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Universality classes of 2D hyperbolic Riemannian manifolds

And analogies with universality classes of planar maps

Bachelorthesis Physics

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1 Motivation

One of the big open problems we are currently facing in physics is the unification of Quantum Mechanics and General Relativity. The problem lies in getting a description of gravity that is consistent with both theories.
Quantum mechanics describes the electromagnetic, strong and weak interaction forces as fields, and states that a particle is described by a wave function [1]. The probability of the particle being in a specific place is given by the square of the wave function. The particle is thus described by a superposition of states, meaning that quantum mechanics is a non-deterministic theory, as one can not predict the particle's trajectory [2].
General relativity on the other hand describes the fourth force: gravity. It states that the universe has four dimensions: three space dimensions and one time dimension [3]. Gravity is described as the curvature of this space-time by energy, and therefore also mass. For example, heavy objects such as stars curve space-time leading to the trajectories of light seeming to curve around the star [4]. The geometry of space-time is therefore dependent on the matter fields described by quantum field theories. We know general relativity is not complete, as it is a theory with deterministic equations of motions [5].
On the one hand, we have a non-deterministic theory, quantum field theory, and on the other hand, we have a deterministic theory, General relativity. Both theories have a high observational strength [6, p. 1-24]. From quantum field theory, for example, vacuum polarization has been detected [7] [8]. From general relativity gravitational waves have been detected by LIGO [9]. The unification of these two theories is still an open problem, especially in four dimensions. One possibility for unifying these theories is by finding a quantum field theory that describes the geometry of spacetime (with or without matter). This can be done by creating a quantum mechanical description of the geometry of spacetime. More specifically, the superposition from quantum mechanics can be interpreted as the probability distribution of the geometry of your space. This field of research is called "Random geometry".

1.1 Random geometry

As there is a probability of your spacetime being in one state or another, or specifically, to have one geometry or another, we need to deal with this randomness of your spacetime. This can be done analogously to the principles used in the path integral for trajectories of particles. This principle can then be generalized for quantum fields. Combining this with the Einstein-Hilbert action we can introduce the partition function for random geometry. From this partition function, one can define a probability distribution on the geometry of the space.

1.1.1 Path integrals

The path integral is based on the quantum mechanical superposition and is dependent on the action of the system. Firstly, note that from classical mechanics we have Hamilton's principle, which states that the path a particle follows between points 1 and 2, in the time interval $[t_1, t_2]$ is such that the action integral

$$S = \int_{t_1}^{t_2} \mathcal{L}\ dt \tag{1}$$

is stationary. Where $\mathcal{L}$ is the Lagrangian of the system. This principle leads to the Euler-Lagrange equations, which we can use to describe the path of the particle [10].
In quantum mechanics we can not just take one trajectory, as particles are described by superpositions. We therefore introduce the path integral. The path integral is a generalization of the classical mechanical action $S[a, b]$ principle [2, p. 503-511]. Let $P(a, b)$ be the probability density of going from point a to b in the time interval $[t_a, t_b]$. Let $K(a, b)$ be the total amplitude for traveling from a to b, where $|K(a, b)|^2 = P(a, b)$. Each trajectory $x(t)$ contributes to the total amplitude, but

only at specific phases, leading to the formulation $K(a,b) = \sum_{x(t) \text{ path from } a \text{ to } b} \phi\,[x(t)]$. Meaning ϕ determines the contribution of a path. It is defined by $\phi\,[x(t)] = c \cdot e^{(i/\hbar)S[x(t)]}$ with $c \in \mathbb{R}$ [11, p. 31-39]. This awkward notation of the total amplitude is usually replaced by

$$K(a,b) = \int_a^b e^{(i/\hbar)S[x(t)]}\mathcal{D}x(t), \tag{2}$$

where the functional integral $\int_a^b \mathcal{D}x(t)$ is the notation for integrating over all paths [12, p. 275-314].
Quantum field theory builds upon this idea of path integrals. The probability amplitude of a state ϕ_a of a field at time a to evolve to a state ϕ_b at time b is defined by

$$K(a,b) = \int_a^b e^{(i/\hbar)S[\phi(t)]}\mathcal{D}\phi(t). \tag{3}$$

This means that we define the probability amplitude by summing over all fields ([1] page 105-127). In equation (3), we have looked at the time evolution of the field. Note however that fields are dependent on your manifold, because they are functions from your manifold M to $\mathbb{R}$. This means that the manifold/space is fixed and the field is defined on points of the manifold [6, p. 39-55].

1.1.2 Einstein-Hilbert action

Before we can consider the quantum gravity partition function, we need to also take a look at the action from General Relativity. This action is called the Einstein-Hilbert action and give us the Einstein field equations [3, p. 123-139]. It is given by

$$S_{\mathrm{EH}} = \frac{c^2}{16\pi G}\int \sqrt{-g}\,(R - 2\Lambda)\,d^4x, \tag{4}$$

where the integral is over the whole spacetime, c is the speed of light, G is the gravitational constant, g is the determinant of the metric tensor, R is the Ricci scalar and Λ is the cosmological constant.
At least formally, we can take the path integral approach to get a partition function. This time we sum over the possible metrics [13, p. 68-75]

$$Z = \int e^{(i/\hbar)S_{\mathrm{EH}}[g]}\mathcal{D}g. \tag{5}$$

Equation (5) tries to therefore formulate a quantum field theory of the metric, meaning it unifies General Relativity and Quantum Field Theory. However, it does not contain matter fields ϕ, meaning it unifies the two theories without matter.
In equation (5), we are not considering functions on a manifold, as done in equation (3). Rather, the partition function (5) is a function over the metric tensor fields *of* your manifold, meaning that the geometry of M is not fixed. In equation (3), the geometry is fixed.

1.1.3 Quantum gravity partition function with matter

As we want to unify general relativity and quantum field theory with matter, we need to consider an action that incorporates equations (5) and (3). In the most general form, we can define the partition function by [14, p. 3-10]

$$Z = \int e^{(i/\hbar)S[g,\phi]}\mathcal{D}g\mathcal{D}\phi. \tag{6}$$

Meaning we integrate over the possible fields ϕ *on* your manifold and metrics g *of* your manifold. Where the phase of the contribution for each possible pair of quantum field and metric is defined by the action. We can use this partition function to try to define a probability distribution on the space

of fields and geometries.
The most difficult part of equation (6) is the summing over all metrics. We therefore want to consider models that simplify this part of the equation. One simplification is only considering two-dimensional spacetime, instead of the more realistic four-dimensional spacetime. However, we are still working with an infinite-dimensional space of geometries when we consider two-dimensional spacetime, as one can locally vary the geometry at every point of the surface. Mathematical analysis for infinite-dimensional space is very difficult, so we need to introduce methods to simplify the problem.

1.2 Discrete two-dimensional theories

Discrete two-dimensional theories for quantum gravity are based on regularization. The idea is to create models for finite discrete surfaces which in the limit transform into models for compact continuous surfaces [15]. These finite discrete surfaces are built by glueing together a finite number of building blocks, leading to the space of geometries being finite. This means that we can still consider continuous surfaces (in the limit), while having the benefit of having a finite-dimensional space of geometries, as this space is $0-$dimensional.

1.2.1 Two-dimensional dynamical triangulations

An example of a discrete two-dimensional theory, which in the limit converges to a model for continuous surfaces, is the theory of two-dimensional dynamical triangulations. The building blocks of the surface are equilateral triangles, and the created surfaces are called triangulations. The number of possible geometries is finite, as the 'glueing' process is finite, meaning that the number of possible triangulations is finite [16]. For an intuition how the glueing process looks like, see Figure 1.
A probability distribution one can consider on the space consisting of triangulations built from n triangles, is the uniform probability distribution. This is because there are only finitely many surfaces one can create. Taking the scaling limit $n \to \infty$, while shrinking the side lengths with a factor of $n^{1/4}$, we get a continuous surface [17]. The uniform probability distribution converges, in this limit, towards a probability distribution of continuous geometries, called the Brownian Sphere [18] and [19]. This is a specific model, which is called pure gravity. This limit is nice, because we already know how to do quantum gravity without matter fields in this limit. The geometry of the two-dimensional pure gravity is the Brownian sphere, also called the Brownian Map [20].
In this model, the surfaces created have fractal characteristics. This means that they have a rough geometric shape which can be split into parts that resemble the original surface. The fractal characteristics of a surface are mathematically represented by critical exponents. One example of such a critical exponent is the Hausdorff-dimension, which is a measure of the fractal dimension. It measures the local surface area, with respect to the distance between two points on the surface [21]. It is possible to get different kinds of surfaces with different critical components in the limit. However, one needs to consider other discrete surfaces than dynamical triangulations to get them. This will be discussed further in the next section.

1.2.2 Universality classes

As stated previously, critical exponents can represent the fractal characteristics of a surface. More generally, critical exponents represent some macroscopic characteristics of the surface considered [22]. A natural question to ask, is whether changing some microscopic details from the surface influences these macroscopic characteristics. For example, can we get the same Hausdorff dimension in the scaling limit for quadrangulations? In this case, the building blocks are squares instead of equilateral triangles. The general answer to such questions is yes, given some restrictions on the model [23]. We will come back to these restrictions later.
It is therefore possible to change the microscopic details of your model, such as its building blocks, while not changing the scaling limit of the surface and therefore keeping the same macroscopic characteristics. If we have two models where this is the case, we say that the two models are in the same universality

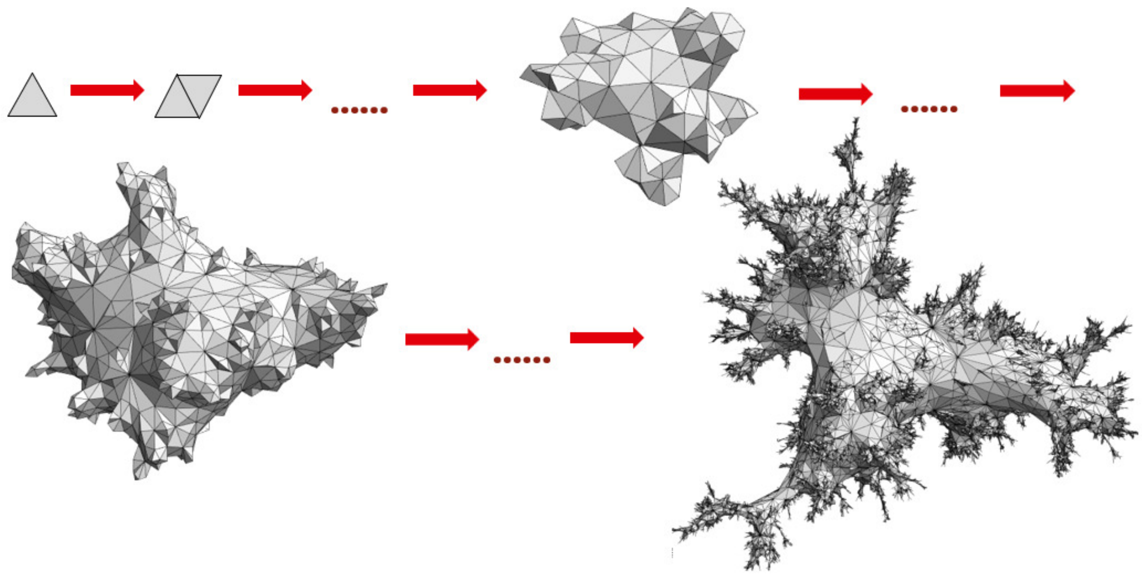

Figure 1: A representation of the glueing process for triangulations. The dots represent omitted steps. The pictures are not to scale, but it should be clear that surfaces later in the glueing process are larger, as represented. The triangulations becomes more fractal, when more triangles are added, as can be seen by comparing the last two triangulations. Source: T. Budd

class. If changing the microscopic details of your model does change the macroscopic details of the surface given by your model, one says that the models are in a different universality class [22].

1.2.3 Planar maps

As stated, it is possible to use squares instead of equilateral triangles and still get the same Hausdorff dimension in the limit. One can ask whether it is possible to generalize this idea even further. Is it possible to get the same results from glueing together any polygon? Or more generally, can we get the same results from planar maps? Mathematically, planar maps are graphs embedded into the plane. Conceptually, they can be thought of as surfaces whose building blocks are different kinds of polygons. As one can have a collection of different building blocks, we can introduce constraints on these building blocks which determine the likelihood of a specific building block being added to the surface in the 'glueing process'. The constraint on your building blocks is given by the weight sequence. Just as for quadrangulations, one can find the same critical exponents as for dynamical triangulations models, given that the constraints on your building blocks are correct.
In this thesis, we will take a look at specific kinds of planar maps: bipartite planar maps. These planar maps are built from polygons with an even number of sides and edges. The reason bipartite maps are considered, is because it is easier to find a recursive formulation for these kinds of surfaces [24]. Using these recursive formulations, one can find patterns for how these surfaces grow depending on the probability certain polygons get added. This can be used to find critical exponents in the limit where the size of the surfaces goes to infinity. This limit can therefore be used to find different kinds of universality classes.

1.3 Continuous two-dimensional theories

So far, we have only discussed discrete two-dimensional surfaces. Another natural question one can ask, is whether there exist models for continuous surfaces which are in the same universality classes as many models for discrete surfaces. As outlined before, one needs to limit the kind of surfaces considered, because otherwise we will be working with an infinite-dimensional space of geometries.

1.3.1 Hyperbolic surfaces

An example of continuous surfaces one can consider, are hyperbolic surfaces. More specifically they are Riemannian surfaces with constant scalar curvature of -2 [25]. They can also be thought of as surfaces, which locally behave like the hyperbolic plane. This is to say, these surfaces are locally isometric to the hyperbolic plane. The set of these surfaces, the Moduli space, is finite-dimensional and has a specific volume which is given by the Weil-Peterson volume [26]. This makes it easier to analyze hyperbolic surfaces, rather than pseudo-Riemannian surfaces, as one can define a partition function for possible surfaces using the Weil-Peterson volume.
To find critical exponents and compare the hyperbolic surfaces with the discrete surfaces, we want to take a limit. It is not evident how one can take a limit of a sequence of 'just' hyperbolic surfaces. We can therefore allow surfaces to have defects in the form of geodesics holes with specified lengths. This allows us to create a probability distribution by putting weights on the defects, depending on the length of the geodesic curve. This is done by the weight function, which is analogous to the weight sequence. The boundary components also makes it possible to take a limit, as one can take the limit where a specified boundary component's length goes to infinity. As we will see in this thesis, it is possible to get the same critical exponents for models of hyperbolic surfaces as for models of planar maps.

1.3.2 JT-gravity

We have already given a mathematical reason for using hyperbolic surfaces: Weil-Peterson volumes are defined for hyperbolic surfaces. However, we also get a reason from physics to consider these kinds of surfaces. This has to do with a specific quantum gravity theory called Jackiw–Teitelboim gravity. Jackiw–Teitelboim gravity, also called JT-gravity, is defined by an action that is similar to Einstein-Hilbert action from equation (4). The JT-gravity action is

$$S = \frac{c^2}{16\pi G} \int d^2x\sqrt{-g}\ \Phi\left(S\right), \left(R - \Lambda\right) \tag{7}$$

where c is the speed of light, G is the gravitational constant, g is the determinant of the metric tensor, $\Phi\left(S\right)$ is the scalar field, R is the Ricci scalar and $\Lambda = -2$ is the cosmological constant [27]. The action from equation (7) uses Lorentzian metrics, as the determinant is negative and the action is still defined for pseudo-Riemannian metrics [28]. When one solves the Euler Lagrange equations for this model, one sees that the Lorentzian metrics are specified to be Anti-de Sitter metrics, also called AdS_2-surfaces. The Euler Lagrange equation

$$\frac{\partial S}{\partial\ \Phi\left(S\right)} = R + 2 = 0 \tag{8}$$

implies that $R = -2$. This means that we are indeed working with Anti-de Sitter metrics, as the curvature is constant and negative [29]. Lorentzian metrics are pseudo-Riemannian metrics, which are more general than Riemannian metrics, as every Riemannian metric is a pseudo-Riemannian metric [25]. Hyperbolic surfaces are Riemannian manifolds with a metric that has a negative determinant and a Ricci scalar $R = -2$, meaning that they are solutions for the Euler Lagrange equation (8) which follows from the JT-gravity action (7).
There has been an increasing interest in JT-gravity, because it can be useful for analyzing black holes [30]. Black holes have an event horizon, which denotes a regime of the black hole where even light can not escape the gravity of the black hole. In certain two-dimensional regimes, the geometry is similar to the geometries JT-gravity gives. This means that it appears as a near horizon theory of near extremal higher black holes [31].
The second reason for the increasing interest is due to the holographic principle. If you have a

holographic duality in your theory, you are able to describe quantum gravity in the 3+1-dimensional universe by a theory on the three-dimensional boundary of the universe without quantum gravity. The requirement is that the system has a good time evolution, e.g. meaning that it has a unitary evolution. This means that one can describe a $d+1$-dimensional theory on a manifold as a d-dimensional theory without gravity on the boundary of the manifold [32]. If one considers 1+1-dimensional JT-gravity on a manifold, it can be described by a one-dimensional model without gravity. This theory on the boundary is called the SYK-model, which has a unitary evolution [33].
A link has already been made between JT-gravity and dynamical triangulations. This is via matrix models, which have already been studied in the 90's [34]. The generating function of Weil-Peterson volumes give us Korteweg–De Vries (KdV) equations [35]: the string equation and Dilatton equation. These equations also appear in the context of partition functions in the matrix models and discrete surface models [36]. However, this link is purely algebraic and gives no clear geometric interpretation. In this thesis, we will find a link between hyperbolic surfaces and planar maps, without using matrix models. We will instead focus on the geometric link.

1.4 Goal and outline of thesis

The goal of this research is twofold. Firstly, to find out under what condition on the weight functions a model for hyperbolic surfaces belongs to the universality class of planar maps. Secondly, to see whether it is possible to use different weight functions to create different universality classes for models with hyperbolic surfaces.

In this thesis, we will firstly take a look at bipartite planar maps. This work will be largely based on works by N. Curien [22] and T. Budd [37] [38] [39]. We will look at different universality classes, by defining different forms of criticalities. These criticalities will be defined by restrictions on the weight sequence, which determines the likely hood of having a polygon in the planar map. We will then define the partition functions for different kinds of planar maps. This makes it possible to define generating functions where the coefficients are partition functions. These generating functions can be studied, using recursive formulations for the partition functions called the Tutte-equations. We can then find the leading terms of the partition functions. These leading terms are dependent on the kind of (critical) weight sequence one considers. Lastly, we will consider the different kinds of surfaces and critical exponents one gets from the different forms of criticalities, meaning we consider the different kinds of universality classes.

For the hyperbolic surface case, we will have a similar setup. Firstly, Riemannian surfaces will be introduced and the volume of the space of hyperbolic surfaces, also called the Weil-Peterson volume, will be discussed. We then introduce the weight function, which determines the probability of having geodesic holes with a specific length in the surface. These geodesic holes are called boundary components. The weight function on the boundary component length makes it possible to define partition functions for different kinds of hyperbolic surfaces. Further analysis of these partition functions will follow after having defined different forms of criticalities. These criticalities are defined by restrictions on the weight function. We will then also look at the leading terms of the partition functions, depending on the different forms of criticalities. This will make it possible to get the critical exponent, corresponding to different forms of criticalities. Having found these critical exponents, we will look at the different kinds of universality classes and the kind of surfaces they have.

2 Bipartite planar maps

It is assumed that the reader has some elementary grasp of graphs and definitions associated with them. A short overview of definitions can be found in the appendix, see Appendix A.
As stated in Section 1, one can create two-dimensional gravity models using the 'glueing' of equilateral triangles to create surfaces. In this section this idea will be generalized to glueing together polygons, where we will allow different polygons with different number of sides. These surfaces are called planar maps.

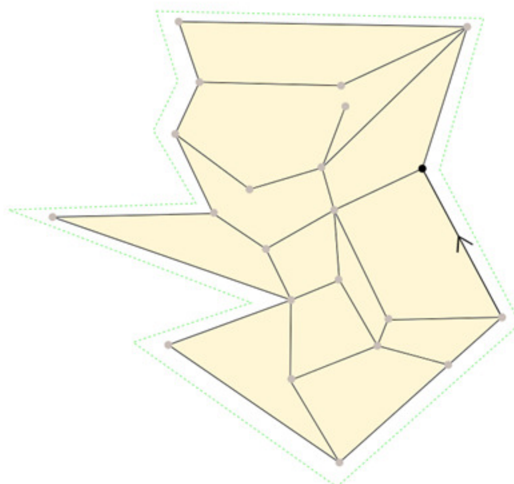

Figure 2: A figure of a planar map with a root face of degree 12. The outer edge of the map is marked by the green dotted lines. The map has a marked vertex, denoted by its black colour, and a distinguished edge with an orientation, denoted by the arrow.

2.1 Introduction to bipartite planar maps

A **planar map** $\mathfrak{m}$ is a non-intersecting graph, meaning that it is a graph which can be embedded in the plane or in the sphere. Alternatively, a planar map can be seen as a gluing of finitely many polygons. Lets consider a map $\mathfrak{m}$. As it is a graph, it has a set of vertices which we will denote by $\mathcal{V}(\mathfrak{m})$. It also has a set of edges, which we express with $\mathcal{E}(\mathfrak{m})$. This means that we can write $\mathfrak{m} = (\mathcal{V}(\mathfrak{m}), \mathcal{E}(\mathfrak{m}))$. The cardinality of the set $\mathcal{E}(\mathfrak{m})$ is also called the **volume of a map** $\mathfrak{m}$, meaning $|\mathcal{E}(\mathfrak{m})| = |\mathfrak{m}|$. A map is therefore called **finite** if $|\mathcal{E}(\mathfrak{m})| < \infty$. Lastly, we introduce the set of faces, denoted by $\mathcal{F}(\mathfrak{m})$. This can be thought of as the set of labeled polygons which are glued together to get the planar map.
In this thesis, only **bipartite** planar maps will be considered. A map being bipartite means that one can label the planar map with two labels, such that two neighbouring vertices do not get the same label [40]. Equivalently, this means that the faces of the map are of even degree [41, p. 8]. The reason for only considering bipartite maps, is that only for bipartite maps one gets a 'clean' recursive formulation (see Section 2.5.1 and Section 2.5.2).
The **root edge**, is an edge $e \in \mathcal{E}(\mathfrak{m})$, which is oriented and marked. If a planar map has a root edge, it is called a **rooted planar map**. The face to the right of the root edge is called the **root face** $f_{\mathbf{r}}$ or alternatively the **external face**. The contour of the root face is also called the **boundary** of the map. An example of a boundary can be seen in Figure 2. The degree of the root face, and therefore the degree of the boundary, is called the **perimeter**. In this thesis, we only consider maps with a perimeter of even length, as we only consider bipartite maps [22].
For easier enumeration, one can distinguish a specific face of degree $2p$ in the map which is not the

root face. Notation wise, in the case of a marked vertex, we say that is a distinguished face of degree 0. The distinguished face is called the **target face**. A map with a distinguished face of degree $2p$ will be denoted by $\mathfrak{m}_{2p}$. In the case of a distinguished face of degree 0, it will be denote by $\mathfrak{m}_{\bullet}$.
The set of maps $\mathfrak{m}$ with a perimeter of length $2l$ with an oriented marked edge is denoted by $\mathcal{M}^{2l}$. Analogously, the set of maps $\mathfrak{m}_{2p}$ with a perimeter of length $2l$ with an oriented marked edge and a target face of degree $2p$ is denoted by $\mathcal{M}_{2p}^{2l}$. Lastly, we define $\mathcal{M}_{2p}^{2l}[n]$ as the set of planar maps with perimeter $2l$, distinguished face of degree $2p$, a root edge and consist of n edges, meaning $|\mathfrak{m}| = n$.

2.2 Weight sequence q

Before we can define the partition function, we need to introduce a **weight sequence**. A weight sequence is a sequence $\mathbf{q} = (q_k)_{k\in\mathbb{N}}$, where $q_k \in \mathbb{R}_{\geq 0}$. The entry q_k determines the likelihood of having a face with degree $2k$ in a map. The weight sequence is therefore a parameter one can introduce to determine the kind of surfaces one looks at.
As an example, consider the weight sequence where we set $q_2 = 1$ and for all other $i \in \mathbb{N}$ we set $q_i = 0$. With this weight sequence, one only considers maps with faces of only degree 4. These surfaces are also called quadrangulations.

2.2.1 First formulation of partition function $W_{2p}^{(2l)}$

Let us consider a map $\mathfrak{m}_{2p} \in \mathcal{M}_{2p}^{2l}$. Let $\mathbf{q} = (q_k)_{k\in\mathbb{N}}$ be a weight sequence. One can define the partition function of maps with perimeter $2l$ and target face $2p$ by

$$W_{2p}^{(2l)} = 1 + \sum_{\mathfrak{m}_{2p}\in\mathcal{M}_{2p}^{2l}} \prod_{f\in\mathcal{F}(\mathfrak{m}_{2p})\setminus f_{\mathbf{r}}} q_{deg(f)}, \tag{9}$$

where the 1 is added for planar maps with $2l = 0$ and $2p = 0$. In equation (9), $\mathcal{M}_{2p}^{2l}$ is the set of maps $\mathfrak{m}_{2p}$ with a perimeter of length $2l$, an oriented marked edge and a target face of degree $2p$ and $\mathcal{F}(\mathfrak{m}_{2p})$ is the set of faces of $\mathfrak{m}_{2p}$. Note that planar maps where $2l = 0$ and $2p > 0$ are not counted in the partition function.
Analogously, we can define $W_{2p}^{(2l)}[n]$, which is the partition function with the added constraint that the number of edges is n. This is done by swapping $\mathcal{M}_{2p}^{(2l)}$ to $\mathcal{M}_{2p}^{2l}[n]$ in equation (9). One can of course also define $W^{(2l)}$. In this case the 1 in equation (9) is added for planar maps with $2l = 0$.

2.2.2 Admissibility of the weight sequence

To define the partition function in equation (9) the weight sequence $\mathbf{q}_k$ was introduced. It is natural to only consider $\mathbf{q}_k$, such that $W_{2p}^{(2l)} < \infty$, for any $l \in \mathbb{N}$ and any $p \in \mathbb{N}$. Weight sequences that fulfil this characteristic are called **admissible**.
The weight sequence $\mathbf{q}$ being admissible is equivalent to $f_{\mathbf{q}}(x) - x = 0$ having a solution for $x \in \mathbb{R}_{\geq 0}$, where $f_{\mathbf{q}}(x)$ is defined by [42]

$$f_{\mathbf{q}}(x) = 1 + \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} x^k. \tag{10}$$

The smallest positive solution to equation (10) is defined as $Z_{\mathbf{q}}$. We also define the constant $c_{\mathbf{q}}$, by $c_{\mathbf{q}} = 4 \cdot Z_{\mathbf{q}}$. These terms will come back later when considering the explicit formulation of the partition functions, such as $W_{2p}^{(2l)}$.

2.2.3 Probability measure

Using the weight sequence, one can define a measure $w_{\mathbf{q}}$ on the set of all finite planar maps, denoted by $\mathcal{M}$ [37]. This can be used to define a probability measure, which can tell us how surfaces will probably look like in certain limits. One of such limits is the the limit where the perimeter goes to infinity.

Theorem 2.1. *Let $\boldsymbol{q}$ be a weight sequence such that: $\exists i \in \mathbb{N} : q_i \neq 0$ and $\exists l \in \mathbb{N} : W^{2l} < \infty$. The set of all finite planar maps $\mathcal{M}$ has a $\sigma-$finite measure, defined by*

$$w_{\boldsymbol{q}}(\mathfrak{m}) = \prod_{f \in \mathcal{F}(\mathfrak{m}) \backslash f_r} q_{deg(f)/2}. \tag{11}$$

For the definition of a $\sigma-$finite measure and the proof of this theorem, see Section D. Note that the probability distribution on the space of planar maps, which follows from this measure is, as outlined in Section 1, is the probability distribution of your space-time.

2.3 Explicit formulation of the partition functions

To analyze how planar maps $\mathfrak{m}$ with a root edge looks like in the limit where the perimeter goes to infinity, we need to have more explicit formulation of the partition function W_{2p}^{2l} and W^{2l}. This will be done by first getting an explicit formulation for $W_{2p}^{2l}[n]$, which will give us the ability to get a more explicit formulation of both W_{2p}^{2l} and W^{2l}.

2.3.1 Explicit formulation of the partition function $W_{2p}^{2l}[n]$

The idea is to express the partition function $W_{2p}^{2l}[n]$ more explicitly then in equation (9) by introducing a new function $g_{\mathbf{q}}$. Using $f_{\mathbf{q}}(x)$ introduced in equation (10), we define

$$g_{\mathbf{q}}(z) = \frac{f_{\mathbf{q}}(z \cdot Z_{\mathbf{q}})}{f_{\mathbf{q}}(Z_{\mathbf{q}})}. \tag{12}$$

The $g_{\mathbf{q}}(z)$ in equation (12) is normalized: $g_{\mathbf{q}}(1) = 1$. Note also that $g_{\mathbf{q}}(z)$ gives us a generating function for a probability measure $\xi_{\mathbf{q}}(k-1)$ on $\mathbb{N}$ [22]

$$g_{\mathbf{q}}(z) = \sum_{k=0}^{\infty} \xi_{\mathbf{q}}(k-1) z^k. \tag{13}$$

The probability measure $\xi_{\mathbf{q}}(k-1)$ can then be used to define a probability measure $\xi_{\mathbf{q}}(k)$ on $\mathbb{N} \cup \{-1\}$. See appendix D.5 for the definition of a probability measure. The probability measure $\xi_{\mathbf{q}}(k)$ can then define a (one-dimensional) random walk Y_n. The finer details are shown in [22, A.2-A.15]. What matters is that one can get an explicit formulation for $W_{2p}^{(2l)}[n]$

$$\begin{aligned} W_{2p}^{2l}[n] &= \frac{l}{2}\binom{2l}{l}\binom{2p}{p}(Z_{\mathbf{q}})^{l+p} \cdot \frac{1}{n}\mathbb{P}[Y_n = -l-p] \\ &= \frac{l}{2(l+p)}\binom{2l}{l}\binom{2p}{p}(Z_{\mathbf{q}})^{l+p} \cdot \mathbb{P}[\tau_{-l-p} = n], \end{aligned} \tag{14}$$

where Z_q has been defined in Section 2.2.2, $\mathbb{P}[Y_n = -2l-2p]$ is the probability of the random walk Y_n being at $-2l-2p$ after n steps and τ_{-2l-2p} is the hitting time of $-2l-2p$ by the random walk Y_n. For an explicit proof for this theorem, see [22, p. 51-53].

2.3.2 Explicit formulation of the partition function $W_{2p}^{(2l)}$

In Section 2.3.1 we derived an explicit equation for $W_p^{2l}[n]$. Summing over the number of edges in the planar map, gives us the partition function

$$\begin{aligned} W_{2p}^{(2l)} &= \sum_{n=0}^{\infty} W_{2p}^{2l}[n+1] = \frac{l}{2\,(l+p)}\binom{2l}{l}\binom{2p}{p}(Z_{\mathbf{q}})^{l+p} \cdot \sum_{n=0}^{\infty} \mathbb{P}[\tau_{-l-p} = n] \\ &= 2^{-2l-2p-1}\binom{2l}{l}\binom{2p}{p}\frac{l}{l+p}c_{\mathbf{q}}^{l+p}, \end{aligned} \tag{15}$$

where $c_{\mathbf{q}}$ is as defined in Section 2.2.2. The details of the proof can be found in [22, p. 51-53], the important part to note is that the calculation for the partition function is dependent on the random walk defined by $g_{\mathbf{q}}$. The result however, is only dependent on the parameters l and p, and $c_{\mathbf{q}}$, which is defined using the admissibility of the weight sequence $\mathbf{q}$.

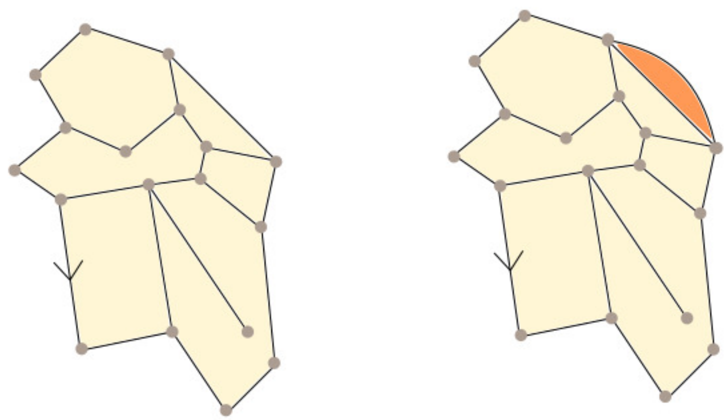

Figure 3: To the left is a map with a root edge and 23 edges without a marked face. A new map with 24 edges is created, as seen to the right, by adding one edge and marking the newly created face of degree 2. This face is coloured orange.

2.3.3 Explicit formulation of the partition function $W^{(2l)}$

Analogously to Section 2.3.3 we want to define $W^{(2l)}$ by summing over $W_2^{2l}[n]$. This can be done by using $W_2^{2l}[n+1] = n \cdot W^{2l}[n]$. This holds, as from any map without a target face and n edges, one can create n maps with $n+1$ edges and a target face of degree 2 by adding one edge along another edge and making the newly created face the target face. This can be done in n ways as there were n edges of the maps. See Figure 3 for an example of this process.
Filling in $p = 1$ in equation (14) and using the above formulation, we get

$$\begin{aligned} W^{(2l)} &= \sum_{n=0}^{\infty} \frac{1}{n} W_2^{2l}[n] = 2^{-2l-1}\binom{2l}{l}\frac{l}{l+1}c_{\mathbf{q}}^{l+1} \cdot \mathbb{E}\left[\frac{1}{\tau_{-l-p}-1}\right] \\ &= \frac{l}{l+1}\binom{2l}{l}\int_0^{Z_{\mathbf{q}}} ds\ \left(f_{\mathbf{q}}(s) - s f_{\mathbf{q}}(s)\right) s^{l-1}. \end{aligned} \tag{16}$$

Again, for the details of the proof, we refer to [22, p. 53-54]. The importance still lays in the dependence of the calculation on the random walk created by $g_{\mathbf{q}}$ and the (explicit) dependence of the result on the perimeter l and the function $f_{\mathbf{q}}$.

2.4 Criticalities

As stated in Section 1, we want to consider surfaces which microscopically differ, but give the same macroscopic characteristics. In this section we will introduce constraints on the microscopic level. Models for surfaces that fulfill these constraints will share some macroscopic characteristics, like the critical exponent. The details surround the macroscopic characteristics will be discussed in later sections.

2.4.1 Definitions of criticalities

The microscopic details we will put constraints on is the weight sequence $\mathbf{q}$ via the function $f_{\mathbf{q}}(x)$. Let $f_{\mathbf{q}}(x)$ be given by equation (10). Recall that $Z_{\mathbf{q}}$ is the smallest positive solution to $f_{\mathbf{q}}(x) = x$. Using the convention from [42], we define a weight sequence $\mathbf{q} = \{q_k\}_{k>0}$ to be **critical** if

$$f'_{\mathbf{q}}(Z_{\mathbf{q}}) = \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} Z_{\mathbf{q}}^{k-1} k = 1. \tag{17}$$

Using the definition of criticality we define when the weight sequence is not critical. Furthermore, we will make a further distinction between critical weight sequences, categorizing them in generic critical weight sequences and non-generic critical weight sequences. The weight sequences that we are interested in can therefore be classified into three cases:

- The weight sequence $\mathbf{q}$ is called **subcritical** if $f'_{\mathbf{q}}(Z_{\mathbf{q}}) \neq 1$.
- The weight sequence $\mathbf{q}$ is called **generic critical** if $f'_{\mathbf{q}}(Z_{\mathbf{q}}) = 1$ and the $f''_{\mathbf{q}}(Z_{\mathbf{q}}) < \infty$
- The weight sequence $\mathbf{q}$ is called **non-generic critical** if $f'_{\mathbf{q}}(Z_{\mathbf{q}}) = 1$ and for $s \nearrow Z_{\mathbf{q}}$:

$$f_{\mathbf{q}}(s) = Z_{\mathbf{q}} - (Z_{\mathbf{q}} - s) + \kappa (Z_{\mathbf{q}} - s)^{\alpha - 1/2} + o\left((Z_{\mathbf{q}} - s)^{\alpha-1/2}\right) \tag{18}$$

  with $\alpha \in (3/2, 5/2)$ and $\kappa \in \mathbb{R}$.

2.4.2 Critical exponents

Using the definitions from Section 2.4.1, we will introduce the concept of critical exponents. These exponents characterise the different kinds of weight sequences. We will define the exponents using $f_{\mathbf{q}}(x)$ from equation (10).

Note that for the non-generic critical case, we had an approximation for $f_{\mathbf{q}}(x)$ for s near $Z_{\mathbf{q}}$. We can get a similar approximation for $f_{\mathbf{q}}(x)$ in the subcritical or generic critical case using the Taylor expansion. Let $s \nearrow Z_{\mathbf{q}}$. In the subcritical case, we have

$$\begin{aligned} f_{\mathbf{q}}(s) &= Z_{\mathbf{q}} + f'_{\mathbf{q}}(Z_{\mathbf{q}})(s - Z_{\mathbf{q}}) + \frac{f''_{\mathbf{q}}(Z_{\mathbf{q}})}{2}(s - Z_{\mathbf{q}})^2 + o\left((s - Z_{\mathbf{q}})^2\right) \\ &= Z_{\mathbf{q}} - (1-\kappa)(Z_{\mathbf{q}} - s) + \frac{f''_{\mathbf{q}}(Z_{\mathbf{q}})}{2}(Z_{\mathbf{q}} - s)^2 + o\left((Z_{\mathbf{q}} - s)^2\right). \end{aligned} \tag{19}$$

It is easy to see, that we can get equation (19) from equation (18) by setting $\alpha = 3/2$. In equation (19), $\kappa \in (0, 1)$ holds, because $f'_{\mathbf{q}}(Z_{\mathbf{q}}) < 1$ [22].

In the generic critical case we get the following approximation for $s \nearrow Z_{\mathbf{q}}$

$$\begin{aligned} f_{\mathbf{q}}(s) &= Z_{\mathbf{q}} + f'_{\mathbf{q}}(Z_{\mathbf{q}})(s - Z_{\mathbf{q}}) + \frac{f''_{\mathbf{q}}(Z_{\mathbf{q}})}{2}(s - Z_{\mathbf{q}})^2 + o\left((s - Z_{\mathbf{q}})^2\right) \\ &= Z_{\mathbf{q}} - (Z_{\mathbf{q}} - s) + \kappa (s - Z_{\mathbf{q}})^2 + o\left((s - Z_{\mathbf{q}})^2\right). \end{aligned} \tag{20}$$

As one can see, equation (20) also follows from equation (18). This time it follows when we set $\alpha = 5/2$. Using the approximation of $f_{\mathbf{q}}(s)$ for $s \nearrow Z_{\mathbf{q}}$, we set the critical exponents for the different criticalities as follows: the subcritical exponent is $3/2$, the generic critical exponent is $5/2$ and the non generic critical exponent can take values in $(3/2, 5/2)$.

2.5 Tutte-equations

As we are working with bipartite planar maps, we can easily find a recursive formulations for the partition functions. These recursive formulations are called the Tutte-equations [43] and will be used to analyze the $W^{(2l)}$ in the $l \to \infty$ limit, which in turn should tell us how surfaces will look like depending on their critical exponents.

2.5.1 Tutte-equation for $W_\bullet^{(2l)}$

In this section only bipartite planar maps with a root edge, perimeter of $2l$ and a marked vertex will be used. As we consider maps with a target face of degree 0, we can set p to 0, meaning we will analyse $W_0^{(2l)} = W_\bullet^{(2l)}$.
Let $\mathfrak{m}_\bullet \in \mathcal{M}_\bullet^{2l}$ be non-trivial. Note that a root edge e is a marked oriented edge of the perimeter, which is oriented anti-clockwise. When this root edge is removed, there are two possibilities: either the map is still connected, see Figure 4, or it is separated into two different maps, see Figure 5. In the first case, we now have a perimeter $2l + 2k - 2$, where $2k$ is the degree of the face incident to the deleted edge. In the second case, we have two separate connected maps with perimeters $2l - 2 - 2l_1$ and $2l_1$. Note that only one of these maps has the marked vertex. This leads to the Tutte-equation

$$W_\bullet^{(2l)} = \sum_{k=1}^{\infty} q_k W_\bullet^{(2l+2k-2)} + 2 \sum_{l_1=0}^{l-1} W_\bullet^{(2l-2l_1-2)} W^{(2l_1)}, \tag{21}$$

where q_k is the weight of a face with degree $2k$; $W_\bullet^{2i}$ is the partition function of bipartite planar maps with a marked vertex, root edge, and root face with degree $2i$ and W^{2j} is the partition function of bipartite planar maps with a root edge and root face with degree $2j$.

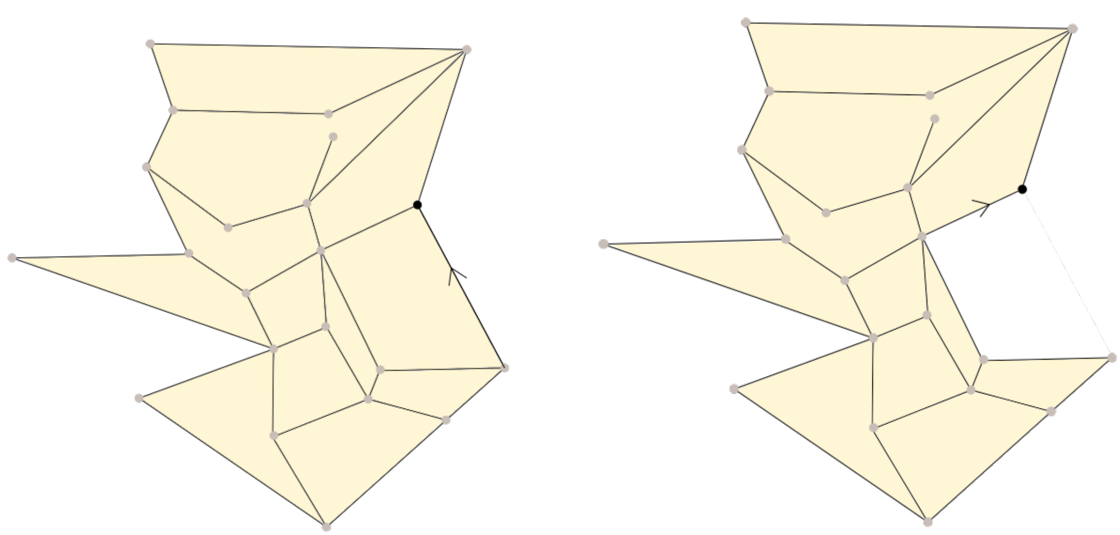

Figure 4: The first possibility: deleting the oriented edge will keep the map connected. The perimeter now has length $2l + 2k - 2$.

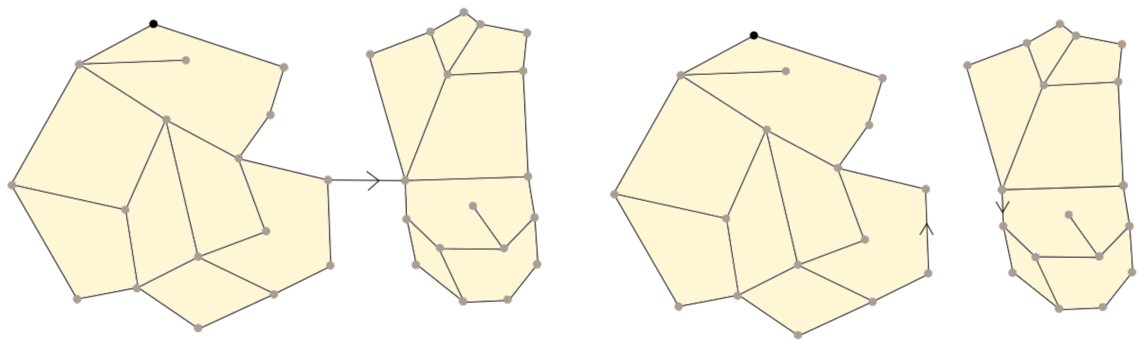

Figure 5: The second possibility: deleting the edge causes the map to fall into two disconnected maps. Note that only one of the maps has the market point and that the deleted edges counts for two in the perimeter: once for the left to the right, once for the right to the left.

2.5.2 Tutte-equation for $W^{(2l)}$

Analogously to the case of $W_\bullet^{(2l)}$ discussed in Section 2.5.1, we can get a generating function for the partition function $W^{(2l)}$. In this case we just consider bipartite planar maps with a root edge and a perimeter of length $2l$. For these planar maps we get

$$W^{(2l)} = \sum_{k=1}^{\infty} q_k W^{(2l-2k-2)} + \sum_{l_1=1}^{l-1} W^{(2l-2l_1-2)} W^{(2l_1)}. \tag{22}$$

Unlike in equation (21), there is no factor 2 in front of the second sum. This is because no distinction can be made when dealing with the second possibility, depicted in the $W_\bullet^{(2l)}$ case by Figure 5, as there is no marked vertex.

2.6 Generating functions

Given the different groups of weight sequences specified in Section 2.7, we can analyze the generating functions defined using the partition functions defined in Section 2.3. This will give us more abilities to characterize the universality classes, by giving us more analysis on the critical exponents for the different groups of weight sequences.
It is assumed that the reader has some familiarity with generating functions. For a short introduction of generating functions, see Appendix B.

2.6.1 Generating function $W_\bullet(z)$

The generating function $W_\bullet(z)$ for bipartite planar maps with perimeter $2l$, a marked root edge and a marked vertex is defined by

$$W_\bullet(z) = \sum_{l=0}^{\infty} W_\bullet^{(2l)} z^{-2l-1}. \tag{23}$$

This is a relatively standard generating function, meaning that one can easily find the analytical formulation for it.

Theorem 2.2. *Let $\boldsymbol{q}$ be an admissible weight sequence and let $z > c_{\boldsymbol{q}}$. The generating function $W_\bullet(z)$ for bipartite planar maps with perimeter $2l$, a marked root edge and a marked vertex given by equation*

(23) *is analytically given by*

$$W_{\bullet}(z) = \frac{1}{2\sqrt{(z^2 - c_{\boldsymbol{q}})}}. \tag{24}$$

The proof of Theorem 2.2 can be found in the Appendix B.1.

2.6.2 Generating function $W(z)$

Analogously to Section 2.6.1, we define the generating function for bipartite planar maps with perimeter $2l$ and a marked root edge by

$$W(z) = \sum_{l=0}^{\infty} W^{(2l)} z^{-2l-1}. \tag{25}$$

Looking at equation (16), one can see that finding an analytical form for $W(z)$ will not be as straightforward as for $W_{\bullet}(z)$. We will therefore express $W(z)$ using $W_{\bullet}(z)$ and the Tutte-equation (21) for $W_{\bullet}^{2l}$. Firstly, we define $U'(z)$ and $M(z)$ similarly as done in [38]

$$U'(z) = \sum_{k=1}^{\infty} q_k z^{2k-1} \tag{26}$$

$$M(z) = -1 + \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} q_k . W_{\bullet}^{(2k-2l-4)} z^{2l}. \tag{27}$$

Using the Tutte-equation (21) for $W_{\bullet}^{(2l)}$ we find the relation

$$[z - U'(z) - 2W(z)]\, W_{\bullet}(z) = -M(z). \tag{28}$$

Rewriting equation (28), and using equation (24), we get for $z > c_{\mathbf{q}}$,

$$W(z) = \frac{1}{2}\left[z - U'(z) + 2M(z)\sqrt{(z^2 - c_{\mathbf{q}})}\right]. \tag{29}$$

This gives us an alternative expression of $W(z)$, as it is a generating function for a solution to the Tutte-equation (22) [38, p. 6-9]. We will use this expression in the next sections.

2.6.3 Coefficients of $W(z)$

Using the expression of $W(z)$ given in equation (29), one can analyse the $W^{(2l)}$ in the $l \to \infty$ for the noncritical, generic critical and non-generic critical case.

Theorem 2.3. *Let $\boldsymbol{q}$ be an admissible weight sequence and let $z > c_{\boldsymbol{q}}$. Let the generating function $W(z)$ for bipartite planar maps with perimeter $2l$, a marked root edge and a marked vertex be given by equation* (25)*. Then, the following holds:*

i. If the weight sequence $\boldsymbol{q}$ is chosen such that $M\left(\sqrt{c_{\boldsymbol{q}}}\right) \neq 0$, then $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-3/2} c_{\boldsymbol{q}}^{-n+1/2}$, where κ is a constant in $\mathbb{R}$.

ii. If the weight sequence $\boldsymbol{q}$ is a finite sequence and $M\left(\sqrt{c_{\boldsymbol{q}}}\right) = 0$, then $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-5/2} c_{\boldsymbol{q}}^{-n+3/2}$, where κ is a constant in $\mathbb{R}$.

iii. If the weight sequence $\boldsymbol{q}$ is chosen such that $M\left(\sqrt{c_{\boldsymbol{q}}}\right) = 0$ and has leading term $\left(z^2 - c_{\boldsymbol{q}}\right)^{\gamma}$ where $\gamma \in (0,1]$, then $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-\gamma-1/2} c_{\boldsymbol{q}}^{n+\gamma+1/2}$, where κ is a constant in $\mathbb{R}$.

Proof. First lets do some generic analysis for $W(z)$. It is easy to see from equations (26) and (27), that the most important part of equation (29) is $g(z) = M(z)\sqrt{(z^2 - c_{\mathbf{q}})}$, as that is the only term which can cause a divergence. The derivative of $g(z)$ is

$$g'(z) = M'(z)\sqrt{(z^2 - c_{\mathbf{q}})} + \frac{M(z)z}{\sqrt{(z^2 - c_{\mathbf{q}})}}. \tag{30}$$

Equation (30) will be used to analyze the growth of the coefficients of $W(z)$.

Proof for i. Let $M\left(\sqrt{c_{\mathbf{q}}}\right) = d \neq 0$. Equation (30) shows that the convergence radius of $g(z)$ is $\sqrt{c_{\mathbf{q}}}$, as the first singularity of $g(z)$ is $\sqrt{c_{\mathbf{q}}}$. Rewriting $g(z)$, we get

$$g(z) = \sqrt{c_{\mathbf{q}}} M(z) \sqrt{\frac{z^2}{c_{\mathbf{q}}} - 1}. \tag{31}$$

Using the transfer theorem G.1 (see Appendix G also for an explanation and definitions of the notations used) and equation (24) one then sees that

$$[\![g(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-3/2} c_{\mathbf{q}}^{-n+1/2}. \tag{32}$$

From equation (23), we see that this implies that $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-3/2} c_{\mathbf{q}}^{-n+1/2}$, with $\kappa \in \mathbb{R}$.

Proof for ii. Let $M\left(\sqrt{c_{\mathbf{q}}}\right) = 0$. The derivative of $g(z)$ at $\sqrt{c_{\mathbf{q}}}$ becomes

$$g'(\sqrt{c_{\mathbf{q}}}) = M'(\sqrt{c_{\mathbf{q}}})\sqrt{(\sqrt{c_{\mathbf{q}}}^2 - c_{\mathbf{q}})} = 0. \tag{33}$$

The second derivative of $g(z)$ is given by

$$g''(z) = M''(z)\sqrt{(z^2 - c_{\mathbf{q}})} + \frac{M'(z)z}{\sqrt{(z^2 - c_{\mathbf{q}})}}. \tag{34}$$

Note that equation (34) therefore shows that $g(z)$ still has a singularity at $\sqrt{c_{\mathbf{q}}}$ as $M'\left(\sqrt{c_{\mathbf{q}}}\right)\sqrt{c_{\mathbf{q}}} \neq 0$. Consider the case in which only finitely many values for q_k are considered, say up to $k = d$. The second term in equation (27) then becomes

$$\sum_{k=2}^{d}\sum_{l=0}^{k-2} q_k W_{\bullet}^{(2k-2l-4)} z^{2l}. \tag{35}$$

If one has $M\left(\sqrt{c_{\mathbf{q}}}\right) = 0$, one gets

$$-1 + \sum_{k=2}^{d}\sum_{l=0}^{k-2} q_k W_{\bullet}^{(2k-2l-4)} z^{2l} = \left(z^2 - c_{\mathbf{q}}\right) \cdot P(z). \tag{36}$$

Where $P(z)$ is a polynomial in z. Note that one can not have higher powers of $\left(z^2 - c_{\mathbf{q}}\right)$ as $M(z)$ needs to be a convex polynomial for equation (28) to hold. Also note that from equation (36) we can not have a power α, such that $0 < \alpha < 1$. Therefore, $g(z)$ gets the form

$$\begin{aligned} g(z) &= c_{\mathbf{q}}^{3/2}\left(\frac{z^2}{c_{\mathbf{q}}} - 1\right) \cdot P(z)\sqrt{\frac{z^2}{c_{\mathbf{q}}} - 1} \\ &= c_{\mathbf{q}}^{3/2}\left(\frac{z^2}{c_{\mathbf{q}}} - 1\right)^{3/2} P(z). \end{aligned} \tag{37}$$

Using the transfer theorem G.1 on equation (37), one gets

$$[\![g(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-5/2} c_{\mathbf{q}}^{-n+3/2}. \tag{38}$$

This implies that for finitely many q_k, we have $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-5/2} c_{\mathbf{q}}^{-n+3/2}$, with $\kappa \in \mathbb{R}$.

Proof for iii. As seen by equation (34), the first singularity of $g(z)$ is still $\sqrt{c_{\mathbf{q}}}$. Let $M(z)$ for $z \nearrow \sqrt{c_{\mathbf{q}}}$ be given by

$$M(z) = \lambda \left(z^2 - c_{\mathbf{q}}\right)^{\gamma} + o\left(\left(z^2 - c_{\mathbf{q}}\right)^{\gamma}\right), \tag{39}$$

where $\gamma \in (0, 1]$ and $\lambda \in \mathbb{R}$.
Using the expression for $M(z)$, we get for $g(z)$

$$\begin{aligned} g(z) &= \left[\lambda \left(z^2 - c_{\mathbf{q}}\right)^{\gamma} + o\left(\left(z^2 - c_{\mathbf{q}}\right)^{\gamma}\right)\right] \sqrt{(z^2 - c_{\mathbf{q}})} \\ &= \left[\lambda\, c_{\mathbf{q}}^{\gamma+1/2} \left(\frac{z^2}{c_{\mathbf{q}}} - 1\right)^{\gamma+1/2} + o\left(\left(\frac{z^2}{c_{\mathbf{q}}} - 1\right)^{\gamma+1/2}\right)\right]. \end{aligned} \tag{40}$$

Using the transfer theorem G.1 on equation (40), one sees that

$$[\![g(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-\gamma-3/2} c_{\mathbf{q}}^{-n+\gamma+1/2}. \tag{41}$$

This implies that we have $[\![W(z)]\!]_{z^{-n}} \backsim \kappa\, n^{-\gamma-3/2} c_{\mathbf{q}}^{-n+\gamma+3/2}$, with $\gamma \in (0, 1]$ and $\kappa \in \mathbb{R}$. □

Note that if we have infinite many q_k, we don't get the restriction on the $(z^2 - c_{\mathbf{q}})$ exponent in the second term of $M(z)$. This means that we can have $(z^2 - c_{\mathbf{q}})^{\alpha}$ term, where $0 < \alpha \leq 1$ hold. Also note that in the $\gamma = 1$ case, we have that equation (39) is the Taylor expansion of $M(z)$.

2.6.4 Critical exponents and the form of $M(z)$

In Section 2.6.2, we introduced $M(z)$ in equation (27). In Section 2.6.3, we found a way to analyze the growth of the coefficients of $W(z)$ depending on some constraints on $M(z)$. In this section, we will introduce the link between the different groups of weight sequences outlined in Section 2.4.1 and $M(z)$.

Theorem 2.4. *Let $\boldsymbol{q}$ be an admissible weight sequence and let $z \to \infty$. Let $W_{\bullet}^{(2l)}$ be the partition function for bipartite planar maps with perimeter $2l$, a marked root edge and a marked vertex. For $M(z)$ defined equation (27) the following holds:*

i. The weight sequence $\boldsymbol{q}$ is a subcritical weight sequence, if and only if $M\left(\sqrt{c_{\boldsymbol{q}}}\right) \neq 0$

ii. The weight sequence $\boldsymbol{q}$ is a generic critical weight sequence, if and only if $M\left(\sqrt{c_{\boldsymbol{q}}}\right) = 0$ and the leading term of $M(z)$ for $z \nearrow \sqrt{c_{\boldsymbol{q}}}$ is $(z - c_{\boldsymbol{q}})$

iii. The weight sequence $\boldsymbol{q}$ is a non-generic critical weight sequence, if and only if $M\left(\sqrt{c_{\boldsymbol{q}}}\right) = 0$ and the leading term of $M(z)$ for $z \nearrow \sqrt{c_{\boldsymbol{q}}}$ is $(z - c_{\boldsymbol{q}})^{\gamma}$ where $\gamma \in (0, 1)$

This proof will not be given, as it is based on peeling algorithms on the map. The idea is that using peeling algorithms, one can define a measure ν on $\mathbb{Z}$, which makes it possible to define a random walk S. From analysis of this random walk and the measure that defines it, Theorem 2.4 follows, as can be seen in [39, p. 9].

From Theorem 2.3 and Theorem 2.4, Corollary 2.4.1 follows.

Corollary 2.4.1. *Let* $\boldsymbol{q}$ *be an admissible weight sequence and let* $z \to \infty$. *Let* $W_{\bullet}^{(2l)}$ *be the partition function for bipartite planar maps with perimeter* $2l$*, a marked root edge and a marked vertex. The following holds:*

i. If the weight sequence $\boldsymbol{q}$ *is a subcritical weight sequence, then* $W_{\bullet}^{(2l)} \backsim \kappa\ l^{-3/2} c_{\boldsymbol{q}}^{-2l+1/2}$*, with* $\kappa \in \mathbb{R}$.

ii. If the weight sequence $\boldsymbol{q}$ *is a generic critical weight sequence, then* $W_{\bullet}^{(2l)} \backsim \kappa\ l^{-5/2} c_{\boldsymbol{q}}^{2l+5/2}$ *, with* $\kappa \in \mathbb{R}$.

iii. If the weight sequence $\boldsymbol{q}$ *is a non generic critical weight sequence, then* $W_{\bullet}^{(2l)} \backsim \kappa\ l^{-\gamma-3/2} c_{\boldsymbol{q}}^{2l+\gamma+1/2}$*, where* $\gamma \in (1,2)$ *and* $\kappa \in \mathbb{R}$.

Proof. Using the definition (25) of $W(z)$, we see that $[\![W(z)]\!]_{z^{-n-1}} = W^n$. From Theorem 2.3we therefore conclude that if $M\left(\sqrt{c_{\mathbf{q}}}\right) \neq 0$, then $W^n \backsim \kappa\ (n+1)^{-3/2} c_{\mathbf{q}}^{-n-1/2}$. If $M\left(\sqrt{c_{\mathbf{q}}}\right) = 0$ and we allow only finite many non-zero entries in $\mathbf{q}$, we have $W^n \backsim \kappa\ (n+1)^{-5/2} c_{\mathbf{q}}^{-n+1/2}$. If we allow infinitely many entries in $\mathbf{q}$ and have $M\left(\sqrt{c_{\mathbf{q}}}\right) = 0$ and its leading term is $\left(z^2 - c_{\mathbf{q}}\right)^{\gamma}$, where $\gamma \in (0,1]$, then $W^n \backsim \kappa\ (n+1)^{-\gamma-1/2} c_{\mathbf{q}}^{-n+\gamma+3/2}$. The corollary now follows from Theorem 2.4. □

2.7 Universality classes

Combining all the previous sections, we are able to analyze the different universality classes. We will label the universality classes by the critical exponent α from equation (18), which we have defined for different groups of weight sequences in Section 2.4.2. As we have seen in Section 2.6.4, these critical exponents are the same exponents for the partition function of W^{2l} if the perimeter goes to infinity. In this section, we will also show how surfaces from the different universality classes will look like in this limit.

2.7.1 The $\alpha = 3/2$ universality class

The $\alpha = 3/2$ universality class corresponds to a model with a **subcritical** weight sequence. From theorems 2.3 and 2.4, we get the following corollary:

Corollary 2.4.2. *Let* $\boldsymbol{q}$ *be an admissible weight sequence. The following are equivalent:*

i. The weight sequence $\boldsymbol{q}$ *is subcritical*

ii. For $s \nearrow Z_{\boldsymbol{q}}$ *we have* $f_{\boldsymbol{q}}(s) = Z_{\boldsymbol{q}} - (1-\kappa)(Z_{\boldsymbol{q}} - s) + o\left(Z_{\boldsymbol{q}} - s\right)$

iii. $\log\left[\frac{W^{2l}}{c_q}\right] \backsim -3/2 \log[l]$ *as* $l \to \infty$.

Note that the value for α in (18) for the subcritical case, agrees with the exponent of l in the W^{2l} term for the $l \to \infty$ limit.
In theorem 2.3, we made a crude estimate of the growth of the coefficients of $W(z)$. The $l^{-3/2}$ term holds, but a more specific coefficient can be found in [22, p. 80]

$$W^{2l} \backsim \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{2l+2} l^{-3/2}, \tag{42}$$

for $l \to \infty$, where

$$p_{\mathbf{q}} = \frac{\kappa Z_{\mathbf{q}}^{\alpha-3/2}}{2\sqrt{\pi}} \Gamma\left(\alpha + \frac{1}{2}\right). \tag{43}$$

We can derive another characteristic of $W^{(2l)}$ in this case, from the fact that the weight sequence $\mathbf{q}$ is not critical.

Theorem 2.5. *Let $\boldsymbol{q}$ be an admissible weight sequence and let $\mathfrak{m}_{2p} \in M_{2p}^{(2l)}$ be a random planar map with perimeter $2l$, an oriented edge and a target face of degree $2p$. The weight sequence $\boldsymbol{q}$ is subcritical if and only if $\mathbb{E}(|\mathfrak{m}_{2p}|) < \infty$.*

The proof is based on the random walk defined by $g_{\mathbf{q}}$ and can be found in [22, p. 77-78]. For notes on Theorem 2.5 and the link to the σ-measure $w_{\mathbf{q}}$ defined in section 2.2.3 see appendix D.5.1. In this section the physical interpretation of this theorem will be discussed.
Remember from Section 2.1 that $|\mathfrak{m}|$ is the number of edges. In the subcritical case, the expected number of edges is finite, even if $l \to \infty$. This means, that it wants to minimize the number of edges, while the number of edges in the perimeter goes to infinity. This means that the number of faces, which can be interpreted as the surface area, is minimized. This causes the surface to get a tree-like form as seen in Figure 6.

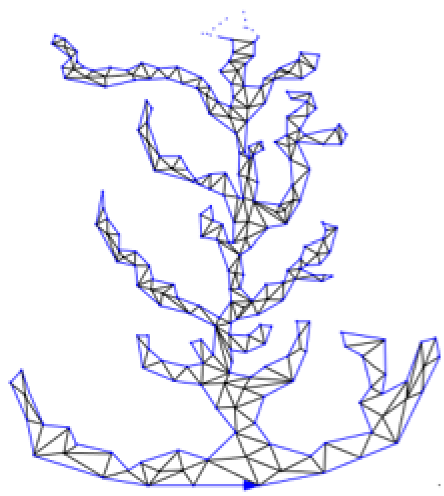

Figure 6: An artist sketch of the subcritical case in the $l \to \infty$ case. The surface is minimized, while the perimeter, coloured in blue, becomes infinitely long. This is why the planar map has a tree-like structure: the only way to minimize the number of edges, while still having the perimeter going to infinity is by minimizing its surface. Source: G. Ray [44].

2.7.2 The $\alpha = 5/2$ universality class

The $\alpha = 5/2$ universality class corresponds to a model with a **generic critical** weight sequence. From previous analysis of $f_{\mathbf{q}}$, Theorem 2.3 and Theorem 2.4 we get the next corollary

Corollary 2.5.1. *Let $\boldsymbol{q}$ be an admissible weight sequence. The following are equivalent:*

i. The weight sequence $\boldsymbol{q}$ is generic critical

ii. For $s \nearrow Z_{\boldsymbol{q}}$ we have $f_{\boldsymbol{q}}(s) = Z_{\boldsymbol{q}} - 1(Z_{\boldsymbol{q}} - s) + \kappa(Z_{\boldsymbol{q}} - s)^2 + o\left((Z_{\boldsymbol{q}} - s)^2\right)$

iii. $\log\left[\frac{W^{2l}}{c_q}\right] \backsim -5/2\log[l]$ as $l \to \infty$

The more specific description for W^{2l} in the limit where $l \to \infty$ given in [22, p. 80-81] is

$$W^{2l} \backsim \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{2l+2} l^{-5/2}, \tag{44}$$

where $p_{\mathbf{q}}$ is still defined by equation (43). Note that again, the value for α from from equation (20) agrees with the exponent of l in the W^{2l} term for the $l \to \infty$ limit.
As the weigh sequence considered is critical, theorem 2.5 gives us that the number of expected edges is infinite. This means that, unlike in the subcritical case, there is no surface area minimization in the $l \to \infty$ limit. This causes the surface to get not take the form of the tree, but rather as the surface shown in Figure 7. The Figure also shows that the surface is fractal, as there are repeating forms when one zooms into the surface. Specifically, one sees these stems-like parts of the surface in different sizes. Note that there are no holes in the surface.

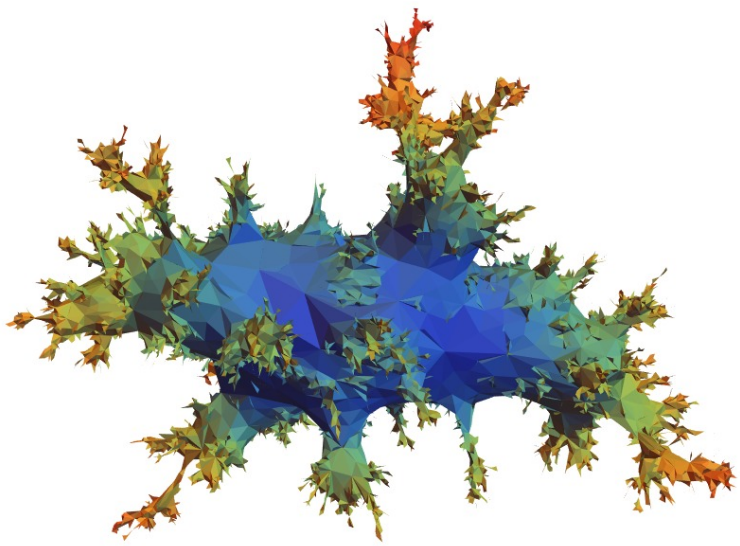

Figure 7: A figure of how a planar map should look like in the $l \to \infty$ limit, if the weight sequence is generic critical. The surface area is not minimized and the surface is fractal. Source: T. Budd [45]

2.7.3 The $\alpha \in (3/2, 5/2)$ universality class

The $\alpha \in (3/2, 5/2)$ universality class corresponds to a model with a **non-generic critical** weight sequence. Using again theorems 2.3 and 2.4, we get the following corollary:

Corollary 2.5.2. *Let $\boldsymbol{q}$ be an admissible weight sequence. The following are equivalent:*

i. The weight sequence $\boldsymbol{q}$ is non-generic critical

ii. For $s \nearrow Z_{\boldsymbol{q}}$ we have $f_{\boldsymbol{q}}(s) = Z_{\boldsymbol{q}} - 1(Z_{\boldsymbol{q}} - s) + \kappa(Z_{\boldsymbol{q}} - s)^{\alpha-1/2} + o\left((Z_{\boldsymbol{q}} - s)^{\alpha-1/2}\right)$

iii. $\log\left[\frac{W^{2l}}{c_q}\right] \backsim -\alpha \log[l]$ as $l \to \infty$

Here the exponent α of l in the W^{2l} term for the $l \to \infty$ limit (seen in the third item of theorem 2.5.2) is the same α from equation (18). Note that $i \leftrightarrows ii$ is trivial by the definition of non-generic critical as seen in Section 2.4.1.
In [22, p. 81-82], the following more specific relation has been found for the partition function W^{2l}:

$$W^{2l} \backsim \frac{p_{\mathbf{q}}}{2} c_{\mathbf{q}}^{2l+2} l^{-\alpha}. \tag{45}$$

The surfaces in the non-generic critical case share some characteristics with surfaces from the generic critical case. Firstly, there is no minimization of the surface area in the case that $l \to \infty$. This means

that surfaces in the non-generic critical model will not look like trees. Secondly, the surfaces are fractal, as they can be split into parts that resemble the original surface. An example of a non-generic critical surface can be found in Figure 8.
Where surfaces from the non-generic critical case differ is the fact that they have holes in them, as can be seen in Figure 8. In the $l \to \infty$ limit, there are holes in the surface that are large enough to still be notable [37].

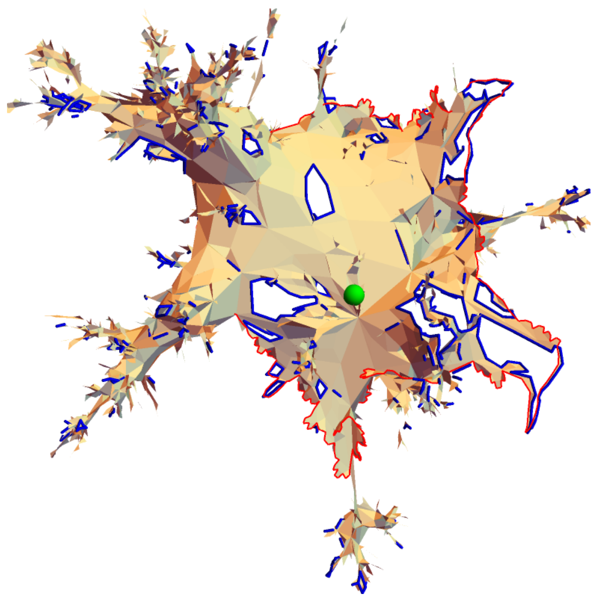

Figure 8: A figure of how a planar map in the non-generic critical case would look like in the $l \to \infty$ limit. The perimeter is highlighted in red. The surface does not have a tree like form and has holes in the surface. These holes are outlined in blue. Source: T. Budd [45].

The interval $(3/2, 5/2)$ can actually be split into two intervals and one point: $(3/2, 2)$, $(2, 5/2)$ and 2. The first interval $(3/2, 2)$ is called the dense phase [37, p. 24-34], while the second interval $(2, 5/2)$ is the dilute phase [37, p. 20-24]. There is therefore a phase transition at $\alpha = 2$. In the dense phase the volume V of a ball with radius r, measured with the graph distance, is exponential in r . In the dilute phase the distance of the ball is of the order r^d, where $d = \frac{a-\frac{1}{2}}{\alpha-2}$ [37]. This means that the surfaces in the two phases have different macroscopic characteristics.
Planar maps from the dense phase also have relatively more holes than planar planar maps from the dilute phase [22]. For a visualisation see Figure 9. However, as they still share the same characteristic of having holes in the surface which are large enough to still be seen in the $l \to \infty$ limit, no distinguishment has been made in this thesis.

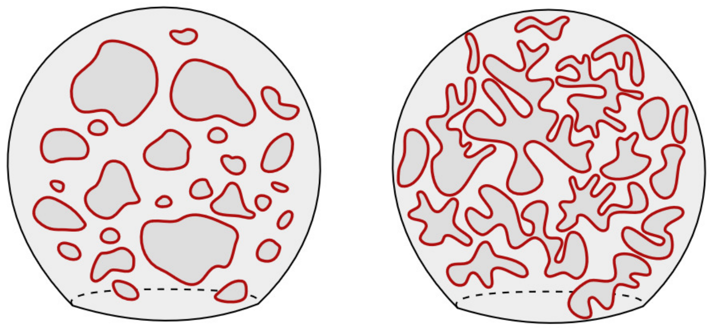

Figure 9: Two sketches of planar maps for the different phases for the non-generic critical model. The planar map to the left is a sketch of a surface from the dilute phase. The planar map to the right is a sketch of a surface in the dense phase. Source: T. Budd [37]

3 Hyperbolic surfaces

In this chapter, hyperbolic surfaces will be analysed to see if one can find a model for continuous surfaces which is analogous to the models for planar maps. Elementary definitions regarding (Riemannian) manifolds can be found in Appendix E.

3.1 Hyperbolic surfaces

Before we can analyze partition functions, the critical exponents, etc. like we did in the planar map case, we need to be clear about what kind of surfaces we are looking at. Hyperbolic surfaces are two-(real)dimensional Riemannian manifolds with constant negative curvature and a conformal structure [46]. Alternatively, hyperbolic surfaces can be considered as a connected one-(complex)dimensional manifold with constant negative curvature [25]. Hyperbolic surfaces are closed and orientable [47] (see Appendix E for definitions).

Every closed orientable surface X is diffeomorphic to a connected sum of a 2-sphere and a finite number of tori [26]. This means that $X \cong \mathbb{S}^2 \ \#_{i=1}^{g} \ \mathbb{T}^2$. We analyze a more general type of hyperbolic surfaces, so-called surfaces **of finite type**. A surface of finite type is a surface one can obtain from a closed surface by removing a finite number of smooth open discs and points. Therefor, one can classify a surface S by three numbers: (g, b, n) [48]:

i. g stands for genus and is the number of tori in the connected sum that the surface S is diffeomorphic to

ii. b stands for the number of disks removed from a closed surface X to get the surface S

iii. n stands for the number of punctures removed from a closed surface X to get the surface S

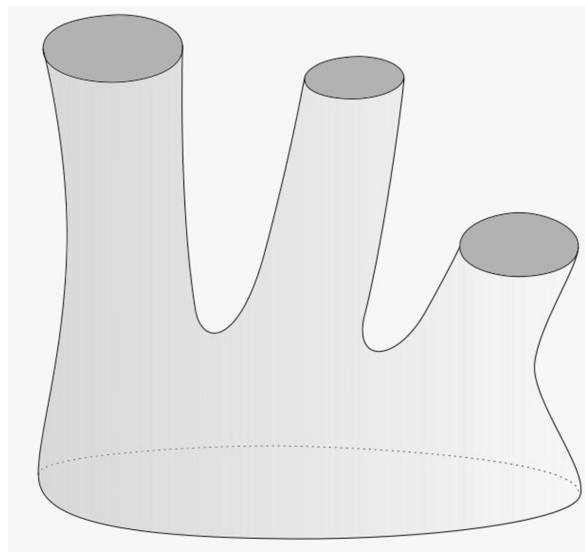

Figure 10: A figure of a hyperbolic surface with four boundary components. The top three are denote by a darker colouring, while the lower one is denoted by the dotted line.

Notation wise, we introduce the idea of a **signature** of a surface, which is conventionally denoted by $\Sigma_{(g,b,n)}$ [26].

The removal of discs corresponds to the surfaces S having **boundary components**. These boundary components are geodesics on the surface and these geodesics will be denoted using $\beta_1, ..., \beta_n$. Note that one can use the Riemannian metric to get the length of the boundary components. We will allow boundary components of lengths 0, which are punctures or alternatively get called **cusps**. See Figure 10 for an example of a surface with four boundary components.

In this thesis, only surfaces with $g = 0$ will be discussed. This implies that $S \cong \mathbb{S}^2 \setminus \{\beta_1, ..., \beta_n\}$, which means that the surfaces discussed will have the topology of the sphere with holes and punctures.

3.2 Weil-Peterson volume

Before we can introduce the partition function of hyperbolic surfaces, we need to discuss the Weil-Peterson volumes. This is an integration over all possible surfaces with a given set of boundary components with specified lengths. The volume will be introduced by first explaining pants decomposition, Teichmüller space, the Moduli space and introducing Fenchel-Nielsen coordinates.

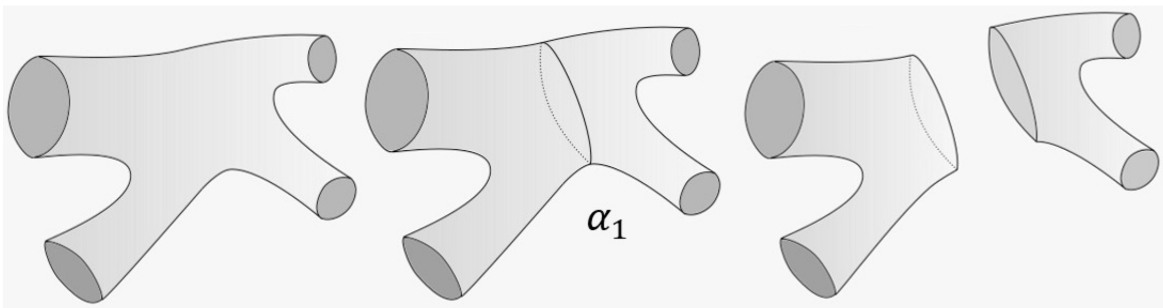

Figure 11: Example of a pants decomposition of a surface S which is diffeomorphic to $\Sigma_{0,4,0}$. The removal of the geodesic α_1 leads to two pairs of pants.

3.2.1 Pants decomposition

The idea of a pants decomposition is to divide a given surface by creating (sub)surfaces with three boundary components (where boundary components of length 0 are allowed). These **pairs of pants**, denoted by P_i, are diffeomorphic to $\Sigma_{0,3,0}$. A **pants decomposition** $\mathcal{P} = \{\alpha_1, ..., \alpha_k\}$ of a complete, orientable surface S of finite area, is a collection of pairwise disjoint simple closed geodesics α_i in S, such that $S \setminus \left(\cup_{i=0}^{k} \alpha_i\right)$ is a collection of pairs of pants [49]. For an example of a pants decomposition, see Figure 19. It has been shown that every complete, orientable hyperbolic surface S of finite area admits a pants decomposition [26, p. 43-46].

3.2.2 Teichmüller space and Moduli space of hyperbolic surfaces

Before we can integrate over surfaces, we need to define a space of hyperbolic surfaces. This space is the Moduli space and it will be defined using the Teichmüller space.
Let $\Sigma_{g,b,n}$ be a surface with a hyperbolic metric. Let $\beta_1, ..., \beta_b$ be the boundary components with respective length $L_1, ..., L_b$. The **Teichmüller space** of $\Sigma_{g,b,n}$ with boundary components $L_1, ..., L_b$ is denoted by $\mathcal{T}_{g,b,n}(L_1, ..., L_b) = \{(X, f) \,|X \text{ is a Riemannian Surface} \wedge f : \Sigma_{g,b,n} \to X\} / \backsim$, where f is an orientation preserving diffeomorphism, such that $L_g(f(\beta_i)) = L_i$. This means that the length of the boundary components is preserved by f. The equivalent relation is defined as follows: $(X, f) \backsim (Y, h)$ if and only if $\exists m : X \to Y$ isometry, such that $h^{-1} \circ m \circ f : \Sigma_{g,b,n} \to \Sigma_{g,b,n}$ is homotopic to the identity [50, p. 21-23]. An alternative shorter notation for the TeichmÜller is $\mathcal{T}(\Sigma_{g,b,n})$. If S is a hyperbolic surface diffeomorphic to $\Sigma_{g,b,n}$, we may also write $\mathcal{T}(S)$ instead.
Having defined Teichmüller space, we want to get to the Moduli space for the same surface $\Sigma_{g,b,n}$, as this is where the Weil-Peterson volumes work on. Let S_0 be a compact hyperbolic surface of finite type such that $\exists\ \Sigma \subset S_0$ which is a finite set such that $\Sigma_{g,b,n} = S_0 \setminus \Sigma$. Having introduced this notation, we define the **mapping class group** $\mathrm{MCG}(\Sigma_{g,b,n})$ of the surface $\Sigma_{g,b,n}$ by

$$\mathrm{MCG}(\Sigma_{g,b,n}) = \mathrm{Diff}^+(\Sigma_{g,b,n}, \delta\Sigma_{g,b,n}, \Sigma) / \mathrm{Diff}_0^+(\Sigma_{g,b,n}, \delta\Sigma_{g,b,n}, \Sigma). \tag{46}$$

In equation (46), the set $\mathrm{Diff}^+(\Sigma_{g,b,n}, \delta\Sigma_{g,b,n}, \Sigma) =$
$\{f : \Sigma_{g,b,n} \to \Sigma_{g,b,n} | f \text{ orientation, boundary component and punctures preserving diffeomorphism}\}$,

where f preserves boundary components set-wise, and punctures point-wise. The set $\text{Diff}_0^+(\Sigma_{g,b,n}, \delta\Sigma_{g,b,n}, \Sigma) = \left\{f : \Sigma_{g,b,n} \to \Sigma_{g,b,n} | f \in \text{Diff}^+(\Sigma_{g,b,n}, \delta\Sigma_{g,b,n}, \Sigma) \wedge f \text{ homotopic to Id}\right\}$ [51]. Now, one can define the **Moduli space** $\mathcal{M}_{g,b,n}(L_1, ..., L_b)$

$$\mathcal{M}_{g,b,n}(L_1, ..., L_b) = \mathcal{T}_{g,b,n}(L_1, ..., L_b) / \text{MCG}(\Sigma_{g,b,n}). \tag{47}$$

Note that $\text{MCG}(S)$ leaves boundary components and punctures 'marked', meaning that if surface S and S' are isometric, but all isometries permute the boundary components or punctures, they represent two different points in $\mathcal{M}_{g,b,n}(L_1, ..., L_b)$ [26, p. 65].

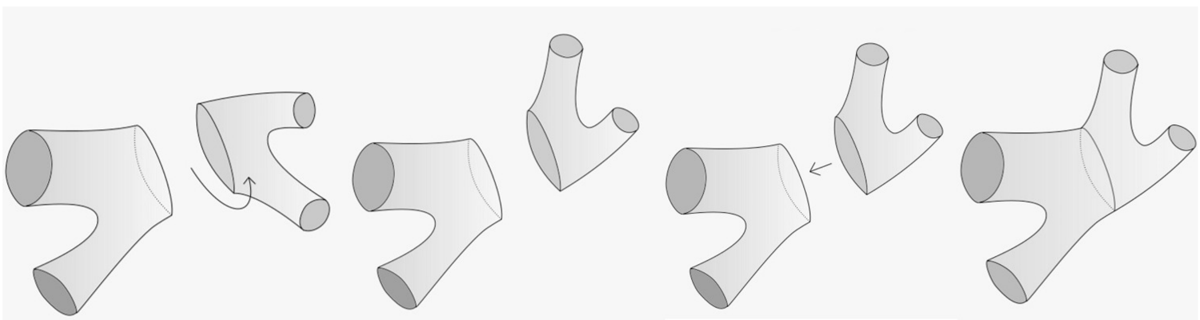

Figure 12: Example of a creation of a new surface from a hyperbolic surface with four boundary components with given lengths, using pants decomposition. Using the pair of pants decomposition, one can get two different pair of pants. Twisting one of the pair of pants and then 'glueing' the pants together gives us a surface that is different from the original surface. However is is still diffeomorphic to $\Sigma_{0,4,0}$, and has boundary components with the same lengths as the previous surface.

3.2.3 Fenchel-Nielsen coordinates

The intuition behind the Weil-Peterson volume and Frenchel-Nielsen coordinates can be seen in Figure 12. Using a pants decomposition $\mathcal{P}$ for a hyperbolic surface S, one can 'twist' the pairs of pants in $S \setminus \mathcal{P}$ to create new surfaces. Combining the twist with changing the lengths between the boundary components of the pair of pants, we get different surfaces even though they have boundary components with the same lengths.

More formally, one needs to introduce so-called Fenchel-Nielsen coordinates in order to get the Weil-Peterson volume. The coordinates are given by the **length function** $l_\alpha : \mathcal{T}(S) \to \mathbb{R}^{3g-3+n}$ and the **twist function** $\tau_\alpha : \mathcal{T}(S) \to \mathbb{R}$, where α is a simple closed geodesic that is part of a pants decomposition $\mathcal{P}$ of S. The volume is defined by integrating using these coordinates. In the following sections the functions will be derived for a surface S of finite type without boundary that admits a complete metric, given any pants decomposition $\mathcal{P} = \{\alpha_1, ..., \alpha_k\}$.

Let α be an essential closed curve on S. Note that every $[X, f] \in \mathcal{T}(S)$ can be seen as a marked hyperbolic surface. Remember that $f : \Sigma_{g,b,n} \to X$ is an orientation and boundary component length preserving diffeomorphism. There is therefore a unique geodesic δ_α in the homotopy class of $f(\alpha)$, which is of minimal length [26, p. 43-45]. We now define the length function l_α by

$$l_\alpha([X, f]) = (L_g(\delta_\alpha))_{\alpha \in \mathcal{P}}, \tag{48}$$

were g is the hyperbolic metric.

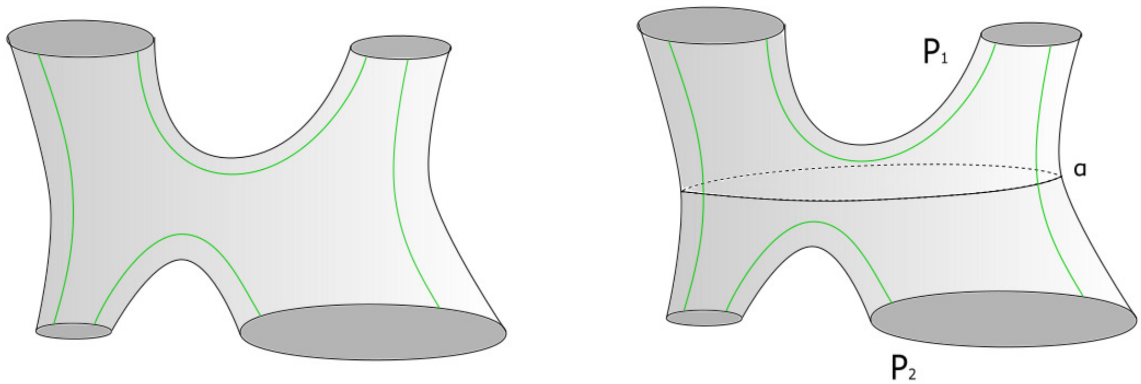

Figure 13: An example of a Riemannian surface of of signature $\Sigma_{0,4,0}$ with a collection of disjoint simple closed curves Γ coloured in green. The surface to the left can be decomposed into two pairs of pants P_1 and P_2 by the curve α, shown in black on the surface to the right. The intersection of $\Gamma \cap P_i$ connects the boundary components of of pair of pants P_i.

For the construction of the twist function, we introduce $\Gamma = \{\gamma_1, ..., \gamma_k\}$, which is a collection of disjoint simple closed curves, such that for every pair of pants $P_i \in S \setminus \mathcal{P}$: $\Gamma \cap P_i = \{\tilde{\gamma}_{i,1}, \tilde{\gamma}_{i,2}, \tilde{\gamma}_{i,3}\}$, where the curves $\tilde{\gamma}_{i,k}$ connect different pairs of boundary components of P_i. For an example of these curves $\tilde{\gamma}_{i,k}$, see the green lines on the surface on the right in Figure 13.
Take an $\alpha \in \mathcal{P}$, and for simplicity sake, assume that α bounds two pairs of pants P_1 and P_2. Let $f : S \to X$ be an orientation preserving diffeomorphism between the surface S and a hyperbolic surface X. As f is orientation preserving, it maps the pants decomposition $\mathcal{P}$ to another pants decomposition $f(\mathcal{P})$. Furthermore, for any $\hat{\gamma} \in (P_1 \cup P_2) \cap \Gamma$, $f(\hat{\gamma})$ is an arc between two boundary components of $f(P_1)$ with length L_1 and $f(P_2)$ with length L_2. Let $\delta_{f(\alpha)}$ be the unique geodesic in the free homotopy class of $f(\alpha)$ on X, introduced in Section 3.2.2. Let η_1, η_2 be the unique perpendiculars between the boundary components with length L_1 or L_2 and δ. The relative boundary of $f(P_1 \cup P_2)$, the arc $f(\hat{\gamma})$ is freely homotopic to $\eta_2 \circ \delta^k \circ \eta_1$, for a specific $k \in \mathbb{Z}$. The twist τ_α along α for $[X, f] \in \mathcal{T}(\Sigma_{g,b,n})$ is defined by [26, p. 86-87]

$$\tau_\alpha([X, f]) = k \cdot l_\alpha([X, f]) \pm d(p_1, p_2), \tag{49}$$

where $p_i = \eta_i \cap \delta$, and the orientation of the sign is positive if the twist along $\delta \in X$ is to the left with respect to the orientation of X and negative if it's to the right. Note that $\tau_\alpha([S, f]) \in \mathbb{R}$. See Figure 14 for a visualisation of the different kind of curves.

We can now define the **Fenchel-Nielsen map** $\mathrm{FN}_\mathcal{P}$ given a pants decomposition $\mathcal{P}$ for a hyperbolic surface with signature $\Sigma_{g,b,n}$. The map is given by

$$\begin{aligned} \mathrm{FN}_\mathcal{P} : \mathcal{T}(\Sigma_{g,b,n}) &\to \mathbb{R}_+^{3g-3+b+n} \times \mathbb{R}^{3g-3+b+n} \\ ([X, f]) &\mapsto (l_\alpha([X, f]), \tau_\alpha([X, f]))_{\alpha \in \mathcal{P}}. \end{aligned} \tag{50}$$

This map is a bijection, giving us a topology on $\mathcal{T}_{g,b,n}(L_1, ..., L_n)$ and $\mathcal{M}_{g,b,n}(L_1, ..., L_n)$. This also shows that the space of hyperbolic surfaces, the Moduli space, is finite-dimensional, meaning that it is indeed useful to consider hyperbolic surfaces, as outlined in Section 1 [52, p. 320-328].

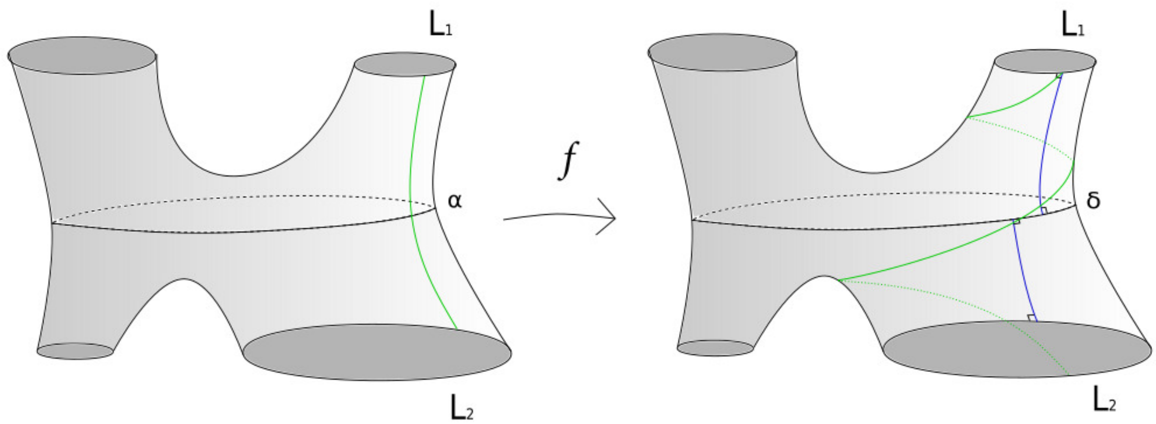

Figure 14: The surface to the left is decomposed into two pairs of pants by the curve α, shown in black. The line between L_1 and L_2 is a curve $\hat{\gamma} \in (P_1 \cup P_2) \cap \Gamma$ is the green line on the left surface. The surface to the right is decomposed in pairs of pants by δ, shown in black. The curve $\hat{\gamma}$ is sent to $f(\hat{\gamma})$, which is represented by the green line. The unique perpendicular geodesics from δ to L_1 or L_2 are shown by blue the blue lines.

3.2.4 The Weil-Peterson formula

For this section some basic knowledge of differential forms is assumed. Definition of terms can be found in Section E.7.
Having defined the Fenchel-Nielsen coordinates, giving us a topology on the Teichmüller and Moduli space, one can get a natural Weil-Peterson sympletic form on the Teichmüller space $\mathcal{T}_{g,b,n}(L_1, ..., L_n)$ given by [53, p. 8-9]

$$\omega_{\mathrm{WP}} = \frac{1}{2} \sum_{\alpha \in \mathcal{P}} dl_\alpha \wedge d\tau_\alpha. \tag{51}$$

The volume form that follows is defined as

$$d\mathrm{vol}_{\mathrm{WP}} = \frac{2^{3g+n-3}}{(3g+n-3)!} \wedge^{3g+n-3} \omega_{\mathrm{WP}}. \tag{52}$$

This sympletic form can in turn gives us an integral by

$$V_{g,n}(L_1, ..., L_n) = \int_{\mathcal{M}_{g,b,n}(L_1,...,L_n)} d\mathrm{vol}_{\mathrm{WP}}. \tag{53}$$

We therefore have an explicit formulation of the **Weil-Peterson volume** $V_{g,n}(L_1, ..., L_n)$ for surfaces with genus g and n boundary components of length $L_1, ..., L_n$. There is recursive formulation of the Weil-Peterson volumes found by Mirzhakani [54]. In this thesis, we will use this recursion implicitly in Section 3.3.3 to analyze the partition function for hyperbolic surfaces.

3.3 Partition function $F_0^{\mathbf{WP}}$ for hyperbolic surfaces

To be able to analyze critical exponents and therefore universality classes, we need to define the partition function for hyperbolic surfaces with specified boundary components. Using Section 3.2, we will introduce the function in this section.

3.3.1 Weight function

To be able to control the kind of surfaces we are looking at and to define the partition function, we need to define the probability of a surface having a boundary component of length L. This is done by introducing the **weight function** $\mu(L)$. This is a continuous analogue for the discrete weight sequence introduced in the planar map case in Section 2.2. As done in the planar map case, we only consider weight functions which are admissible, meaning that the partition function is finite.

3.3.2 Explicit formula

Using sections 3.2 and 3.3.1 we now define the partition function F_0^{WP} as

$$F_0^{\mathrm{WP}} = \sum_{n=3}^{\infty} \frac{2^{2-n}}{n!} \int_0^{\infty} dL_1 ... \int_0^{\infty} dL_n \mu(L_1) ... \mu(L_n) V_{0,n}(L_1, ..., L_n). \tag{54}$$

In equation (54) $V_{0,n}(L_1, ..., L_n)$ is the Weil-Peterson volume for hyperbolic surfaces with n boundary components with length $L_1, ..., L_n$ and $\mu(L)$ is the weight of the boundary component of length L. In equation (54) the 0 in the indexes are there to denote the fact that we only consider surfaces with genus 0. The $\frac{1}{n!}$ is introduced to counter over counting, as there are $n!$ ways to enumerate the boundary components. The 2^{2-n} factor is there for convention. It has been introduced to the partition function to show similarities with the generating function for 2π-irreducible maps [55].

3.3.3 The generating function and the string equation

In the planar map case, we had introduced a new formulation of the partition functions to analyze them. For the hyperbolic surface case we will do the same. There is a generating function for the partition function defined in equation (54) [55, p. 26]

$$F_0^{\mathrm{WP}} = \frac{1}{4} \int_0^R dr Z_\mu(r)^2, \tag{55}$$

where the function $Z_\mu(r)$ is given by

$$Z_\mu(r) = \frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r}) - \int_0^{\infty} \mu(L) I_0(L\sqrt{r}) dL. \tag{56}$$

In equation (56) J_1 and I_0 are Bessel functions (see Section F for definitions), L is the length of a boundary component and $\mu(L)$ is the weight of a boundary component of length L.
We have not yet defined the R term in equation (55). It is (implicitly) defined using $Z_\mu(r)$ from equation (56) in the following way

$$Z_\mu(R) = 0. \tag{57}$$

Where R is the first solution for this equation. Equation (57) is called the **String equation** and is one of the KdV-equations outlined in Section 1.3.2. The string equation shows that R is (implicitly) dependent on the weight function $\mu(L)$, which is why we introduce the notation $R = R_\mu$.
Having defined $Z_\mu(r)$ in equation (56), we can define admissibility for $\mu(L)$ a bit more explicitly. The weight function $\mu(L)$ is called **admissible** if $Z_\mu(r) = 0$ has a solution.

3.4 Marked hyperbolic surfaces

Having introduced the generating function F_0^{WP} of Weil-Peterson volumes in Section 3.3.3, one can define the partition function for hyperbolic surfaces with marked boundary components and analyze the adapted form of the generating function from equation (55).
The partition function for bipartite planar maps was calculated using random walks, where the chance

of making a step was implicitly dependent on the function $f_{\mathbf{q}}$, which was defined using the weight sequence. This used the discrete nature of planar maps. In the hyperbolic surfaces model, one is dealing with non-discrete surfaces, meaning that there is no clear analogue for the random walk. We therefore need to resort to a different way to get the partition function.

In the hyperbolic surface case, the marking of a boundary component of length L' coincides with taking the functional derivative of F_0^{WP} with respect to $\mu(L')$ [55]. This implies that the explicit formulation of the partition function is the same as in equation (54), except the boundary components considered are the non-marked boundary components.

The generating function for surfaces with one marked boundary can not be found easily and will be done in the Section 3.4.2. As we will see, it is easier to take two functional derivatives, meaning we will consider the partition function for hyperbolic surfaces with two marked boundary components. The generating function for the two marked boundary components case is called the **cylinder function**, while the generating function for the one marked boundary component case is called the **disc function**.

3.4.1 Cylinder function $\mathcal{W}(L_i, L_j)$

The cylinder function $\mathcal{W}(L_i, L_j)$ is the generating function for hyperbolic surfaces with two marked boundary components of length L_i and L_j and is given by

$$\mathcal{W}(L_i, L_j) = \sum_{n=3}^{\infty} \frac{2^{2-n}}{(n-2)!} \int_0^{\infty} dL_3 ... \int_0^{\infty} dL_n \mu(L_3) ... \mu(L_n) L_i L_j V_{0,n}(L_i, L_j, L_3, ..., L_n). \tag{58}$$

It is clear from equation (59), that $\mathcal{W}(L_i, L_j)$ the functional derivative of F_0^{WP} from equation (54) with respect to $\mu(L_i)$ and $\mu(L_j)$ is. Analogously, the generating function $\mathcal{W}_\bullet(L)$ is a partition for hyperbolic surfaces with one marked boundary component of length L and one marked cusp and is defined as

$$\mathcal{W}_\bullet(L') = \sum_{n=3}^{\infty} \frac{2^{2-n}}{(n-2)!} \int_0^{\infty} dL_3 ... \int_0^{\infty} dL_n \mu(L_3) ... \mu(L_n) L' V_{0,n}(0, L', L_3, ..., L_n). \tag{59}$$

As we have another formulation of F_0^{WP} given by equation (55), we can calculate a different formulation of $\mathcal{W}(L_i, L_j)$ and $\mathcal{W}_\bullet(L')$. We will do this, using some calculations can be found in Appendix F.4. They will be explicitly referred to if used. For example, in the following equations, the calculations from Appendix F.4.1 are used. The functional derivative of F_0^{WP} with respect to $\mu(L')$ is given by

$$\begin{aligned} \frac{\partial F_0^{\mathrm{WP}}}{\partial \mu(L')} = \partial_{\mu(L')} F_0^{\mathrm{WP}} &= \frac{1}{4} \partial_{\mu(L')} \int_0^R dr Z_\mu(r)^2 \\ &= \frac{1}{4} \cdot Z_\mu(R)^2 \cdot \partial_{\mu(L')} R + \frac{1}{4} \int_0^R dr \cdot -2 \ Z_\mu(r) \cdot \partial_{\mu(L')} Z_\mu(r) \\ &= -\frac{1}{2} \int_0^R dr \ Z_\mu(r) I_0(L' \sqrt{r}). \end{aligned} \tag{60}$$

Taking the second functional derivative and using equation (189) we see that

$$\begin{aligned} \partial_{\mu(L_j)} \partial_{\mu(L_i)} F_0^{\mathrm{WP}} &= -\frac{1}{2} \ \partial_{\mu(L_j)} \int_0^R dr \ Z_\mu(r) I_0(L_i \sqrt{r}) \\ &= -\frac{1}{2} \cdot Z_\mu(r) I_0(L_i \sqrt{r}) \cdot \partial_{\mu(L_j)} R + -\frac{1}{2} \int_0^R dr I_0(L_i \sqrt{r}) \ \partial_{\mu(L_j)} Z_\mu(r) \\ &= \frac{1}{2} \int_0^R dr \ I_0(L_j \sqrt{r}) I_0(L_i \sqrt{r})) \\ &= \left\{ \begin{array}{ll} \frac{\frac{\sqrt{R}}{2}\left[L_i I_1(L_i\sqrt{R}) I_0(L_j\sqrt{R}) - L_j I_1(L_j\sqrt{R}) I_0(L_i\sqrt{R})\right]}{L_i^2 - L_j^2} & \text{if } L_i \neq L_j \\ \frac{R}{2}\left[I_0(L_i\sqrt{R})^2 - I_1(L_i\sqrt{R})^2\right] & \text{if } L_i = L_j \end{array} \right\}. \end{aligned} \tag{61}$$

Having calculated the derivatives of F_0^{WP}, we can now define the cylinder function

$$\begin{aligned}\mathcal{W}(L_i, L_j) &:= L_j L_i \partial_{\mu(L_j)} \partial_{\mu(L_i)} F_0^{\mathrm{WP}} \\ &= L_j L_i \cdot \left\{ \begin{array}{ll} \frac{\frac{\sqrt{R}}{2}\left[L_i I_1(L_i\sqrt{R}) I_0(L_j\sqrt{R}) - L_j I_1(L_j\sqrt{R}) I_0(L_i\sqrt{R})\right]}{L_i^2 - L_j^2} & \text{if } L_i \neq L_j \\ R\left[I_0(L_i\sqrt{R})^2 - I_1(L_i\sqrt{R})^2\right] & \text{if } L_i = L_j \end{array} \right\}.\end{aligned} \tag{62}$$

Using equation (62), we can define the generating function for hyperbolic surfaces with one marked boundary component of length L' and one marked cusp as

$$\begin{aligned}\mathcal{W}_\bullet(L') &:= \frac{\mathcal{W}(L', L_j)}{L_j}\Big|_{L_j=0} = L' \frac{\frac{\sqrt{R}}{2}\left[L' I_1(L'\sqrt{R})\right]}{L'^2} \\ &= \frac{\sqrt{R}}{2}\, I_1(L'\sqrt{R}).\end{aligned} \tag{63}$$

The generating function $\mathcal{W}(L_i, L_j)$ is the hyperbolic surface analogue for the planar map partition function $W_{2p}^{(2l)}$, while $\mathcal{W}_\bullet(L')$ is the analogue for the partition function $W_\bullet^{(2l)}$.

3.4.2 Disc function $\mathcal{W}(L')$

The disc function $\mathcal{W}(L')$ is the generating function for hyperbolic surfaces with one marked boundary component of length L'. We define $\mathcal{W}(L')$ in the same manner as $\mathcal{W}(L_i, L_j)$, using equation (60)

$$\begin{aligned}\mathcal{W}(L') &= L' \partial_{\mu(L')} F_0^{\mathrm{WP}} = \frac{-L'}{2} \int_0^R dr Z_\mu(r) I_0(L'\sqrt{r}) \\ &= \frac{-L'}{2} \int_0^R dr \left[\frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r}) - \int_0^\infty \mu(L) I_0(L\sqrt{r}) dL\right] I_0(L'\sqrt{r}).\end{aligned} \tag{64}$$

It is clear from equation (64), that the calculation for $\mathcal{W}(L')$ is not as easy as the $\mathcal{W}(L_i, L_j)$ case. To make it easier for our self, we split the calculation of equation (64) into two parts.

There are multiple ways to calculate the first term in equation (64). Only one is discussed in this section, a second way can be found in Appendix H. Naturally, they give the same solution. Using partial integration and equation (189) one gets

$$\begin{aligned}\int_0^R \frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r}) I_0(L'\sqrt{r}) dr &= \frac{\sqrt{R}}{\pi} J_1(2\pi\sqrt{R}) \frac{2\sqrt{R}}{L'} I_1(L'\sqrt{R}) - \int_0^R \frac{2\sqrt{r}}{L'} J_0(2\pi\sqrt{r}) I_1(L'\sqrt{r}) dr \\ &= \frac{R}{\pi L'} J_1(2\pi\sqrt{R}) I_1(L'\sqrt{R}) \\ &\quad - \frac{2}{L'} \frac{\partial}{\partial L'} \frac{R\left\{L' J_0(2\pi\sqrt{r}) I_1(L'\sqrt{R}) - 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\right\}}{L'^2 - 4\pi^2}.\end{aligned} \tag{65}$$

Calculating the derivative in equation (65), one sees that

$$
\begin{aligned}
&\frac{\partial}{\partial L'} \frac{\sqrt{R}\{L' J_0(2\pi\sqrt{r}) I_1(L'\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\}}{L'^2 + 4\pi^2} = \\
&- \frac{4L'\sqrt{R}\big[L' J_0(2\pi\sqrt{R}) I_1(L'\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 + 4\pi^2)^2} \\
&+ 2\sqrt{R}\Bigg[\frac{I_1(L'\sqrt{R}) J_0(2\pi\sqrt{R})}{L'^2 + 4\pi^2} + \frac{\frac{1}{2}L'\sqrt{R}\ \{I_0(L'\sqrt{R}) + I_2(L'\sqrt{R})\}\ J_0(2\pi\sqrt{R})}{L'^2 + 4\pi^2} \\
&+ \frac{2\pi\sqrt{R} I_1(L'\sqrt{R}) J_1(2\pi\sqrt{R})}{L'^2 - 4\pi^2}\Bigg] = \\
&- \frac{4L'\sqrt{R}\big[L' J_0(2\pi\sqrt{R}) I_1(L'\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 + 4\pi^2)^2} \\
&+ \frac{2\sqrt{R}\big[L'\sqrt{R} I_0(L'\sqrt{R}) J_0(2\pi\sqrt{R}) + 2\pi\sqrt{R} I_1(L'\sqrt{R}) J_1(2\pi\sqrt{R})\big]}{L'^2 + 4\pi^2}.
\end{aligned} \tag{66}
$$

The second term of equation (64) is given by

$$
\begin{aligned}
\frac{L'}{2}\int_0^R dr \left[\int_0^\infty \mu(L) I_0(L\sqrt{r}) dL\right] I_0(L'\sqrt{r}) &= \frac{L'}{2}\int_0^\infty \mu(L) dL \int_0^R I_0(L'\sqrt{r}) I_0(L\sqrt{r}) dr \\
&= L' \int_0^\infty \mu(L)\ dL\ \ \partial_{\mu(L)}\partial_{\mu(L')} F_0^{\mathrm{WP}} \\
&= \int_0^\infty \mu(L)\ dL\ \ \frac{\mathcal{W}(L', L)}{L}.
\end{aligned} \tag{67}
$$

Combining equations (64), (67) and (65) to fill in equation (64), one gets

$$
\begin{aligned}
\mathcal{W}(L') &= \frac{L'}{2}\int_0^R dr Z_\mu(r) I_0(L'\sqrt{r}) \\
&= \frac{R}{\pi} J_1(2\pi\sqrt{R}) I_1(L'\sqrt{R}) + \frac{4L'\sqrt{R}\big[L' J_0(2\pi\sqrt{R}) I_1(L'\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 + 4\pi^2)^2} \\
&- \frac{2\sqrt{R}\big[L'\sqrt{R} I_0(L'\sqrt{R}) J_0(2\pi\sqrt{R}) + 2\pi\sqrt{R} I_1(L'\sqrt{R}) J_1(2\pi\sqrt{R})\big]}{L'^2 + 4\pi^2} - \int_0^\infty \mu(L)\ dL\ \ \frac{\mathcal{W}(L', L)}{L}.
\end{aligned} \tag{68}
$$

The generating function $\mathcal{W}(L')$ is the hyperbolic surface analogue generating function $W^{(2l)}$ from the planar maps model.

3.5 Criticalities

We want to introduce criticality conditions to analyze similar universality classes that were outlined in the planar map models (see Section 2.7). This will be done in this section, by first taking a closer look at R_μ from Section 3.3.3. We will then, just like in the planar map case, study the critical components for the different criticalities. This will later help us characterise how the surfaces look like for different groups of weight functions.

3.5.1 Analysis of R_μ

In Section 3.3.3 we have implicitly defined R_μ in the String equation (57). In this section, we will take a deeper look into R_μ as we will be classifying when a weight function is subcritical, generic critical

or non-generic critical.
Equation (63) gives us a hint for the physical interpretation of R_μ. Let us consider the generating function F_0^{WP} for hyperbolic surfaces with two marked cusps

$$\begin{aligned}
\partial_{\mu(0)}\partial_{\mu(0)}F_0^{\mathrm{WP}} &= \frac{1}{4}\partial_{\mu(0)}\int_0^R drZ_\mu(r)\cdot 2\left\{-\int_0^\infty \partial_{\mu(0)}\mu(L)I_0(L\sqrt{r})dL\right\} \\
&= \frac{1}{2}\partial_{\mu(0)}\int_0^R drZ_\mu(r)\cdot\left\{-\int_0^\infty \delta\left(L\right)I_0(L\sqrt{r})dL\right\} = \frac{1}{2}\partial_{\mu(0)}\int_0^R drZ_\mu(r)I_0(0\cdot\sqrt{r}) \\
&= \frac{1}{2}\int_0^{R_\mu} dr[I_0(0)\ I_0(0)] = \frac{R_\mu}{2}.
\end{aligned} \tag{69}$$

As one can see from equation (69), R_μ is the generating function for hyperbolic surfaces with two marked cusps, where the cusps are no yet labeled differently from each other. For an example of such a surface, see Figure 15. Note that equation (54) implies that all coefficients of R_μ, are positive, as the coefficients of the partition function of hyperbolic surfaces is positive.

There is another way to analyse R_μ, which is by studying the string equation (57). Note again, that R_μ is a function depending on the weight function. Combining the chain rule, the Taylor expansion and equation (56) gives us

$$\frac{\mathrm{d}}{\mathrm{d}\mu(L')}\left[Z_\mu(R_\mu)\right] = \left[\frac{\partial Z_\mu\left(r\right)}{\partial\mu(L)}\right]\Bigg|_{r=R_\mu} + Z'_\mu(R_\mu)\cdot R'_\mu = 0. \tag{70}$$

In equation (70), we use the notation $Z'_\mu(R_\mu) = \left(\partial_r Z_{x,\mu}\left(r\right)\right)\big|_{r=R_\mu}$ and $R'_\mu = \partial_{\mu(L)}R_\mu$. Rewriting equation (70) and using equation (56) shows that

$$\partial_{\mu(L)}R_\mu = \frac{\left[\frac{\partial Z_\mu(r)}{\partial\mu(L)}\right]\Big|_{r=R_\mu}}{Z'_\mu(R_\mu)} = \frac{I_0(L\sqrt{R_\mu})}{J_0(2\pi\sqrt{R_\mu}) - \int_0^\infty dL'\mu(L')I_1(L'\sqrt{R_\mu})\frac{L'}{2\sqrt{R_\mu}}}. \tag{71}$$

This gives us and explicit formulation of the derivative of R_μ with respect to μ. However, it not easily solvable as there are still R_μ terms inside the Bessel functions on the right side of equation (71). Using this relation, we will define the different forms of criticality.

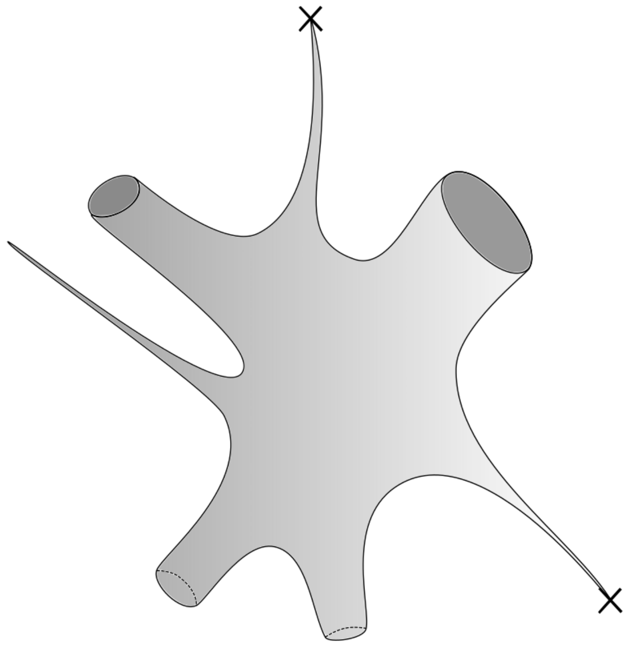

Figure 15: An example of a hyperbolic surface with two marked cusps, one unmarked cusp and four unmarked boundary components. The marked cusps are denoted by the black crosses and they are not labelled any number.

3.5.2 Definitions of criticality

Analogously to the planar map case, we want to define different forms of criticality. Let R_μ be the first solution to equation (57), we define for $\mu(L)$ the following:

- The weight function $\mu(L)$ is called **subcritical** if $Z'_\mu(R_\mu) \neq 0$.
- The weight function $\mu(L)$ is called **generic critical** if $Z'_\mu(R_\mu) = 0$ and the radius of convergence of $Z_\mu(r)$ is larger than R_μ
- The weight function $\mu(L)$ is called **non-generic critical** if $Z'_\mu(R_\mu) = 0$ and $Z_\mu(r)$ for r around R_μ has leading term $(R_\mu - r)^{\alpha-1/2}$, where $\alpha \in (3/2, 5/2)$

. One can see from equation (71), that if $Z'_\mu(R_\mu) \to 0$, then $\partial_{\mu(L)} R_\mu \to \infty$. This means that if the weight function $\mu(L)$ is chosen such that $Z'_\mu(R_\mu) = 0$, then $\partial_{\mu(L')} R_\mu$ diverges. As seen above, this happens when the weight functions are critical. In the case that the weight function is chosen such that $Z'_\mu(R_\mu) \nrightarrow 0$, we have a weight function that is subcritical. This means that the criticality condition for the weight function $\mu(L)$ is

$$Z'_\mu(R_\mu) = J_0(2\pi\sqrt{R_\mu}) - \int_0^\infty dL\mu(L)I_1(L\sqrt{r})\frac{L}{2\sqrt{R_\mu}} = 0. \tag{72}$$

Consider how we have defined criticality for the planar map case (see Section 2.4). Note that for the hyperbolic surface case, we have defined criticality very similar to how we did in the planar map case. Firstly, the weight function/sequence determines a term (R_μ and $Z_{\mathbf{q}}$ respectively) for which a function gives a zero ($Z_\mu(r)$ and $f_{\mathbf{q}}(x) - x$ respectively). Secondly, the form of criticality determines the leading terms for that function. They even share similar exponents in these leading terms.

3.5.3 Criticality analysis for $x \cdot \mu(L)$

We want to be able to describe the behaviour of the coefficients in R_μ that are linked to hyperbolic surfaces with n boundary components, where the marked cusps are not included. This is done by adding a weight x to every boundary component. This is equivalent to changing $\mu(L)$ to $\mu_x(L) = x \cdot \mu(L)$. The goal of this section is to analyse $[\![R_{x,\mu}]\!]_{x^n}$, as this gives us the contribution of hyperbolic surfaces with n boundary components.
Let's consider an adapted form for the function $Z_\mu(r)$ and its derivative. They become

$$Z_{x,\mu}(r) = \frac{\sqrt{r}}{\pi} J_1\left(2\pi\sqrt{r}\right) - x\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{r}\right) \tag{73}$$

$$Z'_{x,\mu}(r) = J_0(2\pi\sqrt{r}) - x\int_0^\infty dL\mu(L) I_1(L\sqrt{r})\frac{L}{2\sqrt{r}}. \tag{74}$$

Analogously to the R_μ case, we want to define a $R_{x,\mu}$ for the $x \cdot \mu(L)$ case. In this context, $R_{x,\mu}$ is the first solution to $Z_{x,\mu}(R_{x,\mu}) = 0$. It is clear from equations (73) and (74), that one gets the old equations back by filling in $x = 1$. We therefore have $R_{1,\mu} = R_\mu$ from Section 3.5.1.

Note that further calculation with respect to the cylinder function and disc function calculated in Section 3.4 remain the same in this case. This is because changing the functional derivative of the changed partition function F_0^{WP} to be with respect to $\mu_x(L)$ gives the same results. This shows that adding x is just a mere mathematical tool which helps us analyze the partition function for hyperbolic surfaces with two undistinguished marked cusps.

3.5.4 Coefficients of $R_{x,\mu}$

In this section, we will look at the coefficients of $R_{x,\mu}$ in the subcritical, generic critical and non-generic critical case.

Theorem 3.1. *Let $R_{x,\mu}$ be the generating function of hyperbolic surfaces with two marked cusps for the $x \cdot \mu(L)$ case, where $R_{x,\mu}$ is the first solution for the String equation with $Z_{x,\mu}$ given by equation (73). Let $\mu(L)$ be an admissible weight function such that $\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_\mu}\right) \neq 0$. Let $x \in \mathbb{R}_{>0}$. The following hold:*

i. If $\mu(L)$ is subcritical, then $\sum_{n=1}^\infty [\![R_{x,\mu}]\!]_{x^n} c^n < \infty$ for a constant $c \in \mathbb{R}_{>1}$.

ii. If $\mu(L)$ is generic critical, then $[\![R_{x,\mu}]\!]_{x^n} \backsim c\ n^{-3/2}$, where $c \in \mathbb{R}_{>0}$.

iii. If $\mu(L)$ is non-generic critical, then $[\![R_{x,\mu}]\!]_{x^n} \backsim c\ n^{-1/\gamma-1}$, where $\gamma \in (1,2)$ and $c \in \mathbb{R}_{>0}$.

Proof. First we will do some general analysis of $R_{x,\mu}$, which will be used to prove the theorem for the different forms of criticalities. Let $R_{x,\mu}$ and $\mu(L)$ be given as described above. Let ρ be such that $Z_{\rho,\mu}$ is an analytical function at $R_{\rho,\mu}$. We can have a Taylor expansion of $Z_{x,\mu}(R_{x,\mu})$ in x around ρ. This is given by

$$\begin{aligned} Z_{x,\mu}(R_{x,\mu}) &= Z_{\rho,\mu}(R_{\rho,\mu}) + \frac{\mathrm{d}Z_{\rho,\mu}(R_{\rho,\mu})}{\mathrm{d}x}(x-\rho) + o\left((x-\rho)^2\right) \\ &= \left[\frac{\partial Z_{x,\mu}(R_{x,\mu})}{\partial x} + \frac{\partial Z_{x,\mu}(R_{x,\mu})}{\partial R_{x,\mu}}\frac{\partial R_{x,\mu}}{\partial x}\right]\Bigg|_{x=\rho}(x-\rho) + o\left((x-\rho)^2\right). \end{aligned} \tag{75}$$

We want to analyze the $x-$dependency of $R_{x,\mu}$ to be able to analyze the coefficients of x^n. Using equation (75) and the fact that $Z_{x,\mu}(R_{x,\mu}) = 0$, we get

$$\frac{\mathrm{d}Z_{x,\mu}(R_{x,\mu})}{\mathrm{d}x} = \frac{\partial Z_{x,\mu}(R_{x,\mu})}{\partial x} + \frac{\partial Z_{x,\mu}(R_{x,\mu})}{\partial R_{x,\mu}}\frac{\partial R_{x,\mu}}{\partial x} = 0. \tag{76}$$

Rewriting equation (76) gives us the possibility to express the derivative $\frac{\partial R_{x,\mu}}{\partial x}$, like we did in equation (71),

$$
\begin{aligned}
\frac{\partial R_{x,\mu}}{\partial x} &= -\frac{\partial Z_{x,\mu}\left(R_{x,\mu}\right)}{\partial x} / \frac{\partial Z_{x,\mu}\left(R_{x,\mu}\right)}{\partial R_{x,\mu}} \\
&= \frac{\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_{x,\mu}}\right)}{J_0(2\pi\sqrt{R_{x,\mu}}) - x\int_0^\infty dL' \mu(L') I_1(L'\sqrt{R_{x,\mu}})\frac{L'}{2\sqrt{R_{x,\mu}}}}.
\end{aligned} \tag{77}
$$

We therefore see that $R_{x,\mu}$ is analytical up to x^*, where x^* is such that $\left(\partial_r Z_{x^*,\mu}\left(r\right)\right)\big|_{r=R_\mu(x^*)} = 0$. This means that x^* is the convergence radius of $R_{x,\mu}$.
Having found the (implicit) convergence radius of $R_{x,\mu}$, we can express x as a function of $R_{x,\mu}$ using equation (73)

$$
x\left(R_{x,\mu}\right) = \frac{\frac{\sqrt{R_{x,\mu}}}{\pi} J_1\left(2\pi\sqrt{R_{x,\mu}}\right)}{\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_{x,\mu}}\right)}. \tag{78}
$$

Note that x is an analytical function up to x^* if $\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_\mu}\right) \neq 0$, which holds by assumption on $\mu\left(L\right)$. This implies that we can look at the derivative of x with respect to $R_{x,\mu}$, for $x < x^*$

$$
x'\left(R_{x,\mu}\right) = \frac{J_0(2\pi\sqrt{R_{x,\mu}})\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_{x,\mu}}\right) - \frac{\sqrt{R_{x,\mu}} J_1\left(2\pi\sqrt{R_{x,\mu}}\right)}{\pi}\int_0^\infty dL \mu(L)\frac{L\ I_1(L\sqrt{R_{x,\mu}})}{2\sqrt{R_{x,\mu}}}}{\left[\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_{x,\mu}}\right)\right]^2}. \tag{79}
$$

Note that $x'\left(R_{x,\mu}\right) = 0$ if and only if

$$
\begin{aligned}
J_0(2\pi\sqrt{R_{x,\mu}})\int_0^\infty dL\ \mu(L) I_0\left(L\sqrt{R_{x,\mu}}\right) &= \frac{\sqrt{R_{x,\mu}}}{\pi} J_1\left(2\pi\sqrt{R_{x,\mu}}\right)\int_0^\infty dL \mu(L) I_1(L\sqrt{R_{x,\mu}})\frac{L}{2\sqrt{R_{x,\mu}}} \\
J_0(2\pi\sqrt{R_{x,\mu}}) &= x\left(R_{x,\mu}\right)\int_0^\infty dL \mu(L) I_1(L\sqrt{R_{x,\mu}})\frac{L}{2\sqrt{R_{x,\mu}}}.
\end{aligned} \tag{80}
$$

Looking at equation (80) and equation (74), we see $x'\left(R_{x,\mu}\right) = 0$ if and only if $\left(\partial_r Z_{x,\mu}\left(r\right)\right)\big|_{r=R_{x,\mu}} = 0$. Looking at equation (77), we see that this implies that $\frac{\partial R_{x,\mu}}{\partial x}$ diverges at the point x where $x'\left(R_{x,\mu}\right) = 0$. Thus, equation (79) show us two things. Firstly it gives us an explicit formulation of the convergence radius x^* of $R_{x,\mu}$, which is

$$
x^* = \frac{J_0\left(2\pi\sqrt{R_{x,\mu}}\right)}{\int_0^\infty dL \mu\left(L\right) I_1\left(L\sqrt{R_{x,\mu}}\right)\frac{L}{2\sqrt{R_{x,\mu}}}}\Bigg|_{x=x^*}. \tag{81}
$$

Secondly, it shows that $x\left(R_{x,\mu}\right)$ is monotonically increasing up to $x = x^*$.

Proof for i. Let $\mu(L)$ be subcritical. This means that $\left(\partial_r Z_{x,\mu}\left(r\right)\right)\big|_{r=R_{x,\mu}} \neq 0$, which implies that $x'\left(R_{x,\mu}\right) \neq 0$, meaning $x < x^*$. Remember that we should get R_μ from $R_{x,\mu}$ by setting $x = 1$. This means that $Z_{x,\mu}\left(R_{x,\mu}\right) = 0$, and therefore equation (77), should still hold for the case where $x = 1$. This shows us that $x^* > 1$. We can therefore pick a $c \in \mathbb{R}$ such that $x < c < x^*$ and $1 < c$. For such a $c \in \mathbb{R}$, we have $\sum_{n=1}^\infty [\![R_{x,\mu}]\!]_{x^n} c^n < \infty$, as it is smaller than x^*, which is the convergence radius of $R_{x,\mu}$.

Proof for ii. Let $\mu(L)$ be generic critical. We will take a look at equation (75), where we consider $\rho = 1$. We now get, using $Z_{x,\mu}(r)\big|_{x=1} = Z_\mu(r)$, the following expression

$$Z_{x,\mu}(r) = Z_{1,\mu}(r) + \left[\frac{\partial Z_{x,\mu}(r)}{\partial x}\right]\Bigg|_{x=1} (x-1) = Z_\mu(r) + \frac{\partial Z_{x,\mu}(r)}{\partial x}\Bigg|_{x=1} (x-1) + o\left((x-1)\right). \tag{82}$$

By the fact that $\mu(L)$ is generic critical, there is a radius of convergence P of $Z_{x,\mu}(r)$, such that $P > R_\mu$. This implies that $Z_{x,\mu}(r)$ is analytical for $r < P$, meaning $\forall r \in \mathbb{R}$, such that $0 < r < P$ $\forall n \in \mathbb{N} : Z^{(n)}_{x,\mu}(r) < \infty$, implying we have a convergent Taylor expansion for r. Combining the Taylor expansion with the fact we are in the generic critical case, we see that for r around R_μ

$$Z_\mu(r) = c_1 + c_2(R_\mu - r) + c_3(R_\mu - r)^2 + o\left((R_\mu - r)^2\right), \tag{83}$$

where $c_i \in \mathbb{R}$. Combining equations (82) and (83), we get for x around 1 and r around R_μ that

$$Z_{x,\mu}(r) = \frac{\partial Z_{x,\mu}(r)}{\partial x}\Bigg|_{x=1} (x-1) + o\left((x-1)\right) + c_1 + c_2(R_\mu - r) + c_3(R_\mu - r)^2 + o\left((R_\mu - r)^2\right). \tag{84}$$

Considering equation (78), we see that

$$x(r) = \frac{d_1 + d_2(R_\mu - r) + d_3(R_\mu - r)^2 + o\left((R_\mu - r)^2\right)}{d_4 + d_5(R_\mu - r) + d_6(R_\mu - r)^\gamma + o\left((R_\mu - r)^\gamma\right)}. \tag{85}$$

Note that $Z_{x,\mu}(R_{x,\mu}) = 0$ implies that $d_1 = d_4$ and that $\left(\partial_r Z_{x,\mu}(r)\right)\big|_{R_{x,\mu}} = 0$ implies that $d_2 = d_5$. Equivalently, it implies that $c_1 = c_2 = 0$. This means that

$$\begin{aligned} x(r) &= 1 - \frac{(d_5 - d_6)(R_\mu - r)^2 + o\left((R_\mu - r)^2\right)}{d_3 + d_4(R_\mu - r) + d_5(R_\mu - r)^2 + o\left((R_\mu - r)^2\right)} \\ &= 1 - b(R_\mu - r)^2\left(1 + o(R_\mu - r)\right), \end{aligned} \tag{86}$$

with $d_i \in \mathbb{R}$. Note that for x around 1, we have that $R_{x,\mu}$ is around R_μ, as $R_{1,\mu} = R_\mu$. Filling in $r = R_{x,\mu}$, we see that

$$\begin{aligned} x - 1 &= -b(R_\mu - R_{x,\mu})^2 + o\left((R_\mu - r)^2\right) \\ R_{x,\mu} &= R_\mu - \left(\frac{1-x}{b}\right)^{1/2}\left(1 + o(1)\right). \end{aligned} \tag{87}$$

Using the Transfer theorem G.1, we see that

$$[\![R_{x,\mu}]\!]_{x^n} \backsim c\, n^{-3/2}, \tag{88}$$

where $c \in \mathbb{R}$.

Proof for iii. Let $\mu(L)$ be non-generic critical. The proof will be an adaptation for the proof for the generic critical case. Again, let x be around 1. Note that, for r around R_μ we have that $Z_\mu(r)$ has leading term $(R_\mu - r)^\gamma$, where $\gamma \in (1, 2)$. Let r be around R_μ. The approximation of $Z_{x,\mu}(r)$ is now given by

$$Z_{x,\mu}(r) = (x-1)\frac{\partial Z_{x,\mu}(r)}{\partial x} + c_3(R_\mu - r)^\gamma + o\left((R_\mu - r)^\gamma\right), \tag{89}$$

with $c_3 \in \mathbb{R}$. Note that the exponent γ is dependent on the criticality, and therefore on $x\mu(L)$. This means that $x \int_0^\infty dL\ \mu(L) I_0(L\sqrt{r})$ has leading term $(R_\mu - r)^\gamma$. We therefore get that x is given by

$$x(r) = \frac{d_1 + d_2(R_{x,\mu} - r) + o\left((R_{x,\mu} - r)^2\right)}{d_3 + d_4(R_{x,\mu} - r) + d_5(R_{x,\mu} - r)^\gamma + o\left((R_{x,\mu} - r)^\gamma\right)}, \tag{90}$$

where $d_i \in \mathbb{R}$. Note that $Z_{x,\mu}(R_{x,\mu}) = 0$ implies that $d_1 = d_3$ and that $\left(\partial_r Z_{x,\mu}(r)\right)\big|_{R_{x,\mu}} = 0$ implies that $d_2 = d_4$. This means that

$$\begin{aligned} x(r) &= 1 - \frac{d_5(R_{x,\mu} - r)^\gamma}{d_3 + d_4(R_{x,\mu} - r) + d_5(R_{x,\mu} - r)^\gamma + o\left((R_{x,\mu} - r)^\gamma\right)} \\ &= 1 - b(R_{x,\mu} - r)^\gamma \left(1 + o(R_{x,\mu} - r)\right), \end{aligned} \tag{91}$$

with $b \in \mathbb{R}$. By setting $r = R_\mu$, we express $R_{x,\mu}$ in x in the following way

$$R_{x,\mu} = R_\mu - \left(\frac{1-x}{b}\right)^{1/\gamma} (1 + o(1)). \tag{92}$$

Using the Transfer theorem G.1, we see that

$$[\![R_{x,\mu}]\!]_{x^n} \backsim c\, n^{-1/\gamma - 1}, \tag{93}$$

where $\gamma \in (1,2)$ and $c \in \mathbb{R}$. $\square$

Having analyzed criticalities, we will consider two specific examples for illustration purposes.

3.5.5 Example: $\mu(L) = c_L\ \delta(L' - L)$

Let $\mu(L)$ be $c_L \cdot \delta(L' - L)$, where $\delta(L' - L)$ is the delta function and c_L is a constant in $\mathbb{R}$. In this case the coefficient c_L for a boundary component of length L is going to determine whether the weight function is critical. We therefore introduce the notation $R_\mu = R_\mu(c_L)$. Remember that the weights are critical when $\partial_{\mu(L)} R_\mu(c_L)$ diverges. The coefficient for which this holds, is denoted by c_L^*. Filling in equation (71), we get

$$\begin{aligned} \partial_{\mu(L)} R_\mu(c_L) &= \frac{I_0(L\sqrt{R_\mu})}{J_0(2\pi\sqrt{R_\mu}) - \int_0^\infty dL' c_L\ \delta(L' - L)\, I_1(L'\sqrt{R_\mu}) \frac{L'}{2\sqrt{R_\mu}}} \\ &= \frac{I_0(L\sqrt{R_\mu(c_L)})}{J_0(2\pi\sqrt{R_\mu(c_L)}) - c_L I_1(L\sqrt{R_\mu(c_L)}) \frac{L}{2\sqrt{R}}}. \end{aligned} \tag{94}$$

To simplify equation (94), we consider the specific case where $L = 0$. The $\partial_{\mu(L)} R_\mu(c_L)$ then simplifies to

$$\begin{aligned} \partial_{\mu(L)} R_\mu(c_0) &= \frac{I_0(0 \cdot \sqrt{R_\mu(c_0)})}{J_0(2\pi\sqrt{R_\mu(c_0)}) - c_0 I_1(0\sqrt{R_\mu(c_0)}) \frac{0}{2\sqrt{R_\mu(c_0)}}} = \frac{1}{J_0(2\pi\sqrt{R_\mu(c_0)}) - c_0 \cdot 1 \cdot 0} \\ &= \frac{1}{J_0(2\pi\sqrt{R_\mu(c_0)})}. \end{aligned} \tag{95}$$

From equation (95), we see that if $J_0(2\pi\sqrt{R_\mu(c_0)}) \to 0$, then $\partial_{\mu(0)} R_\mu(c_0) \to \infty$. From Section 3.5.1, we know that the values c_0 for which this happens are critical. We therefore want to analyse $\sqrt{R_\mu(c_0)}$

such that $J_0(2\pi\sqrt{R_\mu\left(c_0\right)}) \to 0$. This is the case for $R_\mu\left(c_0\right) \to \frac{j_{0,1}{}^2}{4\pi^2} = \mathbf{r}$. Where $j_{0,1}$ is the first value x, such that $J_0(x) = 0$. Note that by construction of R_μ and equation (57) we get

$$Z\left(R_\mu\left(c_0^*\right)\right) = 0 = \frac{\sqrt{R_\mu\left(c_0^*\right)}}{\pi} J_1\left(2\pi\sqrt{R_\mu\left(c_0^*\right)}\right) - c_0^*. \tag{96}$$

It is clear to see that equation (96) implies that

$$c_0^* = \frac{\sqrt{R_\mu\left(c_0^*\right)}}{\pi} J_1\left(2\pi\sqrt{R_\mu\left(c_0^*\right)}\right) = \frac{j_{0,1}}{2\pi^2} J_1(j_{0,1}). \tag{97}$$

Equation (95) implies that $R_\mu(c_0)$ is analytical up to where $J_0(2\pi\sqrt{r}) = 0$. Combining this with equation (97), the convergence radius ρ of R_μ is the constant c_0^* defined in equation (97).
The Taylor expansion of $J(z) = \frac{\sqrt{z}}{\pi} J_1(2\pi\sqrt{z})$ around $\mathbf{r}$ is given by

$$\begin{aligned} J(z) = \frac{\sqrt{z}}{\pi} J_1(2\pi\sqrt{z}) &\approx J(\mathbf{r}) + \frac{J'(\mathbf{r})}{1!}(z-\mathbf{r}) + \frac{J''(\mathbf{r})}{2!}(z-\mathbf{r})^2 + o((z-\mathbf{r})^2) \\ &= c_0^* + \frac{J''(\mathbf{r})}{2!}(z-\mathbf{r})^2 + o((z-\mathbf{r})^2). \end{aligned} \tag{98}$$

Rewriting equation (98) and setting $z = R_\mu(c_0)$, we see that

$$\begin{aligned} c_0 &= J(R_\mu\left(c_0\right)) \\ &\approx c_0^* + \frac{J''(\mathbf{r})}{2!}(R_\mu\left(c_0\right) - R_\mu\left(c_0^*\right))^2 + o\left((R_\mu\left(c_0\right) - R_\mu\left(c_0^*\right))^2\right) \end{aligned} \tag{99}$$

When we rewrite equation (99), we get

$$R_\mu(c_0) \approx R_\mu\left(c_0^*\right) - \sqrt{\frac{2(c_0^* - c_0)}{-J''(\mathbf{r})}} + o(\sqrt{c_0 - c_0^*}). \tag{100}$$

Using the transfer theorem G.1 and equation (100), it follows that

$$\begin{aligned} [\![R_\mu(c_0)]\!]_{(c_0)^n} &= \sqrt{\frac{2c_0^*}{-J''(\mathbf{r})}} \frac{n^{-3/2}}{\Gamma(\frac{-1}{2})}(c_0^*)^{-n}(1+o(1)) \\ &\backsim \kappa\, n^{-3/2}(c_0^*)^{-n+1/2}, \end{aligned} \tag{101}$$

with $\kappa \in \mathbb{R}$. From Section 3.5.2, we see that if $c_0 = c_0^*$, meaning it fulfills the condition laid out in equation (97), we are in the generic critical case. If $c < c_0^*$, we are in the subcritical case.

3.5.6 Example: $\mu(L) = L \cdot \exp\{-cL\}$

Another example of a weight function is $\mu(L) = L \cdot \exp\{-cL\}$, where $c \in \mathbb{R}_{\geq 0}$. We will find out the circumstances when $\mu(L)$ is a critical weight function, by trying to find conditions on c such that $Z'_\mu(R_\mu) = 0$. Calculating the second part of equation (56) for $\mu(L) = L \cdot \exp\{-cL\}$, we get [56]

$$\int_0^\infty \mu\left(L\right) I_0\left(L\sqrt{r}\right) dL = \int_0^\infty L \cdot \exp\{-cL\} I_0\left(L\sqrt{r}\right) dL = \frac{c}{\left(c^2 - r\right)^{3/2}}. \tag{102}$$

Filling equation (102) into equation (56) and setting $r = R_\mu$, we see that

$$\begin{aligned} Z_\mu\left(R_\mu\right) &= \frac{\sqrt{R_\mu}}{\pi} J_1\left(2\pi\sqrt{R_\mu}\right) - \int_0^\infty \mu\left(L\right) I_0\left(L\sqrt{R_\mu}\right) dL \\ &= \frac{\sqrt{R_\mu}}{\pi} J_1\left(2\pi\sqrt{R_\mu}\right) - \frac{c}{\left(c^2 - R_\mu\right)^{3/2}} = 0. \end{aligned} \tag{103}$$

Equation (71) tells us that we should also analyse $Z'_\mu(R_\mu)$. So first let us calculate the second part of equation (72) [56]

$$\begin{aligned}\int_0^\infty dL\ \mu(L)\, I_1\left(L\sqrt{R_\mu}\right)\frac{L}{R_\mu} &= \int_0^\infty dL\ L\cdot\exp\{-cL\} I_1\left(L\sqrt{R_\mu}\right)\frac{L}{2\sqrt{R_\mu}} \\ &= \frac{1}{2\sqrt{R_\mu}}\frac{4c\sqrt{R_\mu}\Gamma\left(\frac{5}{2}\right)}{\left(c^2-R_\mu\right)^{5/2}} = \frac{3c}{2\left(c^2-R_\mu\right)^{5/2}},\end{aligned} \tag{104}$$

such that

$$\begin{aligned}Z'_\mu(R_\mu) &= J_0\left(2\pi\sqrt{R_\mu}\right) - \int_0^\infty dL\ \mu(L)\, I_1\left(L\sqrt{R_\mu}\right)\frac{L}{2\sqrt{R_\mu}} \\ &= J_0\left(2\pi\sqrt{R_\mu}\right) - \frac{3c}{2\left(c^2-R_\mu\right)^{5/2}} = 0.\end{aligned} \tag{105}$$

This means that we now have two (indirect) expressions of R in c. Using Mathematica, one get numerical values for R_μ and c: $R_\mu = 0.145095...$, $c = 4.00363....$ Note that we know that for this value of c, the weight function is generic critical by the fact that equation (102) is analytical.

3.6 The disc function in the $L' \to \infty$ limit

In Section 3.5, we have defined what it means to be subcritical, generic critical and non-generic critical in the hyperbolic surface case. In this part, we will look at what this means for the disk function $\mathcal{W}(L')$ in the $L' \to \infty$ limit for these different kinds of weight function. Taking the $L' \to \infty$ limit is analogous to taking the perimeter l to infinity for the planar map case, which we analyzed in Section 2.6.3.

Theorem 3.2. *Let $\mu(L)$ be an admissible weight function. Let $\mathcal{W}(L)$ be the partition of hyperbolic surfaces with a marked boundary component of length L given by equation (68). If the marked boundary component $L \to \infty$, then the leading term of the partition function $\mathcal{W}(L)$ from equation (68) takes the form*

$$\mathcal{W}(L) \backsim \kappa e^{cL}\left\{\begin{array}{lr} L^{-\frac{3}{2}} & \text{for } \mu(L) \text{ subcritical} \\ L^{-\frac{5}{2}} & \text{for } \mu(L) \text{ generic critical} \\ L^{-\alpha} & \text{for } \mu(L) \text{ non generic critical}\end{array}\right\}, \tag{106}$$

where $\alpha \in (\frac{3}{2}, \frac{5}{2})$ and $c, \kappa \in \mathbb{R}$.

For ease of reading, we will first prove two lemmas. They will each consider a part of the large equation (68), which gives us $\mathcal{W}(L)$.

Lemma 3.3. *Let $\mu(L)$ be an admissible weight function. Let $\mathcal{W}(L)$ be given by equation (68). If the marked boundary component $L \to \infty$, then the leading terms of the first three terms in equation (68) will take the form*

$$\begin{aligned}&e^{L'\sqrt{R}}\Bigg\{\frac{R}{\pi}J_1(2\pi\sqrt{R})\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\left(1+\frac{1}{L'\sqrt{R}}+\frac{29}{70\left(L'\sqrt{R}\right)^2}\right)\right] \\ &+\frac{4\left(L'\right)^2\sqrt{R}J_0(2\pi\sqrt{R})}{\left(L'^2+4\pi^2\right)^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\right]-\frac{2L'RJ_0(2\pi\sqrt{R})}{L'^2+4\pi^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\left(1+\frac{-1}{L'\sqrt{R}}\right)\right] \\ &-\frac{4\pi RJ_1(2\pi\sqrt{R})]}{L'^2+4\pi^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\right]+O\left(\left(\frac{1}{L'}\right)^{5/2}\right)\Bigg\}.\end{aligned} \tag{107}$$

Proof. To get an approximation of the first three terms of $\mathcal{W}(L')$, one needs to get an approximation of $I_n(L'\sqrt{R})$ for $L' \to \infty$ [57]

$$I_n(L\sqrt{R}) \approx \frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(n)}{\left(L\sqrt{R}\right)^s} \pm e^{\pm n\pi i} \frac{e^{-L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(n)}{\left(L\sqrt{R}\right)^s}, \tag{108}$$

where

$$a_s(n) = \frac{\prod_{i=0}^{s}(4n^2 - (2i+1)^2)}{(s+1)!} \sum_{j=0}^{s} \frac{1}{4n^2 - (2i+1)^2}. \tag{109}$$

As $L \to \infty$, we see that $e^{-L\sqrt{R}} \to 0$, meaning equation (108) simplifies to

$$I_n(L\sqrt{R}) \approx \frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(n)}{\left(L\sqrt{R}\right)^s}. \tag{110}$$

Using equation (110) for the first three terms in (68), one gets

$$\begin{aligned}
\mathcal{W}(L)_{1,2,3} &= \frac{R}{\pi} J_1(2\pi\sqrt{R}) I_1(L\sqrt{R}) \\
&+ \frac{4L\sqrt{R}\big[LJ_0(2\pi\sqrt{R})I_1(L\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R})I_0(L\sqrt{R})\big]}{(L^2+4\pi^2)^2} \\
&- \frac{2\sqrt{R}\big[L\sqrt{R}I_0(L\sqrt{R})J_0(2\pi\sqrt{R}) + 2\pi\sqrt{R}I_1(L\sqrt{R})J_1(2\pi\sqrt{R})\big]}{L^2+4\pi^2} \\
&= \frac{R}{\pi} J_1(2\pi\sqrt{R}) \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(1)}{\left(L\sqrt{R}\right)^s}\right] \\
&+ \frac{4\,(L)^2\,\sqrt{R}J_0(2\pi\sqrt{R})}{\left(L^2+4\pi^2\right)^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(1)}{\left(L\sqrt{R}\right)^s}\right] \\
&+ \frac{8\pi L\sqrt{R}J_1(2\pi\sqrt{R})}{\left(L^2+4\pi^2\right)^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(0)}{\left(L\sqrt{R}\right)^s}\right] \\
&- \frac{2LRJ_0(2\pi\sqrt{R})}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(0)}{\left(L\sqrt{R}\right)^s}\right] \\
&- \frac{4\pi RJ_1(2\pi\sqrt{R})\big]}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(1)}{\left(L\sqrt{R}\right)^s}\right].
\end{aligned} \tag{111}$$

Note that as $L \to \infty$

$$\begin{aligned}
\frac{1}{L^2+4\pi^2} &\backsim \kappa\ \frac{1}{L^2}, \\
\frac{L}{L^2+4\pi^2} &= \frac{1}{L+\frac{4\pi^2}{L}} \backsim \kappa\ \frac{1}{L}, \\
\frac{L}{(L^2+4\pi^2)^2} &= \frac{1}{L^3+8\pi^2 L+\frac{6\pi^4}{L}} \backsim \kappa\ \frac{1}{L^3}, \\
\frac{L^2}{(L^2+4\pi^2)^2} &= \frac{L^2}{L^4+8\pi^2L^2+16\pi^4} = \frac{1}{L^2+8\pi^2+\frac{6\pi^4}{L^2}} \backsim \kappa\ \frac{1}{L^2},
\end{aligned} \tag{112}$$

where $\kappa \in \mathbb{R}$. Using the approximations in (112) and ignoring terms decreasing faster then $\left(\frac{1}{L}\right)^{5/2} e^{L\sqrt{R}}$ in equation (111), one gets

$$\begin{aligned}
\mathcal{W}(L)_{1,2,3} = {} & \frac{R}{\pi} J_1(2\pi\sqrt{R}) \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \left(a_0(1) + -1 \cdot \frac{a_1(1)}{L\sqrt{R}} + \frac{a_2(1)}{\left(L\sqrt{R}\right)^2} \right) \right] \\
& + \frac{4\,(L)^2 \sqrt{R} J_0(2\pi\sqrt{R})}{\left(L^2+4\pi^2\right)^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} a_0(1) \right] \\
& - \frac{2LRJ_0(2\pi\sqrt{R})}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \left(a_0(0) + -1 \cdot \frac{a_1(0)}{L\sqrt{R}} \right) \right] \\
& - \frac{4\pi R J_1(2\pi\sqrt{R})]}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} a_0(1) \right] + O\left(\left(\frac{1}{L}\right)^{5/2} e^{L\sqrt{R}} \right) \\
= {} & \frac{R}{\pi} J_1(2\pi\sqrt{R}) \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{1}{L\sqrt{R}} + \frac{29}{70\left(L\sqrt{R}\right)^2} \right) \right] \\
& + \frac{4\,(L)^2 \sqrt{R} J_0(2\pi\sqrt{R})}{\left(L^2+4\pi^2\right)^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \right] - \frac{2LRJ_0(2\pi\sqrt{R})}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{-1}{L\sqrt{R}} \right) \right] \\
& - \frac{4\pi R J_1(2\pi\sqrt{R})]}{L^2+4\pi^2} \left[\frac{e^{L\sqrt{R}}}{(2\pi L\sqrt{R})^{\frac{1}{2}}} \right] + O\left(\left(\frac{1}{L}\right)^{5/2} e^{L\sqrt{R}} \right).
\end{aligned} \tag{113}$$

□

The second lemma will be concerned with the last term in equation (68), by analyzing the cylinder function.

Lemma 3.4. *Let $\mu(L)$ be an admissible weight function. Let $\mathcal{W}(L, L')$ be defined by equation (62). If the marked boundary component $L' \to \infty$, then the leading terms of the cylinder function $\mathcal{W}(L, L')$ will take the form*

$$\begin{aligned}
& e^{L'\sqrt{R}} \Bigg\{ L \frac{\sqrt{R}(L')^2 I_0(L\sqrt{R})}{L'^2 - L^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{1}{L'\sqrt{R}} + \frac{29}{70\left(L'\sqrt{R}\right)^2} \right) \right] \\
& - L \frac{L'\sqrt{R} L I_1(L\sqrt{R})}{L'^2 - L^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{-1}{L'\sqrt{R}} \right) \right] \Bigg\}.
\end{aligned} \tag{114}$$

Proof. Using definition (62) and the approximation given by equation (110) one gets

$$\begin{aligned}
\mathcal{W}(L, L') = {} & L'L \frac{\sqrt{R}\left[L' I_1(L'\sqrt{R}) I_0(L\sqrt{R}) - L I_1(L\sqrt{R}) I_0(L'\sqrt{R}) \right]}{L'^2 - L^2} \\
\approx {} & L \frac{\sqrt{R}(L')^2 I_0(L\sqrt{R})}{L'^2 - L^2} \left[\frac{e^{L'\sqrt{R}}}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(1)}{\left(L'\sqrt{R}\right)^s} \right] \\
& - L \frac{L'\sqrt{R} L I_1(L\sqrt{R})}{L'^2 - L^2} \left[\frac{e^{L'\sqrt{R}}}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \sum_{s=0}^{\infty} (-1)^s \frac{a_s(0)}{\left(L'\sqrt{R}\right)^s} \right].
\end{aligned} \tag{115}$$

As $L' \to \infty$, we will use the following equations

$$\begin{aligned}
\frac{L'}{L'^2 - L^2} &= \frac{1}{L' - \frac{L^2}{L'}} \backsim \kappa \, \frac{1}{L'}, \\
\frac{(L')^2}{L'^2 - L^2} &= \frac{1}{1 - \frac{L^2}{(L')^2}} \backsim 1,
\end{aligned} \tag{116}$$

with $\kappa \in \mathbb{R}$. Using the approximations in (116) and ignoring terms decreasing faster then $\left(\frac{1}{L'}\right)^{5/2} e^{L'\sqrt{R}}$ in equation (115), one gets

$$\begin{aligned}
\mathcal{W}(L, L') \approx{}& L \frac{\sqrt{R}(L')^2 I_0(L\sqrt{R})}{L'^2 - L^2} \left[\frac{e^{L'\sqrt{R}}}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{1}{L'\sqrt{R}} + \frac{29}{70\left(L'\sqrt{R}\right)^2} \right) \right] \\
&- L \frac{L'\sqrt{R} L I_1(L\sqrt{R})}{L'^2 - L^2} \left[\frac{e^{L'\sqrt{R}}}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{-1}{L'\sqrt{R}} \right) \right].
\end{aligned} \tag{117}$$

□

Having proven both lemma 3.3 and lemma 3.4, we can now proof Theorem 3.2.

Proof. Let $\mu(L)$ be an admissible weight function and let $\mathcal{W}(L')$ be given by equation (68). Combining equations (113) and (117) to fill in equation (68), we get

$$\begin{aligned}
\mathcal{W}(L') ={}& \mathcal{W}(L')_{1,2,3} - \int_0^\infty \mu(L)\, dL \; \frac{\mathcal{W}(L, L')}{L} \\
={}& e^{L'\sqrt{R}} \Bigg\{ \frac{R}{\pi} J_1(2\pi\sqrt{R}) \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{1}{L'\sqrt{R}} + \frac{29}{70\left(L'\sqrt{R}\right)^2} \right) \right] \\
&+ \frac{4\left(L'\right)^2 \sqrt{R} J_0(2\pi\sqrt{R})}{\left(L'^2 + 4\pi^2\right)^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \right] - \frac{2L' R J_0(2\pi\sqrt{R})}{L'^2 + 4\pi^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{-1}{L'\sqrt{R}} \right) \right] \\
&- \frac{4\pi R J_1(2\pi\sqrt{R})]}{L'^2 + 4\pi^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \right] + O\left(\left(\frac{1}{L'} \right)^{5/2} \right) \Bigg\} \\
&- \int_0^\infty \mu(L)\, dL \;\; e^{L'\sqrt{R}} \Bigg\{ \frac{\sqrt{R}(L')^2 I_0(L\sqrt{R})}{L'^2 - L^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{1}{L'\sqrt{R}} + \frac{29}{70\left(L'\sqrt{R}\right)^2} \right) \right] \\
&- \frac{L'\sqrt{R} L I_1(L\sqrt{R})}{L'^2 - L^2} \left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}} \left(1 + \frac{-1}{L'\sqrt{R}} \right) \right] \Bigg\}.
\end{aligned} \tag{118}$$

Let's consider the coefficients of $\frac{e^{L'\sqrt{R}}}{(L')^{1/2}}$ in $\mathcal{W}(L')$. For legibility reason we use the notation $\left[\frac{e^{L'\sqrt{R}}}{L'^{1/2}}\right] \mathcal{W}(L')$ to denote the coefficients.

$$\begin{aligned}
\left[\frac{e^{L'\sqrt{R}}}{L'^{1/2}} \right] \mathcal{W}(L') &\approx \frac{R}{\pi} J_1(2\pi\sqrt{R}) \frac{1}{(2\pi\sqrt{R})^{\frac{1}{2}}} - \int_0^\infty dL\; \mu(L) I_0(L\sqrt{R})\sqrt{R} \frac{1}{(2\pi\sqrt{R})^{\frac{1}{2}}} \\
&= \frac{\sqrt{R}}{(2\pi\sqrt{R})^{\frac{1}{2}}} \left[Z(R) \right] = 0.
\end{aligned} \tag{119}$$

In equation (119), we have used the string equation (57). Note that equation (118) then simplifies to

$$
\begin{aligned}
\mathcal{W}(L') = e^{L'\sqrt{R}} \Bigg\{ & \frac{4\,(L')^2\sqrt{R}J_0(2\pi\sqrt{R})}{(L'^2+4\pi^2)^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\right] - \frac{2L'RJ_0(2\pi\sqrt{R})}{L'^2+4\pi^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\left(1+\frac{-1}{L'\sqrt{R}}\right)\right] \\
& - \frac{4\pi RJ_1(2\pi\sqrt{R})]}{L'^2+4\pi^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\right] + O\left(\left(\frac{1}{L'}\right)^{5/2}\right)\Bigg\} \\
& - \int_0^\infty \mu(L)\,dL\;\; e^{L'\sqrt{R}}\Bigg\{-\frac{L'\sqrt{R}LI_1(L\sqrt{R})}{L'^2-L^2}\left[\frac{1}{(2\pi L'\sqrt{R})^{\frac{1}{2}}}\left(1+\frac{-1}{L'\sqrt{R}}\right)\right]\Bigg\}.
\end{aligned}
\tag{120}
$$

Let's now consider the coefficients of $\frac{e^{L'\sqrt{r}}}{(L')^{3/2}}$ in $\mathcal{W}(L')$,

$$
\begin{aligned}
\left[\frac{e^{L'\sqrt{r}}}{L'^{3/2}}\right]\mathcal{W}(L') &\approx \frac{-1}{(2\pi\sqrt{R})^{\frac{1}{2}}}J_0(2\pi\sqrt{R})2R + \int_0^\infty dL\;\mu(L)L\sqrt{R}I_1(L\sqrt{R})\frac{1}{(2\pi\sqrt{R})^{\frac{1}{2}}} \\
&= \frac{-2R}{(2\pi\sqrt{R})^{\frac{1}{2}}}\left[Z'(R)\right].
\end{aligned}
\tag{121}
$$

From equation (121) and equation (71) one sees that $\mathcal{W}(L') \backsim \kappa\;\left(\frac{1}{L'}\right)^{3/2}$ for some $\kappa \in \mathbb{R}$ if and only if $Z'(R) \neq 0$, meaning that the leading term of $\mathcal{W}(L')$ is $\left(\frac{1}{L'}\right)^{3/2}$ in the subcritical case.
It is also clear that in the generic critical case, that is to say, when $Z'(R) = 0$ and the leading term of $Z_\mu(r)$ is $(R_\mu - r)^2$, that the leading term of $\mathcal{W}(L')$ is $\left(\frac{1}{L'}\right)^{5/2}$
Now to check for the non-generic critical case. In this case, we have a non-analytical integral $\int_0^\infty \mu(L)I_0(L\sqrt{r})dL$, leading to a $c_1\,(R_\mu - r)^{\alpha-1/2}$ term where $\alpha \in (3/2, 5/2)$. For ease of reading set $\gamma = \alpha - 1/2 \in (1, 2)$. Consider therefore equation (64), ignoring larger terms than $(R_\mu - r)^\gamma$

$$
\begin{aligned}
\mathcal{W}(L) &= \frac{-L}{2}\int_0^R drZ_\mu(r)I_0(L\sqrt{r}) = \frac{-L}{2}\int_0^R dr\;c_1\,(R_\mu - r)^\gamma\,I_0(L\sqrt{r}) \\
&= \frac{-L}{2}c_1\;\gamma R_\mu^{\gamma+1}\Gamma\,(\gamma)\;\;{}_0\tilde{F}_1\left(\gamma+2, \frac{1}{4}L^2R_\mu\right),
\end{aligned}
\tag{122}
$$

where ${}_0\tilde{F}_1\left(\gamma+2, \frac{1}{4}L^2R_\mu\right)$ is the regularized confluent hypergeometric function. Another way to write it is

$$
\begin{aligned}
{}_0\tilde{F}_1\left(\gamma+2, \frac{1}{4}L^2R_\mu\right) &= 2^{\gamma+1}\left(L\sqrt{R}\right)^{-\gamma-1}\left(L^2R\right)^{\frac{(-\gamma-1)}{2}+\frac{\gamma+1}{2}}\;I_{\gamma+1}\left(L\sqrt{R}\right) \\
&= 2^{\gamma+1}\left(L\sqrt{R}\right)^{-\gamma-1}\;I_{\gamma+1}\left(L\sqrt{R}\right).
\end{aligned}
\tag{123}
$$

Note that by the previous analysis from this section the leading term of $I_{\gamma+1}\left(L\sqrt{R}\right)$ is $e^{L\sqrt{R}}\frac{L^{-1/2}}{\sqrt{2\pi R}}$.
We therefore see that the leading term of $\mathcal{W}(L)$ is

$$
\mathcal{W}(L) \backsim \kappa\;e^{L\sqrt{R}}\frac{L^{1-\gamma-1-1/2}}{\sqrt{2\pi R}} = \kappa\;e^{L\sqrt{R}}\frac{L^{-\gamma-1/2}}{\sqrt{2\pi R}},
\tag{124}
$$

for some $\kappa \in \mathbb{R}$. Or even more simple, we see that $\mathcal{W}(L) \backsim \kappa\;e^{L\sqrt{R}}L^{-\alpha}$, where $\alpha \in (3/2, 5/2)$ and $\kappa \in \mathbb{R}$. □

We therefore see that in the hyperbolic surface case, we get similar critical exponents in the leading terms of $\mathcal{W}(L)$ as $L \to \infty$ for subcritical, generic critical and non-generic critical weight functions, as we get for W^{2l} in the $l \to \infty$ limit in Section 2.7.

3.7 Laplace transform $\mathcal{W}_\bullet(z)$

In the planar map model, the generating functions $W_\bullet(z)$ and $W(z)$ for respectively $W_\bullet^{(2l)}$ and $W^{(2l)}$ were introduced in Section 2.6. This was mostly done to show the growth of the partition function $W^{(2l)}$ in the $l \to \infty$ case. In this section, we will introduce the Laplace transform $\mathcal{W}_\bullet(z)$ (see Appendix C) for $\mathcal{W}_\bullet(L)$. The Laplace Transform can be seen as a continuous analogue for a generating function, meaning we can use it to show familiarity between $\mathcal{W}_\bullet(z)$ and the analytical form of $W_\bullet(z)$ from the planar map models.

Theorem 3.5. *Let $\mu(L)$ be an admissible weight function. The Laplace transform of the cylinder function $\mathcal{W}_\bullet(L)$ is given by*

$$\mathcal{W}_\bullet(z) = \frac{R_\mu}{2\left(z^2 - R_\mu\right)^{-3/2}}. \tag{125}$$

Proof. Assume $\mu(L)$ is an admissible weight function. Using $\mathcal{W}_\bullet(L)$ defined by equation (63) and [56], we get

$$\begin{aligned}
\mathcal{W}_\bullet(z) &= \int_0^\infty \exp(-L\cdot z)\mathcal{W}_\bullet(L)dL = \int_0^\infty \exp(-L\cdot z)\ L\partial_{\mu(L)}F_0^{\mathrm{WP}}dL \\
&= \int_0^\infty \exp(-L\cdot z)L\frac{\sqrt{R_\mu}}{2}\ I_1(L\sqrt{R_\mu})\ dL \\
&= \frac{\sqrt{R_\mu}}{2}2^1\pi^{-1/2}\Gamma\left(\frac{3}{2}\right)\sqrt{R_\mu}\left(z^2-\sqrt{R_\mu}^2\right)^{-3/2} \\
&= \frac{R_\mu}{2\left(z^2-R_\mu\right)^{-3/2}}.
\end{aligned} \tag{126}$$

□

Comparing equations (125) and (24), we indeed see that they have a similar form, except for the multiplication by a constant R_μ and an extra $\left(z^2 - R_\mu\right)$ term in the denominator in the hyperbolic surface case .
It is clear from equation (68) and equation (62) that taking the Laplace transform of $\mathcal{W}(L)$ or $\mathcal{W}(L_i, L_j)$ is not as easy as for $\mathcal{W}_\bullet(L)$. One might be able to get the Laplace transform using topological recursion, but that is outside the scope of this thesis to consider.

3.8 Probability densities and expected values

To introduce and analyze universality classes and how hyperbolic surfaces will look like, we will introduce probability densities and expected value functions. These will build upon the partition functions of hyperbolic surfaces with marked boundary components introduced in Section 3.4 and the definitions for the different possible weight functions from Section 3.5.2.

3.8.1 Probability density for unmarked surfaces

For this section it is assumed that $\mu(L)$ is an admissible weight function. To get a feel for what kind of surfaces we have in the different universality classes, we can define the probability of getting a hyperbolic surface with n boundary components and the expected number of boundary components. This will of course be done using the weight function $\mu(L)$ and the partition function F_0^{WP} from Section 3.3. The probability density is defined for $n \geq 3$ as

$$p_n\left(L_1, ..., L_n\right) = \frac{2^{2-n}}{n!}\frac{\prod_{i=1}^n \mu(L_i)}{F_0^{\mathrm{WP}}}V_{0,n}(L_1, ..., L_n), \tag{127}$$

where n is the number of boundary components, $L_1, ..., L_n$ are lengths of the boundary component, $\mu(L)$ is the weight function, $V_{0,n}(L_1, ..., L_n)$ is the Weil-Peterson volume of the surface and F_0^{WP} is the partition function of hyperbolic surfaces with boundary components. The $n \geq 3$ constraint follows from the Weil-Peterson volume.
The probability of having a surface with $n \geq 3$ boundary components is given by

$$\begin{aligned}\mathbb{P}(n) &= \int_0^\infty ... \int_0^\infty p_n\left(L_1, ..., L_n\right) dL_1 ... dL_n \\ &= \frac{2^{2-n}}{n!} \frac{\prod_{i=1}^n \int_0^\infty dL_i \ \mu(L_i) V_{0,n}(L_1, ..., L_n)}{F_0^{\mathrm{WP}}}\end{aligned} \tag{128}$$

Using $\mathbb{P}(n)$ defined in equation (128), one can define the expected number of boundary components

$$\begin{aligned}\mathbb{E}(n) &= \sum_{n=3}^\infty \mathbb{P}(n) \cdot n \\ &= \sum_{n=3}^\infty \frac{2^{2-n}}{(n-1)!} \frac{\prod_{i=1}^n \int_0^\infty dL_i \ \mu(L_i) V_{0,n}(L_1, ..., L_n)}{F_0^{\mathrm{WP}}}.\end{aligned} \tag{129}$$

If $\mathbb{P}(n)$ scales with, for example, $n^{-\alpha}$, then it is easy to see from equation (129) that the term inside the sum scales with $n^{-\alpha+1}$. This implies that for the right values of α, the sum is divergent, meaning that $\mathbb{E}(n) = \infty$. In this case, the expected numbers of boundary components would be infinite, which corresponds to a fractal hyperbolic surface. The way to get α such that $\mathbb{P}(n) \backsim \kappa \ n^{-\alpha}$, where $\kappa \in \mathbb{R}$, is to study F_0^{WP}. Note that we have only done analysis for the growth of $R_{x,\mu}$. We will therefore consider the case of surfaces with two cusps in Section 3.8.3.

3.8.2 Example: $\mu(L) = \sum_{i=1}^d x_i \delta(L - L_i)$

To illustrate how the expected number of boundary components looks like, we introduce a weight function similar the the one discussed in Section 3.5.5. Let $\mu(L) = \sum_{i=1}^d x_i \delta(L - L_i)$, where $\delta(x)$ is the delta function, and x_i are coefficients in $\mathbb{R}$, such that $\mu(L)$ is admissible. The probability density in equations (127) and the probability form equation (128) become

$$p(n, L_1, ..., L_n) = \frac{2^{2-n}}{n!} \frac{\prod_{j=1}^n \sum_{i=1}^d x_i \delta(L_j - L_i)}{F_0^{\mathrm{WP}}} V_{0,n}(L_1, ..., L_n), \tag{130}$$

$$\begin{aligned}\mathbb{P}(n) &= \frac{2^{2-n}}{n!} \frac{\prod_{j=1}^n \int_0^\infty dL_j \sum_{i=1}^d x_i \delta(L_j - L_i) V_{0,n}(L_1, ..., L_n)}{F_0^{\mathrm{WP}}} \\ &= \frac{2^{2-n}}{n!} \frac{\prod_{j=1}^n \sum_{i=1}^d x_i L_j}{F_0^{\mathrm{WP}}} V_{0,n}(L_1, ..., L_n).\end{aligned} \tag{131}$$

In equations (130) and (131), n is the number of boundary components, $L_1, ..., L_n$ are lengths of the boundary component, x_i is the weight of the boundary component of length L_i, $V_{0,n}(L_1, ..., L_n)$ is the Weil-Peterson volume of the surface and F_0^{WP} is the partition function of hyperbolic surfaces with boundary components.
The expected number of boundary components is therefore

$$\mathbb{E}(n) = \sum_{n=3}^\infty \frac{2^{2-n}}{(n-1)!} \frac{\prod_{j=1}^n \sum_{i=1}^d x_i L_j}{F_0^{\mathrm{WP}}} V_{0,n}(L_1, ..., L_n). \tag{132}$$

It is clear that we can't analyze equation (132) without getting more information about the expansion of F_0^{WP}. We will therefore come back to an adapted version of this case in Section 3.8.4.

3.8.3 Probability density for surfaces with two marked cusps

As shown in Section 3.8.1, the probability density for surfaces without marked boundary components doesn't give us a lot of information. We will therefore look at the case where the surfaces have two marked cusps.
Using these definitions with the results for $R_{x,\mu}$ we got in Section 3.5.3, we get the following result

Theorem 3.6. *Let $\mu(L)$ be an admissible weight function. Then the following holds for surfaces with two marked cusps:*

i. If $\mu(L)$ is subcritical, the expected number of boundary components is finite.

ii. If $\mu(L)$ is critical, the expected number of boundary components is infinite.

Proof. Let $\mu(L)$ be an admissible weight function. The probability density $p_n(L_1, ..., L_n)$ given by equation (127), where F_0^{WP} is changed into R_μ, meaning

$$p_n(L_1, ..., L_n) = \frac{2^{2-n}}{(n-2)!} \frac{\prod_{i=3}^{n} \mu(L_i) V_{0,n}(0,0,L_3,...,L_4)}{R_\mu}. \tag{133}$$

Note that the generating function for hyperbolic surfaces with two marked cusps is R_μ. Using equation (54) we see

$$\begin{aligned}
\partial_{\mu(0)}\partial_{\mu(0)}F_0^{\mathrm{WP}} &= \partial_{\mu(0)} \sum_{n=3}^{\infty} \frac{2^{2-n}}{n!} \int_0^\infty dL_1 ... \int_0^\infty dL_n \partial_{\mu(0)} \mu(L_1) ... \mu(L_n) V_{0,n}(L_1, ..., L_n) \\
&= \partial_{\mu(0)} \sum_{n=3}^{\infty} \frac{2^{2-n}}{n!} n \int_0^\infty dL_1 ... \int_0^\infty dL_n \delta(L_1 - 0) \mu(L_2) ... \mu(L_n) V_{0,n}(L_1, L_2, ..., L_n) \\
&= \partial_{\mu(0)} \sum_{n=3}^{\infty} \frac{2^{2-n}}{(n-1)!} \int_0^\infty dL_2 ... \int_0^\infty dL_n \mu(L_2) ... \mu(L_n) V_{0,n}(0, L_2, ..., L_n) \\
&= \sum_{n=3}^{\infty} \frac{2^{2-n}}{(n-2)!} \left(\prod_{k=3}^{n} \int_0^\infty dL_k \ \mu(L_k) \right) V_{0,n}(0,0,L_3,...,L_n). \\
&= R_\mu/2.
\end{aligned} \tag{134}$$

Equation (134) therefore gives us

$$\begin{aligned}
\partial_{\mu(L)} R_\mu &= 2\partial_{\mu(L)}\partial_{\mu(0)}\partial_{\mu(0)}F_0^{\mathrm{WP}} \\
&= \sum_{n=3}^{\infty} \frac{2^{3-n}}{(n-3)!} \left(\prod_{k=4}^{n} \int_0^\infty dL_k \ \mu(L_k) \right) V_{0,n}(0,0,L,L_4,...,L_n).
\end{aligned} \tag{135}$$

Note that the expected number of boundary components becomes

$$\begin{aligned}
\mathbb{E}(n) &= \frac{1}{R_\mu} \sum_{n=3}^{\infty} \frac{2^{3-n}}{(n-2)!} \left(\prod_{i=3}^{n} \int_0^\infty dL_i \ \mu(L_i) \right) V_{0,n}(0,0,L_3,...,L_n) \\
&= \frac{1}{2R_\mu} \int_0^\infty dL \mu(L) \partial_{\mu(L)} R_\mu.
\end{aligned} \tag{136}$$

Looking at Section 3.5.1 and the definition of criticality, we see that the theorem holds, as $partial_{\mu(L)} R_\mu$ diverges if and only if $\mu(L)$ is critical. □

Analogously we can analyze the expected total length of the boundary components, defined by

$$\begin{aligned}\mathbb{E}(L_{\text{tot}}) &= \sum_{n=3}^{\infty} p(n, L_1, ..., L_n) \cdot [L_1 + ... + L_n] \\ &= \sum_{n=3}^{\infty} \frac{2^{2-n}}{n!} \frac{\prod_{i=1}^{n} \int_0^{\infty} dL_i \ \mu(L_i)}{F_0^{\text{WP}}} V_{0,n}(L_1, ..., L_n) \ [L_1 + ... + L_n]\end{aligned} \tag{137}$$

Corollary 3.6.1. *Let $\mu(L)$ be an admissible weight function. Then the following holds for surfaces with two marked cusps:*

i. If $\mu(L)$ is subcritical, the expected total length of the boundary components is finite

ii. If $\mu(L)$ is critical, the expected total length of the boundary components is infinite

The proof is analogues to the proof of theorem 3.6.

3.8.4 Example: $\mu(L) = c_{L'}\delta(L - L')$

Let us illustrate the previous results. We will consider the weight function $\mu(L) = c_{L'} \ \delta(L - L')$ from Section 3.5.5. Specifically chose $L' = 0$. Remember that we have already know the value of c_0^*, the value at which the weight function is generic critical.
Using equations (128) and (101), we can get the probability of getting a hyperbolic surface with only n cusps,

$$\mathbb{P}(n) = \frac{\sum_{n=3}^{\infty} \frac{2^{3-n}}{(n-2)!} \left(\prod_{i=3}^{n} \int_0^{\infty} dL_i \ \mu(L_i)\right) V_{0,n}(0, 0, L_3, ..., L_n)}{R_\mu} = \frac{[\![R_{x,\mu}]\!]_{c_0^n} \ (c_0^*)^n}{R_\mu} \backsim \kappa \ n^{-3/2}, \tag{138}$$

with $\kappa \in \mathbb{R}$. This implies that $\mathbb{E}(n) = \sum_{n=3}^{\infty} \left(n^{-1/2} + o(n^{-1/2})\right)$, which is a divergent sum, meaning $\mathbb{E}(n) = \infty$. This means that the number of expected cusps is infinite if $c_o = c_0^*$ from equation (97).

3.9 Universality classes

In this section we will give a brief overview of the results from previous sections. We will also show how surfaces from different universality classes would look like, just like we did in the planar map case (see Section 2.7). The universality classes are labeled by the critical exponents from Theorem 3.2.

3.9.1 The 3/2 universality class

Let's consider the 3/2 universality class, which corresponds to a weight function $\mu(L)$ that is subcritical. The next corollary holds:

Corollary 3.6.2. *Let $\mu(L)$ be a subcritical weight function. The following holds:*

i. For the generating function $R_{x,\mu}$ of hyperbolic surfaces with two marked cusps from equation (92) we have $\sum_{n=1}^{\infty} [\![R_{x,\mu}]\!]_{x^n} c^n < \infty$ for a $c \in \mathbb{R}_{>1}$.

ii. For the partition function for hyperbolic surfaces with one marked boundary component $\mathcal{W}(L)$ from equation (68) one gets $\mathcal{W}(L) \backsim \kappa \ \exp\{cL\} L^{-\frac{3}{2}}$ for $L \to \infty$, where $c, \kappa \in \mathbb{R}$.

iii. The expected number of boundary components and the expected total boundary length for surfaces with two marked cusps is finite.

At the moment the marked boundary component length is taken to the $L \to \infty$ limit, we still have that the expected number of boundary components is finite. From the analysis of subcritical planar map models (see Section 2.7.1), we know that the surface area in this context should be minimized, as these hyperbolic surfaces are in the same universality class. This is why the surfaces from the $\alpha = 3/2$ are not fractal and look like the surfaces on the left in Figure 16.

3.9.2 The $5/2$ universality class

If the $5/2$ universality class is considered, the weight function $\mu(L)$ is generic critical. For this universality class, the following corollary holds:

Corollary 3.6.3. *Let $\mu(L)$ be a generic critical weight function. The following holds:*

i. For the generating function $R_{x,\mu}$ of hyperbolic surfaces with two marked cusps from equation (92) we have $[\![R_{x,\mu}]\!]_{x^n} \backsim c\, n^{-3/2}$, where $c \in \mathbb{R}_{>0}$.

ii. For the partition function for hyperbolic surfaces with one marked boundary component $\mathcal{W}(L)$ from equation (68) one gets $\mathcal{W}(L) \backsim \kappa\, \exp\{cL\}L^{-\frac{5}{2}}$ for $L \to \infty$, where $c, \kappa \in \mathbb{R}$.

iii. The expected number of boundary components and the expected total boundary length for surfaces with two marked cusps is infinite.

If we consider the marked boundary component length is taken to the $L \to \infty$ limit, we still have that the expected number of boundary components is infinite. Using the generic critical planar map models from Section 2.7.2, we see that the surface area is not minimized. This is why the surfaces from the $\alpha = 5/2$ case are fractal and look like the artist impression in the middle of Figure 16.

3.9.3 The $\alpha \in (3/2, 5/2)$ universality class

Let α be from the interval $(3/2, 5/2)$. The corresponding universality classes consider hyperbolic surfaces with a weight function $\mu(L)$ that is non-generic critical.

Corollary 3.6.4. *Let $\mu(L)$ be a generic critical weight function. The following holds:*

i. For the generating function $R_{x,\mu}$ of hyperbolic surfaces with two marked cusps from equation (92) we have $[\![R_{x,\mu}]\!]_{x^n} \backsim c\, n^{-1/\gamma-1}$, where $\gamma \in (1,2)$ and $c \in \mathbb{R}_{>0}$.

ii. For the partition function for hyperbolic surfaces with one marked boundary component $\mathcal{W}(L)$ from equation (68) one gets $\mathcal{W}(L) \backsim \kappa\, e^{cL}L^{-\alpha}$ for $L \to \infty$, where $(3/2, 5/2)$ and $c, \kappa \in \mathbb{R}$.

iii. The expected number of boundary components and the expected total boundary length for surfaces with two marked cusps is infinite.

Similarly to the surfaces from the $\alpha \in (3/2, 5/2)$ universality class in the planar maps model, the surfaces have holes. The holes one gets in these kind of surfaces are not diffeomorphic to a torus. They are rather holes like the boundary components of the surface. Analogously to the previous section, we expect the surface are to not be minimized if we consider the marked boundary component length is taken to the $L \to \infty$ limit. This is why surfaces from this universality class look like the artist impression on the right of Figure 16.

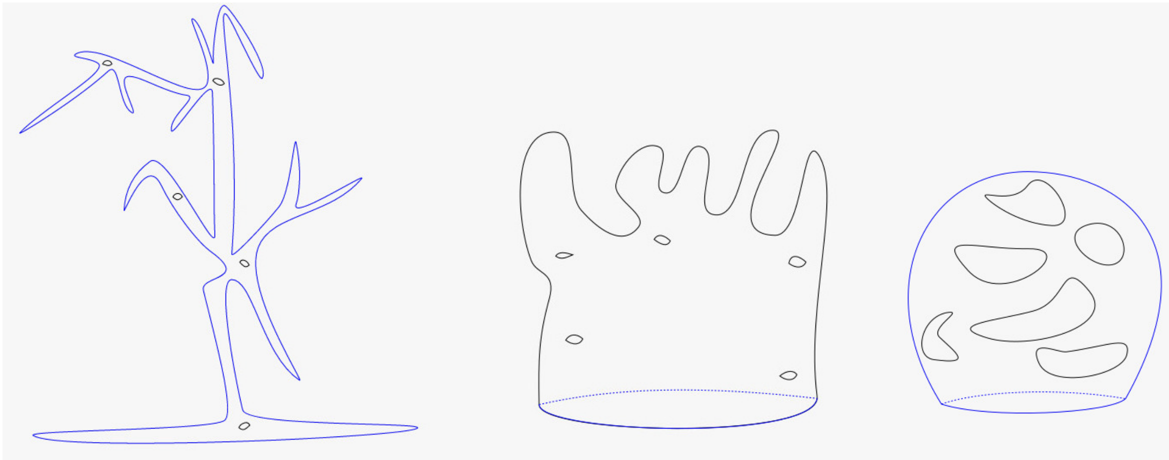

Figure 16: Artist impressions of hyperbolic surfaces for different criticalities in the $L \to \infty$ case. The boundary component whose length is going to infinity is coloured blue in all the three cases. The surface on the left is the subcritical case, the surface in the middle is the generic critical case and the surface to the right is the non-generic critical case. In the subcritical case, the surfaces try to minimize its area, causing a tree like structure. In the generic critical case, there is no minimization of surface are and the holes in the surfaces (which are different boundary components) are small relative to the blue boundary component. In the non-generic case, their is also no minimization of surface area and the holes in the surface are large compared to the boundary component coloured in blue.

4 Conclusion and discussion.

Concluding, we are able to get the same universality classes for hyperbolic surfaces as for planar maps. This is done by having restrictions on the weight function $\mu(L)$, just like we had restrictions on our weight sequence $\mathbf{q}$ for the planar maps model. We see similar behaviour and characteristics, such as (non)fractal behaviour, in the discrete model and continuous model we considered given that the models are from the universality class.
If $\mu(L)$ is chosen, such that $Z'_\mu(R_\mu) \neq 0$, then the model lives in the $\alpha = 3/2-$universality class. In this case, the expected number of boundary components is finite, meaning that we don't have a fractal surface. In this universality class, we have that discrete surfaces are volume minimizing, meaning they get tree like structures if the perimeter goes to infinity. For the hyperbolic surfaces from this universality class, we expect the same behaviour when one of the boundary components goes to infinity.
If for $\mu(L)$ holds that $Z'_\mu(R_\mu) = 0$ and the radius of convergence of $Z_{x,\mu}(r)$ is larger then R_μ, then the model is part of the $\alpha = 5/2-$universality class. In this case, the expected number of boundary components is infinite, meaning that the hyperbolic surfaces are fractal. For the planar map surfaces, the expected volume is infinite. This means that the surfaces one gets, when the perimeter goes to infinity, is not tree like. In hyperbolic surface model that live in this universality class, we expect the same behaviour when one of the boundary components goes to infinity.
If $\mu(L)$ has the property that $Z'_\mu(R_\mu) = 0$ and $Z_\mu(r)$ for r around R_μ has leading term is $(R_\mu - r)^{\alpha-1/2}$, where $\alpha \in (3/2, 5/2)$, then we are considering models from the $\alpha \in (3/2, 5/2)-$universality class. In both the hyperbolic and planar map surfaces we get holes in the surface. For the hyperbolic surface, we get fractal behaviour from the fact that the number of expected boundary components ins infinite. We also see fractal behaviour in the planar map models from this universality class, as the number of expected edges is infinite.

Due to the limited scope of this thesis, some details were not discussed. The point of the following sections is to highlight these details and give some possibilities for future research.

4.1 Four-dimensional theories

One limitation of this thesis is that we have only considered two-dimensional theories for quantum gravity. We know from general relativity that our universe is described by four dimensions, one time dimension and three space dimension. We therefore want our principle to be generalized to higher dimensions. It is however unclear how one would, for example, create four-dimensional building blocks and 'glueing' processes and get the right continuum limit. Also, the Weil-Peterson volume and the generating function for the partition function is defined for hyperbolic (two-dimensional) surfaces. Finding four-dimensional analogues would be difficult, as one would, for example, need to find an analogue for pairs of pants decomposition and four-dimensional boundary components.

4.2 Higher genus models

In this thesis, only models with genus zero were considered. Both in the case of maps and hyperbolic surface, one could consider higher genus models. In the maps case, you would fore example get an embedding of bipartite graphs on the tori if the genus is one [58]. In this case, if you represent the map on a flat plane, you would allow specific overlapping of edges. In the case of hyperbolic surfaces, you can still use the Weil-Peterson volume, as pants decomposition and Frenchel-Nielssen coordinates are still well defined for Riemannian surfaces with higher a higher genus [26]. The generating function for the volumes however, would be much more difficult, except for the $g = 1$ model [55]. In this case $F_1^{\mathrm{WP}} = -\frac{1}{24}\log\left[\partial_{\mu(0)}R_\mu\right]$. For higher genus models however, the marking of one boundary component and taking its length to infinity, will be much harder to calculate.

4.3 Tree bijections

The way the bipartite planar maps were analyzed, was based on analysis from random walks created by the function $g_{\mathbf{q}}$ or the measure $w_{\mathbf{q}}$. There is however another way to analyze bipartite planar maps, which uses trees. There is a bijection between bipartite planar maps and a specific subset of trees, which was discovered by Bouttier, Di Fransesco and Guiter [59]. This bijection is called the BdFG-bijection or BFG-bijection. It gives an explicit algorithm which can be used to go from bipartite planar maps to mobiles and it gives an algorithm for the other way around.
The specific subset of trees are called mobiles. They are trees, such that the vertices are either labeled with positive integer labels or no labels, each edge connects two vertices of different kind, and there is at least one vertex of degree 1. Besides, for any unlabeled vertex the adjacent two labeled vertices labels n, m consecutively clockwise follow the relation: $m \geq n - 1$ [59].
This bijection can in turn be generalized to be between generalized mobiles and Eulerian maps [59]. Eulerian maps are rooted planar maps where the degree of the vertices is even. Generalized mobiles are trees, such that every vertex is either coloured black, coloured white or labeled a positive integer number. Every edge is either labeled on both sides with a positive integer number and connects two differently coloured vertices, or is unlabeled and connects a labeled vertex with a white vertex. For a black vertex, the edges connected to it have labels which are non-increasing when read clockwise and two consecutive edges have non-decreasing labels. For a visualisation see Figure 17. For a white vertex, the edges connected to it and neighbouring labeled vertices have labels which are non-decreasing when read clockwise; at each edge crossing it is non-decreasing; decreasing by one after each labeled vertex and stationary between a labeled edge and the next label. Lastly, there is a at least one side of an edge which is labeled zero. For a visualisation see Figure 18.

The previously outlined bijections can be used to get more general results for planar maps. For example, using the more general bijection, one can analyze and rewrite $W_\bullet(z)$ [37]. Using the partition functions of mobiles, combinatorial equations and definitions of generating functions, one gets

$$W_\bullet(z) = \frac{1}{\sqrt{(z - c_+)(z - c_-)}}. \tag{139}$$

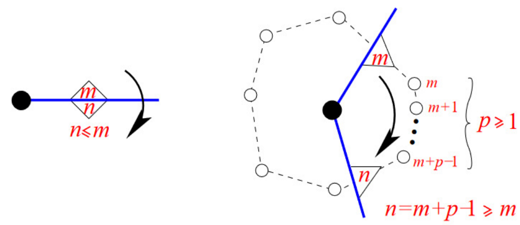

Figure 17: A visualisation of the restrictions on the labeling of the vertices and sides of the edges when considering it from the viewpoint of a black vertex. All letters n, m, p represent some positive integer assigned to either a labeled vertex or labeled side of an edge. Source: Bouttier, Di Fransesco and Guiter [59].

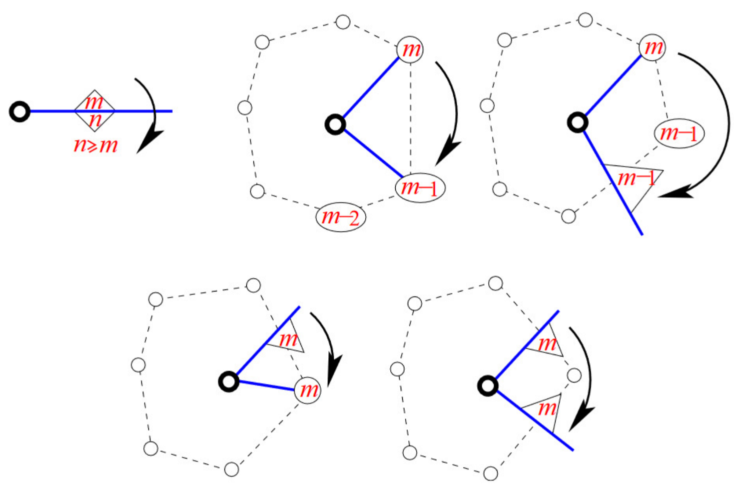

Figure 18: A visualisation of the restrictions on the labeling of the vertices and sides of the edges when considering it from the viewpoint of a white vertex. All letters n, m represent some positive integer assigned to either a labeled vertex or labeled side of an edge. Source: Bouttier, Di Fransesco and Guiter [59].

It holds that $c_+ = -c_-$ if and only if $\mathbf{q}$ is bipartite. This means that we get the generating function for bipartite planar maps back when $c = c_+ = -c_-$.

There are multiple uses for either bijection. Firstly, it could be used to analyze more general (discrete) surfaces models and their analogies between hyperbolic surface surfaces. Secondly, it might be possible to find a bijection between a specific set of trees and hyperbolic surfaces. One possibility could be the creation of trees created from 'shooting' geodesics from the boundary components of a hyperbolic surface. Instead of labeling the vertices, as for the trees created for Eulerian maps, we could label the angles between edges. This encoding of hyperbolic surfaces as trees, might lead to new results with respect to the partition function. These trees might then, analogously to the planar map case using the BDFG-bijection, lead to a random walk. This could tell us how hyperbolic surfaces from different universality classes look like in certain limits.

However, it is clear that the more general the planar maps are that you consider, the more difficult it becomes to use the bijection. This follows from the fact, that there are more specific restrictions on the labeling of the vertices and edges of your trees. This can make it harder to find, for example, results for the created random walks in specific limits you want to consider.

4.4 Irreducible planar maps

In this thesis, the length of every edge of the planar map is taken to be the same length. If one adds lengths to the edge, one considers metric planar maps. These lengths could differ per edge and are therefore continuous, meaning we have introduces a continuous parameter. Irreducible maps are maps where cycles of specific length are not allowed. This gives restrictions on the perimeter of the building blocks, as any building block has a cycle. These irreducible maps are interesting, as the generating function of the Weil-Peterson volume for hyperbolic surfaces is the same as the generating function for $2\pi-$irreducible planar maps [55]. Also, there is a way to go from irreducible maps to planar maps without the irreducibility constraint. This implies that we could study the relation between specific planar maps and hyperbolic surfaces, without considering criticality. Further research could be done between irreducible maps and hyperbolic surfaces, by analyzing the partition functions. This is necessary as the calculations with respect to the partition function $W^{(2l)}$ and $W_{2p}^{(2l)}$ change.

4.5 Peeling process

In this thesis, we haven't discussed the peeling process of (finite) planar maps. The peeling process is given by a peeling algorithm $\mathfrak{A}$, see for example [38] or [22, p. 61-75]. This peeling algorithm gives a sequence of maps, by the glueing of, pinching of or adding of edges to a map. The probability of getting a map is dependent on the weight sequence, as the chance of encountering certain faces is determined by the weight sequence. The peeling process is a Markov chain, meaning that the probability of something happening is only depended on the context left by the previous step.
The peeling process that Curien uses in [22] to show the critical exponents of the partition function W^{2l}, has transition probabilities that are explicitly dependent on the partition function $W_{\bullet}^{2l}$, W^{2l} and the weight sequence $\mathbf{q}$. Using probability analysis, one can define a probability measure ν on $\mathbb{Z}$. This probability measure defines a random walk. This random walk can in turn be used to analyze the critical exponents of W^{2l} and how the planar maps look like in the $l \to \infty$ limit.
There is no real clear continuous surface analogue to the peeling process of planar maps. The closest we have is the Mirzhakani recursion which sends geodesics form boundary components and analyzes the length of the geodesic when it hits another boundary component (if the boundary component length is 0, one can use horrocycles). Using this recursion , one might be able to use the same logic steps to analyze the hyperbolic surfaces (in specific limits).

4.6 Pseudo-Riemannian manifolds

Solutions to the JT-gravity action from equation (7) are technically $1 + 1$-dimensional surfaces [27]. This means that the solution has one time dimension. In this thesis we have treated the time dimension as a space dimension. We know from special relativity that this should not be the case, as there is a restriction on the structure of the surfaces. As nothing can travel faster than the speed of light, the possibilities for the worldlines of particles is limited. These possibilities can be presented by lightcones, as seen in Figure 19. The fact that the worldlines of two objects can't cross the worldline of a photon, means there is a constraint on causality, as two objects can only meet at the points where their lightcones overlap [60]. There is therefore a dependency of your space dimension on your time dimension. If one uses $\mathbb{R}^4$ and a Riemannian metric, all regions are accessible for every path, meaning the lightcone structure is not taken into account [29].
Therefore, we can't use Riemannian manifolds to describe the solutions. This is because you need a negative sign for the time coordinate to have a geometry which is compatible with special relativity. The lightcones consequently lead to scalar products to not always be positive. The tangent spaces T_pM (see Appendix E) therefore need to be divided into time-like, null and spaccelike vectors [61]. This

implies that we should use pseudo-Riemannian manifolds. These are smooth manifolds with a pseudo-Riemmannian metric $\tilde{g}$. These pseudo-Riemannian manifolds are more general then Riemannian manifolds [28] and as discussed in Section 1, the JT-action is still well defined if we consider pseudo-Riemannian manifolds.
Note that one then needs to find an alternative formulation for the Weil-Peterson volume, as it is dependent on having a Riemannian metric. Consequently, one can't use the generating function for Weil-Peterson volumes, which was used to analyze hyperbolic (Riemannian) surfaces. This means that further research could focus on trying to find a Weil-Person like volume and recursive formulation. Another possibility for research is trying to find a specific link or bijection between pseudo-Riemannian surfaces and (specific subsets of) planar maps.

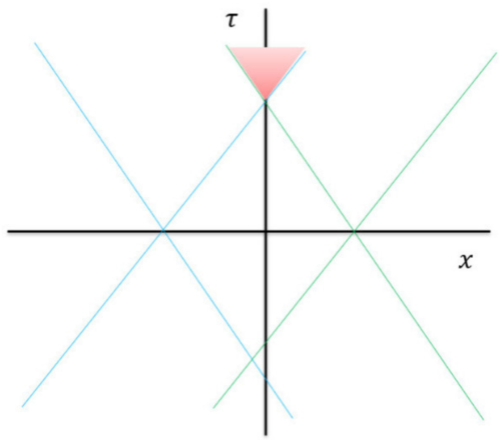

Figure 19: The x-axis represents one space dimension. The y-axis shows the time. Two observers have limited possibilities for their worldline as the can't go faster then the speed of light. The cones that limit their worldlines are shown in blue and green respectively. The observers can only meet at the points where the two cones overlap. This area is shown in red.

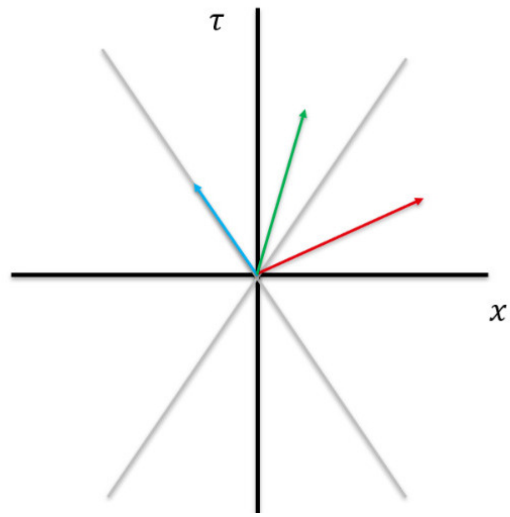

Figure 20: Three different kind of vectors. The x-axis represents one space dimension. The y-axis shows the time. The light cone is depicted by the two grey lines. The red vector is a time-like vector. The blue line is a null vector. The green vector is a space-like vector.

Appendix

The appendix contains calculations, definitions and background information used in my thesis.

A Graph theory

In this short section, some elementary definitions and notations with regards to graphs will be introduced. Most of it is based on the work of Schrijver[40] .

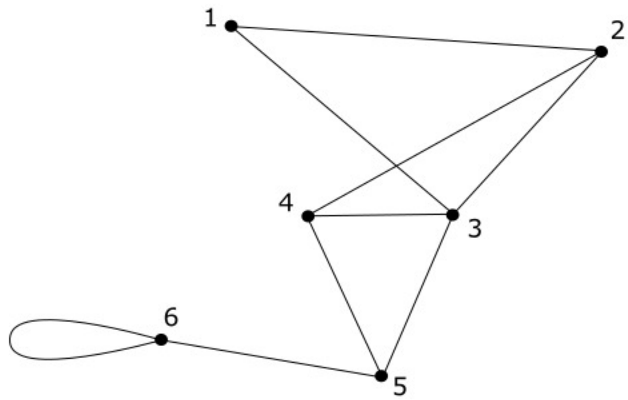

Figure 21: To the left is the graph $\mathfrak{g}$, with $\mathcal{V}(\mathfrak{g}) = \{1,2,3,4,5,6\}$ and $\mathcal{E}(\mathfrak{g}) = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{5,6\},\{6,6\}\}$. Note that $\{6,6\}$ is called a loop.

A graph $\mathfrak{g}$ consists of two sets $\mathcal{V}(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g})$. The first set, $\mathcal{V}(\mathfrak{g}) = \{v_1, v_2, \ ... \ \}$, consists of vertices v_i which can be thought of as points. The second set, $\mathcal{E}(\mathfrak{g}) = \{\{v_i, v_j\} \, | v_i, v_j \in \mathcal{V}(\mathfrak{g})\}$, consists of edges $e_k = \{v_i, v_j\}$ between two vertices for $\mathcal{V}(\mathfrak{g})$ and can be thought of as lines between two points. When $\{v_i, v_j\} \in \mathcal{E}(\mathfrak{g})$ we say that v_i is **connected** to v_j (and vice-versa) or that v_i and v_j are each others neighbours. An edge in a graph $\mathfrak{g}$ is **oriented** from v_i towards v_j if the edge has a direction. These oriented edges are denoted by $(v_i, v_j) \in \mathcal{E}(\mathfrak{g})$. In this thesis we allow so-called **loops**, these are edges that connect one vertex with itself, that is to say, these edges are given by $\{v_i, v_i\}$. See Figure 21 for an example of a graph with a loop.
Two graphs $\mathfrak{g}$ and $\mathfrak{g}'$ are called **isomorphic**, if there is a bijection $h : \mathcal{V}(\mathfrak{g}) \to \mathcal{V}(\mathfrak{g}')$, such that: $\{v_i, v_j\} \in \mathcal{E}(\mathfrak{g}) \leftrightarrows \{h(v_i), h(v_j)\} \in \mathcal{E}(\mathfrak{g}')$. This means that there is a bijection, which preserves the quality of two vertices being neighbours. For an example of isomorphic graphs, see Figure 22. In this thesis we consider graphs which are unique up to isomorphism.
The **degree** of a vertex, $\deg(v_i)$, is the number of vertices v_i is connected to. Or equivalently: $\deg(v_i) = |\{v_j \in \mathcal{V}(\mathfrak{g})| \{v_i, v_j\} \in \mathcal{E}(\mathfrak{g})\}|$. For an example, look at Figure 21, where one sees that $\deg(v_3) = 3$. In this thesis, all vertices are of finite degree.
A **path** is an ordered set of vertices $(v_0, v_1, ..., v_k)$, such that: $\forall i, j \in \mathbb{N} : [[i \neq j] \to [v_i \neq v_j]] \wedge \forall i : [[0 \leq i < k] \to [\{v_i, v_{i+1}\} \in \mathcal{E}(\mathfrak{g})]]$. The length of the path is $k-1$. A path of length $k-1$ where $v_0 = v_k$ is called a **circuit**. A circuit is thus a path that starts and ends at the same vertex. A graph is called a **tree** if it has no circuits. The **distance** $d(v_i, v_j)$ between two vertices $v_i, v_j \in \mathcal{V}(\mathfrak{g})$ is the length of the minimal path between the two vertices. For an example, consider Figure 22, where $d(3,4) = 1$ and $d(3,6) = 1$.
A graph $\mathfrak{g}$ is **non-intersecting** if it can be drawn on a flat surface without any edges overlapping except at the begin and endpoints of the edges. If this is the case, it is said that the graph can be embbeded into the plane or into the sphere. Note that it can be possible that a graph that is initially

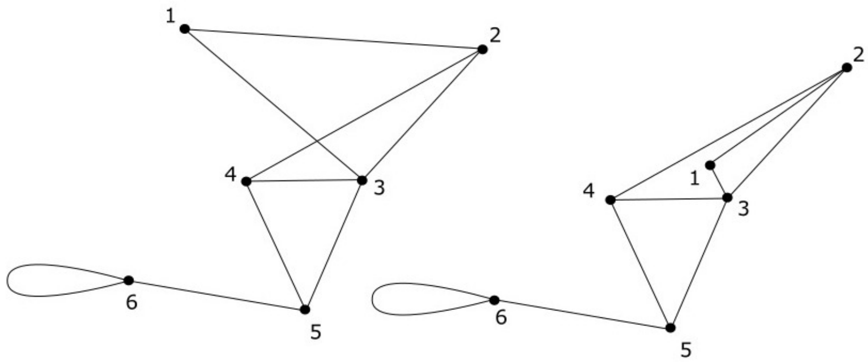

Figure 22: To the left is the graph $\mathfrak{g}$, with $\mathcal{V}(\mathfrak{g}) = \{1,2,3,4,5,6\}$ and $\mathcal{E}(\mathfrak{g}) = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{5,6\},\{6,6\}\}$. To the right is the non-intersecting graph $\mathfrak{g}'$, with $\mathcal{V}(\mathfrak{g}') = \mathcal{V}(\mathfrak{g})$ and $\mathcal{E}(\mathfrak{g}') = \mathcal{E}(\mathfrak{g})$. Note that $\mathfrak{g}$ and $\mathfrak{g}'$ are clearly isomorphic via $h = Id$.

drawn as intersecting, can actually be drawn as non-intersecting. See Figure 22 for an example. Consider an edge $\{v_i, v_j\} \in \mathcal{E}(\mathfrak{g})$. We can make it an oriented edge: (v_i, v_j). If the area to its left or right is encircled by a circuit $(v_0, v_1, ..., v_{k-1}, v_0)$ of minimal length k, the area is called a **face**. The circuit is of minimal length if it is not possible to create another circuit that starts with the same two vertices, is oriented the same way and has a smaller length. A face will be denoted by f, and we say that the face is of degree k if the length of the minimal circuit is k. Equivalently, we can say that the degree of a face is the number of edges in the minimal circuit. For an example, see Figure 23.

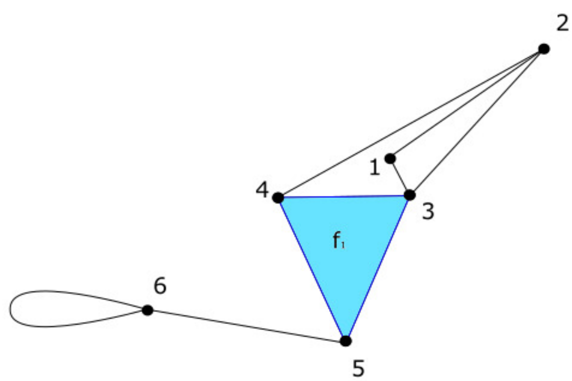

Figure 23: To the left is the graph $\mathfrak{g}$, with $\mathcal{V}(\mathfrak{g}) = \{1,2,3,4,5,6\}$ and $\mathcal{E}(\mathfrak{g}) = \{\{1,2\},\{1,3\},\{2,3\},\{3,4\},\{3,5\},\{4,5\},\{5,6\},\{6,6\}\}$. An example of a circuit which encompasses a face is coloured in dark blue: $(3,4,5,3)$. The face f_1 encircled is highlighted with a light blue colour. Note that $\deg(f_1) = 3$.

A graph can be coloured by assigning a colour to every vertex of a graph, such that two neighbouring vertices do not get the same colour. **Bipartite graphs** are graphs which can be coloured with two colours. Figure 24 shows an example of bipartite graph. In the context of non-intersecting graphs, the graph being bipartite is equivalent to having only faces with even degree.

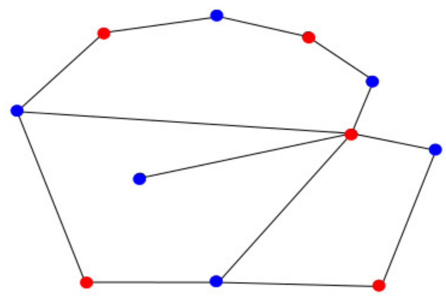

Figure 24: An example of a bipartite graph/bipartite planar map. Note that every pair of adjacent vertices don't have the same colour and all the faces are of even degree . Here a face of degree 0 is taken to also be a face of even degree.

B Generating functions

Generating functions are used in this theses to analyse sequences, such as the sequence of partition functions $W_\bullet^{2l}$. The definitions of this section are based on the book by Mazur [62].
Consider a number sequence $\{a_k\}_{k\geq 0}$, such that $\forall i \in \mathbb{N}: \quad a_i \in \mathbb{R}$. The **generating function** of this sequence is defined as $\sum_{k\geq 0} a_k x^k$. The convergence radius $\rho \in \mathbb{R}$ is the value such that if $x \in \mathbb{R}$ is chosen such that the sum of the generating function converges, then $|x| < \rho$. Note that not every generating function might have a convergence radius. If the sum converges, the form to which it converges is called the **analytical form** of the generating function. A well-known example is $\{a_k\}_{k\geq 0} = \{1\}_{k\geq 0}$, whose generating function is

$$\sum_{k\geq 0} x^k = \frac{1}{1-x}, \tag{140}$$

which holds for $|x| < 1$. The radius of convergence is thus 1. The form $\frac{1}{1-x}$ is the analytical form of the generating function $\sum_{k\geq 0} x^k$. As we see in sections 2.6.1, one is able to analyze the sequences by looking at the analytical form.
Another way to use a generating function is to solve a recurrence relation. If the sequence $\{a_k\}_{k\geq 0}$ is given by a regressive formulation, one needs to know $\{a_i\}_{i<n}$ to be able to get a_n. For large n, this can be too tedious to do by straightforward calculations of all a_i for $i < n$. Using the generating function for this recursive sequence, one is sometimes able to get an analytical form. This analytical form is often linked with a generating function for which the number sequence is known. One is then able to get the value of a_n for any $n \in \mathbb{N}$. This is why the Tutte-equations from Section 2.5 are so powerful.
An example of this technique is the sequence given by $a_0 = 1, a_k = 3a_{k-1}$. The generating function is then given by

$$\begin{aligned} f(x) = \sum_{k\geq 0} a_k x^k = \left(\sum_{k\geq 1} a_k x^k\right) + a_0 = \left(\sum_{k\geq 1} 3a_{k-1} x^k\right) + 1 \\ = 3x\left(\sum_{k\geq 1} a_{k-1} x^{k-1}\right) = 3xf(x) - 1 \end{aligned} \tag{141}$$

This implies $f(x) = \frac{1}{1-3x}$. As one can see, the other generating formula which has the same analytical form is given by substituting $x = 3x$ in equation (140). This means that $a_n = 3^n$. We will introduce the following notation: $a_n = [\![f(x)]\!]_{x^n}$, where $f(x)$ is the analytical form. Alternatively it can be used if $f(x)$ can be approximated by a power series. In this context $c_n = [\![f(x)]\!]_{x^n} = [x^n]\, f(x)$, meaning that it is the coefficient of x^n of the power series. In the case discussed in equation (141) it would be: $a_n = [\![\frac{1}{1-3x}]\!]_{x^n} = 3^n$. This method of finding the coefficients will be used to analyze the partition function $W^{(2l)}$ in Section 2.6.4.

Note that defining the generating function $\mathcal{W}(L_i, L_j)$ of hyperbolic surfaces with two marked boundary components of length L_i and L_j as a (functional) derivative of the more general generating function F_0^{WP} makes sense. Let's take a random element from the generating function F_0^{WP}: $[\![F_0^{\mathrm{WP}}]\!]_{\mu(0)^n}$ gives the number of surfaces with n cusps. After taking the derivative with regard to $\mu(0)$, there is now an n term in front of the coefficient, standing for the n ways one cusp can be chosen to be marked. The $n-1$ in the exponent of $\mu(0)$ now stands for the number of unmarked cusps.

B.1 Proof of Theorem 2.2

Proof. Let $\mathbf{q}$ be an admissible weight sequence. This means that $W_\bullet^{(2l)} < \infty$, meaning we will be able to use equation (15), where $p = 0$. We therefore have

$$\begin{aligned} W_\bullet(z) &= \sum_{l=0}^{\infty} W_\bullet^{(2l)} z^{-2l-1} \\ &= \sum_{l=0}^{\infty} 2^{-2l-1} \binom{2l}{l} c_{\mathbf{q}}^l z^{-2l-1} \\ &= \frac{1}{2z} \sum_{l=0}^{\infty} \binom{2l}{l} \left(\frac{4z^2}{c_{\mathbf{q}}}\right)^{-l} \\ &= \frac{1}{2z} \frac{1}{\sqrt{\frac{z^2 - c_{\mathbf{q}}}{z^2}}} \\ &= \frac{1}{2\sqrt{z^2 - c}}, \end{aligned} \tag{142}$$

where the following standard identity for the generating function is used

$$\sum_{i=0}^{\infty} \binom{2i}{i} x^{-i} = \frac{1}{\sqrt{\frac{x-4}{x}}}, \tag{143}$$

which holds for $|x| > 4$. This means that we need to show that $z > 2$. This follows from the fact that $z > c_{\mathbf{q}}$. Note that $c_{\mathbf{q}} = 4Z_{\mathbf{q}}$, and $Z_{\mathbf{q}}$ is the smallest solution for

$$1 + \sum_{k=1}^{\infty} q_k \binom{2k-1}{k} x^k. \tag{144}$$

This implies that $Z_{\mathbf{q}} > 1$, meaning $z > 4$. □

C Laplace transform

The Laplace transform is in essence just a basis transformation [63]. It transforms a function with a real variable (t in equation (145)) to a complex valued function (denoted in equation (145) by s). In the general case, this is written as

$$F(s) = (\mathcal{L}(f))(s) = \int_0^{\infty} \exp(-s \cdot t) f(t) ds \tag{145}$$

The Laplace transform can be seen as a continuous analogue for the generating function and we will therefore use to compare the hyperbolic surface models with planar map models in Section 3.7.

D Measure theory

A small introduction to measure theory will be given in this section, so one can define the σ-algebra on $\mathcal{M}$ using the weight sequence $\mathbf{q}$ in Section 2.2 and show the intuition behind Theorem 2.5. The idea of a measure is to give a volume to subsets A_i of a space X. The definitions are based on the book by Cohen [64].

D.1 Algebras

Let X be a set. A collection of subsets of X, denoted by $\mathcal{A} = \{A_i | A_i \subset X\}$, is called an **algebra** on the set X if:

$$X \in \mathcal{A} \tag{146}$$
$$\forall A_i \in \mathcal{A} : A_i^c \in \mathcal{A} \tag{147}$$
$$\forall n < \infty : [[A_1, ..., A_n \in \mathcal{A}] \rightarrow [\cup_{i=1}^n A_i \in \mathcal{A}]] \tag{148}$$
$$\forall n < \infty : [[A_1, ..., A_n \in \mathcal{A}] \rightarrow [\cap_{i=1}^n A_i \in \mathcal{A}]] \tag{149}$$

The collection of sets $\mathcal{A}$ is called a $\sigma-$**algebra** if in equation (148) and equation (149) n is replaced with ∞.

Lemma D.1. *Let $\mathcal{M}$ be the set of all finite planar maps , the set $\mathcal{A} = \mathcal{P}(\mathcal{M})$ is a $\sigma-$algebra on $\mathcal{M}$.*

This follows trivially from the definition of a power set.

D.2 Measure

Let X be a set, let $\mathcal{A}$ be a (σ-)algebra on the set X. A **measure** is a function $\nu : \mathcal{A} \rightarrow [0, +\infty]$, that is countably additive and satisfies $\nu(\emptyset) = 0$. A function is countably additive if for disjoint $A_1, A_2, ... \in \mathcal{A}$

$$\nu\left(\cup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \nu\left(A_i\right). \tag{150}$$

A measure ν is called $\sigma-$finite if $X = \cup_{i=1}^{\infty} A_i$ for some disjoint $A_1, A_2, ... \in \mathcal{A}$, where $\forall i \geq 1$ we have that $\nu(A_i) < \infty$.
An example of a measure is $w_{\mathbf{q}}(\mathfrak{m}) = \prod_{f \in \mathcal{F}(\mathfrak{m})} q_{deg(f)/2}$, where for two different maps $\mathfrak{m}$ and $\mathfrak{m}'$ the following holds

$$w_{\mathbf{q}}(\mathfrak{m} \cup \mathfrak{m}') = \prod_{f \in \mathcal{F}(\mathfrak{m})} q_{deg(f)/2} + \prod_{f \in \mathcal{F}(\mathfrak{m}')} q_{deg(f)/2}. \tag{151}$$

More generally, let $\mathcal{A}' = \cup_{i=1}^{\infty} A_i$ for disjoint sets $A_1, A_2, ... \in \mathcal{A}$, then the following holds

$$w_{\mathbf{q}}\left(\bigcup_{\mathfrak{m} \in \mathcal{A}'} m\right) = \sum_{i=0}^{\infty} \sum_{\mathfrak{m} \in A_i} \prod_{f \in \mathcal{F}(\mathfrak{m})} q_{deg(f)/2} \tag{152}$$

D.2.1 Proof of Theorem 2.1

Proof. Let $w_{\mathbf{q}}(\mathfrak{m})$ be given as in equation (11). Let $\mathcal{A}$ be given as in lemma D.1.
Firstly, note that it is clear from the definition of $w_{\mathbf{q}}$ that $w_{\mathbf{q}}(\emptyset) = 0$. Secondly, it clear that the countably additive requirement from equation (150) holds, as for disjoint $A_1, A_2, ... \in \mathcal{A}$, we define $\mathcal{A}' = \cup_{i=1}^{\infty} A_i$ and see that

$$\begin{aligned} w_{\mathbf{q}}\left(\cup_{i=1}^{\infty} A_i\right) &= w_{\mathbf{q}}\left(\cup_{i=1}^{\infty} \cup_{\mathfrak{m} \in A_i} \mathfrak{m}\right) = w_{\mathbf{q}}\left(\cup_{\mathfrak{m} \in \mathcal{A}'} \mathfrak{m}\right) \\ &= \sum_{i=0}^{\infty} \sum_{\mathfrak{m} \in A_i} \prod_{f \in \mathcal{F}(\mathfrak{m})} q_{deg(f)/2} = \sum_{i=1}^{\infty} w_{\mathbf{q}}\left(A_i\right). \end{aligned} \tag{153}$$

Meaning it is clear that $w_{\mathbf{q}}$ is a measure.
To proof that it is a $\sigma-$finite measure, note that $\mathcal{M} = \bigcup_{l=0}^{\infty} \mathcal{M}^{(2l)}$, where $\forall l \in \mathbb{N} : \mathcal{M}^{(2l)} \in \mathcal{P}(\mathcal{M})$. By assumption there is an l such that $W^{(2l)} < \infty$. Equation (9) then implies that $w_{\mathbf{q}}\left(\mathcal{M}^{(2l)}\right) < \infty$. From [22, p. 55], we know that $\exists l \in \mathbb{N} : W^{(2l)} < \infty$ implies that $\forall l \in \mathbb{N} : W^{(2l)} < \infty$. Using this fact, we get $\forall l \in \mathbb{N} : w_{\mathbf{q}}\left(\mathcal{M}^{(2l)}\right) < \infty$, which means that $w_{\mathbf{q}}$ is a $\sigma-$finite measure. $\square$

D.3 Measure space

The combination of a set X, an $(\sigma-)$ algebra $\mathcal{A}$, and a $(\sigma-)$measure ν is called a $(\sigma-)$ measure space $(X, \mathcal{A}, \nu)$. A measure space is finite if $\nu(X) < \infty$. For finite measure spaces, one has a measure ν, such that $\nu : \mathcal{A} \to [0, +\infty)$.

Corollary D.1.1. *Let $\mathcal{M}$ be the set of all bipartite planar map and let $w_{\boldsymbol{q}}(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{M}$ be given by equation* (11)*, then $(\mathcal{M}, \mathcal{P}(\mathcal{M}), w_{\boldsymbol{q}})$ is a σ-measure space.*

Proof. This follows from combining lemma D.1 and theorem 2.1, with the fact that $\mathcal{M} = \bigcup_{l=0}^{\infty} \mathcal{M}^{(2l)}$, as $\forall l \in \mathbb{N} : \mathcal{M}^{(2l)} \in \mathcal{P}(\mathcal{M})$. $\square$

D.4 Integration

We want to be able to analyze how planar maps will look like by getting a formulation of the expected number of edges in the planar map. This will be done by firstly introducing integration and then defining the concept of a probability measure.
Before we can define integration, we need to introduce the concept of measurable functions. Let $(X, \mathcal{A})$ be a measure space and let $A \in \mathcal{A}$. A function $f : A \to [-\infty, \infty]$ is called measurable with respect to $\mathcal{A}$ if $\forall\, t \in \mathbb{R} : \{x \in A | f(x) < t\} \in \mathcal{A}$.
Let ν be a measure on $(X, \mathcal{A})$. Let $f : X \to \mathbb{R}$ be an $\mathcal{A}-$measurable function, such that $f = \sum_{i=1}^{m} c_i \chi_{A_i}$. Here $c_i \in \mathbb{R}_{\geq 0}$, $A_1, ..., A_m \in \mathcal{A}$ are disjoint subsets of X and $\chi_{A_i} : X \to \mathbb{R}$ is the characteristic function of A_i defined by

$$\chi_{A_i}(x) = \left\{ \begin{array}{ll} 1 & \text{if } x \in A_i \\ 1 & \text{if } x \notin A_i \end{array} \right\}. \tag{154}$$

We can define the concept of an integral of f with respect to ν by

$$\int f d\nu = \sum_{i=1}^{m} c_i\ \nu(A_i). \tag{155}$$

D.5 Probability measure

A **probability space** is a measure space $(X, \mathcal{A}, \nu)$, such that $\nu(X) = 1$. A measure ν for which this holds, is called a **probability measure** [65].

Corollary D.1.2. *Let $\mathcal{M}_{2p}^{(2l)}$ be the set of all rooted bipartite planar maps, with target face of degree $2p$ and perimeter of length $2l$. Let $w_{\boldsymbol{q}}(\mathfrak{m})$ for $\mathfrak{m} \in \mathcal{M}_{2p}^{(2l)}$ be given by equation* (11)*. Let $\mathbb{P}_{2p}^{(2l)}(\mathfrak{m})$ be defined by*

$$\mathbb{P}_{2p}^{(2l)}(\mathfrak{m}) = \frac{w_{\boldsymbol{q}}(\mathfrak{m})}{W_{2p}^{2l}}. \tag{156}$$

Then $\left(\mathcal{M}_{2p}^{(2l)}, \mathcal{P}\left(\mathcal{M}_{2p}^{(2l)}\right), \mathbb{P}_{2p}^{(2l)}\right)$ is a probability space.

Proof. It is clear that $\mathcal{P}\left(\mathcal{M}_{2p}^{(2l)}\right)$ is a σ-algebra for $\mathcal{M}_{2p}^{(2l)}$.
From corollary D.1.1, we know that $(\mathcal{M}, \mathcal{P}(\mathcal{M}), w_{\mathbf{q}})$ is a σ-measure space. This implies that $w_{\mathbf{q}}$ is measure on $\mathcal{M}$, thus

$$\mathbb{P}_{2p}^{(2l)}(\emptyset) = \frac{w_{\mathbf{q}}(\emptyset)}{W_{2p}^{2l}} = 0. \tag{157}$$

Let $A_1, A_2, ... \in \mathcal{P}\left(\mathcal{M}_{2p}^{(2l)}\right) \subseteq \mathcal{P}(\mathcal{M})$ be disjoint, using theorem 2.1 we know that

$$\mathbb{P}_{2p}^{(2l)}\left(\cup_{i=1}^{\infty} A_i\right) = \frac{w_{\mathbf{q}}\left(\cup_{i=1}^{\infty} A_i\right)}{W_{2p}^{2l}} = \frac{\sum_{i=1}^{\infty} w_{\mathbf{q}}\left(A_i\right)}{W_{2p}^{2l}} = \sum_{i=1}^{\infty} \mathbb{P}_{2p}^{(2l)}\left(A_i\right), \tag{158}$$

meaning $\mathbb{P}_{2p}^{(2l)}\left(\mathcal{M}_{2p}^{(2l)}\right)$ is also a measure. To show that it is a σ-finite measure, one can follow the proof of Theorem 2.1 while substituting $\mathcal{M}$ with $\mathcal{M}_{2p}^{(2l)}$ and $\mathcal{M}^{(2l)}$ with $\mathcal{M}_{2p}^{(2l)}[n]$.
As $\left(\mathcal{M}_{2p}^{(2l)}, \mathcal{P}\left(\mathcal{M}_{2p}^{(2l)}\right), \mathbb{P}_{2p}^{(2l)}\right)$ is a σ-measure space, we know that

$$\begin{aligned}
\mathbb{P}_{2p}^{(2l)}\left(\mathcal{M}_{2p}^{(2l)}\right) &= \frac{w_{\mathbf{q}}\left(\mathcal{M}_{2p}^{(2l)}\right)}{W_{2p}^{2l}} = \frac{w_{\mathbf{q}}\left(\cup_{n=0}^{\infty}\mathcal{M}_{2p}^{(2l)}[n]\right)}{W_{2p}^{2l}} = \frac{\sum_{n=0}^{\infty} w_{\mathbf{q}}\left(\mathcal{M}_{2p}^{(2l)}[n]\right)}{W_{2p}^{2l}} \\
&= \frac{\sum_{n=0}^{\infty}\sum_{m_{2p}[n]\in\mathcal{M}_{2p}^{(2l)}[n]}\prod_{f\in\mathcal{F}(\mathfrak{m}_{2p}[n])\setminus f_{\mathbf{r}}} q_{deg(f)}}{W_{2p}^{2l}} = \frac{W_{2p}^{2l}}{W_{2p}^{2l}} = 1,
\end{aligned} \tag{159}$$

meaning that the corollary holds. □

One can analogously define $\mathbb{P}^{(2l)}$ for $\mathcal{M}^{(2l)}$ and $\mathbb{P}^{(2l)}[n]$ for $\mathcal{M}^{(2l)}[n]$.

D.5.1 Notes on Theorem 2.5

Having defined a probability measure on appendix D.5, we are able to define the expected value of edges in a planar map. This is done by using integration defined in Appendix D.4. Let $\mathfrak{m} \in \mathcal{M}_{2p}^{(2l)}$ be a rooted bipartite planar map with a target face of degree $2p$ and perimeter $2l$, the expected number of edges is given by

$$\mathbb{E}\left(|\mathfrak{m}|\right) = \int d\mathbb{P}_{2p}^{2l}(\mathfrak{m}) \cdot |\mathfrak{m}|. \tag{160}$$

It is clear that $|.| : M_{2p}^{(2l)} \to \mathbb{R}_{\geq 0}$ is measurable with respect to $\mathcal{P}\left(\mathcal{M}_{2p}^{(2l)}\right)$, meaning the integral in equation 160 is well defined. Besides, from corollary D.1.2 we know that $\mathbb{P}_{2p}^{2l}(\mathfrak{m})$ is a probability measure, so we know that the integral gives the expected number of edges for the planar map $\mathfrak{m} \in \mathcal{M}_{2p}^{(2l)}$. The formulation of theorem 2.5 is therefore a reformulation of theorem D.2.

Theorem D.2. *Let* $\boldsymbol{q}$ *be an admissible weight sequence. Let* $\mathfrak{m} \in M_{2p}^{(2l)}$. *The weight sequence* $\boldsymbol{q}$ *is critical if and only if* $\int d\mathbb{P}_{2p}^{2l}(\mathfrak{m}) \cdot |\mathfrak{m}| = \infty$

The formulation of theorem D.2 makes it intuitively clear how the proof of theorem 2.5 is dependent on the random walk defined by $g_{\mathbf{q}}$. The criticality condition on $f_{\mathbf{q}}$ gives a condition on $g_{\mathbf{q}}$, which translates to a condition for the random walk Y_n. From the random walk Y_n, one is able to analyze the behaviour of W_{2p}^{2l} and therefore $\mathbb{P}_{2p}^{2l}(\mathfrak{m})$.

E Riemannian manifolds

In this section, we will recall some definitions and propositions from differential geometry. This is necessary to define lengths of curves on surfaces, orientation preserving maps, integration on a manifold, etc. Fur further reading, see [25] [46] and [47].

E.1 Topology

Before we are able to define a manifold, we need to introduce the concept of a topology. Let M be a set. A **topology on M** is a collection $\mathcal{T} = \{U_i | U_i \subseteq M\}$ of subsets of M, such that

$$M, \emptyset \in \mathcal{T}, \tag{161}$$
$$\forall \mathcal{F} = \{U_i | U_i \in \mathcal{T}\} : \cup_{U_i \in \mathcal{F}} U_i \in \mathcal{T}, \tag{162}$$
$$\forall n \in \mathbb{N} : [[U_1, ..., U_n \in \mathcal{T}] \to [\cap_{i=1}^{n} U_i \in \mathcal{T}]] . \tag{163}$$

The topology $\mathcal{T}$ of M is the set consisting of the open sets of M. When $\mathcal{T}$ fulfills the above requirements, we say that $(M, \mathcal{T})$ is a **topological space** [66].
We will introduce the concept of a basis for the topology so we can define manifolds in Appendix E.2. A collection $\mathcal{B} = \{B_i | B_i \in \mathcal{T}\}$ is called a **basis** for the topology on M, if

$$\bigcup_{B_i \in \mathcal{B}} B_i = M, \tag{164}$$
$$\forall B_1, B_2 \in \mathcal{B} : [[x \in B_1 \cap B_2] \to [\exists B_3 \in \mathcal{B} : x \in B_3 \subseteq B_1 \cap B_2]] . \tag{165}$$

For the introduction of the Weil-Peterson volume the concepts of homotopy and equivalence relations need to be revisited. Let $(M, \mathcal{T}_M), (N, \mathcal{T}_N)$ be two topological spaces. A useful tool in our kit would be generalizing the idea of continuity on $\mathbb{R}$ to topological spaces. A function $f : M \to N$ is continuous if $\forall\, U \in \mathcal{T}_N : f^{-1}(U) \in \mathcal{T}_M$. This means that f, reaches only open subsets of the set N if the input is an open subset of the set M. Consider two of such continuous maps between topological spaces $f_0, f_1 : X \to Y$. A **homotopy** from f_0 to f_1 is a continuous map $F : X \times [0, 1] \to Y$, where $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. If such an F exists between f_0 and f_1, f_0 and f_1 are called **homotopic** [67]. This means that one can continuously map one function to another function. For an example see Figure 25. A relation $\backsim$ on a topological space X is called an **equivalence relation** if

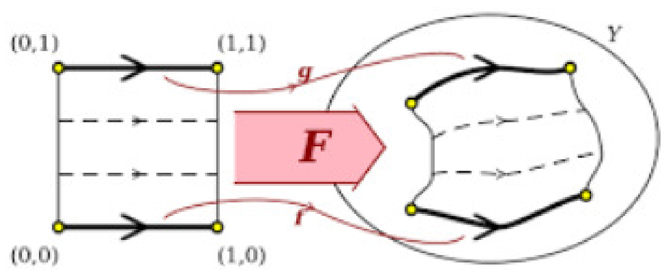

Figure 25: An example of a homotopy F between two continuous functions $g, f : X \to Y$. Source: Smeets [66]

i. $\forall x \in X : x \backsim x$,

ii. $\forall x, y \in X : x \backsim y \implies y \backsim x$,

iii. $\forall x, y, z \in X : [x \backsim y \wedge y \backsim z] \implies x \backsim z$.

An equivalence relation partitions X into **equivalence classes** $[x] = \{y \in X | x \backsim y\}$. The collection of these equivalence classes is called the **quotient** and is denoted by $X/\backsim$ [68].
Note that by the first characteristic of an equivalence relation, we have that $X/\backsim$ is a partition of X. This means that the equivalence relation can be used to group together points from a topological space, that due to some shared characteristics are considered to be too similar to be considered on

their own, in some context. In the context of the Teichmüller space of hyperbolic surfaces, two pairs $(X, f), (Y, h)$ of a hyperbolic surface and an orientation preserving diffeomorphism from $\Sigma_{g,b,n}$ to the surface are seen as 'too similar', when one can have an isometry between the two surfaces X and Y, such that mapping $\Sigma_{g,b,n}$ to X using f followed by mapping X to Y using m and mapping Y back to $\Sigma_{g,b,n}$ can be continuously mapped to the identity. In the context of Weill-Peterson volume we also use **homotopy classes** of continuous functions, which are equivalence classes where the equivalence relation between two function is that they are homotopic to each other.

E.2 (Smooth) Manifolds

A **topological manifold of dimension n** is a topological space such that

i. M is a **Hausdorff space**: $\forall p, q \in M \ \exists\ U, V \in \mathcal{T} : [U \cap V = \emptyset \wedge p \in U \wedge q \in V]$.

ii. M is **second-countable**: $\exists\ \mathcal{B} = \{B_i\}_{i\in\mathbb{N}} : \mathcal{B}$ is a basis for the topology of M.

iii. M is **locally Euclidean with dimension n**: $\forall p \in M \ \exists\ U \in \mathcal{T}$:
$\left[p \in U \wedge \exists\ \widehat{U} \subseteq \mathbb{R}^n \ \exists\ \phi : U \to \widehat{U} : \phi \text{ is a homeomorphism}\right]$.

Where $\phi : U \to \widehat{U}$ being a **homeomorphism**, means that ϕ is a continuous bijective map, with a continuous inverse $\phi^{-1} : \widehat{U} \to U$. These definitions give us the ability to characterize some of the properties of manifolds, such as when a set is open or compact [46]. However, one can not yet do calculus on these manifolds. To do this, one needs to introduce a 'smooth' structure. This will be done in the next paragraph.

A **(coordinate) chart** (U, ϕ) on a manifold M consists of an open subset U of M and a homeomorphism $\phi : U \to \phi(U) \subseteq \mathbb{R}^n$. A collection of charts whose domain covers M is called an **atlas** $\mathfrak{A}$. An atlas is **smooth** if any two charts are smoothly compatible. Two charts $(U, \phi), (V, \psi)$ are **smoothly compatible** if and only if $U \cap V = \emptyset$ or $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism. A map $\psi \circ \phi^{-1}$ being a **diffeomorphism** means that $\psi \circ \phi^{-1}$ is a smooth homeomorphism with a smooth inverse. The pair $(M, \mathfrak{A})$ is a **smooth manifold** if M is a manifold and $\mathfrak{A}$ is a smooth atlas on M.

E.3 Vector fields

Before we can introduce the Riemannian metric and integration, we need to define vector fields using the tangent space. The **tangent space** T_pM of $p \in M$, also called the set of derivatives at $p \in M$, is defined by: $T_pM = \{(p, v) \,|v : C^\infty(M) \to \mathbb{R} \wedge \forall f, g \in C^\infty(M) : [v(fg)(p) = v(f)g(p) + f(p)v(g)]\}$. Alternatively, we write $(p, v) = v_p$. The tangent space of M is defined by $TM = \bigsqcup_{p\in M} T_pM$ and consists of finite-dimensional real vector spaces [47]. Note that the tangent space itself is *not* a vector space. Defining the dual space TM^* of the tangent space is therefore done by taking the dual vector space of every T_pM. Also note that there is a natural projection of TM on M given by $\pi : TM \to M$, where $\pi(p, v) = p$.

A **(smooth) vector field** on M, often denoted by X, is a (smooth) section of the natural projection of TM on M. The fact that X is a (smooth) section, means that X is a continuous (smooth) map $X : M \to TM$, such that $\pi \circ X = \mathrm{Id}_M$. Notation-wise, we can set $X(p) = X_p$. The set of these vector fields, $\mathfrak{X}(M) = \{X : M \to TM | X \text{ is a vector field}\}$ is a vector space under point-wise addition and scalar multiplication, and it's a module over the ring $C^\infty(M)$ [46].

E.4 Riemannian manifolds

In this section we introduce the Riemannian metric, which allows us to define the geometry of the manifold. This will be done by defining an inner product for any tangent space.

Having defined the tangent space in the previous section, we can define a set of covariant 2-tensors on its cotangent bundle TM^*: $T^2(TM^*) = \bigsqcup_{p\in M} T^2(T_pM)$, where $T^2(T_pM) = \{\alpha_p : T_pM \otimes T_pM \to \mathbb{R}\}$, with α_p being a multi-linear real-valued function. Lastly, we define a space of sections of the covariant 2-tensor sets: $\Gamma\left(T^2\left(TM^*\right)\right) = \{A : M \to T^2\left(TM^*\right) | A \text{ is a smooth section}\}$. Where A is a smooth section with respect to $\pi : T^2\left(TM^*\right) \to M$, where $\pi\left(\alpha_p\right) = p$. Now we can define a **Riemannian metric** on a smooth manifold M. A smooth covariant 2-tensor field $g : M \to T^2\left(TM^*\right)$ is a Riemannian metric, if $g\left(p\right) = g_p : T_pM \otimes T_pM \to \mathbb{R}$ is an inner product. This means that it satisfies

$$\forall v, w \in T_pM : g_p\left(v, w\right) = g_p\left(w, v\right), \tag{166}$$

$$\forall v \in T_pM : g_p\left(v, v\right) \geq 0, \tag{167}$$

$$\forall v \in T_pM : \left[g_p\left(v, v\right) = 0 \leftrightarrows v = 0 \in T_pM\right]. \tag{168}$$

A **Riemannian manifold** is a pair (M, g), where M is a smooth manifold and g is a **Riemannian metric** [25].

Using g one can define the length of a vector. Take $(p, v) \in T_pM$: $|v|_g := g_p\left(v, v\right)^{1/2}$. Let $\gamma : I \to M$ be a smooth curve on M with $I = [a, b] \subseteq \mathbb{R}$. This means that $\forall t \in I\ \exists\ \gamma'\left(t\right) \in T_{\gamma(t)}M$. The length of the curve is defined by: $L_g\left(\gamma\right) := \int_a^b |\gamma'\left(t\right)|_g dt$. One can define the distance between two points on the manifold using the definition of the length of curve:

$d_g\left(p, q\right) = \inf\left\{L_g(\gamma) \mid \gamma \text{ is an admissible curve starting at } p \text{ and ending at } q\right\}$. A smooth curve is called a **geodesic**, if the distance between any two points on the curve is the same as the length of the curve between two points: $\forall s, t \in I : d_g\left(\gamma\left(s\right), \gamma\left(t\right)\right) = L_g(\gamma|_{[s,t]})$ [25].

E.5 Curvature

To define what *hyperbolic* manifolds are, we need to define curvature. To do this, we need to introduce the **connection**, which is a map $\nabla : \mathfrak{X}\left(M\right) \times \mathfrak{X}\left(M\right) \to \mathfrak{X}\left(M\right)$, such that

$$\forall f, g \in C^\infty\left(M\right)\ \forall\ X_1, X_2, Y \in \mathfrak{X}\left(M\right) : \nabla_{fX_1 + gX_2} Y = \nabla_{fX_1} Y + \nabla_{gX_2} Y, \tag{169}$$

$$\forall a, b \in \mathbb{R}\ \forall X, Y_1, Y_2 \in \mathfrak{X}\left(M\right) : \nabla_X\left(aY_1 + bY_2\right) = a\nabla_X Y_1 + b\nabla_X Y_2, \tag{170}$$

$$\forall f \in C^\infty\left(M\right)\ \forall X, Y \in \mathfrak{X}\left(M\right) : \nabla_X\left(fY\right) = f\nabla_X\left(Y\right) + X\left(fY\right). \tag{171}$$

Conventionally, $\nabla_X\left(Y\right)$ is called the covariant derivative of Y in the direction X. Using this idea of taking a covariant derivative in the direction of a vector field, we introduce a new map R by

$$\begin{aligned} &R : \mathfrak{X}\left(M\right) \times \mathfrak{X}\left(M\right) \times \mathfrak{X}\left(M\right) \to \mathfrak{X}\left(M\right) \\ &R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{XY - YX} Z, \end{aligned} \tag{172}$$

which is a $(1, 3)-$tensor field on M. We can now introduce the **Riemannian curvature tensor** $Rm : \mathfrak{X}\left(M\right) \otimes \mathfrak{X}\left(M\right) \otimes \mathfrak{X}\left(M\right) \otimes \mathfrak{X}\left(M\right) \to \mathbb{R}$, where $Rm\left(X, Y, Z, W\right) = g\left(R\left(X, Y\right)Z, W\right)$, which is a covariant tensor. We say that a manifold is flat, if the Riemannian curvature vanishes, that is to say if: $\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z = \nabla_{XY - YX} Z$ for all $X, Y, Z \in \mathfrak{X}\left(M\right)$.

Having introduced the Riemannian curvature tensor, one can define the **Ricci curvature** Rc. It is a covariant $2-$tensor field $Rc : \mathfrak{X}\left(M\right) \otimes \mathfrak{X}\left(M\right) \to \mathbb{R}$, where $Rc\left(X, Y\right) = Tr\left(\psi_{X,Y}\right)$. Here, $\psi_{X,Y} : \mathfrak{X}\left(M\right) \to \mathfrak{X}\left(M\right)$ is defined by $\psi_{X,Y}\left(Z\right) = R\left(X, Y\right)Z$. The **scalar curvature** S is defined using the Ricci curvature: $S = Tr\left(Rc\right)$. **Hyperbolic manifolds** are manifolds with constant scalar curvature of -2 [25].

E.6 Orientation

The Weil-Peterson volume is defined as an integral over the Moduli space of certain surfaces. To define the signs of the integration and the coordinate system used for the integration, one needs to define orientations on a smooth $n-$dimensional manifold.

A **(smooth) local frame** for M is a tuple of (smooth) vector fields $(E_1, ..., E_k)$ defined on an open set $U \subseteq M$, such that $\forall q \in U : \left(E_1\left(q\right), ..., E_k(E_1\left(q\right)\right)$ is a basis for T_qM. It is a **(smooth) global**

frame if we have $U = M$.
Choose a frame $(E_1, ..., E_k)$ for an open set $U \subseteq M$. Let $p \in U$ and look at T_pM. Another basis $(\tilde{E}_1(p), ..., \tilde{E}_k(p))$ is called called **consistently oriented** if there is a transition matrix B, such that B has positive determinant and $E_i(p) = B_i^j \tilde{E}_j(p)$, where we use the Einstein summation notation. This gives us exactly two equivalence classes on the set of ordered bases of T_pM. The bases belonging to the same equivalence classes as $(E_1, ..., E_k)$ are called **positively oriented**. The bases that do not, are called **negatively oriented**. A **pointwise orientation** of M is therefore defined as the choice of orientation $\forall p \in M$. A pointwise orientation of M is **continuous** if there is an oriented global frame. An **orientation** of M is then a continuous pointwise orientation of M.
A map $F: M \to N$ is called **orientation preserving** if the isomorphism $dF_p : T_pM \to T_{F(p)}M$ takes an positively orientated base of T_pM to anther oriented base in $T_{F(p)}M$. Where $dF_p : T_pM \to T_{F(p)}M$ is defined by: $\forall v \in T_pM \; \forall h \in C^\infty(N) : dF_p(v)(h) = v(h \circ F)$ [46].

E.7 Differential forms and integrating

To define the Weil-Peterson volume, one makes use of wedge products and sympletic forms, as this is used to define integration.
Let $F : V^* \times ... \times V^* \to \mathbb{R}$ be covariant $k-$tensor. The tensor is called **alternating** if the value changes when two different arguments are interchanged, meaning:
$F(v_1^*, ..., v_i^*, ..., v_j^*, ..., v_k^*) = -F(v_1^*, ..., v_j^*, ..., v_i^*, ..., v_k^*)$. The **alternation of F** is defined by

$$(\text{Alt } F)(v_1^*, ..., v_k^*) = \frac{1}{k!} \sum_{\sigma \in S_k} \text{sgn}(\sigma) \; F\left(v_{\sigma(1)}^*, ..., v_{\sigma(k)}^*\right) \;, \tag{173}$$

where $\text{sgn}(\sigma)$ is the **sign** of the permutation σ, which is $+1$ if the permutation can be written as a composition of even number of transpositions, and -1 if it can be in an odd number. Using the alternation, we can define the wedge product. Let $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$. The **wedge product** for these alternating tensor is defined by

$$\omega \wedge \eta = \frac{(l+k)!}{l!k!} \text{Alt}(\omega \otimes \eta). \tag{174}$$

Let M be a smooth manifold of dimension n. An alternating tensor field on M is called a **differential** $k-$**form**. Let (x^i) be smooth coordinates on M. A smooth differential k-form ω can be written as

$$\omega = \sum_{j_1 < ... < j_k} \omega_{j_1 ... j_k} dx^{j_1} \wedge ... \wedge dx^{j_k}. \tag{175}$$

A **sympletic form** is a $2-$form ω that is non-degenerate, meaning: $\forall p \in M \; \forall v_p \in T_pM :$
$[\forall u_p \in T_pM : \omega_p(v_p, u_p) = 0] \to [v_p = 0]$ The **exterior derivative** of a k-form ω is defined by

$$d\omega = \sum_{j_1 < ... < j_k} \sum_{i=1}^{n} \frac{\partial \omega_{j_1 ... j_k}}{\partial x^i} \; dx^{j_1} \wedge ... \wedge dx^{j_k}. \tag{176}$$

Let M be an oriented smooth $n-$dimensional manifold with boundary and let ω be a compactly supported smooth $(n-1)-$ form on M. Let $\{(U_i, \phi_i)\}_{i=1}^k$ be coordinate charts of M, such that the domains cover the support of ω. Let $\{\psi_i\}_{i=1}^k$ be a smooth partition of unity, subordinate to the cover given by $\{(U_i, \phi_i)\}_{i=1}^k$. The **integration** of ω over M is defined by [46]

$$\int_M d\omega = \sum_{i=1}^{k} \int_{\phi_i(U_i)} \left(\phi_i^{-1}\right)^* (\psi_i \; \omega). \tag{177}$$

F Bessel functions

The Bessel equations $J_n(x)$ is the solution for y to the differential equation: $x^2y''+xy'+(x^2-n^2)y=0$ [69]. $J_n(x)$ is called the **Bessel function of first kind** of order n. A **Bessel function of second kind** of order n is denoted by $I_n(x)$ and is related to $J_n(x)$ by

$$\begin{aligned} I_n(x) &= \exp\Big\{i\frac{n\pi}{2}\Big\}J_n(x\cdot\exp\Big\{-i\frac{\pi}{2}\Big\}) \\ &= iJ_n(-ix). \end{aligned} \tag{178}$$

Having defined the Bessel functions, some of its characteristics will be discussed in the following sections.

F.1 Approximations

Let's start with approximations for Assuming $n \in \mathbb{N}$, the following equation holds around $x=0$ [70]

$$J_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!(s+n+1)!}(\frac{x}{2})^{2s+n}. \tag{179}$$

Note that equations (179) and (178) imply

$$I_n(x) = \sum_{s=0}^{\infty} \frac{1}{s!(s+n+1)!}(\frac{x}{2})^{2s+n}. \tag{180}$$

These approximation can be used to find the Laplace transform of the cylinder function and are implicitly used in Section C. There is also an approximation around $x \approx \infty$, which we will use in Section 3.6,

$$I_n(x) = \frac{e^x}{(2\pi x)^{\frac{1}{2}}}\sum_{s=0}^{\infty}(-1)^s\frac{a_s(n)}{x^s}, \tag{181}$$

where

$$a_s(n) = \frac{\prod_{i=0}^{s}(4s^2-(2i+1)^2)}{(s+1)!}\sum_{j=0}^{s}\frac{1}{4s^2-(2i+1)^2}. \tag{182}$$

F.2 Derivatives

Let's consider the derivatives of $J_n(x)$ with regard to x. They will be denoted by $J_n(x)'$. The following properties hold [57]

$$J_n(x)' = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)], \tag{183}$$

$$J_n(x)' = \frac{n}{x}J_n(x) - J_{n+1}(x). \tag{184}$$

Note that equation (184) can be calculated from differentiating equation (179). Equation (183) comes from differentiating $x^{-n}J_n(x)$.
Combining equation (183) and (184), one gets

$$\begin{aligned} \frac{n}{x}J_n(x) - J_{n+1}(x) &= \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \\ \frac{n}{x}J_n(x) &= \frac{1}{2}J_{n-1}(x) + \frac{1}{2}J_{n+1}(x) \\ \frac{2n}{x}J_n(x) &= J_{n-1}(x) + J_{n+1}(x). \end{aligned} \tag{185}$$

Alternatively, this gives us

$$J_n(x)' = \frac{2n}{x} J_n(x) - J_{n+1}(x), \tag{186}$$

which we will use for calculating the disc partition function for hyperbolic surfaces.

F.3 Integration

We also know some integrals [71, p. 659-749], we will mainly use

$$\int_0^z xJ_0(x)dx = zJ_1(z). \tag{187}$$

Note that equation (178) implies

$$\int_0^z xI_0(x)dx = zI_1(z). \tag{188}$$

The following more general integral is also known

$$\int_0^z zF_n(kz)\overline{F_n}(lz)dz = \frac{z\left\{kF_{n+1}(kz)\overline{F_n}(lz) - lF_n(kz)\overline{F_{n+1}}(lz)\right\}}{k^2 - l^2}. \tag{189}$$

In equation (189), F_μ and $\overline{F_\nu}$ are cylinder functions of order n. As Bessel functions of the first and second kind are cylinder functions, equation (189) holds for them as well.

F.4 Calculations

Some small calculations have been repeated in the main section. To make the main section of thesis more readable and stop clutter from piling up, some calculations have been moved to the following subsections.

F.4.1 Calculations for $Z_\mu(r)$

The functional derivative of Z_μ with respect to $\mu(L')$ is given by

$$\begin{aligned}\partial_{\mu(L')}(Z_\mu(r)) = \frac{\partial Z_\mu(r)}{\partial \mu(L')} &= -\frac{\partial}{\partial \mu(L')}\int_0^\infty \mu(L)I_0(L\sqrt{r})dL \\ &= -\int_0^\infty I_0(L\sqrt{r})\frac{\partial \mu(L)}{\partial \mu(L')}dL = -\int_0^\infty I_0(L\sqrt{r})\delta(L-L')dL \\ &= -I_0(L'\sqrt{r}).\end{aligned} \tag{190}$$

Using equation (190), the functional derivative of the square of Z_μ gives us

$$\begin{aligned}\partial_{\mu(L')}(Z_\mu(r)^2) &= 2Z_\mu(r)\partial_{\mu(L')}\left[Z_\mu(r)\right] \\ &= -2Z_\mu(r)I_0(L'\sqrt{r}).\end{aligned} \tag{191}$$

Using equations (57) and (191) then gives us

$$\partial_{\mu(L')}(Z_\mu(R)^2) = 0. \tag{192}$$

F.4.2 Calculations: $\mathcal{W}(L')$

In equation (65), the following calculation has been used

$$\begin{aligned}
\int_0^R \frac{\sqrt{r}}{L'} J_0(2\pi\sqrt{r}) I_1(L'\sqrt{r}) dr &= \int_0^R \frac{1}{L'} J_0(2\pi\sqrt{r}) \frac{\partial}{\partial L'} I_0(L'\sqrt{r}) dr \\
&= \frac{1}{L'} \frac{\partial}{\partial L'} \int_0^R J_0(2\pi x) I_0(L'x) x dx \\
&= \frac{1}{L'} \frac{\partial}{\partial L'} \frac{2\sqrt{R}\left\{L' J_0(2\pi\sqrt{r}) I_1(L'\sqrt{R}) + 2\pi J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R})\right\}}{L'^2 + 4\pi^2},
\end{aligned} \tag{193}$$

where $x = \sqrt{r}$.
In equation (66), the following identity has been used

$$\frac{1}{2} L'\sqrt{R} \left\{I_0(L'\sqrt{R}) + I_2(L'\sqrt{R})\right\} J_0(2\pi\sqrt{R}) = L'\sqrt{R} I_0(L'\sqrt{R}) - I_1(L'\sqrt{R}), \tag{194}$$

which follows from filling in $x = L\sqrt{x}$ into equation (186)

$$\frac{L\sqrt{x}}{2} \left\{I_0(L\sqrt{x}) + I_2(L\sqrt{x})\right\} = L\sqrt{x} I_0(L\sqrt{x}) - I_1(L\sqrt{x}). \tag{195}$$

F.4.3 Calculations: Analyzing R

Using equation (186), one gets

$$\begin{aligned}
\frac{d}{dr}\left[\frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r})\right] &= \frac{1}{2\pi\sqrt{r}} J_1(2\pi\sqrt{r}) + \frac{\sqrt{r}}{2\pi}[J_0(2\pi\sqrt{r}) - J_2(2\pi\sqrt{r})] \cdot 2\pi \cdot \frac{1}{2\sqrt{r}} \\
&= \frac{1}{2}[J_0(2\pi\sqrt{r}) + \frac{2}{2\pi\sqrt{r}} J_1(2\pi\sqrt{r}) - J_2(2\pi\sqrt{r})] \\
&= \frac{1}{2}[2 \cdot J_0(2\pi\sqrt{r})] = J_0(2\pi\sqrt{r}).
\end{aligned} \tag{196}$$

G The transfer theorem

The transfer theorem is used multiple times in this thesis to show what the leading terms are of some functions. The theorem goes as follows [72]:

Theorem G.1. *Let $A(z)$ be a (complex) function that has its smallest singularity at $z = \rho$ and $A(z) = a_0 + a_1(1 - \frac{z}{\rho})^{-\alpha} + o((1 - \frac{z}{\rho})^{-\alpha+1})$, then: $[\![A(z)]\!]_{z^n} = a_1 \frac{n^{\alpha-1}}{\Gamma(\alpha)} \rho^{-n}[1 + o(1)]$*

The small o-notation in theorem G.1, should not be confused with big O-notation. Using small o-notation for a function $f(n)$ to compare it to a function $g(n)$, means that $f(n)$ is ultimately smaller than $g(n)$. Mathematically speaking, this means that if we write $f(n) = o(g(n))$ we mean that $\lim_{n\to\infty}\left[\frac{f(n)}{g(n)}\right] = 0$. In the case where $g(n) = 1$, $\lim_{n\to\infty}\left[\frac{f(n)}{c}\right] = 0$ for a non-zero constant $c \in \mathbb{R}$.

If one uses big O-notation to compare a function $f(n)$ to the function $g(n)$, it means that $f(n)$ is of the same order as $g(n)$. To be more exact, if we write down $f(n) = O(g(n))$ we mean that $\lim_{n\to\infty}\left[\frac{f(n)}{g(n)}\right] = c$ for a constant $c \in \mathbb{R}$. For the case where $g(n) = 1$, we have $\lim_{n\to\infty}\left[\frac{f(n)}{c}\right] = 1$ for a non-zero constant $c \in \mathbb{R}$ [73, p. 537-538].

The previous notation is be used to describe, for example analyze, the disc function in a specific limit (see Section 3.6). In this context we also use the notation $f(n) \backsim g(n)$, which means that $f(n) = g(n) + o(g(n))$ or equivalently $f(n) = g(n)(1 + o(1))$.

G.1 Note on Theorem 2.3

In this theorem we have used the transfer theorem for $z \to \infty$. This is technically not correct, if we look at theorem G.1. However, if we substitute z with $\frac{1}{x}$ and change the radius of convergence to $\frac{1}{c_{\mathbf{q}}}$, we get the same results as stated in the proof, while being consistent with the wording of the Transfer theorem G.1.

H Second way calculating $\mathcal{W}(L')$

There is a second way to calculate equation (64). Unlike in Section 3.4.2, we do not use partial integration. Instead, we substitute the variable $L_2 = 2\pi i$ in equation (65). With equations (178), (189) and (196), one gets

$$\begin{aligned}
\int_0^R \frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r}) I_0(L'\sqrt{r}) dr &= \int_0^R \frac{2\sqrt{r}}{2\pi} \left[-iI_1\left(2\pi i\sqrt{r}\right)\right] I_0(L'\sqrt{r}) dr = 2\int_0^R \frac{\sqrt{r}}{L_2} I_1(L_2\sqrt{r}) I_0(L'\sqrt{r}) dr \\
&= \frac{2}{L_2}\frac{\partial}{\partial L_2}\int_0^R I_0(L_2\sqrt{r}) I_0(L'\sqrt{r}) dr \\
&= \frac{2}{L_2}\frac{\partial}{\partial L_2}\left[\frac{2\sqrt{R}\{L' I_0(L_2\sqrt{r}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\}}{L'^2 - {L_2}^2}\right].
\end{aligned} \tag{197}$$

Just like we did in equation (66), we calculate the derivative using equation (195)

$$\begin{aligned}
&\frac{\partial}{\partial L_2}\left[\frac{2\sqrt{R}\{L' I_0(L_2\sqrt{r}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\}}{L'^2 - {L_2}^2}\right] = \\
&\frac{4L_2\sqrt{R}\big[L' I_0(L_2\sqrt{R}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 - {L_2}^2)^2} \\
&+ 2\sqrt{R}\left[\frac{L'\sqrt{R} I_1(L'\sqrt{R}) I_1(L_2\sqrt{R})}{L'^2 - {L_2}^2} - \frac{\frac{1}{2} L_2\sqrt{R}\ \{I_0(L_2\sqrt{R}) + I_2(L_2\sqrt{R}\}\ I_0(L'\sqrt{R})}{L'^2 - {L_2}^2}\right. \\
&\left. - \frac{I_0(L'\sqrt{R}) I_1(L_2\sqrt{R})}{L'^2 - {L_2}^2}\right] = \\
&\frac{4L_2\sqrt{R}\big[L' I_0(L_2\sqrt{R}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 - {L_2}^2)^2} \\
&+ 2\sqrt{R}\left[\frac{L'\sqrt{R} I_1(L'\sqrt{R}) I_1(L_2\sqrt{R}) - L_2\sqrt{R} I_0(L_2\sqrt{R}) I_0(L'\sqrt{R})}{L'^2 - {L_2}^2}\right],
\end{aligned} \tag{198}$$

implying

$$\begin{aligned}
\frac{2}{L_2}\frac{\partial}{\partial L_2}&\left[\frac{\sqrt{R}\{L' I_0(L_2\sqrt{r}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\}}{L'^2 - {L_2}^2}\right] = \\
&-\frac{8\sqrt{R}\big[L' I_0(L_2\sqrt{R}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2 + 4\pi^2)^2} \\
+\frac{4\sqrt{R}}{L_2}&\left[\frac{L'\sqrt{R} I_1(L'\sqrt{R}) I_1(L_2\sqrt{R}) - L_2\sqrt{R} I_0(L_2\sqrt{R}) I_0(L'\sqrt{R})}{L'^2 + 4\pi^2}\right].
\end{aligned} \tag{199}$$

Using equations (64), (67) and (199), one gets

$$
\begin{aligned}
\mathcal{W}(L') &= \frac{L'}{2}\int_0^R dr Z_\mu(r) I_0(L'\sqrt{r}) \\
&= \frac{L'}{2}\left\{\int_0^R \frac{\sqrt{r}}{\pi} J_1(2\pi\sqrt{r}) I_0(L'\sqrt{r}) dr + \int_0^R I_0(L'\sqrt{r})\Big[\int_0^\infty \mu(L) I_0(L\sqrt{r}) dL\Big] dr\right\} \\
&= +\frac{4L'\sqrt{R}\big[L' I_0(L_2\sqrt{R}) I_1(L'\sqrt{R}) - L_2 I_1(L_2\sqrt{R}) I_0(L'\sqrt{R})\big]}{(L'^2+4\pi^2)^2} \\
&+ \frac{2L'\sqrt{R}}{L_2}\left[\frac{L'\sqrt{R} I_1(L'\sqrt{R}) I_1(L_2\sqrt{R}) - L_2\sqrt{R} I_0(L_2\sqrt{R}) I_0(L'\sqrt{R})}{L'^2+4\pi^2}\right] \\
&- \int_0^\infty \mu(L)\, dL\ \ \mathcal{W}(L', L).
\end{aligned}
\tag{200}
$$

Numerically checking equations (68) and (200) shows that the results are the same, giving us a proof that the expressions from the two methods are the same. This gives us evidence that the expression given in equation (68) is indeed correct.
This belief is strengthened by the fact that the results also comply with similar results given in the literature, more specifically from [55, p. 9]

$$
\begin{aligned}
L'\partial_{x_1} F_0^{\mathrm{WP}} = L'\Big\{ &-4\frac{2\pi\sqrt{R} J_1(2\pi\sqrt{R}) I_0(L'\sqrt{R}) + L' J_0(2\pi\sqrt{R}) I_1(L'\sqrt{R})}{(L'^2+4\pi^2)^2} \\
&+ \frac{2\pi R J_0(2\pi\sqrt{R}) I_0(L'\sqrt{R}) - L' R J_1(2\pi\sqrt{R}) I_1(L'\sqrt{R})}{\pi(L'^2+4\pi^2)} + \sum_{i=0}^{d} x_i \partial_{x_1}\partial_{x_i} F_0^{\mathrm{WP}}\Big\}.
\end{aligned}
\tag{201}
$$

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