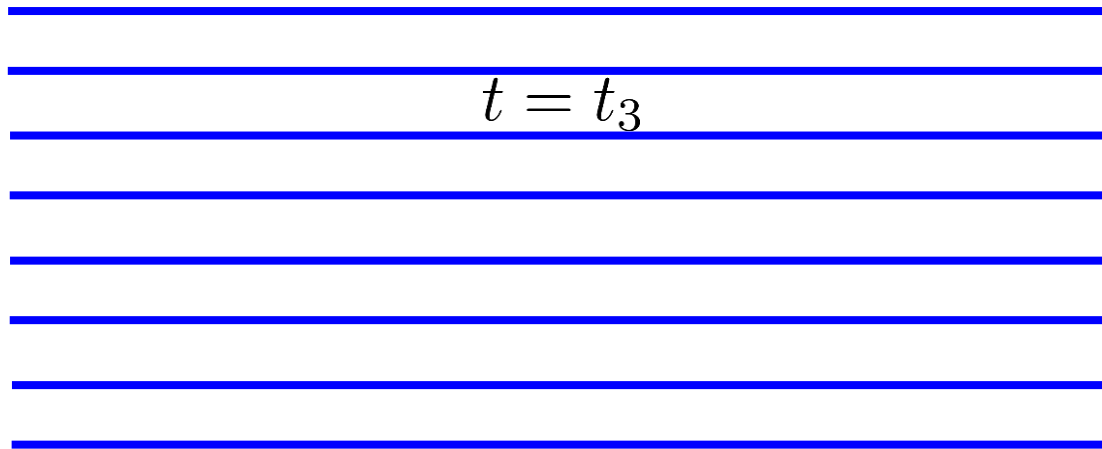
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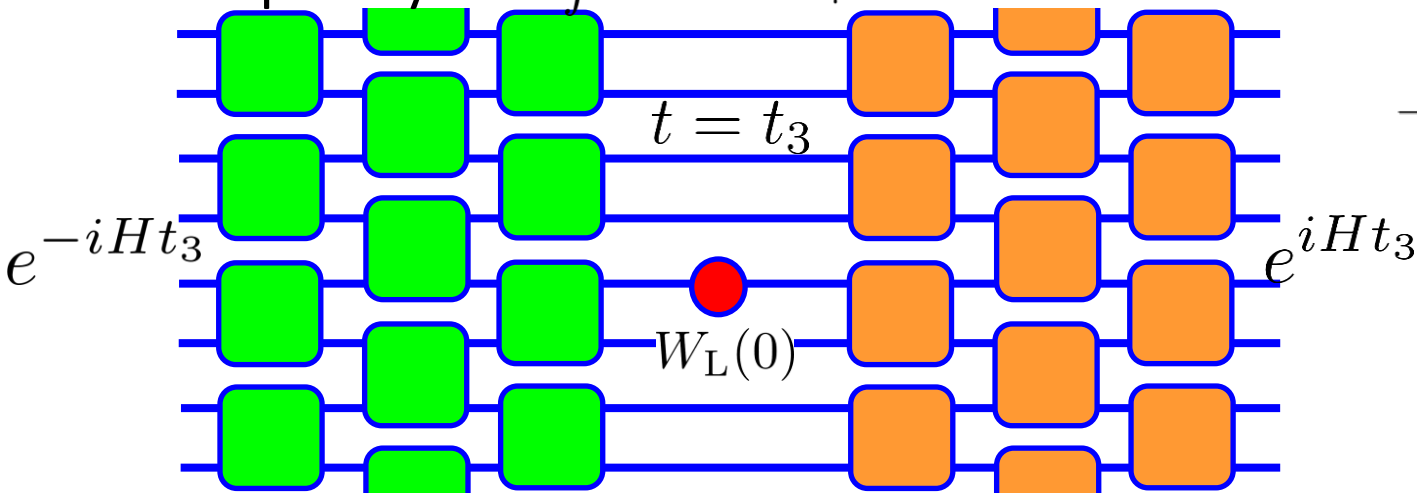
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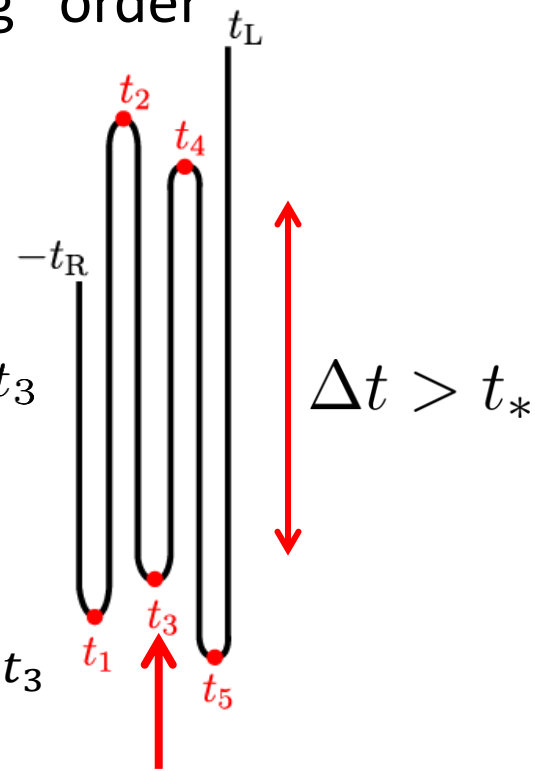
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with simple $W_L(0)$ (only affects few qubits), still get partial cancellation

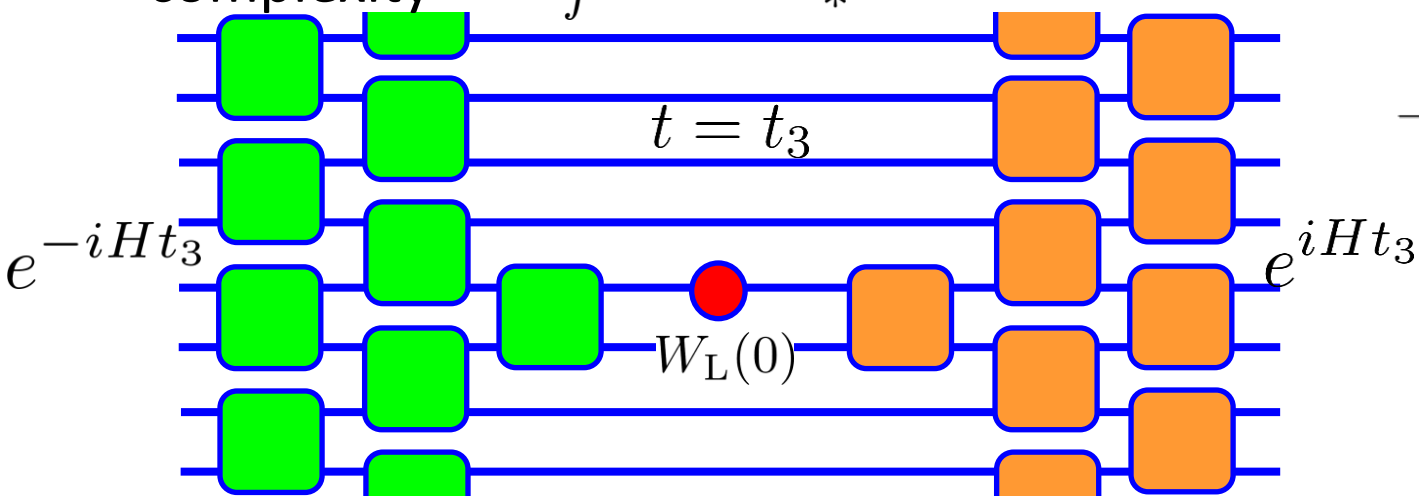
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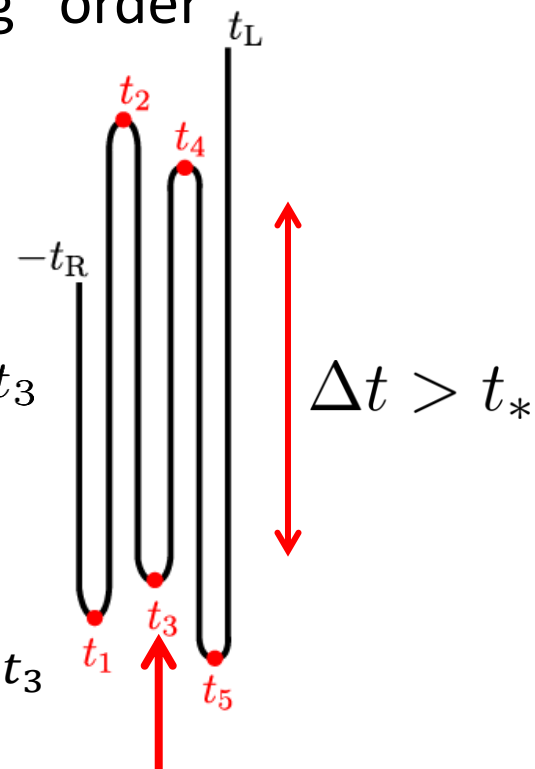
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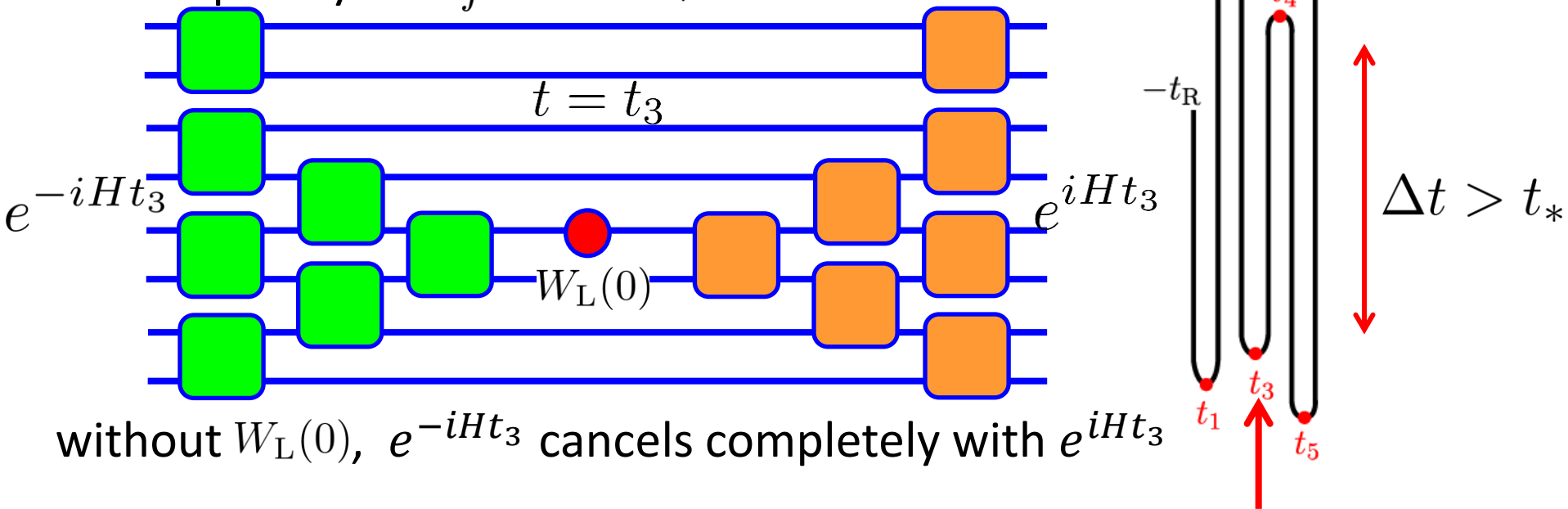
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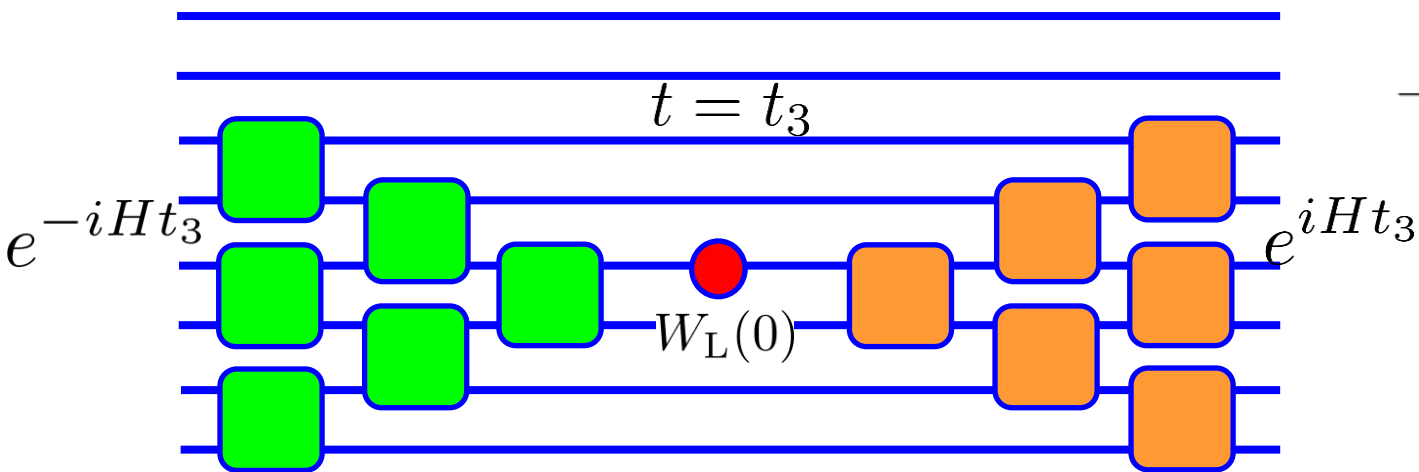
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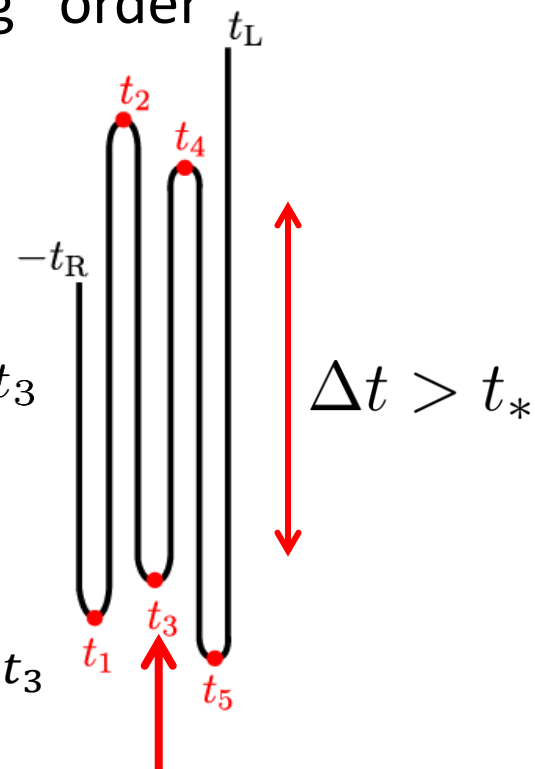
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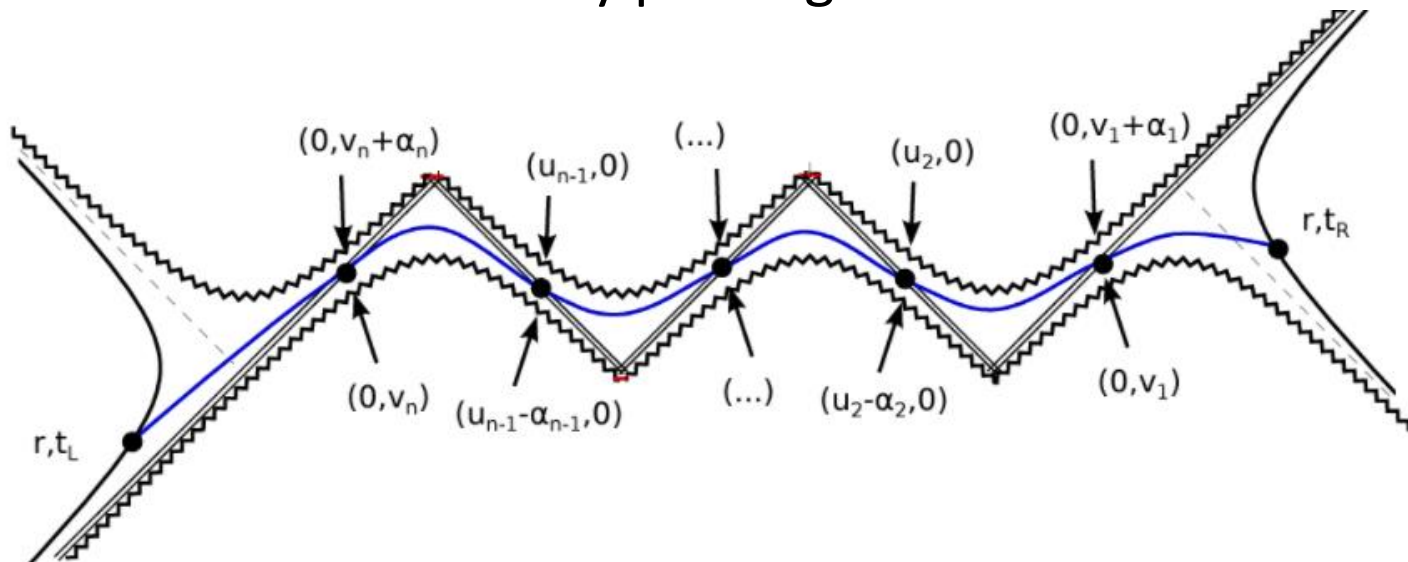
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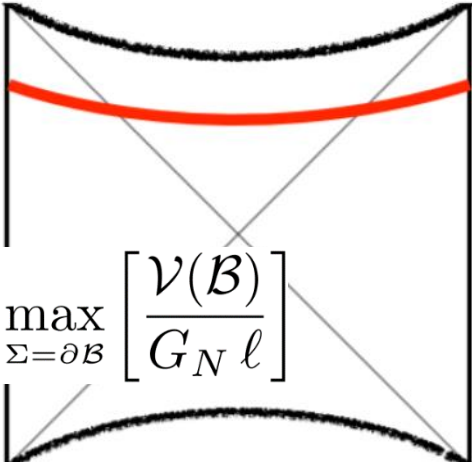
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- complexity $\propto t_f - 2n t_*$
- “switchback effect” is $-2n t_*$ due to partial cancellation of forward and backward evolutions along the time folds
- reproduce this behaviour by probing black hole with shock waves



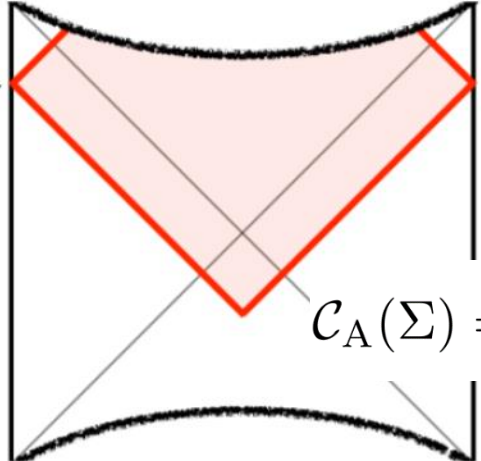
Holographic Complexity:

Complexity = Volume



$$\mathcal{C}_V(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N \ell} \right]$$

Complexity = Action



$$\mathcal{C}_A(\Sigma) = \frac{I_{\text{WDW}}}{\pi \hbar}$$

WHY COMPLEXITY??

- connection of complexity=volume to AdS/MERA 

- linear growth (at late times)

(d = boundary dimension)

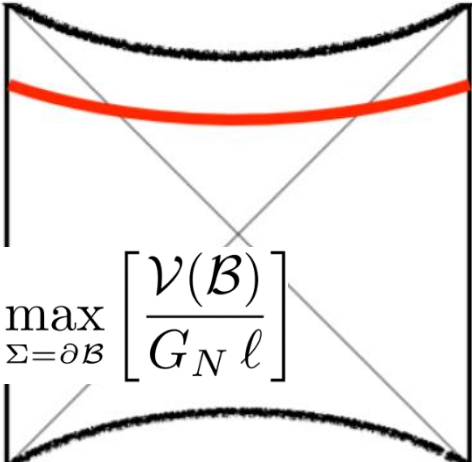
$$\left. \frac{d\mathcal{C}_V}{dt} \right|_{t \rightarrow \infty} = \frac{8\pi}{d-1} M \quad (\text{planar})$$

$$\left. \frac{d\mathcal{C}_A}{dt} \right|_{t \rightarrow \infty} = \frac{2M}{\pi} \quad \img alt="red checkmark" data-bbox="910 790 980 870"/>$$

- “switchback effect” (probe black holes with shock waves) 

Holographic Complexity:

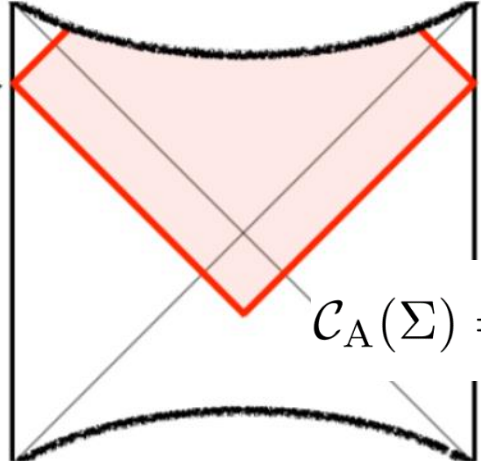
Complexity = Volume



A diagram of a spacetime volume represented as a rectangle with curved top and bottom edges. A red curve, representing a surface Σ , is drawn across the volume. The left and right boundaries are labeled t_L and t_R respectively. The volume is divided into four regions by two diagonal lines crossing at the center.

$$\mathcal{C}_V(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N \ell} \right]$$

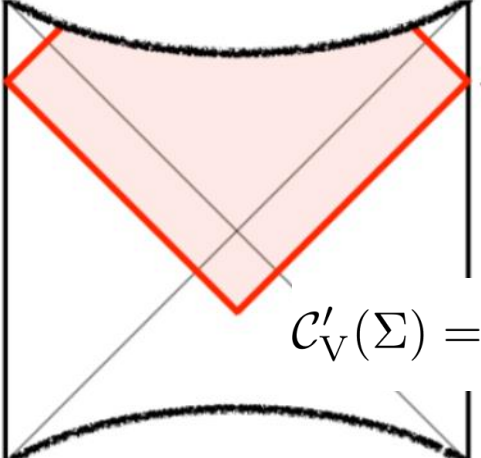
Complexity = Action



A diagram of a spacetime volume similar to the first one, but with a red shaded region bounded by a red curve Σ . The shaded region is a diamond shape with its vertices at the top and bottom boundaries. The left and right boundaries are labeled t_L and t_R .

$$\mathcal{C}_A(\Sigma) = \frac{I_{\text{WDW}}}{\pi \hbar}$$

Complexity = Volume2.0



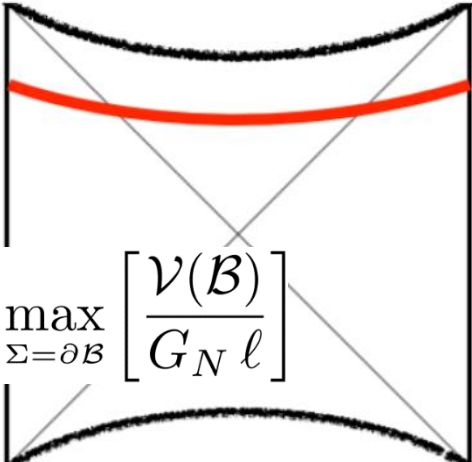
A diagram of a spacetime volume similar to the second one, but with a red shaded region bounded by a red curve Σ . The shaded region is a diamond shape with its vertices at the top and bottom boundaries. The left and right boundaries are labeled t_L and t_R .

$$\mathcal{C}'_V(\Sigma) = \frac{V_{\text{WDW}}}{G_N \ell^2}$$

**WHY Volume
or Action
or Spacetime Volume???**

Holographic Complexity:

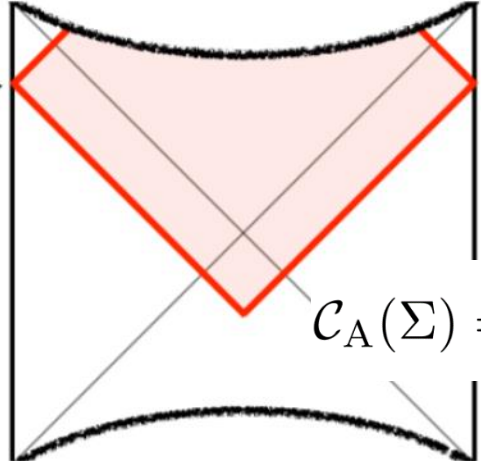
Complexity = Volume



A diagram of a spacetime volume represented as a rectangle with curved top and bottom edges. Two vertical lines on the left and right are labeled t_L and t_R with arrows pointing inward. A red curve, representing a surface Σ , is drawn across the volume. The volume is divided into two regions by this curve. The formula for complexity is given as:

$$\mathcal{C}_V(\Sigma) = \max_{\Sigma=\partial\mathcal{B}} \left[\frac{\mathcal{V}(\mathcal{B})}{G_N \ell} \right]$$

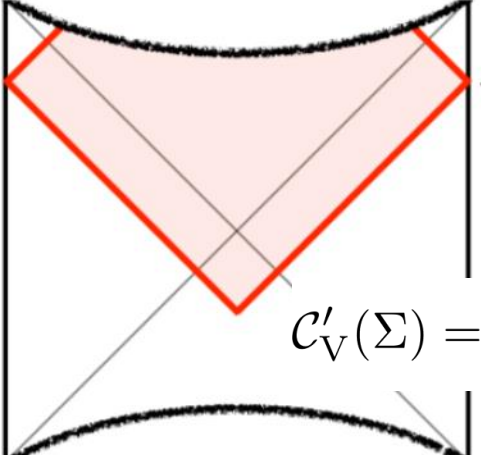
Complexity = Action



A diagram of a spacetime volume similar to the first one, but with a red curve and a shaded region. The shaded region is a V-shape pointing downwards, bounded by the red curve and the top boundary of the volume. The formula for complexity is given as:

$$\mathcal{C}_A(\Sigma) = \frac{I_{\text{WDW}}}{\pi \hbar}$$

Complexity = Volume2.0



A diagram of a spacetime volume similar to the second one, but with a different shaded region. The shaded region is a V-shape pointing downwards, bounded by the red curve and the top boundary of the volume. The formula for complexity is given as:

$$\mathcal{C}'_V(\Sigma) = \frac{V_{\text{WDW}}}{G_N \ell^2}$$

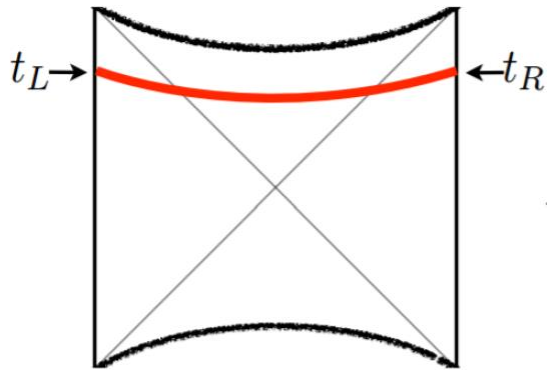
**WHY Volume
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or Spacetime Volume???**

Ambiguities in defining complexity?

Complexity=Volume Revisited:

- complexity=volume: evaluate proper volume of extremal codim-one surface connecting Cauchy surfaces in boundary theory (cf holo EE)

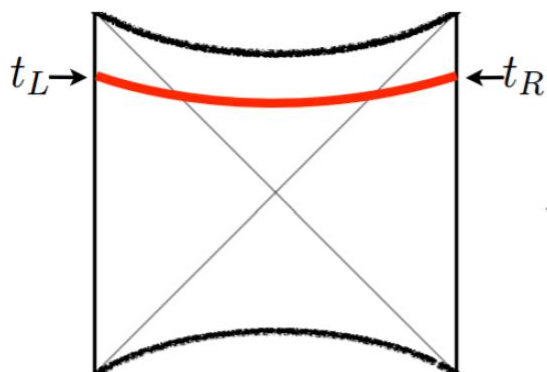
(Stanford & Susskind)



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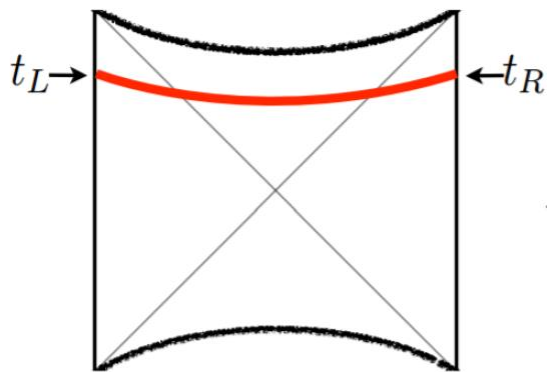
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Two steps: 1) find a special surface

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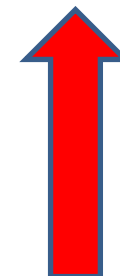


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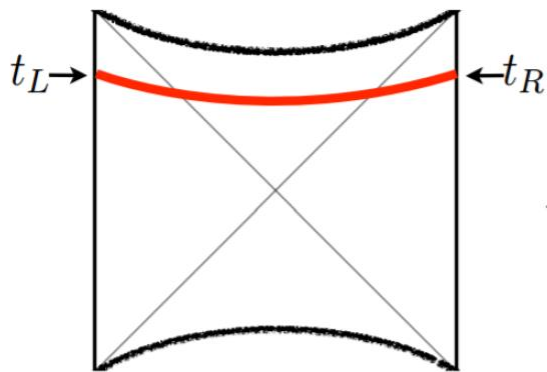
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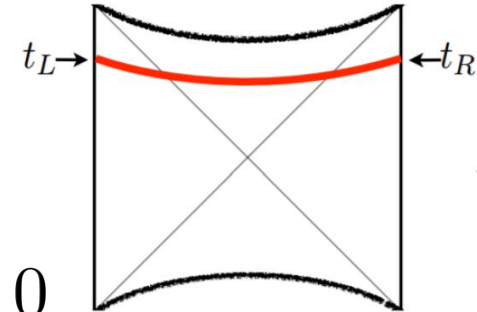
2) Evaluate geometric feature of surface

- yields “nice” diffeomorphism invariant observable

Generalize two step procedure:

1) find a special surface Σ :

$$\delta_X \left(\int_{\Sigma} d^d \sigma \sqrt{h} F_2(g_{\mu\nu}; X^\mu) \right) = 0$$

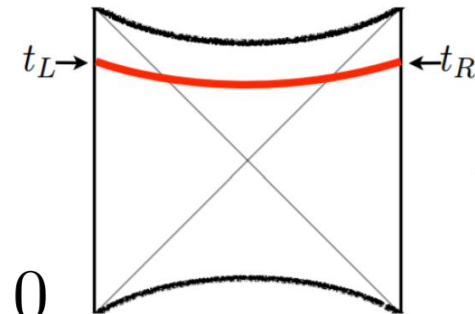


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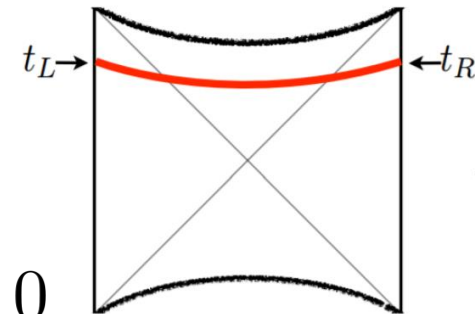
$$O_{F_1, \Sigma_{F_2}}(\Sigma_{CFT}) = \frac{1}{G_N L} \int_{\Sigma_{F_2}} d^d \sigma \sqrt{h} F_1(g_{\mu\nu}; X^\mu)$$

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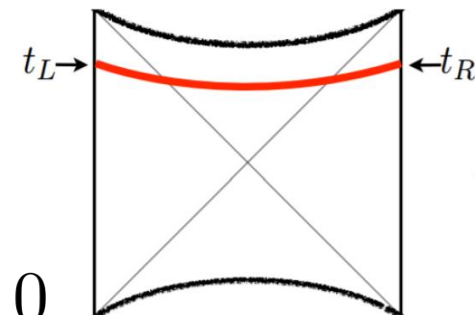
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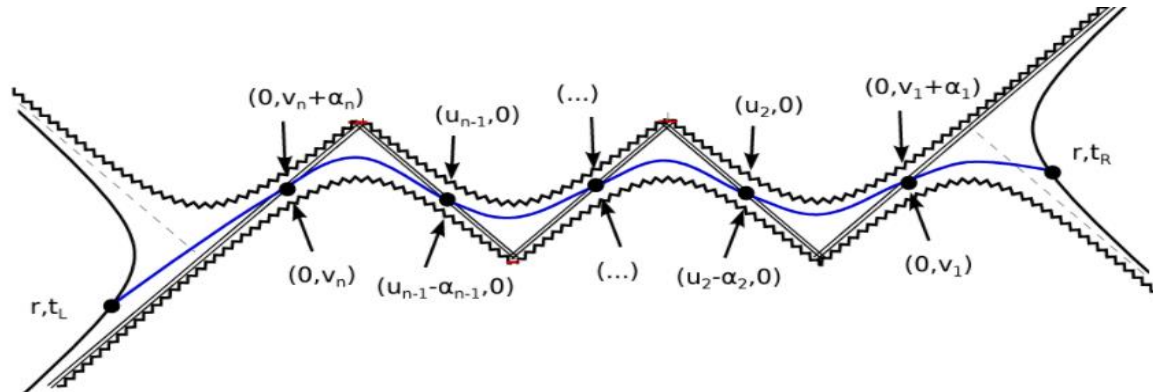
So what?

1) Observables grow linearly with time at late times:

$$\lim_{\tau \rightarrow \infty} O_{F_1, \Sigma_{F_2}}(\tau) \sim P_\infty \tau$$

where in large T limit, the constant $P_\infty \propto \text{mass}$

2) Observables exhibit “switchback effect”, ie, universal time delay in response to shock waves falling into the dual black hole



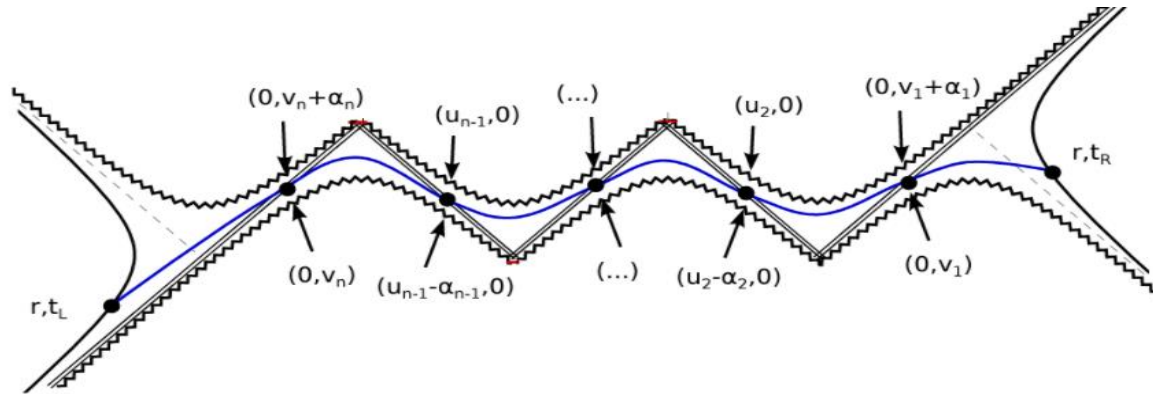
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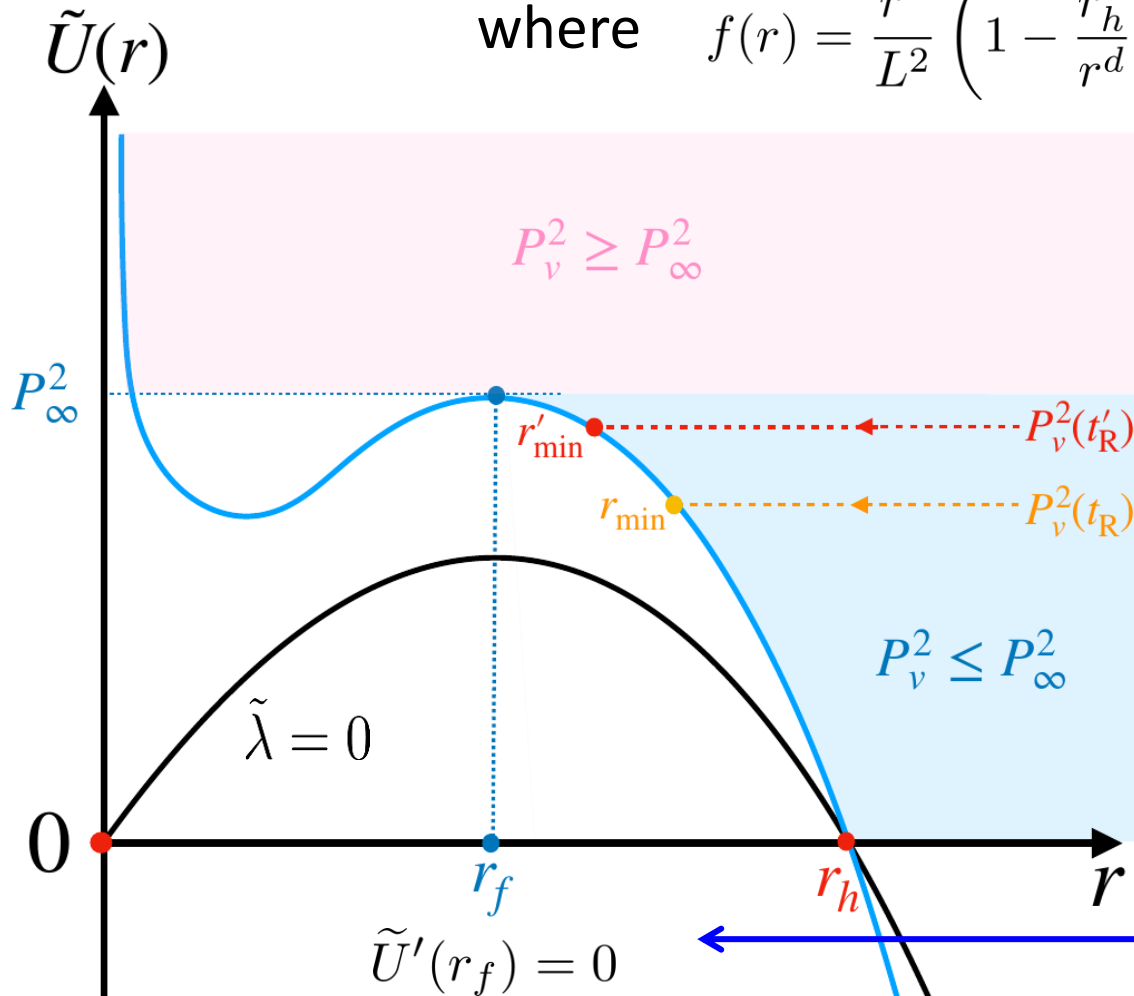
- when F_1 and F_2 constructed with bkgd curvatures, analysis similar to extremal volume

Simple Example: $F_1 = F_2 = 1 + \lambda L^4 C_{abcd} C^{abcd}$

- profile determined by classical mechanics problem

$$\dot{r}^2 + \tilde{U}(r) = P_v^2 \quad \text{with} \quad \tilde{U}(r) = -f(r)a^2(r) \left(\frac{r}{L}\right)^{2(d-1)},$$

where $f(r) = \frac{r^2}{L^2} \left(1 - \frac{r_h^d}{r^d}\right)$ and $a(r) = 1 + \tilde{\lambda} \left(\frac{r_h}{r}\right)^{2d}$



- turning point:
 $P_v^2 = \tilde{U}(r_{min})$

$$P_\infty^2 = U(r_f) = \xi \left(\frac{r_h}{L}\right)^{2d}$$

So what?

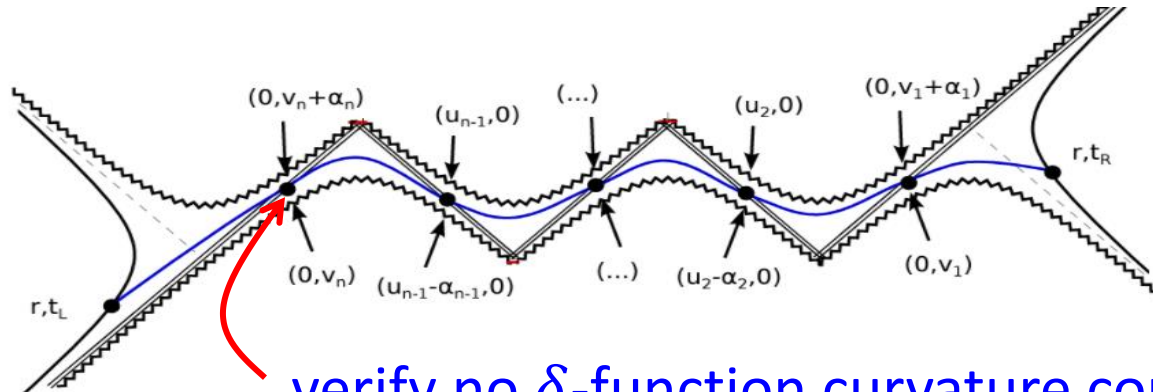
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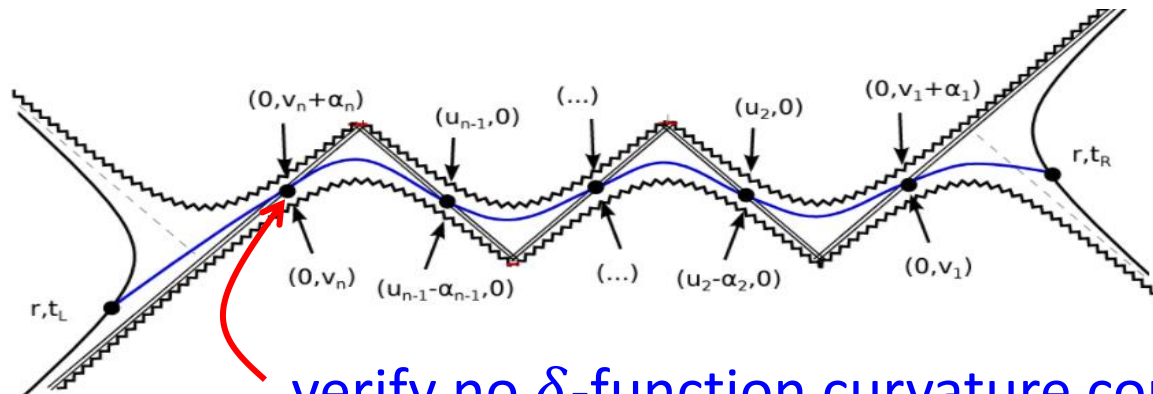
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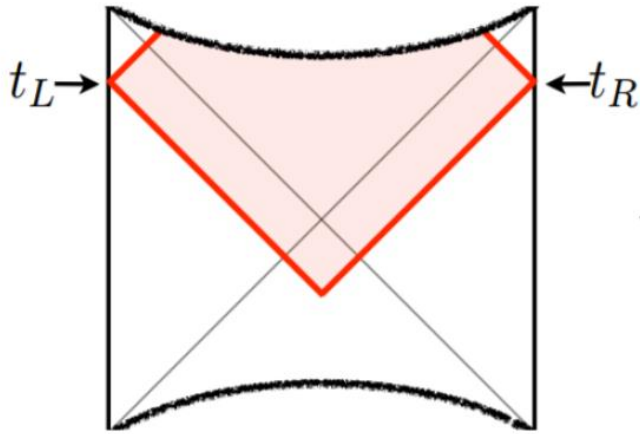
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- **universality displayed observables suggests that all of them are equally viable candidates for holographic complexity!!**

What about extensions of CA and CV2.0?

- complexity=action: evaluate gravitational action for Wheeler-DeWitt patch = domain of dependence of bulk time slice connecting boundary Cauchy slices in CFT (Brown, Roberts, Swingle, Susskind & Zhao)
- complexity=volume2.0: evaluate spacetime volume of WDW patch (Couch, Fischler & Nguyen)



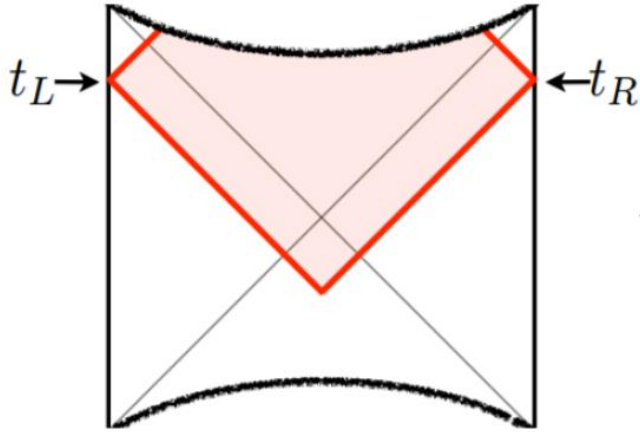
$$\mathcal{C}_A(\Sigma) = \frac{I_{\text{WDW}}}{\pi \hbar}$$

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- no extremization procedure implemented; surfaces bounding volume (ie, codimension-zero region) are light sheets

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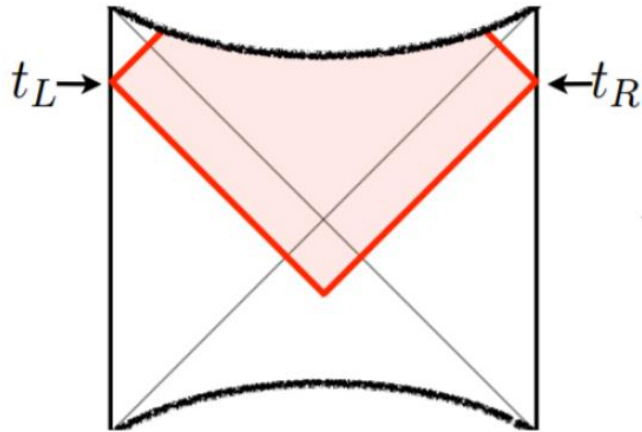
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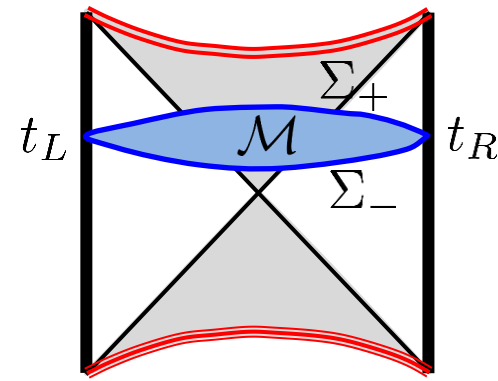
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- 1) find a special surfaces bounding codim.-0 region
 - 2) Evaluate geometric feature of codim.-0 region (& bounding surfaces)

- yields “nice” diffeomorphism invariant observable

Generalized procedure for codim.-0 observables:

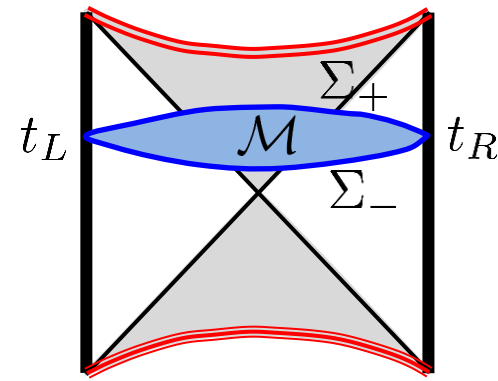


1) find a bounding surfaces Σ_{\pm} :

$$\delta_{\{X_+, X_-\}} \left(\int_{\Sigma_+} d^d \sigma \sqrt{h} F_4(g_{\mu\nu}; X_+^\mu) + \int_{\Sigma_-} d^d \sigma \sqrt{h} F_5(g_{\mu\nu}; X_-^\mu) + \int_{\mathcal{M}} d^{d+1} \sigma \sqrt{-g} F_6(g_{\mu\nu}) \right) = 0$$

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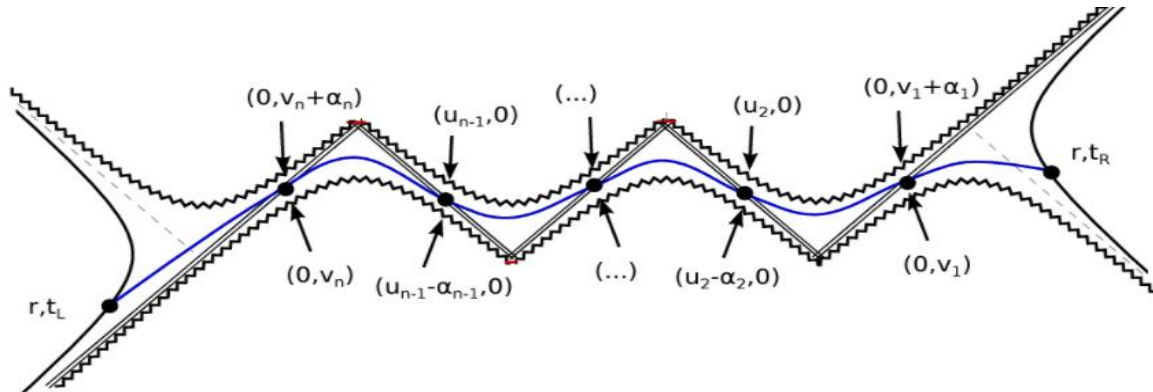
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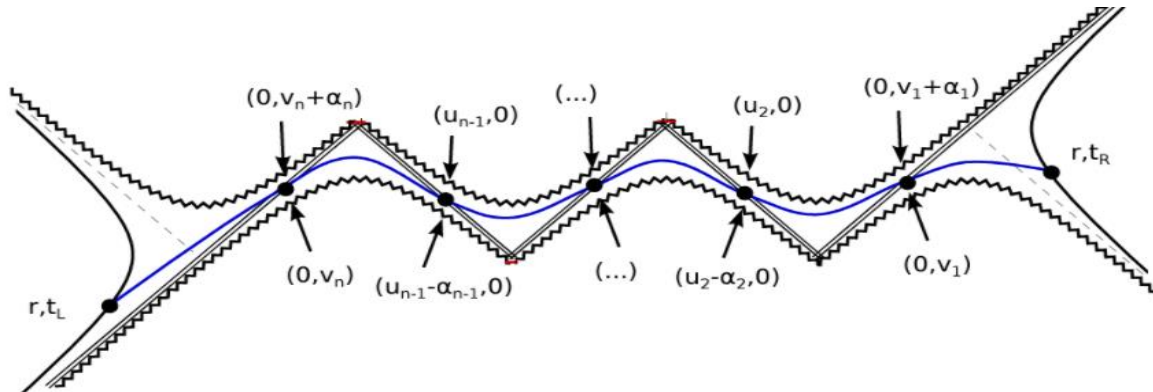
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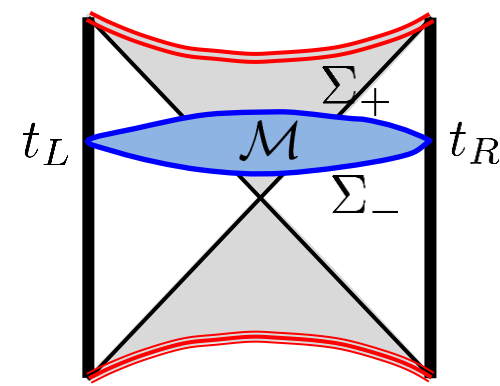


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Simplest Example:

- extremize the functional

$$O(\Sigma_{CFT}) = \frac{\alpha_+}{G_N L} \int_{\Sigma_+} d^d \sigma \sqrt{h} + \frac{\alpha_-}{G_N L} \int_{\Sigma_-} d^d \sigma \sqrt{h} + \frac{1}{G_N L^2} \int_{\mathcal{M}} d^{d+1} \sigma \sqrt{-g}$$



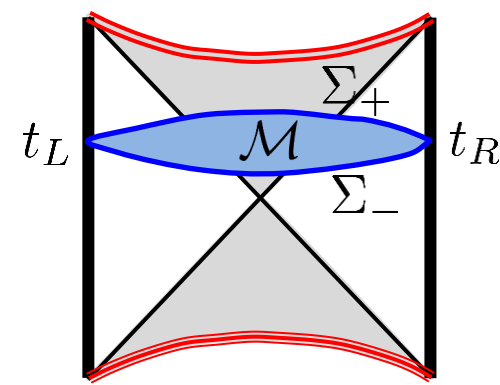
- evaluating the volumes of the bounding surfaces Σ_{\pm} weighted by coefficients α_{\pm} , as well as of volume of codim.-0 region \mathcal{M}
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- in limit $\alpha_{\pm} \rightarrow 0$, these surfaces become the future/past light sheets

→ \mathcal{M} becomes WDW patch!

- evaluate volume (same functional) **→** CV2.0
- evaluate action (including bdy terms) **→** CA

Conclusions/Questions/Outlook:

- simple example but “classical mechanics” analysis readily extends to $F_1(g_{\mu\nu}, \mathcal{R}_{\mu\nu\rho\sigma}, \nabla_\mu)$ and to observables where $F_1 \neq F_2$
- couplings for curvature invariants should not be too large
- similar behaviour appears to hold for functionals including dependence on extrinsic curvature
- **infinite class of holographic observables equally viable candidates for gravitational dual of complexity!!**
- can freedom in constructing gravitational observables be related to freedom in constructing complexity model in boundary QFT
- is there something that singles out maximal volume?
- what is role of extremal solutions which are not global maxima and probe very near to singularity?
- further investigation of codimension-zero observables
- add matter contributions to new observables (eg, CA proposal)

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Lots to explore!

