A more analytic approach to Bell's theorem

Bachelor Thesis

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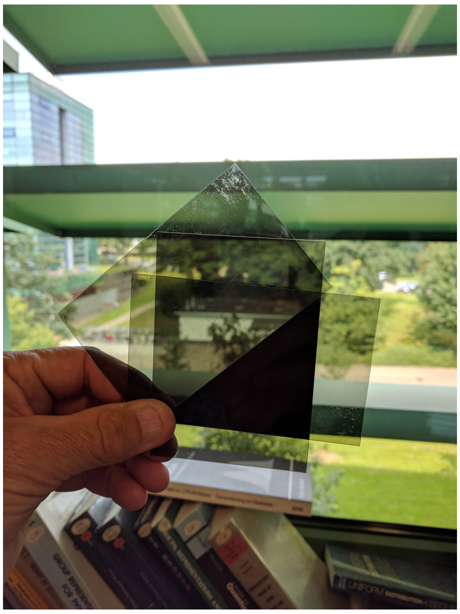

Abstract

In 1964 a famous paper appeared by John Bell[1] which answers the paradox proposed by Einstein Podolsky and Rosen[2] concerning the incompleteness of quantum mechanics and says: *no theory of local hidden variables is compatible with the results of quantum mechanics*. He does so by deriving an inequality relating the results of a two photon experiment. In this text we analyse all two photon (polarisation) states and find a new Bell inequality. Additionally, we deduce that for states with more than two particles, and restricting ourselves to an approach similar to Bell's, it is not possible to derive a Bell inequality.

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Cover image

Three polarisers superimposed show the polarisation property of light: it is absorbed for two perpendicular polarisers (the dark triangle on the right), but passes if a third polariser is placed in between.

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Preface

This text resulted from my research internship at the department of Theoretical High Energy Physics under the supervision of Ronald Kleiss during the academic year of 2016-2017. The internship is the concluding project for undergraduate students in pursue of their bachelor degree.

To my disappointment, a major element in my undergraduate courses was neglected: both my textbooks in quantum mechanics conclude with Bell's theorem, but it was not part of my curriculum. It gives an answer to the paradox proposed by Einstein, Podolsky and Rosen concerning the indeterminacy of quantum mechanics. In particular, it rules out the existence of *local hidden variables*. The final chapters of my textbooks, then, inspired the subject of this thesis and led me to apply Bell's method more extensively.

In the first part of this thesis (the Preliminaries and Chapter 1) we repeat some fundamentals of quantum mechanics to eventually derive Bell's theorem, as is done in many introductory textbooks. I think most readers can omit these sections. In the second part (Chapters 2 and 3), we fully analyse the two particle case of Bell's theorem where the last chapter explains why an inequality for a system of more than two particles is impossible if we abide to a similar method John Bell used in his theorem.

The research mainly took place in the form of conversations between my supervisor and I facing his blackboard. While his explanations seemed completely clear in that office, leaving its door made me doubt the arguments that I was strongly convinced about moments before, and made me realise how hefty I have depended on his guidance. I thank him for his guidance and appreciate his "Socratic" ways of doing theoretical research.

Preliminaries

As this thesis is aimed at undergraduate students, some familiarity with quantum mechanics is assumed. To refresh the reader's memory, we repeat several basic concepts of quantum mechanics using the polarisation properties of light as illustrations. Most readers are recommended to omit this section and skip to Chapter 1.

Postulates of quantum mechanics and the polarisation of photons

Quantum mechanics can be formalised with a set of postulates. The postulates presented below are mostly from the textbook Quantum Mechanics by B. Bransden and C. Joachain [3].

Postulate 1. To a physical system can be associated a state function[1] which contains all the information about the system.

Example 1. Suppose our physical system consists of a photon. Photons posses two properties, one of which is *polarisation*. This property can be made evident by holding an object called a polariser in front of a light source. We observe that the photons from the source either passes or do not pass. We observe also that changing the orientation of the polariser changes the probability that the photon passes. Moreover, if we lay two polarisers onto each other and we ensure they are oriented the same, we observe that if the photon passes the first polariser, it will surely pass the second. Even if we stack many more polarisers and orient them the same, the photon will continue to pass all polarisers whenever it passed the first. We can then say that a photon that passes a polariser oriented at angle θ is in a *definite* state, which we denote by

$$|\theta\rangle \,. \tag{1}$$

[1] Only in certain cases this can be done. There exist *mixed states* for which the state function is not completely known. We will not consider these however.

Postulate 2. (the superposition principle) If $|\psi_1\rangle$ and $|\psi_2\rangle$ are possible state functions of a physical system, then so is

$$c_1 |\psi_1\rangle + c_2 |\psi_2\rangle . \tag{2}$$

Example 2. Let $|0\rangle := |v\rangle$ be a photon in a state that surely passes a polariser oriented vertically, and $|\pi/2\rangle := |h\rangle$ for a polariser oriented horizontally. Suppose we have a system of two photons, one polarised vertically and one horizontally. We denote this system as

$$|1 = v, 2 = h\rangle , \qquad \text{or more compact as} \qquad |vh\rangle .$$

Then

$$\frac{1}{\sqrt{2}}(|vh\rangle - |hv\rangle) \tag{3}$$

is also viable state by the superposition principle. Physically, the notation of this state says: *if someone observes photon 1 to pass a vertically oriented polariser, then a second person with a polariser oriented horizontally will never get photon 2 to pass his*. The state in equation 3 is called *entangled*; the two photons cannot be described independently. A more rigorous definition of entanglement follows.

Definition 1. A multi-particle state is called entangled if it cannot be expressed as a product of one-particle states.

Example 3. The two-particle state from equation 3 is entangled in the sense of definition 1.

Proof. Suppose not. Then

$$|vh\rangle - |hv\rangle = |\phi_1\rangle |\phi_2\rangle \tag{4}$$

for some one particle states $|\phi_1\rangle$ and $|\phi_2\rangle$. Since $\{|v\rangle , |h\rangle\}$[2] provides a basis for the polarisation state of a photon, we have

$$|\phi_1\rangle = a |v\rangle + b |h\rangle , \qquad \text{and} \qquad |\phi_2\rangle = c |v\rangle + d |h\rangle$$

for some complex constants a, b, c and d. Substitution gives

$$|\phi_1\rangle |\phi_2\rangle = ac |vv\rangle + ad |vh\rangle + bc |hv\rangle + bd |hh\rangle . \tag{5}$$

Then equation 4 implies $ac = 0$, $bd = 0$, $ad = 1$ and $bc = -1$. Since $ad = 1 \implies a \neq 0$ and $bc = -1 \implies c \neq 0$, we have a contradiction with $ac = 0$. Thus equation 4 does not hold after all. □

[2] This becomes clear in example 5.

Postulate 3. To every dynamical variable is associated a linear operator, and the only results of a measurement of the dynamical variable is one of the eigenvalues of the linear operator associated to it.

Example 4. The passing (or not passing) of a photon through a polariser is observable, and thus has a linear operator associated to it. If we denote with A the observable that equals 1 if a photon passed a polariser oriented at an angle θ and equals 0 if it didn't, then A has the linear operator $\hat{A}$ given by

$$\hat{A} = |\theta\rangle \langle\theta| \tag{6}$$

associated to it.

Postulate 4. If a series of measurements of a dynamical variable A is made on a system described by $|\psi\rangle$, the expectation value of A is

$$\langle\hat{A}\rangle = \frac{\langle\psi|\hat{A}|\psi\rangle}{\langle\psi|\psi\rangle}. \tag{7}$$

Example 5. Suppose we have a photon in the state $|\theta\rangle$ and a polariser A oriented vertically. By experiment, we find that the probability that the photon passes the polariser is $\langle\hat{A}\rangle = \frac{1}{2}(\cos\theta)^2$. Additionally, we find this probability for a polariser B oriented horizontally to be $\langle\hat{B}\rangle = \frac{1}{2}(\sin\theta)^2$. Since a photon in the state $|v\rangle$ surely passes polariser A, and a photon in the state $|h\rangle$ surely passes B, we conclude (using postulate 4) that the state $|\theta\rangle$ can be given by[3]

$$|\theta\rangle = \cos\theta\,|v\rangle + \sin\theta\,|h\rangle\,. \tag{8}$$

Since every polarisation state $|\theta\rangle$ can be written as a linear combination of the states $|v\rangle$ and $|h\rangle$, they provide a basis.

[3] up to a possible complex phase

Chapter 1

Introduction

One of the key principles of quantum mechanics is its indeterminacy: *When the maximum amount of information about a system is known, it is still impossible to predict all its characteristics.* This distinguishes it from any classical theory. Admittedly, classical examples exist which are described statistically. The roll of dice for example. This system is however indeterminate in a different sense. Although very difficult, if we know all the positions and momenta of the dice's particles, the table it bounces on and the air in between, and so on, we can *in principle* determine the outcome of the roll. The indeterminism of dice arises merely from our lack of knowledge about the system. It is then reasonable that many found the stronger notion of indeterminacy in quantum mechanics bothersome. This has led to the presumption that quantum mechanics is *incomplete*. There should exist some *hidden variables*, of which we are yet unaware, that *do* decide the outcome of experiments with certainty. The indeterminacy of quantum mechanics is just a mirage arising from our lack of knowledge of these hidden variables; seemingly identically prepared systems yield different results because they are ascribed different hidden variables.

1.1 The Einstein Podolsky Rosen paradox

In 1935, Einstein, Podolsky and Rosen questioned the completeness of quantum mechanics by considering a Gedankenexperiment of which I give a more didactically appropriate version[1].

Suppose we are dealing with an atom that decays and emits two photons in the state

[1] that was first discussed by David Bohm

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|vh\rangle - |hv\rangle). \tag{1.1}$$

Suppose we let the photons fly away in opposite directions (at the speed of light) until they are a certain distance apart-10 meters say, or if we like, 10 light years. We then perform a measurement on the first photon by passing it through a polariser oriented vertically. Suppose it passes. Then we immediately know that a person measuring the second photon (which could be 10 light years away) will get the state $|h\rangle$. We have now performed a measurement on the second photon (which is many light years away) even though no information could have reached it on time to inform it about the result of our measurement on photon 1. As no information travels faster than light, this violates *locality*. Being bothered by this, Einstein, Podolsky and Rosen advocated what seemed the only possible explanation for this result: the state in equation 1.1 does not describe the photons' polarisations fully, they must have had some set of instructions ascribed at their creation, which tells them how to behave.

This experiment does not try to show that quantum mechanics is wrong, merely that it is incomplete, and I think any reasonable physicist at the time would agree with their argument.

1.2 Bell's theorem

It took almost 30 years until in 1964 John Bell was able to respond to this paradox. With a clever argument he showed that *any local hidden variable theory is incompatible with the results of quantum mechanics*. His argument is as follows:

Suppose we are dealing with the atom decaying and emitting two photons in state 1.1, flying off in opposite directions as before. Suppose we have the two usual observers Alice and Bob each holding a polariser oriented at angles θ_1 and θ_2 respectively. Alice holds her polariser in the path of photon 1 and measures whether it passes or not, writing down a 1 if it does, and a 0 if not. Bob does the same with his polariser on photon 2. We call the result of one measurement for Alice A and B for Bob. We also denote the product AB of the results. After some iterations of the experiment, the results might look like this:

Alice	0	0	1	1	1	1	0	$\cdots$
Bob	1	1	1	0	1	0	0	$\cdots$
AB	0	0	1	0	1	0	0	$\cdots$

We then calculate the *average* of these products for Alice's polariser oriented at angle θ_1 and Bob's at θ_2, and call this $C(\theta_1, \theta_2)$. It is easy to find this average for the state in equation 1.1: suppose that $|\theta\rangle$ represents a photon in a state that surely passes a polariser oriented at angle θ. Let $\hat{A}$ and $\hat{B}$ be the operators associated to the observable A and B. Because Alice will write down a 1 for a photon in the state $|\theta_1\rangle$, we have

$$\hat{A} = |1 = \theta_1\rangle \langle 1 = \theta_1| \, . \tag{1.2}$$

Likewise, for Bob we have

$$\hat{B} = |2 = \theta_2\rangle \langle 2 = \theta_2| \, . \tag{1.3}$$

Then the average of their products is just the expectation value (by postulate 4)

$$C(\theta_1, \theta_2) = \langle\psi| \hat{A}\hat{B} |\psi\rangle \, . \tag{1.4}$$

Note that since $|\theta\rangle = \cos\theta\, |v\rangle + \sin\theta\, |h\rangle$, then

$$\hat{B}\,|\psi\rangle = \frac{1}{\sqrt{2}}\Big(\sin\theta_2\, |1 = v, 2 = \theta_2\rangle - \cos\theta_2\, |1 = h, 2 = \theta_2\rangle \Big). \tag{1.5}$$

Applying $\hat{A}$ gives

$$\hat{A}\hat{B}\,|\psi\rangle = \frac{1}{\sqrt{2}}\Big(\cos\theta_1 \sin\theta_2\, |1 = \theta_1, 2 = \theta_2\rangle - \cos\theta_2 \sin\theta_1\, |1 = \theta_1, 2 = \theta_2\rangle \Big). \tag{1.6}$$

Multiplying with $|\psi\rangle$ gives

$$\begin{aligned}
\langle\psi| \hat{A}\hat{B}\,|\psi\rangle &= \frac{1}{\sqrt{2}}\,\langle\psi|\,\Big(\cos\theta_1 \sin\theta_2\, |1 = \theta_1, 2 = \theta_2\rangle - \cos\theta_2 \sin\theta_1\, |1 = \theta_1, 2 = \theta_2\rangle \Big) \\
&= \frac{1}{2}\Big((\sin\theta_2 \cos\theta_1)^2 + (\sin\theta_1 \cos\theta_2)^2 - 2\cos\theta_1 \cos\theta_2 \sin\theta_1 \sin\theta_2 \Big) \\
&= \frac{1}{2}\sin(\theta_1 - \theta_2)^2.
\end{aligned}$$

Bell discovered that this result is incompatible with any local hidden variable theory, using a remarkable simple argument.

Assume that the complete state of the two photon system is described by some hidden variable w. The variable w can vary for each pair of photons and is thus distributed with some probability density. Assume also that Alice and Bob are far enough apart to ensure that the outcome of the result at Bob is independent of anything happening at Alice (this is the assumption of locality). Then there exists a function $A(\theta_1, w)$ which determines the result of Alice's measurement and a function $B(\theta_2, w)$ which does the same for Bob, such that

$$A(\theta_1, w) = 0 \text{ or } 1 \quad \text{and} \quad B(\theta_1, w) = 0 \text{ or } 1. \tag{1.7}$$

When both observers orient their polarisers the same ($\theta_1 = \theta_2$), there is a relation between the responses of the polarisers. If Alice's (resp. Bob's) photon passes, Bob's (resp. Alice's) does not, i.e.:

$$A(\theta, w) = 1 - B(\theta, w). \tag{1.8}$$

Let ρ be the probability density by which w is distributed, then

$$\int \rho(w)dw = 1 \text{ and } \rho(w) \geq 0. \tag{1.9}$$

Then the average of the products is given by

$$\begin{aligned} C(\theta_1, \theta_2) &= \int dw\rho(w)A(\theta_1, w)B(\theta_2, w) \\ &= \int dw\rho(w)A(\theta_1, w)\Big(1 - A(\theta_2, w)\Big) \end{aligned} \tag{1.10}$$

We now compare the averages for three similar experiments, but each with different orientations of the polarisers:

$$\begin{aligned} \alpha C(\theta_1, \theta_2) + \beta C(\theta_3, \theta_2) + \gamma C(\theta_1, \theta_3) = \int dw\rho(w)\Big(&\alpha A(\theta_1, w)(1 - A(\theta_2, w)) \\ &+ \beta A(\theta_3, w)(1 - A(\theta_2, w)) + \gamma A(\theta_1, w)(1 - A(\theta_3, w))\Big), \end{aligned}$$

where α, β and γ are real constants. Since each $A(\theta, w)$ is either 0 or 1, there is only eight possible evaluations of the integral (denoted by I in the table below). They are

Let $m = \min\{0, \beta, \alpha + \gamma, \alpha + \beta, \gamma\}$ and $M = \max\{0, \beta, \alpha + \gamma, \alpha + \beta, \gamma\}$. Then clearly

$$m \leq \alpha C(\theta_1, \theta_2) + \beta C(\theta_3, \theta_2) + \gamma C(\theta_1, \theta_3) \leq M. \tag{1.11}$$

If we choose $\alpha = -1$, $\beta = 1$ and $\gamma = 1$, then $m = 0$. This implies that

$$0 \leq -C(\theta_1, \theta_2) + C(\theta_3, \theta_2) + C(\theta_1, \theta_3) \tag{1.12}$$

or

$$C(\theta_1, \theta_2) \leq C(\theta_1, \theta_3) + C(\theta_3, \theta_2), \tag{1.13}$$

$A(\theta_1, w)$	$A(\theta_2, w)$	$A(\theta_3, w)$	I
0	0	0	0
0	0	1	β
0	1	0	0
0	1	1	0
1	0	0	$\alpha + \gamma$
1	0	1	$\alpha + \beta$
1	1	0	γ
1	1	1	0

Table 1.1

which is called the *Bell inequality*. It is valid for all 'values' of w.

But it is easy to find orientations of the polarisers for which the inequality is violated. For example, choose

$$\theta_1 = 0, \qquad \theta_2 = \frac{\pi}{3} \quad \text{and} \qquad \theta_3 = \frac{\pi}{6},$$

then

$$C(\theta_1, \theta_2) = \frac{1}{2}\sin(\frac{\pi}{3})^2 = \frac{3}{8} \tag{1.14}$$

and

$$C(\theta_1, \theta_3) = C(\theta_3, \theta_2) = \frac{1}{2}\sin(\frac{\pi}{6})^2 = \frac{1}{8}. \tag{1.15}$$

But the Bell inequality says

$$\frac{3}{8} \leq \frac{1}{8} + \frac{1}{8}, \tag{1.16}$$

which is clearly false. Then, under the assumption of locality, which Einstein Podolsky and Rosen advocated, the results of quantum mechanics are incompatable with hidden variables. All what was left to do is to test the above experiment. Many were (and still *are* [4]) performed in the '60's and '70's, of which the experiments from Aspect, Grangier and Roger [5] were most significant.

Finally I like to remark that in contrast to what is found in textbooks (such as Griffiths and Bransden [3]), the Bell inequality is seen to arise from the simple case-by-case distinction of table 1.

Chapter 2

The two-particle system

I find it hard to believe that $\frac{1}{\sqrt{2}}(|vh\rangle - |hv\rangle)$ is the only state bearing the ability to contradict the existence of hidden variables (as it has such grand implications for physics). It seems too great of a burden for this single state. It is then natural to ask: *can we find other states which yield a Bell inequality similar to the original one?* In this chapter we analyse all two photon states with this property. We restrict ourselves to a method *similar to Bell's*[1].

To derive an inequality for other two photon states, note that Bell's derivation uses one important fact: there is a definite relation between the responses of the polarisers $A(\theta, w) = 1 - B(\theta, w)$, which holds for all angles. A rotation of both polarisers with respect to the photon source does not affect this relation. If we too like to use any relation between the responses of the polarisers in our derivation of an equality, we must restrict ourselves to states with rotationally invariant polarisations. We will use the spin properties of a photon to find these states.

2.1 Spin and polarisation

Elementary particles, such as photons, carry an intrinsic angular momentum called spin. It so happens that photons are spin 1 particles. This spin angular momentum is due to a rotation of its electric and magnetic field. This is called the circular polarisation of the photon. The polarisation has a direction in space and is thus a vector **S** having components S_x, S_y and S_z. The vector **S** is an observable thus has a linear operator $\hat{\mathbf{S}}$ associated to it by postulate 3.

We can define the z-axis to be in the direction of propagation of the photon. It

[1]Perhaps, an approach different to Bell's could contradict hidden variables as well. I wouldn't know, and am not considering this.

is clear that if **S** points in the direction of propagation, then *the polarisation of the photon is symmetric with respect to rotations along this axis*.
The vector **S** points in this direction if it is an eigenstate of S_z (because we defined the z-axis to be along the direction of propagation). Our job is then to find the eigenstates of $\hat{S}_z$.
The reader may be used to having $2s + 1$ eigenvalues for $\hat{S}_z$, but this is only true for massive particles. For *massless* particles, the eigenvalue 0 is excluded: suppose we have a photon in the spin eigenstate of $\hat{S}_z$ with eigenvalue 0. Then the photon is invariant under rotations around any axis perpendicular to the $z - axis$. This is only possible if the momentum of the photon is zero in its rest frame, because if the momentum was non-zero, any rotation will change the direction of the momentum. For massive particles, we can always find a rest frame. For massless particles, which propagate ate the speed of light, we cannot. Thus the eigenstate of $\hat{S}_z$ with eigenvalue 0 is excluded.

To find the eigenstates of $\hat{S}_z$, we use the technique of lowering and raising operators.

Notation 1. *We call the component of the total spin in the direction of propagation of the photon its helicity. For example, if a photon is an eigenstate of $\hat{S}_z$, then its helicity is either $-\hbar$ or $+\hbar$. We denote a state with total spin s and helicity $\hbar\lambda$ as $|s, \lambda\rangle$. In addition, to make the effect of the lowering operator more clear, we denote $|1, -1\rangle = |-\rangle$ and $|1, +1\rangle = |+\rangle$.*

2.2 The eigenstates of $\hat{S}_z$

Definition 2. The lowering operator S_- i defined as

$$\hat{S}_- := |-\rangle \langle +| \,. \tag{2.1}$$

For we are dealing with a two photon system, and each photon has a helicity of either $-\hbar$ or $+\hbar$, we have the possibilities $\lambda = -2, 0, +2$. Then $s = 0, 2$ (because helicities *add*). Then the total spin $s = 0, 2$ (the helicity can never exceed the total spin, as it is its component along the z-axis).

$$|2, +2\rangle = |++\rangle \,. \tag{2.2}$$

Then, applying the lowering operator to $|++\rangle$, we can find the next eigenstate

$|2,0\rangle$.

$$\begin{aligned}
|2,0\rangle &= \hat{S}_- |{+}{+}\rangle \\
&= (\hat{S}_- |+\rangle) |+\rangle + |+\rangle (\hat{S}_- |+\rangle) \\
&= |{-}{+}\rangle + |{+}{-}\rangle \\
&\to 1/\sqrt{2}(|{-}{+}\rangle + |{+}{-}\rangle) \text{ after normalisation,}
\end{aligned} \tag{2.3}$$

where we have used the Leibniz rule for a linear operator on a product state. Applying the lowering operator again we find

$$\begin{aligned}
|2,-2\rangle &= 1/\sqrt{2}\hat{S}_-(|{-}{+}\rangle + |{+}{-}\rangle) \\
&= 1/\sqrt{2}((\hat{S}_- |-\rangle) |+\rangle + |-\rangle (\hat{S}_- |-\rangle) + (\hat{S}_- |+\rangle) |-\rangle + |+(\hat{S}_- |-\rangle))\rangle \\
&= 2/\sqrt{2}\, |{-}{-}\rangle \\
&\to |{-}{-}\rangle \text{ after normalisation.}
\end{aligned} \tag{2.4}$$

We can rewrite these states using $|+\rangle = 1/\sqrt{2}(|v\rangle + i\,|h\rangle)$ and $|-\rangle = 1/\sqrt{2}(|v\rangle - i\,|h\rangle)$. Then

$$\begin{aligned}
|2,+2\rangle &= 1/2(|v\rangle + i\,|h\rangle)(|v\rangle + i\,|h\rangle) \\
&= 1/2(|vv\rangle + i\,|vh\rangle + i\,|hv\rangle - |hh\rangle) \\
|2,\quad 0\rangle &= \sqrt{2}/2(|v\rangle - i\,|h\rangle)(|v\rangle + i\,|h\rangle) + \sqrt{2}/2(|v\rangle + i\,|h\rangle)(|v\rangle - i\,|h\rangle) \\
&= 1/\sqrt{2}(|vv\rangle + |hh\rangle) \text{ after normalisation} \\
|2,-2\rangle &= 1/2(|v\rangle - i\,|h\rangle)(|v\rangle - i\,|h\rangle) \\
&= 1/2(|vv\rangle - i\,|vh\rangle - i\,|hv\rangle - |hh\rangle)
\end{aligned} \tag{2.5}$$

To find the last state $|0,0\rangle$, there is no use in the lowering operator. We use the closure relation

$$\hat{I} = |2,+2\rangle \langle 2,+2| + |2,0\rangle \langle 2,0| + |2,-2\rangle \langle 2,-2| + |0,0\rangle \langle 0,0| \tag{2.6}$$

instead.

We can write the closure relation in vector notation with respect to the ordered basis $(|vv\rangle, |vh\rangle, |hv\rangle, |hh\rangle)$ as

$$\hat{I} = \begin{pmatrix} 1/2 \\ i/2 \\ i/2 \\ -1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ -i/2 \\ -i/2 \\ -1/2 \end{pmatrix}^T + \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{pmatrix}^T + \begin{pmatrix} 1/2 \\ -i/2 \\ -i/2 \\ -1/2 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ +i/2 \\ +i/2 \\ -1/2 \end{pmatrix}^T + |0,0\rangle\langle 0,0|$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + |0,0\rangle\langle 0,0|.$$

Then

$$|0,0\rangle\langle 0,0| = \hat{I} - \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.7}$$

We have now found the missing rotationally invariant state

$$|0,0\rangle = 1/\sqrt{2}(|vh\rangle - |hv\rangle). \tag{2.8}$$

Of course, this is the state we *should* have found, as it is the state used in Bell's original inequality. Though it is still nice to recover it in the end.
We are now done finding all rotationally invariant states and can move on in our attempt at deriving more Bell inequalities.

2.3 The state $|vv\rangle + |hh\rangle$

From the four states

$$|2,+2\rangle = |+\rangle\,|+\rangle\,, \qquad |2,0\rangle = 1/\sqrt{2}(|vv\rangle + |hh\rangle)$$
$$|2,-2\rangle = |-\rangle\,|-\rangle\,, \qquad |0,0\rangle = 1/\sqrt{2}(|vh\rangle - |hv\rangle)$$

only $|2,0\rangle$ and $|0,0\rangle$ are entangled, because $|2,+2\rangle = |+\rangle\,|+\rangle$ and $|2,-2\rangle = |-\rangle\,|-\rangle$ are products of one particle states. We have already found an inequality for $|0,0\rangle$ in Chapter 1. Thus, to complete the picture, we must derive an inequality for $|2,0\rangle$.

We approach in a similar manner as in Chapter 1. From equation 1.10 we have for the average of the responses of the polarisers

$$C(\theta_1,\theta_2) = \int dw\rho(w)A(\theta_1,w)B(\theta_2,w). \tag{2.9}$$

Again, there exists a relation between the responses of the polarisers. For the state $(1/\sqrt{2})(|vv\rangle + |hh\rangle)$, it is clear that

$$A(\theta,w) = B(\theta,w). \tag{2.10}$$

Substituting this into equation 2.9 gives

$$C(\theta_2,\theta_2) = \int dw\rho(w)A(\theta_1,w)A(\theta_2,w). \tag{2.11}$$

To find any non trivial relation between the averages C for three similar setups of the experiment, we examine the linear combination

$$\begin{aligned} L &:= \alpha C(\theta_1,\theta_2) + \beta C(\theta_2,\theta_3) + \gamma C(\theta_1,\theta_3) \\ &= \int dw\rho(w)\Big(\alpha A(\theta_1,w)A(\theta_2,w) + \beta A(\theta_2,w)A(\theta_3,w) + \gamma A(\theta_1,w)A(\theta_3,w)\Big). \end{aligned}$$

Since each $A(\theta,w)$ is either 0 or 1, all possible evaluations of the integral L are

$A(\theta_1,w)$	$A(\theta_2,w)$	$A(\theta_3,w)$	L
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	β
1	0	0	0
1	0	1	γ
1	1	0	α
1	1	1	$\alpha+\beta+\gamma$

Let $m = \min\{0, \alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ and $M = \max\{0, \alpha, \beta, \gamma, \alpha + \beta + \gamma\}$. Then

$$m \le \alpha C(\theta_1, \theta_2) + \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3) \le M \tag{2.12}$$

must hold. If we can find α, β and γ such that inequality 2.12 is violated for some orientation of the polarisers, then we have found a new Bell inequality. Unfortunately, this is impossible:

Proposition 1. *Suppose $C : [a, b] \times [a, b] \to [0, \frac{1}{2}]$ and suppose α, β, γ are real numbers. Let $M = \max\{0, \alpha, \beta, \gamma, \alpha + \beta + \gamma\}$ and $m = \min\{0, \alpha, \beta, \gamma, \alpha + \beta + \gamma\}$. Then the inequality*

$$m \le \alpha C(\theta_1, \theta_2) + \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3) \le M$$

is always satisfied.

Proof. I prove by case distinction on α, β, γ. Suppose $\alpha, \beta, \gamma \ge 0$. Then $m = 0$ and $M = \alpha + \beta + \gamma$. Since $0 \le C$, we have

$$m = 0 \le \alpha C(\theta_1, \theta_2) + \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3),$$

thus the lower bound holds. Since $C \le 1/2$, we have

$$\alpha C(\theta_1, \theta_2) + \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3) \le 1/2(\alpha + \beta + \gamma) \le \alpha + \beta + \gamma = M,$$

thus the upper bound also holds. For the case $\alpha, \beta, \gamma \le 0$, just use $(\alpha, \beta, \gamma) \mapsto (-\alpha, -\beta, -\gamma)$. Then the inequality signs reverse and $(m, M) \mapsto (M, m)$, which proves the inequality by the argument above.

Now suppose $\alpha \le 0$ and $\beta, \gamma \ge 0$. Then $m = \alpha$. As $C \le 1/2$, we have

$$\alpha \le 0 \Rightarrow \alpha C(\theta_1, \theta_2) \ge \alpha/2 \ge \alpha.$$

Since $\beta, \gamma \ge 0$, the remaining terms $\beta C(\theta_2, \theta_3)$ and $\gamma C(\theta_1, \theta_3)$ are positive or 0. Then the lower bound follows. For the upper bound, notice that

$$\begin{aligned} \alpha C(\theta_1, \theta_2) + \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3) &\le \beta C(\theta_2, \theta_3) + \gamma C(\theta_1, \theta_3) \\ &\le \frac{\beta + \gamma}{2} \\ &\le \max\{\beta, \gamma\} \\ &\le M. \end{aligned}$$

To see this last inequality, notice that $\{\beta, \gamma\} \subseteq \{0, \alpha, \beta, \gamma, \alpha + \beta + \gamma\}$, and adding extra terms to a set only increases its maximum. Thus the upper bound holds.

Lastly, notice that the symmetry of the situation allows us to prove all other cases under correct permutation of α, β, γ. $\square$

2.4 A Bell inequality for $|vv\rangle + |hh\rangle$

It seems now as if we are out of luck and are unable to find any new Bell inequality. I claim, however, that this is still possible in a suprisingly simple way.

Instead of denoting a 1 for a photon passing polariser B, and a 0 for not passing, we reverse our notation:

$$A = \begin{cases} 1 & \text{if photon 1 passes Alice's polariser,} \\ 0 & \text{if not.} \end{cases}$$

and

$$\tilde{B} = \begin{cases} 1 & \text{if photon 1 does not pass Bob's polariser,} \\ 0 & \text{if it does.} \end{cases} \quad .$$

Suppose we now calculate the average of the product $A\tilde{B}$, which we denote by $\tilde{C}(\theta_1, \theta_2)$. It seems at first as if this is just a notation change, but it is not: The relation between $A(\theta, w)$ and $\tilde{B}(\theta, w)$ is

$$A(\theta, w) = 1 - \tilde{B}(\theta, w). \tag{2.13}$$

Then

$$\tilde{C}(\theta_1, \theta_2) = \int dw \rho(w) A(\theta_1, w) \tilde{B}(\theta_2, w)$$
$$\int dw \rho(w) A(\theta_1, w)\Big(1 - A(\theta_2, w)\Big),$$

which is exactly the same as equation 1.10. We then derive the new Bell inequality

$$\tilde{C}(\theta_1, \theta_2) \leq \tilde{C}(\theta_1, \theta_3) + \tilde{C}(\theta_3, \theta_1) \tag{2.14}$$

by the same arguments as in Chapter 1.

This concludes Chapter 2: we have fully analysed the two particle case by characterising all rotationally invariant entangled states. The only states with these properties are the special states $|vh\rangle - |hv\rangle$ and $|vv\rangle + |hh\rangle$, and both yield a Bell inequality.

Chapter 3

The n-particle system

3.1 A system of n particles

It is tempting to try to derive inequalities for an n-particle state in a similar manner as we did for a 2-particle state. We must then, as before, find states which

(1) are rotationally invariant under change of the parameters θ_i's, i.e. being invariant under the function which maps $\theta_i \mapsto \theta_i + \phi$ for all i, and

(2) yield a non-trivial relation between the responses A_i of the polarizers.

A state satisfying condition (1) must be a spin eigenstate. We can expand a spin eigenstate $|s\rangle$ with total spin s as a linear combination of the form

$$|s\rangle = \sum_{\substack{p_{ij} \in \{+,-\} \\ \sum \chi(p_{ij}) = s}} c_{si} \left| p_{i1} p_{i2} \ldots p_{in} \right\rangle , \tag{3.1}$$

where each c_{si} is some complex constant. The function χ maps $+ \mapsto (+1)$ and $- \mapsto (-1)$. The first constraint on the sum means that each term $|p_{i1}p_{i2}\ldots p_{in}\rangle$ in the expansion is some product of length n consisting of $|+\rangle$-states or $|-\rangle$-states. The second constraint on the sum ensures the total helicity of each term equals s.

Concerning condition (2), it seems at first as if there exist many relations between the A_i's. However, since each $A_i = 1$ or 0, and since we can always relabel the indices of the particles, the only relations eligible are of the form

$$(A_1 = A_2 = \ldots = A_k) \neq (A_{k+1} = A_{k+2} = \ldots = A_n), \tag{3.2}$$

for some $1 \leq k \leq n$. Thus a state satisfying (2) reduces to the form

$$|n,k\rangle = a_1 |\underbrace{vv...v}_{\text{k times}} \underbrace{hh...h}_{\text{n-k times}}\rangle + a_2 |\underbrace{hh...h}_{\text{k times}} \underbrace{vv...v}_{\text{n-k times}}\rangle \,, \tag{3.3}$$

which has just two complex (non-zero) parameters a_1 and a_2. We shall call such states H-states.

Question. Can a state that satisfies (2) also satisfy (1)? That is, does there exist an s and a set of constants c_{si} such that

$$|nk\rangle = |s\rangle ? \tag{3.4}$$

To make this more clear, I give an example for a 3 particle state.

Example. Consider a 3 particle system, and suppose the responses of the polarizers A_1, A_2 and A_3 relate as $(A_1) \neq (A_2 = A_3)$ (thus $n = 3$ and $k = 1$). Then

$$|nk\rangle = a_1 |vhh\rangle + a_2 |hvv\rangle \,. \tag{3.5}$$

To check whether this is a pure spin eigenstate (checking if the condition $|nk\rangle = |s\rangle$ holds for some s), we would like to get rid of the v's and h's and expand the state in only $+$'s and $-$'s. That is, mapping $v \mapsto |+\rangle + |-\rangle$ and $h \mapsto (1/i)(|+\rangle - |-\rangle)$. You could to this by hand or, as I did, use some code[1]. The expansion yields

$$\begin{aligned} |n=3,k=1\rangle = (-a_1 + ia_2) & |- - -\rangle \\ & + (-a_1 - ia_2) |+ - -\rangle + (a_1 + ia_2) |- + -\rangle + (a_1 + ia_2) |- - +\rangle \\ & + (a_1 - ia_2) |+ + -\rangle + +(a_1 - ia_2) |+ - +\rangle + (-a_1 + ia_2) |+ - -\rangle \\ & + (-a_1 - ia_2) |+ + +\rangle \end{aligned} \tag{3.6}$$

If we want this state to be a pure spin eigenstate, we must have that (all) the coefficient(s) for all of the spin eigenstates in the expansion vanish, except for one spin eigenstate. Clearly, if the coefficient $(-a_1 - ia_2)$ for the $|+ + +\rangle$-state vanishes, we must have that all the coefficients for the $|+ - -\rangle$, $|- + -\rangle$ and $|- - +\rangle$ states also vanish. Similarly, if the coefficient $(-a_1 + ia_2)$ for the $|- - -\rangle$-state vanishes, we must have that all the coefficients for the $|+ + -\rangle$, $|+ - +\rangle$ and $|- + +\rangle$ states also vanish. Thus we cannot write $|n,k\rangle$ as a pure spin eigenstate. We conclude that for a 3 particle system satisfying $(A_1) \neq (A_2 = A_3)$, there is no rotationally invariant H-state.

[1]The Maple code used is found in the appendix.

We could check many more examples for different values of k and n, and check whether they yield pure spin eigenstates. I tried this for all n and k, in the range $3 \leq n \leq 6$, $1 \leq k \leq n$, but the result of the example remains unchanged: no state satisfying (5) is a spin eigenstate. However, there exists a pattern in the coefficients. Of course *some* pattern ought exist, as the expansion in $+$'s and $-$'s is a straightforward one. Fortunately, the pattern is simple and useful:

Proposition 2. *Let $c_{s1}, c_{s2}, ..., c_{sl}$ be the coefficients in the expansion of $|nk\rangle = \sum |s\rangle$, where $l = \binom{n}{(n+s)/2}$. Then for every s*

$$c_{s1} = 0 \iff c_{s2} = 0 \iff ... \iff c_{sl} = 0. \tag{3.7}$$

Proof. We can write one of the coefficients c_{si} as

$$c_{si} = \langle p_{1i}p_{2i}... + ... - ...p_{ni}| \, (a_1 \, |\underbrace{vv...v}_{\text{k times}} \underbrace{hh...h}_{\text{n-k times}}\rangle + a_2 \, |\underbrace{hh...h}_{\text{k times}} \underbrace{vv...v}_{\text{n-k times}}\rangle), \tag{3.8}$$

where $\langle p_{1i}p_{2i}... + ... - ...p_{ni}|$ is some product of x $\langle +|$-bras and y $\langle -|$-bras, such that $x + y = s$. Since all the different arrangements of the p_{ij}'s occur in the expansion of $|s\rangle$, there exists a coefficient c_{sj} which is identical to c_{si} except for the helicities of two particles, which are switched. As before, we can write down this c_{sj} as

$$c_{sj} = \langle p_{1j}p_{2j}... - ... + ...p_{nj}| \, (a_1 \, |\underbrace{vv...v}_{\text{k times}} \underbrace{hh...h}_{\text{n-k times}}\rangle + a_2 \, |\underbrace{hh...h}_{\text{k times}} \underbrace{vv...v}_{\text{n-k times}}\rangle), \tag{3.9}$$

where $\langle p_{1i}p_{2i}... - ... + ...p_{ni}|$ is another such product, in which the helicity of one pair of particles is switched with respect to the bra $\langle p_{1i}p_{2i}... + ... - ...p_{ni}|$. By switching the helicity of several pairs of particles consecutively, we can obtain all the l separable states with total spin s. Moreover, this ensures the total spin s of the separable state to remain unchanged. The switching of helicities between particles u and w can occur in several cases

(1) if $u, w \leq k$, then we are able to compute c_{si}more explicitly as

$$\begin{aligned} c_{si} = {} & a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ... \langle -|v\rangle ... \langle +|v\rangle ... \langle p_{nj}|h\rangle \\ & + a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ... \langle -|h\rangle ... \langle +|h\rangle ... \langle p_{nj}|v\rangle . \end{aligned}$$

To find c_{sj}, we switch the two $+$'s and the two $-$'s.

$$\begin{aligned}
c_{sj} &= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ... \langle +|v\rangle ... \langle -|v\rangle ... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ... \langle +|h\rangle ... \langle -|h\rangle ... \langle p_{nj}|v\rangle \\
&= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ... \langle -|v\rangle ... \langle +|v\rangle ... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ... \langle -|h\rangle ... \langle +|h\rangle ... \langle p_{nj}|v\rangle \\
&= c_{si},
\end{aligned}$$

since multiplication is commutative.

(2) if $u, w \geq k$ we have $c_{sj} = c_{si}$ by the argument above.

(3) if $u \leq k$ and $w > k$, then

$$\begin{aligned}
c_{si} &= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ... \langle -|v\rangle ... \langle +|h\rangle ... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ... \langle -|h\rangle ... \langle +|v\rangle ... \langle p_{nj}|v\rangle \\
&= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ...(1)...(-i)... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ...(i)...(1)... \langle p_{nj}|v\rangle ,
\end{aligned}$$

while

$$\begin{aligned}
c_{sj} &= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ... \langle +|v\rangle ... \langle -|h\rangle ... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ... \langle +|h\rangle ... \langle -|v\rangle ... \langle p_{nj}|v\rangle \\
&= a_1 \langle p_{1j}|v\rangle \langle p_{2j}|v\rangle ...(1)...(i)... \langle p_{nj}|h\rangle \\
&+ a_2 \langle p_{1j}|h\rangle \langle p_{2j}|h\rangle ...(-i)...(1)... \langle p_{nj}|v\rangle \\
&= -c_{si}.
\end{aligned}$$

(4) if $w \leq k$ and $u > k$, then again $c_{si} = -c_{sj}$ by the argument above.

Then $c_{si} = \pm c_{sj}$, which implies $c_{si} = 0 \iff c_{sj}$, proving the proposition. □

There is a second pattern concerning successive coefficients in the expansion of $|n, k\rangle$ in spin eigenstates $|s\rangle$:

Proposition 3. *The coefficient* $c_{si} = 0$ *if and only if* $c_{s+4,i} = 0$.

Proof. To obtain $c_{s+4,i}$ from c_{si} we must raise the helicity of two particles from a $|-\rangle$ to a $|+\rangle$. That is

$$c_{si} = \langle p_{1i}p_{2i}... - ... - ...p_{ni}| \left(a_1 |vv...vhh...h\rangle + a_2 |hh...hvv...v\rangle\right), \tag{3.10}$$

and

$$c_{s+4,i} = \langle p_{1i}p_{2i}... + ... + ...p_{ni}| \left(a_1 |vv...vhh...h\rangle + a_2 |hh...hvv...v\rangle\right) \tag{3.11}$$

Suppose we raise the helicities of particle u and w. This raising can occur in several cases

- if $u, w \leq k$, then

$$\begin{aligned} c_{si} &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ... \langle -|v\rangle ... \langle -|v\rangle ... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ... \langle -|h\rangle ... \langle -|h\rangle ... \langle p_{nj}|v\rangle \\ &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ...(1)...(1)... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ...(i)...(i)... \langle p_{nj}|v\rangle , \end{aligned}$$

  and

$$\begin{aligned} c_{s+4,i} &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ... \langle +|v\rangle ... \langle +|v\rangle ... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ... \langle +|h\rangle ... \langle +|h\rangle ... \langle p_{nj}|v\rangle \\ &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ...(1)...(1)... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ...(-i)...(-i)... \langle p_{nj}|v\rangle \\ &= c_{si}. \end{aligned}$$

- if $u, w \geq k$, then we have $c_{si} = c_{s+4,i}$ by the argument above.

- if $u \leq k$ and $w > k$, then

$$\begin{aligned} c_{si} &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ... \langle -|v\rangle ... \langle -|h\rangle ... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ... \langle -|h\rangle ... \langle -|v\rangle ... \langle p_{nj}|v\rangle \\ &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle ...(1)...(i)... \langle p_{ni}|h\rangle \\ &\quad + a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle ...(i)...(1)... \langle p_{nj}|v\rangle , \end{aligned}$$

and

$$\begin{aligned}
c_{s+4,i} &= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle \ldots \langle +|v\rangle \ldots \langle +|h\rangle \ldots \langle p_{ni}|h\rangle \\
&+ a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle \ldots \langle +|h\rangle \ldots \langle +|v\rangle \ldots \langle p_{nj}|v\rangle \\
&= a_1 \langle p_{1i}|v\rangle \langle p_{2i}|v\rangle \ldots (1) \ldots (-i) \ldots \langle p_{ni}|h\rangle \\
&+ a_2 \langle p_{1i}|h\rangle \langle p_{2i}|h\rangle \ldots (-i) \ldots (1) \ldots \langle p_{nj}|v\rangle \\
&= -c_{si}.
\end{aligned}$$

- if $w \leq k$ and $u > k$, then again $c_{si} = -c_{s+4,i}$ by the argument above.

In all the cases we have $c_{si} = \pm c_{s+4,i}$, which proves $c_{si} = 0$ if and only if $c_{s+4,i} = 0$. □

We now have the tools (being the two propositions) to answer the question: Does there exist an s and a set of constants c_{si} such that

$$|n,k\rangle = |s\rangle? \tag{3.12}$$

Answer. Note that we can always expand $|n,k\rangle$ as

$$|n,k\rangle = \sum_{s \in \{-n,-n+2,\ldots,n\}} |s\rangle\,. \tag{3.13}$$

For $|n,k\rangle$ to be a pure spin eigenstate, the following condition must hold:

> *For every $s \in \{-n,-n+2,...,n\}$ except one, we must have for every $i \in \{1,...\binom{n}{(n+s)/2}\}$ that $c_{si} = 0$.*

The first proposition ensures that the part "*for every $i \in \{1,...\binom{n}{(n+s)/2}\}$*" in this condition is always satisfied, since if we choose an s, we have $c_{si} = 0$ if and only if $c_{sj} = 0$ for every i and j. This simplifies the condition to

> *for every $s \in \{-n,-n+2,...,n\}$ except one, we must have $c_{si} = 0$ for some i.*

Note that we have $n+1$ different values for s, as $s \in \{-n,-n+2,...,n\}$. Suppose $n > 2$. Then we have 4 or more different values for s. Now using the second proposition we derive:

> *Suppose $c_{si} = 0$ for every s but one, namely $c_{zi} \neq 0$. Then clearly $c_{z\pm4,i} = 0$ as well. Surely, $c_{z\pm4,i}$ exists since we have more than 4 values for s. But the second proposition says precisely; if $c_{z\pm4,i} = 0$ then $c_{z,i} = 0$. This is a contradiction.*

We conclude that for a system of more than 2 particles, there is no H-state which is (1) rotationally invariant and (2) satisfies nontrivial relations for the responses of the polarisers.

If however $n \leq 2$, the coefficient $c_{z\pm4,i}$ does not exist, since we only have 3 or less values for s, and the argument above fails. This shows why a Bell inequality for a two particle experiment is not excluded by the propositions in this chapter.

Afterword

We have concluded Chapter 2 with all possible Bell inequalities for a two photon experiment: there are just two of them. This makes the states $|vh\rangle - |hv\rangle$ and $|vv\rangle + |hh\rangle$ quite remarkable.

In the final chapter, we found that H-states with more than two particles are never rotationally invariant. I therefore consider $|vh\rangle - |hv\rangle$ and $|vv\rangle + |hh\rangle$ as rather important.

I find it important to emphasise that statements such as '*having no Bell inequality*', must be read as having no inequality if we use a similar method as Bell originally did; the reader must not assume that this is the only way local hidden variables are ruled out. Most likely, there exist *some* argumentation which makes local hidden variables incompatible with quantum mechanical results.

Maple code

In Chapter 4, we proved in propositions 2 and 3 that there exists a pattern in the coefficients c_{si}. This pattern might strike the reader as being out of the blue, and he/she might wonder how it was found. It was *discovered* while trying to check whether it is possible to find states which hold a non trivial relation between the responses of the polarisers could be rotationally invariant. Since doing this for more than 3 particles becomes labourious, we used Maple. The following code was written by Ronald Kleiss and edited by me.

As an example, we check the conditions (1) and (2) from Chapter 4, for the 4 particle state

$$a_1 \left|vhhh\right\rangle + a_2 \left|hvvv\right\rangle . \tag{14}$$

```
restart;
a := a1*ket([v, h, h, h])+a2*ket([h, v, v, v]);
```

We define four procedures which map the one particle states of a as

$$\left|h\right\rangle \mapsto i(\left|-\right\rangle - \left|+\right\rangle), \qquad \left|v\right\rangle \mapsto \left|+\right\rangle + \left|-\right\rangle .$$

We then apply the procedure to a and expand the result.

```
w1:=proc(ketje);
   if op(1,ketje)=h then return (-I)*ket(subsop(1=p,ketje))
  + I*ket(subsop(1=m,ketje)) fi;
   if op(1,ketje)=v then return ket(subsop(1=p,ketje))
  + ket(subsop(1=m,ketje)) fi;
end proc;

w2:=proc(ketje);
   if op(2,ketje)=h then return (-I)*ket(subsop(2=p,ketje))
  + I*ket(subsop(2=m,ketje)) fi;
   if op(2,ketje)=v then return ket(subsop(2=p,ketje))
```

```
  + ket(subsop(2=m, ketje)) fi;
end proc;

w3:=proc(ketje);
   if op(3,ketje)=h then return (-I)*ket(subsop(3=p, ketje))
  + I*ket(subsop(3=m, ketje)) fi;
   if op(3,ketje)=v then return ket(subsop(3=p, ketje))
  + ket(subsop(3=m, ketje)) fi;
end proc;

w4:=proc(ketje);
   if op(4,ketje)=h then return (-I)*ket(subsop(4=p, ketje))
  + I*ket(subsop(4=m, ketje)) fi;
   if op(4,ketje)=v then return ket(subsop(4=p, ketje))
  + ket(subsop(4=m, ketje)) fi;
end proc;

b:=expand(eval(subs(ket=w4, eval(subs(ket=w3,
eval(subs(ket=w2, eval(subs(ket=w1,a)))))))));
```

Finally, we ask Maple to evaluate the coefficients for each term in the expansion such that coefficients with the same helicity are displayed together.

```
coeff(b, ket([m, m, m, m]));
                                  a1+a2
coeff(b, ket([p, m, m, m])); coeff(b,ket([m,p,m,m]));
coeff(b, ket([m, m, p, m])); coeff(b, ket([m, m, m, p]));
                                a1 - a2
                                a1 - a2
                                a1 - a2
                                a1 - a2
coeff(b, ket([p, p, m, m])); coeff(b,ket([p, m, p, m]));
coeff(b, ket([p, m, m, p])); coeff(b, ket([m, p, p, m]));
coeff(b, ket([m, p, m, p])); coeff(b,ket([m,m,p,p]));


                                a1 + a2
                                a1 + a2
                                a1 + a2
                                a1 + a2
                                a1 + a2
                                a1 + a2
```

```
coeff(b, ket([m, p, p, p])); coeff(b, ket([p, m, p, p]));
coeff(b, ket([p, p, m, p])); coeff(b, ket([p, p, p, m]));
                                    a1 - a2
                                    a1 - a2
                                    a1 - a2
                                    a1 - a2
coeff(b, ket([p, p, p, p]));
                                    a1 + a2
```

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