Homework 6 - CAO 2024

3. **Subgradient Method**. Let $k < n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \max_{i=1,\dots,k+1} x_i + \frac{1}{2}\|x\|^2$. Notice that f is strongly convex. We are interested on the behaviour of the subgradient method applied to the problem

$$f^* = \min_x f(x). \tag{P}$$

   (a) Find $\partial f(x)$ for all x.

   (b) Show that problem (P) has a unique solution x^* where $x^*_j = -1/(k+1)$ for $j = 1, \dots, k+1$ and $x^*_j = 0$ for $k+1 < j \le n$.

   (c) Let $R = \|x^*\|$. Let $G = 1 + 2/\sqrt{k+1}$. Show that f is G-Lipchitz on $\{x : \|x - x^*\| \le R\}$.

   Now we will show the optimality of the subgradient method among all methods that interactively use subgradients to solve problem P. We assume an oracle that on input x returns the subgradient $g_x = e_{\hat{j}} + x$ where $\hat{j} = \min j : x_j = \max_{i=1,\dots,k+1}$. Consider any sequence starting at $x^0 = 0$ and such that, $x^{k+1} \in x^0 + \operatorname{span}\{g_{x^0}, g_{x^1}, \dots, g_{x^k}\}$.

   (d) Show that for all $i \le k$:

      i. $x^i_j = 0$ for all $i < j \le n$, and

      ii. $f(x^i) \ge \frac{1}{2}\|x^i\|^2 \ge 0$.

   (e) Let $f^*_k = \min_{i=0,1,\dots,k} f(x^i)$. Show that $f^*_k - f^* \ge \frac{GR}{2(2+\sqrt{k+1})}$.

4. **Computing Projection into a ball or an ellipse**. This is a continuation from exercise 1 (HW5). Projecting is a basic operation in many optimization algorithms, thus one needs to compute projections very fast.

   Let $\|\cdot\|_p$ denote the p-norm. Consider the p-Ball $B_p := \{x \in \mathbb{R}^n : \|x\|_p \le 1\}$.

   (a) For which values of p is B_p convex?

   (b) For the values of p when B_p is convex, compute $\Pi_{B_p}(u)$ for any $u \in \mathbb{R}^n$.

   Now let $A \in \mathbb{R}^{m\times n}$. Consider the ellipse $E = \{x \in \mathbb{R}^n : \|Ax\|_2 \le 1\}$.

   (c) [**Remark**: How to answer this problem is rather open]. Write a method/algorithm to compute $\Pi_E(u)$ for any $u \in \mathbb{R}^n$. How efficient is your method?