CIEM1110-1: FEM, lecture 3.2

Path-following for nonlinear FEM

Iuri Rocha, Martin Lesueur

TU Delft

Agenda for today

1. Nonlinear FEM in load control (recap)
2. Displacement-controlled simulations

Recap – linearization

Discretized form:

$$\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Recap – linearization

Discretized form:

$$\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Equilibrium residual:

$$\mathbf{r} = \mathbf{f}_{\text{ext}} - \mathbf{f}_{\text{int}} \left(\mathbf{a} \right)$$

Recap — linearization

Discretized form:

$$\int_{\Omega} \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_{\Omega} \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Equilibrium residual:

$$\mathbf{r} = \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int}}\left(\mathbf{a}\right)$$

1st-order Taylor expansion:

$$\mathbf{r}_{\mathrm{new}} = \mathbf{r} + \frac{\partial \mathbf{r}}{\partial \mathbf{a}} \Delta \mathbf{a} \quad \Rightarrow \quad \mathbf{r}_{\mathrm{new}} = \mathbf{r} - \mathbf{K} \Delta \mathbf{a}$$

Recap — linearization

Discretized form:

$$\int_\Omega \mathbf{B}^T \boldsymbol{\sigma} \, \mathrm{d}\Omega = \int_\Omega \mathbf{N}^T \mathbf{b} \, \mathrm{d}\Omega + \int_{\Gamma_t} \mathbf{N}^T \mathbf{t} \, \mathrm{d}\Gamma$$

Equilibrium residual:

$$\mathbf{r} = \mathbf{f}_{\mathrm{ext}} - \mathbf{f}_{\mathrm{int}} (\mathbf{a})$$

1st-order Taylor expansion:

$$\mathbf{r}_{\mathrm{new}} = \mathbf{r} + \frac{\partial \mathbf{r}}{\partial \mathbf{a}} \Delta \mathbf{a} \quad \Rightarrow \quad \mathbf{r}_{\mathrm{new}} = \mathbf{r} - \mathbf{K} \Delta \mathbf{a}$$

Newton-Raphson iteration ($\mathbf{r}_{\mathrm{new}} = 0$):

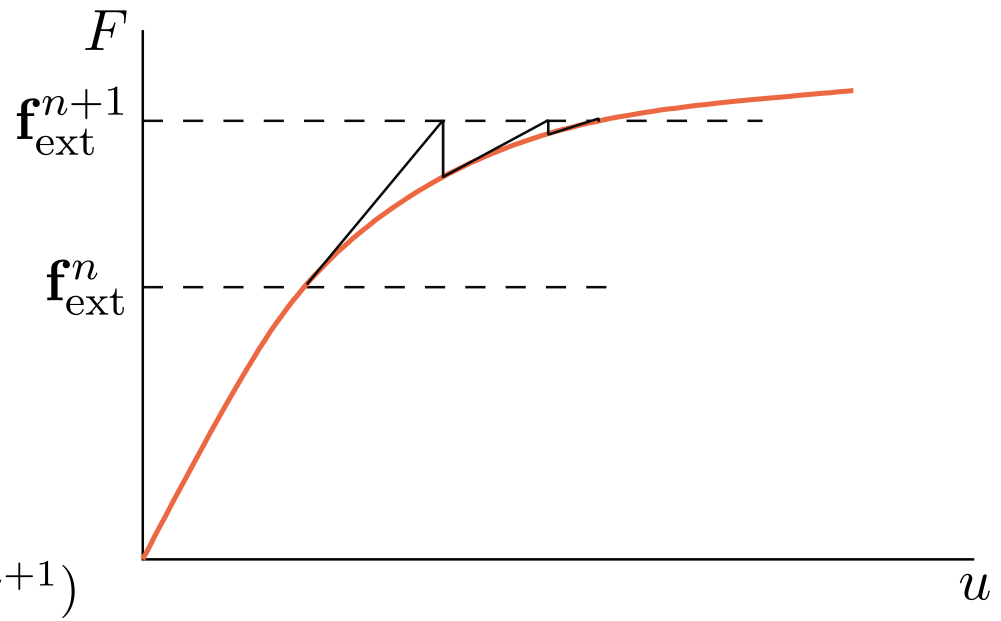
$$\mathbf{K} \Delta \mathbf{a} = \mathbf{r}$$

TU Delft

Recap — load control

Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

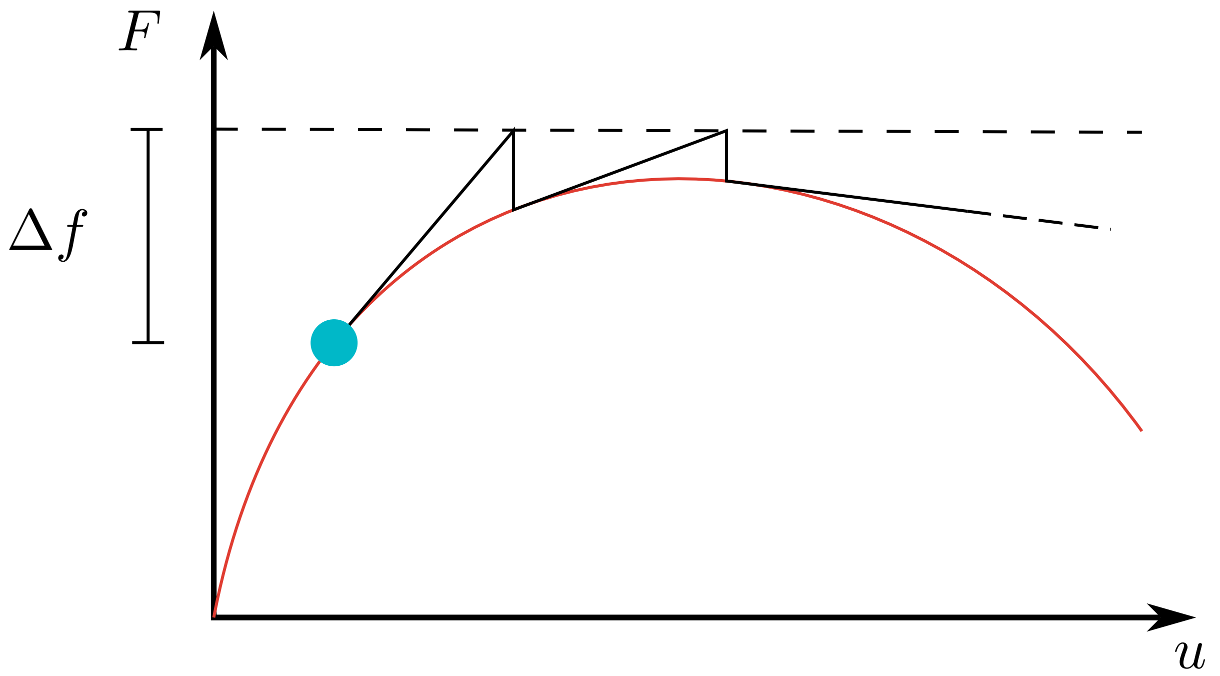
Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps **do**
  Get new external force vector: $\mathbf{f}_{\text{ext}}^{n+1}$
  Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$
  Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
  Evaluate first residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
  repeat
    Solve linear system of equations: $\mathbf{K}^{n+1}\Delta\mathbf{a} = \mathbf{r}$
    Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta\mathbf{a}$
    Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
    Evaluate residual: $\mathbf{r} = \mathbf{f}_{\text{ext}}^{n+1} - \mathbf{f}_{\text{int}}^{n+1}$
  until $|\mathbf{r}| <$ tolerance
  $n = n + 1$
end while



Limitations of load control

Snap-through behavior:

- Sometimes there is no (close) solution on top of the equilibrium path for an increasing load



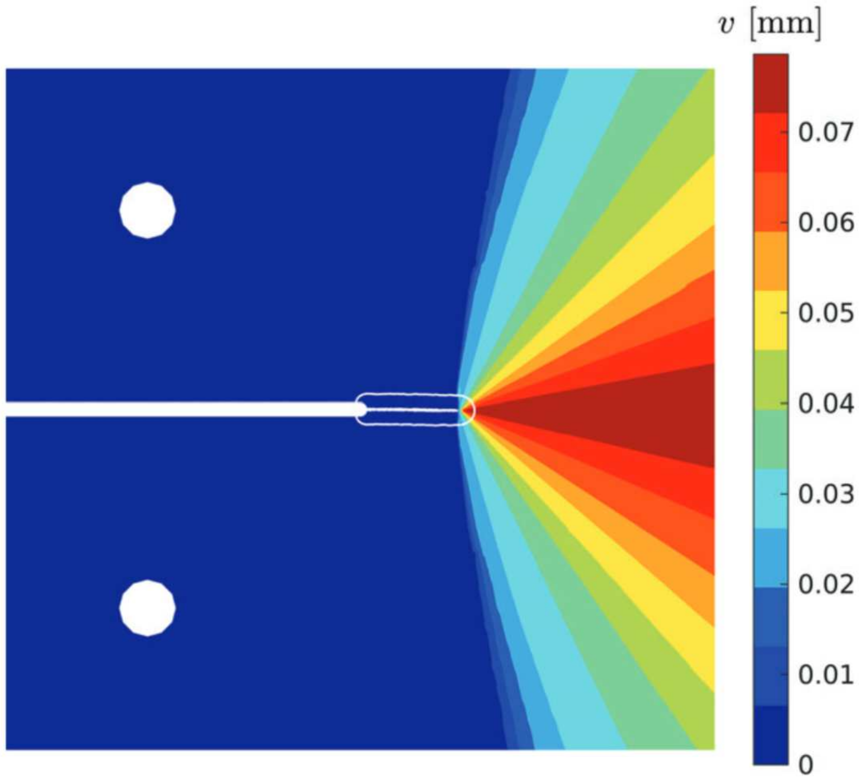
divergence!

Limitations of load control

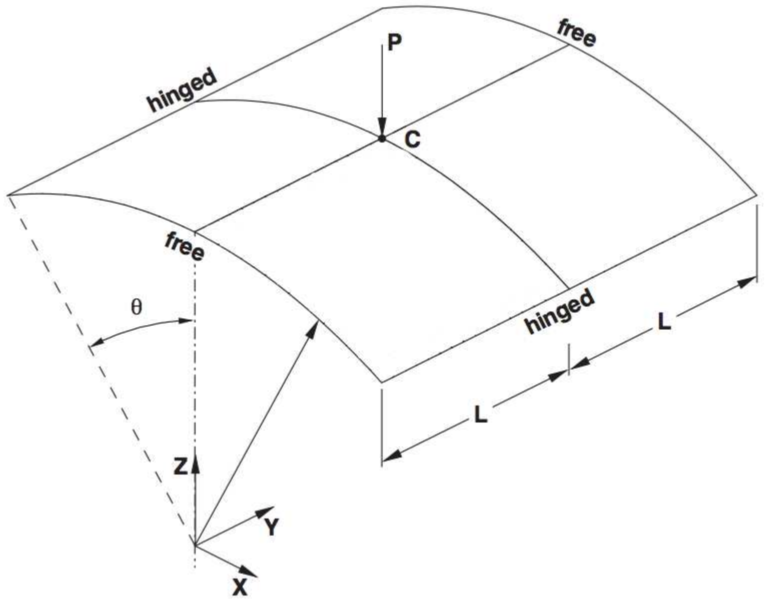
Snap-through behavior:

- Sometimes there is no (close) solution on top of the equilibrium path for an increasing load

Compact tension test (material nonlinearity)



Laminated shallow shell (geometric nonlinearity)

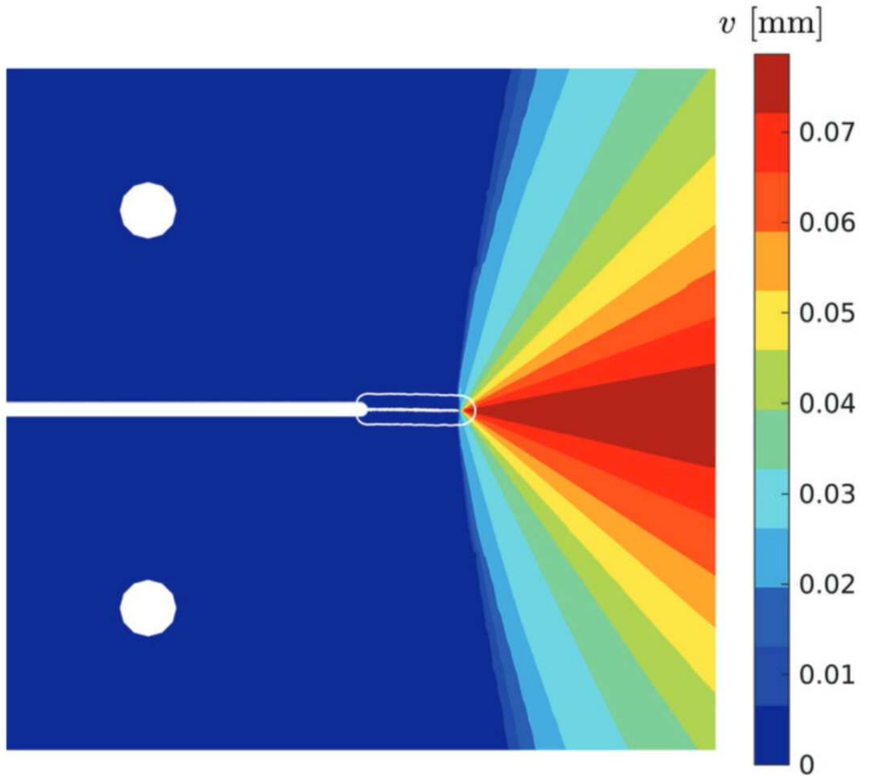


Limitations of load control

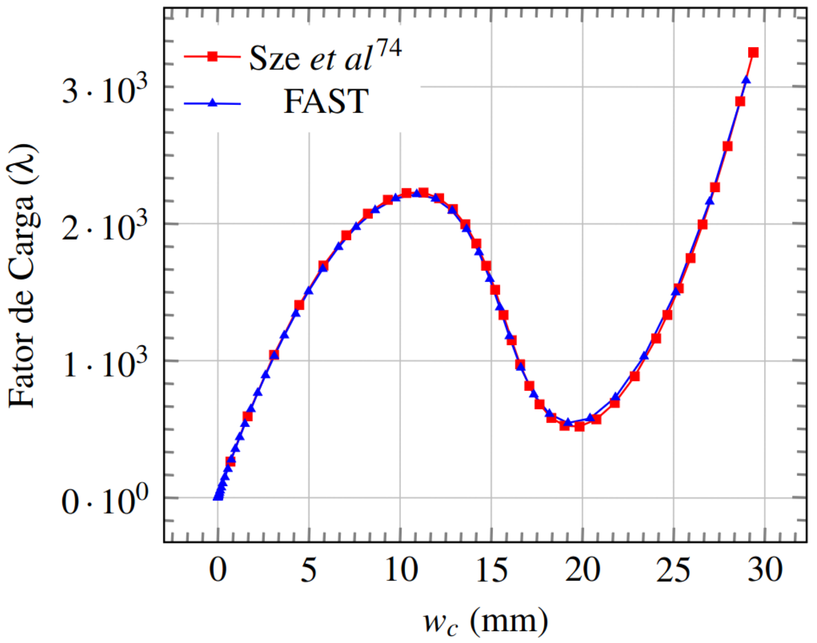
Snap-through behavior:

- Sometimes there is no (close) solution on top of the equilibrium path for an increasing load

Compact tension test (material nonlinearity)



Laminated shallow shell (geometric nonlinearity)



Displacement control

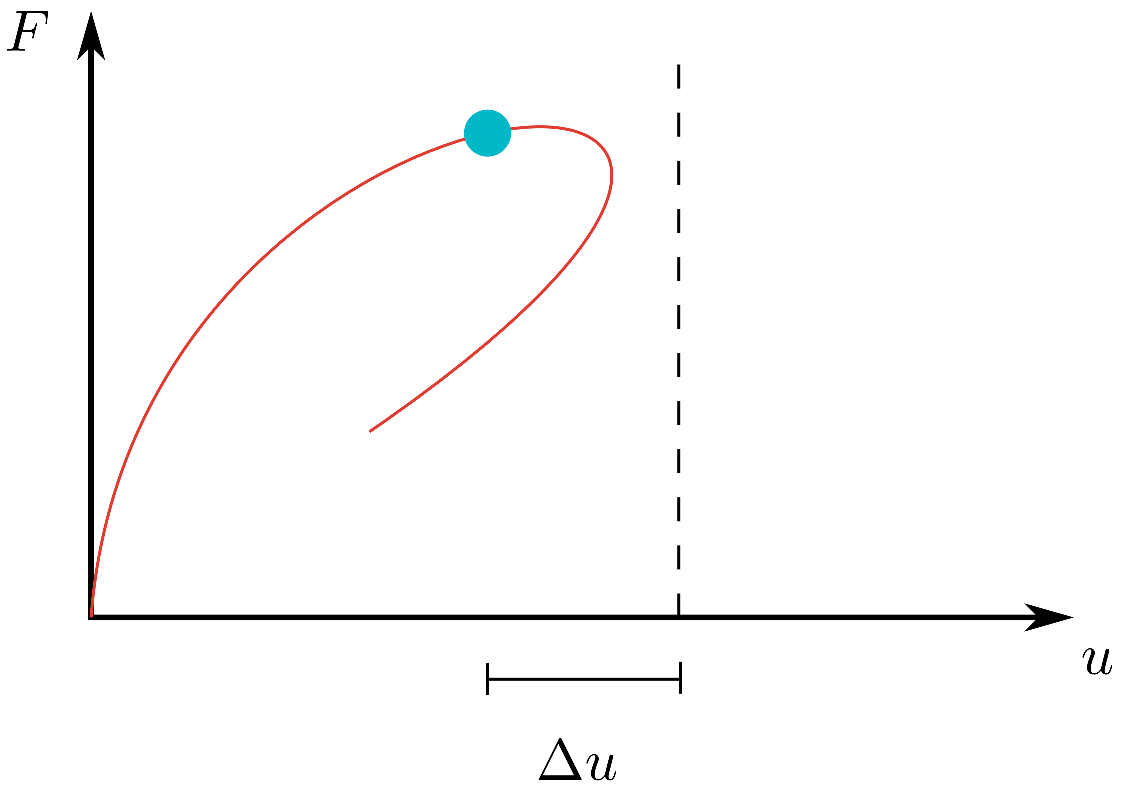
Require: Nonlinear relation $\mathbf{f}_{\text{int}}(\mathbf{a})$ with $\mathbf{K}(\mathbf{a}) = \frac{\partial \mathbf{f}_{\text{int}}}{\partial \mathbf{a}}$

Initialize $n = 0$, $\mathbf{a}^0 = \mathbf{0}$
while $n <$ number of time steps **do**
  Initialize new solution at old one: $\mathbf{a}^{n+1} = \mathbf{a}^n$
  Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
  Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = \bar{\mathbf{a}}^{n+1} - \bar{\mathbf{a}}^n$
  Evaluate first residual: $\mathbf{r} = -\mathbf{f}_{\text{int},f}^{n+1}$
  repeat
    Solve linear system of equations: $\mathbf{K}^{n+1} \Delta \mathbf{a} = \mathbf{r}$
    Update solution: $\mathbf{a}^{n+1} = \mathbf{a}^{n+1} + \Delta \mathbf{a}$
    Compute internal force and stiffness: $\mathbf{f}_{\text{int}}^{n+1}(\mathbf{a}^{n+1})$, $\mathbf{K}^{n+1}(\mathbf{a}^{n+1})$
    Evaluate residual: $\mathbf{r} = -\mathbf{f}_{\text{int},f}^{n+1}$
    Constrain $\mathbf{K}^{n+1}$ so that $\Delta \mathbf{a}_c = 0$
  until $|\mathbf{r}| <$ tolerance
  $n = n + 1$
end while

Limitations of displacement control

Snap-back behavior:

- Sometimes displacements do not increase monotonically

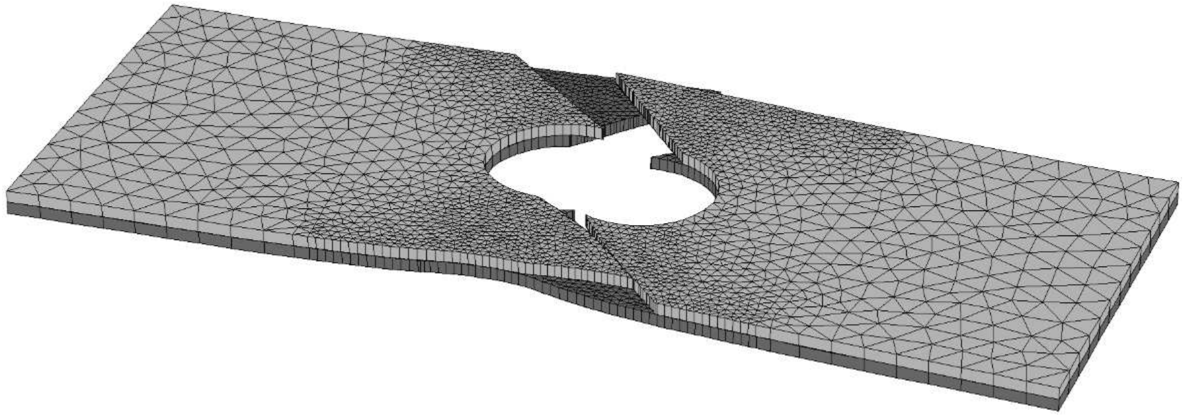


Limitations of displacement control

Snap-back behavior:

- Sometimes displacements do not increase monotonically

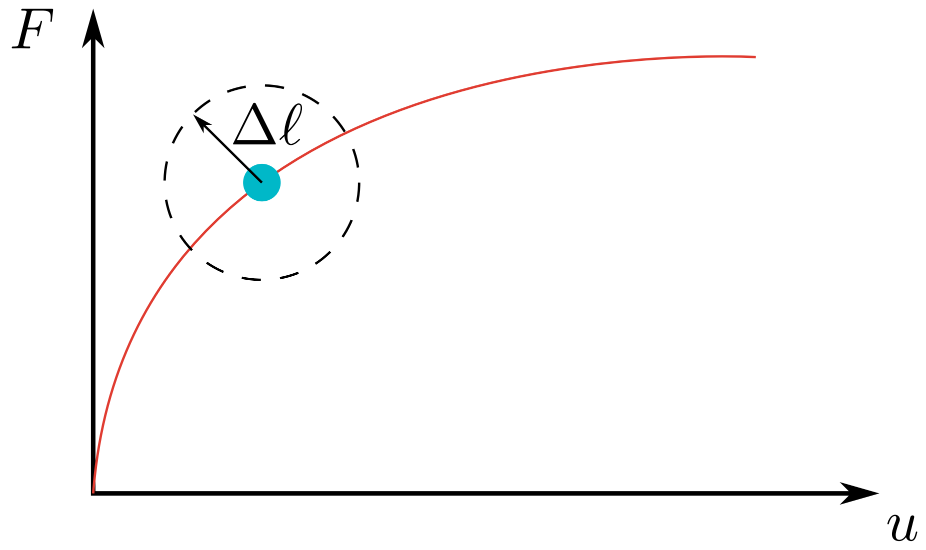
Complex failure in laminates



Arc-length control — linearization

Redefine the external load vector:

$$\mathbf{r}(\mathbf{a}) = \mathbf{f}_{\text{ext}} - \mathbf{f}_{\text{int}}(\mathbf{a}) \quad \Rightarrow \quad \mathbf{r}(\mathbf{a}, \lambda) = \lambda\hat{\mathbf{f}} - \mathbf{f}_{\text{int}}(\mathbf{a})$$



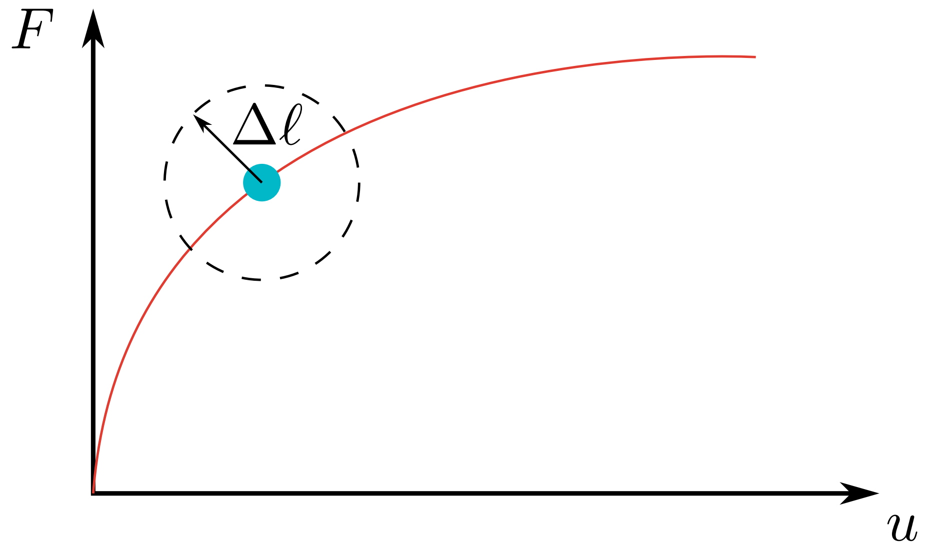
Arc-length control — linearization

Redefine the external load vector:

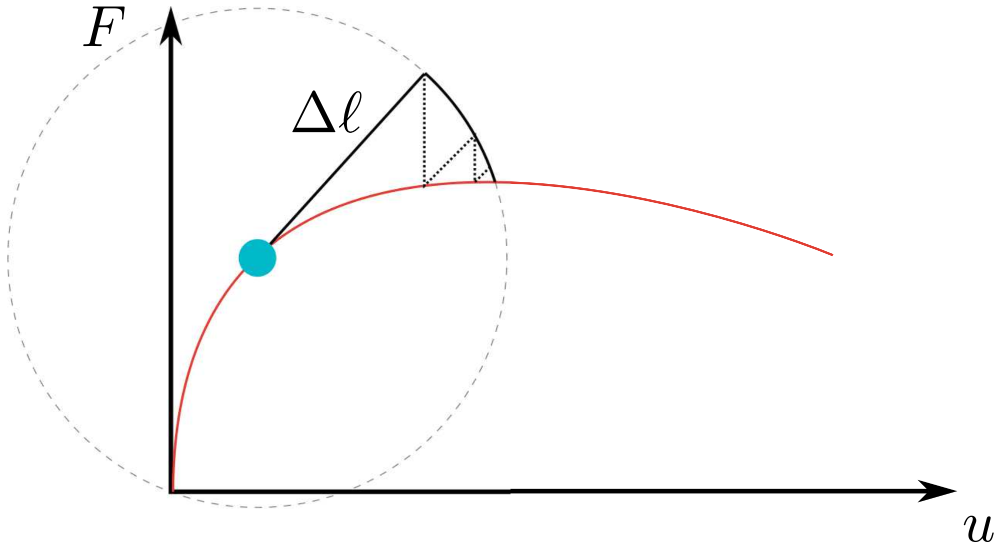
$$\mathbf{r}\left(\mathbf{a}\right) = \mathbf{f}_{\text{ext}} - \mathbf{f}_{\text{int}}\left(\mathbf{a}\right) \quad \Rightarrow \quad \mathbf{r}\left(\mathbf{a}, \lambda\right) = \lambda\hat{\mathbf{f}} - \mathbf{f}_{\text{int}}\left(\mathbf{a}\right)$$

Too many unknowns, so introduce a new constraint equation:

$$g\left(\Delta\mathbf{a}, \Delta\lambda, \Delta\ell\right) = 0$$



Arc-length control — constraints



Recap and outlook

Controlling nonlinear FE simulations:

- Load/displacement cannot trace general equilibrium paths
- Arc-length models come in different flavors $\Rightarrow$ active research topic
- Convergence is always an issue, smart stepping algorithms are a must

Coming up next:

- Tomorrow: zoom into material nonlinearity
- Next week: viscoelasticity theory and pyJive implementation