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Bounding Quantum Gravity by Observations

THESIS BSc PHYSICS AND ASTRONOMY

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Abstract

The current theory of gravity, described by Einstein's general relativity, is two-loop divergent and therefore non renormalisable. The Goroff-Saganotti counterterm in the action counteracts this divergence. This introduces a new, free coupling constant in the gravitational dynamics. For spherically symmetric, vacuum solutions, this produces a modified Schwarzschild metric, with additional quantum gravity correction terms. These extra terms depend on a new coupling constant, which determines how strongly the extra terms contribute to the modified metric. Astrophysical tests, such as the perihelion advance of Mercury or the orbit of photons around a black hole, allow us to test the spacetime geometry experimentally. By investigating different tests of general relativity, we can derive numerical upper bounds for the coupling constant.

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1 Introduction

General relativity is the foundation, on which the most important physical and astrophysical discoveries of the 20th and 21st centuries lie. On the astrophysical side, one example is the discovery, description and visualization of black holes. Gravitational waves, which are a prediction of the theory of general relativity, were measured for the first time in 2015 by the LIGO/VIRGO team, as described in [1]. Very recently, an international collaboration of Pulsar Timing Arrays published results in [2] on the measurement of gravitational waves in very low frequencies. The orbit of planets or stars and the shift of the perihelion of an elliptical orbit are also accurately described by general relativity, as seen in [3].

The theory of general relativity arises from the insight, first stated in [4], that special relativity and Newton's law of gravity are inconsistent. Newton's law states that there is an instant effect caused by the attraction of two objects towards each other. This would mean there is communication faster than the speed of light, which special relativity forbids. The concept of spacetime solves this inconsistency and gives rise to a new theory. Mass curves spacetime, which forms a gravitational field around this mass. The curvature of spacetime tells matter how to move. A different mass in a stable orbit around the larger mass will move on a geodesic - a straight line - in this curved spacetime.

This thesis will start by explaining the theory and structure of general relativity. This gives us a mathematical toolbox to describe the force of gravity. An important mathematical component in general relativity, and an essential part of this thesis, is the concept of a metric. A metric measures the distance between two points in a certain geometry. It gives the equations of motion and the energy of particles moving in it. In general relativity, the curvature of spacetime around a spherically symmetric static object is described by the Schwarzschild metric. The Schwarzschild metric is an exact solution to the Einstein field equation in a vacuum.

The next step, after uniting the theories of Newtonian gravity and special relativity, is to find a theory in line with both general relativity and quantum mechanics. This would be a theory of quantum gravity. This thesis is connected to perturbative quantum gravity. We know that the current theory of gravity is non-renormalisable: evaluating gravitational Feynman diagrams at two-loop level shows that there are infinities, which cannot be absorbed into the coupling constants of general relativity. Removing these infinities requires a new coupling constant that need to be determined from experiments. This new coupling constant also causes a modified Schwarzschild metric, which includes new quantum gravity correction terms. Such an expansion of the Schwarzschild metric into higher-order terms is very similar to an already known framework: the post-Newtonian parameterisation. The post-Newtonian parameterisation uses an approximation where objects move at speeds much slower than the speed of light, and gravitational fields are weak. This is the case for massive objects in our solar system.

The task of determining these numerical coupling constants in the modified Schwarzschild metric can be done by investigating tests of general relativity and spacetime geometry. These tests are measurements of astrophysical phenomena, that test the Schwarzschild metric. Examples of such tests are the perihelion shift of Mercury, the deflection of light by the sun or a black hole, and the gravitational redshift of light. This thesis will go into depth on the perihelion shift of Mercury and other objects - inside and outside the solar system. The orbit of photons around a black hole will also be investigated.

In this thesis, we will calculate new equations for the variables measured in these tests of general relativity, using the modified Schwarzschild metric. These equations will then take quantum gravity corrections into account. This will give us different estimates for the upper bound on the coupling constants in a modified Schwarzschild metric. An important equation that was found in this thesis, is (15) - the Schwarzschild effective potential with a quantum gravity correction. This effective potential is essential to the tests of general relativity. (67) is a equation for the perihelion shift with quantum corrections, which was derived in this thesis. For the orbit of photons around a black hole, a small quantum perturbation on the orbit radius was found in (80). A summary of the most important results found in this thesis are given in Table 4.

2 General Relativity

In this thesis and the derivations, we will use the geometric unit system, where the speed of light c and the gravitational constant G are both set to 1. If eventually needed, units are restored by dimensional analysis.

The following section will give an introduction into the theory of general relativity. We will begin by looking at the mathematical concept of a metric; this can be used to describe the geometry of spacetime. The Schwarzschild metric is a specific metric, which plays an essential role in general relativity and the description of a gravitational field. From the Schwarzschild metric, we can derive the effective potential energy of particles in a gravitational field. Finally, we will look at the incompatibility between the theories of general relativity and quantum mechanics. We will give two arguments that these theories cause inconsistencies and divergences. Certain quantum gravity corrections, specifically the Goroff-Saganotti counterterm, can solve these divergences.

2.1 Spacetime Geometries

In flat, Euclidean space, the distance between two points in space is given by the Euclidean metric. This metric can be described either in Cartesian coordinates,

$$ds^2 = dx^2 + dy^2 + dz^2, \quad (1)$$

or in spherical coordinates,

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin(\theta) d\phi^2, \quad (2)$$

where ds^2 defines the distance between two points, (x, y, z) are the well-known Cartesian coordinates, and (r, θ, ϕ) are the radius, polar angle and azimuthal angle in spherical coordinates. This metric does not include a time coordinate, because the concepts of space and time are separate in Euclidean space. In special relativity, time must be added to this metric to create a description for spacetime. One then gets an equation for the Minkowski metric

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \quad (3)$$

Just like Euclidean space, this space is flat, corresponding to a space with no rest mass or other energy. The Minkowski metric can describe an observer in respect to all other events in spacetime. From the position of an observer in spacetime, we can define a lightcone, which is the path that a photon will take in space and time if it starts from the observer and moves into the future, or starts from the past and arrives at the observer in the present.

All information about the geometric structure of spacetime is encoded in the metric tensor $g_{\mu\nu}$. It gives a interval of the metric, defined as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \quad (4)$$

In Minkowski spacetime, the metric tensor in matrix form is denoted by

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (5)$$

This version of the Minkowski metric uses the signature $(+---)$, which corresponds to the signature used in (3). This is the signature we will use in this whole thesis.

In a metric, we can define the concept of a geodesic. A free-falling particle will move along the geodesic in a metric. Intuitively, we can see a geodesic as a straight line in respect to the geometry of the metric. It is the shortest path between two points on a metric. In the Euclidean metric given in (1) and (2), a geodesic is described as a simple straight line. The geodesic equation gives a definition to straight lines in a curved spacetime. It is given as

$$\frac{d^2 x^\mu}{d\tau} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0. \quad (6)$$

$\Gamma^\mu_{\alpha\beta}$ is called a Christoffel symbol, defined as

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\lambda} (g_{\alpha\lambda,\beta} + g_{\beta\lambda,\alpha} - g_{\alpha\beta,\lambda}), \quad (7)$$

where the notation $g_{\pi\rho,\sigma} = \frac{dg_{\pi\rho}}{dx^\sigma}$. The geodesic equation (6) is written in respect to the proper time τ . This is defined as the time measured along a certain path, by a clock following a particle on that path.

2.2 Schwarzschild Metric

Different demands can be imposed on a metric, to make it describe certain spacetime geometries. Most astrophysical objects that generate a gravitational field, such as stars or black holes, are approximately spherically symmetric and static.¹ A general equation for a spherically symmetric metric is

$$ds^2 = A(r)dt^2 - B(r)dr^2 - r^2(d\theta^2 + \sin^2(\theta)d\phi^2). \quad (8)$$

This metric is set as spherically symmetric by the fact that the angular part, including the polar angle $d\theta$ and the azimuthal angle $d\phi$, is the metric of a 2-sphere such as in (2). In the special relativistic limit, $A(r)$ and $B(r)$ are equal to 1 and the Minkowski metric in (3) is retrieved. This metric can also be encoded in a metric tensor, which is given in matrix form as

$$g_{\mu\nu} = \begin{pmatrix} A(r) & 0 & 0 & 0 \\ 0 & -B(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 1 & 0 & 0 & -r^2 \sin^2(\theta) \end{pmatrix}. \quad (9)$$

The equations of motion of general relativity fix the two functions $A(r)$ and $B(r)$ to the Schwarzschild metric. This describes the gravitational field generated by a spherically symmetric object in a vacuum. In this metric, $A(r)$ and $B(r)$ are related to the Schwarzschild radius of the object, $r_s = 2M$, by

$$A(r) = 1 - \frac{2M}{r}, \quad (10a)$$

$$B(r) = A(r)^{-1} = \left(1 - \frac{2M}{r}\right)^{-1}. \quad (10b)$$

¹An example of a spherically symmetric but non-static metric, for a rotating black hole, is the Kerr metric.

If the vacuum solution is valid at $r = 2M$ one encounters a horizon: $A(r)$ is equal to zero and $B(r)$ diverges to infinity. This shows the existence of black holes in the Schwarzschild metric. The event horizon of a black hole is set at its Schwarzschild radius.

For observers or particles moving in the Schwarzschild metric, there are certain conserved quantities that do not change over proper time. The fact that $A(r)$ and $B(r)$ are time-independent entails the conservation of energy, and the spherical symmetry gives the conservation of total angular momentum of the system. For the general, spherically symmetric metric given by (8), these are given by

$$e = A(r) \frac{dt}{d\tau}, \quad (11a)$$

$$l = r^2 \sin^2(\theta) \frac{d\phi}{d\tau}. \quad (11b)$$

For specifically the Schwarzschild metric, the conserved angular momentum is the same. The conserved energy in the Schwarzschild metric is given by

$$e = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau}. \quad (12)$$

Important to note here is that l is the effective angular momentum, and that this is related to the ‘physical’ angular momentum by $L = Ml$. Also, e is the effective total energy and is related to the ‘physical’ total energy by $e = \frac{E}{Mc^2}$. In these equations, M is an asymptotic mass in the spacetime geometry. The derivation of these conserved quantities can be found in [5].

2.3 Effective Potential

We will now consider the geodesics of particles moving in the Schwarzschild geometry. The conserved quantities provide a convenient way to find the effective potential of a massless or massive particle, moving in an orbit around a massive body such as a black hole or the sun. To do this derivation, we impose the boundary of being in the equatorial plane, where $\theta = \frac{\pi}{2}$. We can always choose a coordinate system where this is the case. First, we take the double derivative to an affine parameter λ of the general metric (8)

$$A(r) \left(\frac{dt}{d\lambda}\right)^2 - B(r) \left(\frac{dr}{d\lambda}\right)^2 - r^2 \left(\frac{d\phi}{d\lambda}\right)^2 = \delta, \quad (13)$$

where δ is equal to 0 for massless particles, and equal to 1 for massive particles. This can be seen from the fact that particles move along geodesics. $\delta = 1$ signifies that a massive particle is moving along a timelike geodesic, and $\delta = 0$ gives a null geodesic for a massless particle. We then substitute the general expressions for e and l , given in (11a) and (11b), into this equation, and then multiply the whole equation by $A(r)$, which gives

$$e^2 - AB \left(\frac{dr}{d\lambda}\right)^2 = A \cdot \left(\delta \frac{l^2}{r^2}\right). \quad (14)$$

Any spherically symmetric metric, such as the Schwarzschild metric, has $A(r)B(r) = 1$. This gives a special case for (14) and this equation can be recast into a new form to determine the effective potential for the Schwarzschild metric

$$\frac{1}{2} \left(\frac{dr}{d\lambda}\right)^2 + V_{\text{eff}}(r) = \mathcal{E}. \quad (15)$$

In this equation, the effective potential energy V_{eff} is equal to

$$V_{\text{eff}}(r) = \frac{1}{2} \left(1 - \frac{2M}{r} \right) \left(\delta + \frac{l^2}{r^2} \right) = \frac{1}{2} \delta - \delta \frac{M}{r} + \frac{l^2}{2r} - \frac{Ml^2}{r^3}, \quad (16)$$

and \mathcal{E} is related to the conserved energy by

$$\mathcal{E} = \frac{e^2 - 1}{2}. \quad (17)$$

The first three terms of the effective potential in (16) are Newtonian, and the last term is a relativistic correction. This effective potential is used to calculate the general relativistic effects in many tests of spacetime geometry. This will be shown in chapter 5.

2.4 General Relativity and Quantum Mechanics

General relativity and quantum mechanics are known to be incompatible. Conceptually, the inconsistency can be described in one sentence: quantum mechanics describes matter and energy as quantized, while general relativity describes them as classical. This problem can be seen in the Einstein field equation, which is a tensor equation describing the relationship between spacetime and the matter within it

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = 8\pi G T^{\mu\nu}. \quad (18)$$

Here $G^{\mu\nu}$ is the Einstein tensor and describes the curvature of spacetime. It is consistent with the conserved quantities of energy and momentum, which were described in section 2.2. $R^{\mu\nu}$ is the Ricci tensor, and can be defined in simple language as the degree to which the geometry of a spacetime differs from the Euclidean metric. In terms of Christoffel symbols, the Ricci tensor is

$$R_{\alpha\beta} = \Gamma^\rho_{\beta\alpha,\rho} - \Gamma^\rho_{\rho\alpha,\beta} + \Gamma^\rho_{\rho\lambda} \Gamma^\lambda_{\beta\alpha} - \Gamma^\rho_{\beta\lambda} \Gamma^\lambda_{\rho\alpha}, \quad (19)$$

where the notation $\Gamma^\rho_{\alpha\beta,\gamma} = \frac{d\Gamma^\rho_{\alpha\beta}}{dx^\gamma}$. The trace of the Ricci tensor is equal to the Ricci scalar R . $g^{\mu\nu}$ is the contravariant version of the metric tensor, which was shown in section 2.1. $T^{\mu\nu}$ is the stress-energy tensor, which describes the density of energy and momentum, and therefore matter, in spacetime.

A more mathematical piece of evidence for the conflict between general relativity and quantum mechanics comes from the divergence of the Einstein-Hilbert action. The action of a system is a scalar quantity, which encodes the equations of motion of a theory. The action is the integral over the Lagrangian of a system and is therefore related to the total energy of a system. In general relativity, the Einstein-Hilbert action gives rise to the Einstein field equation in a vacuum. The Einstein-Hilbert action is defined as

$$S_{EH} = \frac{1}{16\pi G} \int R \sqrt{-\det(g_{\mu\nu})}, \quad (20)$$

where $g_{\mu\nu}$ is the metric tensor and R is the Ricci scalar, which are both also parts of the Einstein field equation. The coupling constant G of the Einstein-Hilbert action in (20) has a dimension of E^{-2} , where E is the total energy of the system. Therefore, to create a dimensionless quantity, G should be multiplied by E^2 . The Feynman diagrams encoding the quantum corrections to (20), include terms of $(G \cdot E^2)^n$. This shows that the current theory of gravity is perturbatively nonrenormalisable. At higher orders of perturbation theory, terms with $(G)^n$ will appear with factors of E^{2n} . The divergence

appearing as $E^2 \rightarrow \infty$ signals the breakdown of (20) as a quantum theory.

For Einstein's equations in a vacuum, $R_{\mu\nu} = 0$. The only static spherically symmetric solution of this equation, is the Schwarzschild metric. One can then calculate which curvature invariants in four dimensions can be non-zero, if you impose this condition on the equations of motion. Curvature invariants are a set of scalars formed from the Riemann, Weyl and Ricci tensors which describe curvature of spacetime. The Ricci tensor was defined earlier, and can be found in (18). The Ricci tensor is a contraction of the Riemann tensor $R_{\alpha\beta\mu\nu}$, so that

$$R_{\mu\nu} = g^{\alpha\beta} R_{\alpha\beta\mu\nu}. \quad (21)$$

The Weyl tensor is defined as the traceless part of the Riemann tensor. The equation relating the Weyl tensor to the Riemann tensor and Ricci scalar is

$$\begin{aligned} C_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} + \frac{1}{n-2} (R_{\alpha\delta}g_{\beta\gamma} - R_{\alpha\gamma}g_{\beta\delta} + R_{\beta\gamma}g_{\alpha\delta} - R_{\beta\delta}g_{\alpha\gamma}) \\ + \frac{1}{(n-1)(n-2)} R (g_{\alpha\gamma}g_{\beta\delta} - g_{\alpha\delta}g_{\beta\gamma}), \end{aligned} \quad (22)$$

where n is the dimension of the geometry. The curvature invariants have an order: an order zero curvature invariant is just "1", order one is the Ricci scalar R , order two is R^2 , $R_{\mu\nu}R^{\mu\nu}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, and so on.

't Hooft and Veltman showed in [6] that the one-loop correction causes all curvature invariants until order two to vanish for Einstein's equation in a vacuum. This means that gravity is one-loop finite. 'One-loop correction' means this is a first order expansion of the action in G . The curvature invariants provide the power of the energy that is needed at this given order: G has a dimension of E^{-2} , so a one-loop correction gives second order curvature invariant terms. For two-loop corrections, it is not the case that all curvature invariants vanish. Goroff and Saganotti calculated in [7] that the only curvature invariant that survives vacuum conditions is one of order three, that is $C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\tau\omega}C_{\tau\omega}{}^{\mu\nu}$. This is a combination of three Weyl tensors with two indexes raised, so $C_{\mu\nu}{}^{\rho\sigma} = g^{\alpha\rho}g^{\beta\sigma}C_{\mu\nu\alpha\beta}$. This signifies that gravity is two-loop divergent. This problem is thought to reoccur to all higher-order corrections.

To generate a new metric which includes the physics effects of the two-loop counterterm, a new action functional is defined. The new action will contain the Goroff-Saganotti counterterm, which will counteract the divergences that occur in the expansion of the action function. The Einstein-Hilbert action receives a correction containing the non-vanishing curvature invariant of order three

$$S = \int \sqrt{-\det(g_{\mu\nu})} \left[\frac{R}{16\pi G} + \kappa C_{\mu\nu}{}^{\rho\sigma}C_{\rho\sigma}{}^{\tau\omega}C_{\tau\omega}{}^{\mu\nu} \right]. \quad (23)$$

In this equation κ is a to-be-determined coupling constant. This coupling constant is no longer dependent on mass, angular momentum, energy or other properties of a system. From this modified action, we can derive new equations of motion. These new equations of motion give us a new metric: a modified Schwarzschild metric. This metric will be given and further discussed in section 4.1.

3 Tests of Spacetime Geometry

Confronting general relativity with observations is a longstanding research line [3]. This is most prevalent in the observations of gravitational waves that have been done in the last decade. The wave form, emitted by the merging of two black holes and detected by the LIGO/VIRGO teams, matched the predictions that were done by the theory of general relativity. Testing general relativity to a high accuracy and finding discrepancies could be the key to solve the gap between the general relativity description of gravity, and a new theory of quantum gravity.

The three classical tests of general relativity were already proposed by Albert Einstein in his famous paper ‘The Foundation of the Generalized theory of Relativity’ from 1916. Observational tests, including the bending of light and the perihelion shift of Mercury are already suggested in the last paragraph of [4]. Astronomers from that time had already measured the perihelion shift as a small correction to the Keplerian and Newtonian laws of gravity. The theoretical value, derived from the theory of general relativity by Einstein, matched the observations that had been done. The perihelion shift of Mercury and corresponding calculations are described in section 3.1 in detail. The other two tests have been experimentally verified by astronomical observations, and give certain bounds to the PPN parameters, that will be discussed in section 4.3. These classical tests and many others are described in large detail in [3].

The three classical tests of general relativity are the deflection of light by the sun, the gravitational redshift of light and the perihelion shift of Mercury. The deflection of light by the sun is based on the principle that massless particles including photons are influenced by gravitational fields. This causes a phenomenon known as gravitational lensing. When an object with a strong gravitational field, such as a star, a black hole or a galaxy, sits in between an observer and a further light source, it will act as a lens for the light traveling towards the observer. Therefore, stars behind the sun will appear in a different position on the night sky than their actual position, for an observer on earth. Light from stars located behind the sun will even become visible. When the source, the lens and the observer are all in one line, the observer will see the source as a ring of light, known as an Einstein ring. When the light source is not directly behind the gravitational lens, this will cause the Einstein ring to only partially appear. Multiple images of the source will appear to the observer. The equations describing gravitational lensing and the Einstein ring are described in p.234 - 243 of [8]. The influence of a gravitational field on photons, specifically for a black hole, will be further shown in section 3.3.

The gravitational redshift of light is shown by the measurement that photons lose energy traveling out of a gravitational field. This causes an increase in wavelength of the light and therefore a redshift. This phenomenon is based on the equivalence principle, one of the basic principles in general relativity. It states that gravitational and inertial mass are equivalent, so a photon should exhibit the same behavior in a uniformly accelerating frame as in a uniform gravitational field. The equations describing gravitational redshift of light can be found in p. 189 - 191 of [8].

In this part of the thesis, the mathematical framework behind a few tests of general relativity, using the Schwarzschild metric, will be shown. This will lay a foundation to test a modified Schwarzschild metric in the same manner.

3.1 Perihelion Shift of Mercury

The orbits of planets around the sun are elliptical. The sun sits in one of the focal points of this ellipse. Newtonian Mechanics gives a sufficiently accurate description of this orbit in many cases. However, already in the 19th century, astronomers noticed a shift of the perihelion of the orbit of Mercury. This shift was not fully explained by the influence of other planets on Mercury. Einstein's theory of general relativity accounted for this difference, and this was a very important supporting argument for the theory.

The measured total precession of the perihelion of Mercury has been found in [9] to be 575.31 ± 0.0015 arcseconds per century. This precession mostly occurs because of the gravitational effects of other planets on Mercury's orbit, this accounts for 532.3035 arcseconds. The general relativity effect of perihelion shift accounts for 42.9799 arcseconds. This was measured using an analysis of the data obtained by the MESSENGER (MErcury Surface, Space ENvironment, GEochemistry, and Ranging) spacecraft in orbit around Mercury.

The mathematical expression for the general relativity perihelion shift of Mercury, or any other object in a stable elliptical orbit, can be determined by finding the change in the azimuthal angle ϕ per full orbit. It is assumed that Mercury is in the equatorial plane of the sun, so $\theta = \frac{1}{2}\pi$. The conserved quantity of angular momentum in (11b) is then given by $l = r^2 \frac{d\phi}{d\tau} = r^2 \frac{d\phi}{dr} \frac{dr}{d\tau}$. Separating $\frac{dr}{d\tau}$ in this formula and combining this with (15) will give a general equation for $\frac{d\phi}{dr}$:

$$\frac{d\phi}{dr} = \pm \frac{l}{r^2} \frac{1}{2(\mathcal{E} - V_{\text{eff}}(r))^{\frac{1}{2}}}. \quad (24)$$

To find the change in the angle ϕ per rotation, we must integrate this expression between the two turning points of the elliptical orbit. For a Newtonian case ϕ would be equal to 2π , because the planet has completed a full orbit. The corrections of general relativity to this value, which cause the shift of the perihelion, make that $\phi > 2\pi$. To find the value of these corrections, one subtracts 2π from ϕ , and this quantity will be written as $\delta\phi = \phi - 2\pi$.

In the process of doing the integral over the angle, a few approximations are made. If we fill in the values for V_{eff} and \mathcal{E} , derived in (16) and (17), into (24), we can decompose the denominator of the fraction into convenient terms

$$2 \cdot (\mathcal{E} - V_{\text{eff}}(r))^{-1/2} = 2 \cdot \left(e^2 - \frac{1}{2} \left(1 - \frac{2M}{r} \right) \left(1 + \frac{l^2}{r^2} \right) \right)^{-1/2}.$$

We substitute this into (24) and also put back in the constants c and G . This is done by substituting $l \rightarrow \frac{l}{c}$ and $M \rightarrow \frac{GM}{c^2}$.

$$\begin{aligned} \frac{d\phi}{dr} &= \pm \frac{l}{cr^2} \cdot \left(e^2 - \frac{1}{2} \left(1 - \frac{2GM}{c^2 r} \right) \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2} \\ &= \pm \frac{l}{r^2} \left(1 - \frac{2GM}{c^2 r} \right)^{-1/2} \left(c^2 e^2 \left(1 - \frac{2GM}{c^2 r} \right)^{-1} - c^2 \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2} \\ &\approx \pm \frac{l}{r^2} \left(1 + \frac{GM}{c^2 r} + \dots \right) \left(c^2 e^2 \left(1 - \frac{2GM}{c^2 r} + \frac{4G^2 M^2}{c^4 r^2} + \dots \right) - c^2 \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2} \end{aligned} \quad (25)$$

Here two approximations are done to cut off the expression at the leading order general relativity terms. The expansion of the term $(1 - \frac{2GM}{c^2 r})^{-1/2}$ is expanded up to $O(r^{-2})$, and the term $(1 - \frac{2GM}{c^2 r})^{-1}$ is expanded up to $O(r^{-3})$. The terms that are neglected later turn out to be sufficiently small compared to the leading order perihelion shift. We will elaborate on the higher order general relativity terms, which are neglected in this calculation, in section 5.1.

The next approximation we will do, is to find an expression for the conserved energy e . We anticipate a correspondence between e and the Newtonian energy E . Therefore we approximate the conserved energy as

$$e = \frac{mc^2 + E}{mc^2} = 1 + \frac{E}{mc^2}. \quad (26)$$

This approximation can be found on p. 194 of [8]. Using this, e^2 turns out to be

$$e^2 = \left(1 + \frac{E}{mc^2}\right)^2 = 1 + \frac{2E}{mc^2} + \frac{E^2}{m^2 c^4}, \quad (27)$$

with E the Newtonian energy of the orbit and m the mass of the smaller body, which in this case is Mercury. Using this expression, we can rewrite the second part of (60) as a polynomial in $\frac{1}{r}$. Here all terms of order $O(c^{-4})$ are negligible.

$$\frac{d\phi}{dr} \approx \pm \frac{l}{r^2} \left(1 + \frac{GM}{c^2 r}\right) \left(a \cdot \frac{1}{r^2} + b \cdot \frac{1}{r} + \eta\right)^{-1/2}, \quad (28)$$

where a , b and η are calculated to be

$$a = l^2 - \frac{4G^2 M^2}{c^2}, \quad (29a)$$

$$b = 2GM + \frac{4EGM}{mc^2}, \quad (29b)$$

$$\eta = 2E + \frac{E^2}{m^2 c^2}. \quad (29c)$$

The roots of this polynomial, using the coordinate transformation $u = \frac{1}{r}$, are

$$u_{1,2} = \frac{b}{2a} \pm \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{\eta}{a}}. \quad (30)$$

Now we have rewritten this into a convenient form to integrate it over r . First we do the previously mentioned coordinate transformation $u = \frac{1}{r}$. This means that $du = -\frac{1}{r^2} dr$, so the whole expression is multiplied by r^2 and the multiplication factor just becomes $\pm l$. Because the integral is symmetric, $\int_{-\infty}^{+\infty} \pm l \dots dr = 2l \int_0^\infty \dots dr$. The polynomial in u that was defined in (28), gives two roots in (30), which are also the two boundaries for the integral.

$$\begin{aligned} \Delta\phi &= 2l \int_{u_1}^{u_2} \left(1 + \frac{GM}{c^2} u\right) (au^2 + bu + \eta)^{-1/2} du \\ &= 2l \int_{u_1}^{u_2} \left(1 + \frac{GM}{c^2} u\right) a^{-1/2} ((u_1 - u)(u - u_2))^{-1/2} du \end{aligned} \quad (31)$$

The integral can now be split up into two parts and solved as two elliptic integrals

$$\Delta\phi_{SS} = 2l \cdot a^{-1/2} \left(\int_{u_1}^{u_2} \frac{1}{\sqrt{(u_1 - u)(u - u_2)}} + \frac{GM}{c^2} \int_{u_1}^{u_2} \frac{u}{\sqrt{(u_1 - u)(u - u_2)}} \right). \quad (32)$$

We also use the approximation

$$a^{-\frac{1}{2}} = \left(l^2 - \frac{4G^2 M^2}{c^2} \right)^{-\frac{1}{2}} \approx \frac{1}{l} \left(1 + 2 \left(\frac{GM}{cl} \right)^2 \right). \quad (33)$$

The two integrals in (32) are standard elliptic integrals that can be solved analytically as

$$\int_{u_1}^{u_2} \frac{1}{\sqrt{(u_1 - u)(u - u_2)}} = 2\pi, \quad (34a)$$

$$\int_{u_1}^{u_2} \frac{u}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{\pi}{2} (u_1 + u_2) = \frac{\pi}{2} \frac{b}{a} = \frac{\pi GM}{l^2} + O(c^{-2}). \quad (34b)$$

This means the final equation for the change in the azimuthal angle ϕ in the Schwarzschild metric is given by

$$\Delta\phi = 2\pi + 6\pi \left(\frac{GM}{cl} \right)^2. \quad (35)$$

This equation with a more condensed derivation and conceptual remarks can also be found in p. 201-204 of [8]. The first term in $\Delta\phi$ is a Newtonian term. In a single Newtonian elliptic orbit, the azimuthal angle ϕ will experience a change of 2π . The orbiting body will then find itself back in the same position after every orbit. The precession of the perihelion in Newtonian Mechanics is therefore given by $\delta\phi = \Delta\phi - 2\pi = 0$. General relativity contributes the second term to equation (35); after an orbit of 2π , the azimuthal angle will ‘shift’ with the value of the general relativity correction, and this causes the advance of the perihelion of elliptical orbits. The precession is then given by

$$\delta\phi = \Delta\phi - 2\pi = 6\pi \left(\frac{GM}{cl} \right)^2. \quad (36)$$

3.2 Perihelion Shift of Stars

Sagittarius A* is the supermassive black hole at the center of our Milky Way galaxy. A group of stars known as ‘S stars’ are in highly elliptical orbits around Sagittarius A*. Because their orientations in space appear random, it is most likely they were brought into these elliptical orbits by individual scattering events, and not as a cluster. Precise measurements of the movements of these stars and more information on them can be found in [10].

General relativity predicts a perihelion shift in the elliptical orbit of these stars, similarly to the orbit of Mercury. Star S2 is one of the most investigated stars in the group, and in 2020 the first measurement of the perihelion shift in the star’s orbit was reported in [11]. S2’s radial velocity and trajectory were monitored for 2.7 decades to show this. Until 2018, it was thought that S2 might be a part of a double star system. This would make it not suitable to test the theory of general relativity. In [12], it was proven from 16 years of earlier measurements that the orbit of S2 is not consistent with that

of a double star system. This makes it a good candidate for tests of spacetime geometry.

However, Sagittarius A* is a rotating black hole. This means its gravitational field is described by the Kerr metric, and not by the Schwarzschild metric. At large distances from the source, the Kerr metric reduces to the Schwarzschild metric. This means for star S2, the Schwarzschild metric is a correct approximation. The Kerr metric does give a different value for the black hole shadow, compared to the Schwarzschild metric. Black holes with spin can have a shadow size of 7.5 % smaller than black holes without a spin. This will not influence the equations for the perihelion shift of star S2. The differences between the Kerr and Schwarzschild metric in analyzing Sagittarius A* are described in [13].

The perihelion shift of S2 can be calculated with the same equations as the perihelion shift of Mercury. Therefore we can use equation (35), with the mass of Sagittarius A* and the angular momentum of the orbit of S2.

3.3 Photon Orbit around a Black Hole

A photon ring is a ring of light found around a black hole, where photons are kept in a circular orbit. This ring is found further out from the black hole than the event horizon at the Schwarzschild radius. The radius of the photon ring can be calculated by minimizing the effective potential with respect to the radius. For this we will use the effective potential calculated in (16):

$$\frac{dV_{\text{eff}}}{dr} = \delta \frac{M}{r^2} - \frac{l^2}{r^3} + \frac{3Ml^3}{r^4}. \quad (37)$$

The effective potential energy is minimized by setting $\frac{dV_{\text{eff}}}{dr} = 0$, this gives

$$\delta Mr^2 - l^2 r + 3Ml^2 = 0. \quad (38)$$

It is now interesting to analyze (38) in the limit of Newtonian Gravity, and for particles with and without mass. In Newtonian gravity, as has been noted for equation (15), the last term in the effective potential disappears. Therefore the last term in (38) containing the angular momentum l is zero. For particles with a mass, we find that stable circular orbits then occur at $r = \frac{l^2}{M}$.² For massless particles, we find that circular orbits are impossible. This is consistent with the theory of Newtonian gravity: massless particles should not feel the effect of a gravitational force. General relativity does describe the effect of a gravitational field on a photon and for this we solve (38) for $\delta = 0$. The result we obtain is

$$r = 3M \quad (39)$$

This means the theoretical prediction is to find a photon ring at $r = \frac{3GM}{c^2}$.

²This result agrees with Kepler's third law, when we retrieve c and G in the equation.

4 Quantum Corrections to the Schwarzschild Metric

In section 2.4, we have explained the Goroff-Saganotti counterterm in the action in (23). From this modified action functional, we can derive quantum corrections to the Schwarzschild metric. In this section we will first explain the free parameters that are introduced in the quantum corrections. Using the modified Schwarzschild metric, we will derive a new equation for the effective potential in (16). This effective potential takes the form of the Schwarzschild effective potential, plus a quantum correction. Finally, we will explain the conception of the post-Newtonian framework. This framework introduces parameterized post Newtonian (PPN) parameters, which bound the terms in a power expansion of the Schwarzschild metric.

4.1 Corrections

The Goroff-Saganotti term added on to equation (23) gives rise to corrections to the Schwarzschild metric. The functions $A(r)$ and $B(r)$ in the Schwarzschild metric are at higher orders of $\frac{1}{r}$. The first terms in this expansion that are non-zero, are the terms containing $\frac{1}{r^6}$ and $\frac{1}{r^7}$. There are more terms after this, but in this thesis, we will analyse the leading order quantum corrections to $A(r)$ and $B(r)$. The modified equations are given by

$$A(r) \approx 1 - \frac{2M}{r} + \frac{a_6}{r^6} + \frac{a_7}{r^7}, \quad (40)$$

$$B(r) \approx \left(1 - \frac{2M}{r} + \frac{b_6}{r^6} + \frac{b_7}{r^7}\right)^{-1} \quad (41)$$

The terms containing $\frac{1}{r^6}$ and $\frac{1}{r^7}$ are introduced with free parameters a_6 , b_6 , a_7 and b_7 . These constants are only dependent on the coupling constant κ in (23) and no longer on the mass, energy or momentum of the body generating the gravitational field. It follows from the derivation of the equations of motion from the modified action that the coupling constants $a_6 \neq b_6$ and $a_7 \neq b_7$.

This modified metric gives rise to a modification of the conserved quantities. The conserved angular momentum is left unchanged compared to (11b), because the modified Schwarzschild metric does not change the angular part of the metric. The modified conserved energy is still given by (11a), but now we will use (40) for $A(r)$ instead of the Schwarzschild version.

4.2 Effective Potential

This new, modified metric also gives rise to a new effective potential, because $A(r)$ is no longer equal to $B(r)^{-1}$. To derive this, we start from equation (14) and divide it by $2B$

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 - \frac{e^2}{2AB} + \frac{l^2}{2Br^2} = -\frac{\delta}{2B}. \quad (42)$$

This is a convenient form to write this equation, because we can now expand certain terms and use this to recast the equation into the form of (15). A change to this equation, compared to the Schwarzschild metric case, is that the e^2 term is modified by a factor $\frac{1}{AB}$. Another difference is that the l^2 and δ terms are both divided by an extra factor of B . To process these differences into a new expression for $V_{\text{eff}}(r)$, we must make a few approximations.

The first approximation is to reduce AB into only its leading order terms. The constants a_7 and b_7 are assumed to be negligibly small and all terms containing $\frac{1}{r^7}$ are thrown away.

$$\begin{aligned}
AB &\approx \left(1 - \frac{2M}{r} + \frac{a_6}{r^6}\right) \left(1 - \frac{2M}{r} + \frac{b_6}{r^6}\right)^{-1} \\
&\approx \left(1 - \frac{2M}{r} + \frac{a_6}{r^6}\right) \left(1 - \frac{2M}{r}\right)^{-1} \left(1 + \frac{b_6}{r^6}\right)^{-1} \\
&\approx 1 + \frac{a_6}{r^6} - \frac{b_6}{r^6} + O\left(\frac{1}{r^7}\right)
\end{aligned} \tag{43}$$

This means that the term $-\frac{e^2}{2AB}$ can be split into a ‘Schwarzschild term’ and a modified term

$$\begin{aligned}
-\frac{e^2}{2AB} &\approx -\frac{e^2}{2} + \frac{e^2}{2} \frac{a_6 - b_6}{r^6} \\
&\approx \mathcal{E} + \frac{e^2}{2} \frac{a_6 - b_6}{r^6}.
\end{aligned} \tag{44}$$

With this we have reduced this term to the Schwarzschild case, plus a quantum gravity correction. The same should be found for the l^2 and δ terms.

$$\begin{aligned}
\frac{1}{2B} \left(\frac{l^2}{r^2} + \delta\right) &= \frac{1}{2} \left(1 - \frac{2M}{r} + \frac{b_6}{r^6}\right) \left(\frac{l^2}{r^2} + \delta\right) \\
&\approx \frac{1}{2} \left(1 - \frac{2M}{r}\right) \left(1 + \frac{b_6}{r^6}\right) \left(\frac{l^2}{r^2} + \delta\right) \\
&= V_{\text{eff,SS}}(r) \left(1 + \frac{b_6}{r^6}\right) \\
&\approx V_{\text{eff,SS}}(r) + \frac{b_6}{2r^6} \delta + O\left(\frac{1}{r^7}\right)
\end{aligned} \tag{45}$$

Now, we can write an equation for the modified effective potential, which is equal to the Schwarzschild effective potential given in (15), plus a correction term

$$V_{\text{eff,QG}}(r) = V_{\text{eff,SS}}(r) + \frac{1}{r^6} \left(\frac{b_6}{2} \delta + e^2(a_6 - b_6)\right). \tag{46}$$

This modified effective potential has a few remarkable qualities. For massless particles, it only gives information on the difference between the constants a_6 and b_6 . For particles with a mass, it could give information on a combination of $a_6 - b_6$ and b_6 . However, e is the effective total energy, meaning the total energy divided by mc^2 . For systems where mass is not converted into energy almost perfectly, e will be much smaller than 1. This is the case for all systems that are analyzed in the tests of spacetime geometry, so we can assume that the term containing $a_6 - b_6$ will be far smaller than the term with only b_6 . We also assume that $a_6 - b_6$ are in the same order as b_6 . This means that tests using massive particles, like Mercury or a star, will give information on the bounds for only b_6 . By combining this with tests of massless particles, a bound on a_6 can also be derived.

4.3 Parameterized Post-Newtonian Formalism

A mathematical formalism to describe small-order deviations from general relativity already exists and is used widely. This is called the parameterized post-Newtonian (PPN) formalism. The PPN parameters are a set of 10 parameters which describe different behaviors of the theory of gravity. They were first formulated by Clifford Will in [14].

The PPN framework uses an isotropic coordinate system for metrics, instead of the usual radial equation. The coordinate transformation used to bring the Schwarzschild metric from Schwarzschild coordinates to isotropic coordinates is given by

$$r = \rho \left(1 + \frac{M}{2\rho} \right)^2. \quad (47)$$

This coordinate transformation turns the metric into its isotropic form.

$$ds^2 = \left(1 + \frac{M}{2\rho} \right) (dx^2 + dy^2 + dz^2) + \frac{\left(1 + \frac{M}{2\rho} \right)^2}{\left(1 - \frac{M}{2\rho} \right)^2} dt^2 \quad (48)$$

The derivation of this coordinate transformation and a calculation of the isotropic metric can be found in Appendix A. It is important to understand this coordinate transformation, so we can find a new coordinate transformation, and with this an isotropic metric, which corresponds to the modified quantum gravity metric.

One can think of a physical system in which motions are slow compared to the speed of light and gravitational fields are weak. This is a correct assumption for solar system tests of general relativity. It then turns out that the metric tensor $g_{\mu\nu}$ can be written as expansion about the Minkowski metric $\eta_{\mu\nu}$ given in (3). This expansion is done in terms of gravitational potentials. For a spherically symmetric, static metric, the only potential that is non-zero is the Newtonian gravitational potential. For the Schwarzschild Metric, the Newtonian Gravitational potential is equal to $U = \frac{GM}{c^2 r}$. A general equation for this potential is

$$U(\mathbf{x}, t) = \int \frac{\rho(x', t)}{|x - x'|} d^3 x'. \quad (49)$$

where x, x' are the positions of two objects with a mass, and $\rho(x', t)$ is the density of rest mass at position x' . The coefficients which are put in front of these gravitational potentials, are called PPN parameters. The values of the PPN parameters is dependent on which theory the metric is describing. The PPN parameters which are relevant to the classical tests of general relativity, are β and γ . β describes the non-linearity of gravity. Intuitively, this parameter may be understood as follows: adding the forces of gravity at a single point should happen linearly. The total force is equal to the sum of all separate acting forces. If there would be an extra term in this sum, meaning a term that is non-linear, β would describe to what degree this term contributes to the theory of gravity. The other relevant PPN parameter γ gives a description of how much space curvature is produced by unit rest mass. In general relativity, both of these parameters are set to 1. The different components of the metric tensor in a isotropic, symmetric and static metric are given by

$$g_{00} = 1 - 2U - 2\beta U^2 + O(U^3), \quad (50a)$$

$$g_{ij} = (1 + 2\gamma U)\delta_{ij} + O(U^2). \quad (50b)$$

As described in p. 222 of [8], ³ substituting this in (50a) and (50b) gives the following expansion for the metric

$$g_{00} = A_{\text{SS,PPN}}(r) = 1 - \frac{2GM}{c^2 r} + 2\beta \left(\frac{GM}{c^2 r} \right)^2 + \dots, \quad (51a)$$

$$g_{rr} = B_{\text{SS,PPN}}(r) = 1 + 2\gamma \left(\frac{GM}{c^2 r} \right) + \dots \quad (51b)$$

Earlier research has found bounds on the parameters β and γ , using tests of spacetime geometry. The current best bounds on β and γ , which are described in [3], are

$$\beta - 1 < 8 \cdot 10^{-5}, \quad (52a)$$

$$\gamma - 1 < 2.3 \cdot 10^{-5}. \quad (52b)$$

The bound (52a) can be found by measuring the perihelion shift of Mercury. (52b) is found by measuring the gravitational redshift of radio waves, or the deflection of light by a heavy object. This is done with measurements of radio waves transmitted from the Cassini space probe. A more in depth discussion of how the bounds on these parameters are obtained, can be found in [15].

As mentioned earlier, general relativity assumes the values $\beta = \gamma = 1$ for these PPN parameters. If one takes these values in (51a) and (51b), $A(r)$ and $B(r)$ return back to the Schwarzschild metric. To analyze a deviating theory of gravity, like quantum gravity, it is appropriate to look at the bounds that have already been found for γ and β . We can describe $A(r)$ and $B(r)$, from the modified metric in (40) and (41), in terms of the PPN parameters.

To do this, we must fill in (41), which is $B(r)$ with quantum corrections, into the differential equation (86). This will give us the equation

$$\frac{d\rho}{\rho} = \frac{dr}{\left(r^2 - 2Mr + \frac{b_6}{r^4}\right)^{1/2}}. \quad (53)$$

We can make an approximation to simplify this equation,

$$\begin{aligned} \frac{dr}{\left(r^2 - 2Mr + \frac{b_6}{r^4}\right)^{1/2}} &\approx \frac{dr}{(r^2 - 2Mr)^{1/2} \left(1 + \frac{b_6}{r^6} + O(r^{-7})\right)^{1/2}} \\ &\approx \left(1 + \frac{b_6}{r^6}\right) \cdot \frac{dr}{(r - 2Mr)^{1/2}} \\ &= \frac{dr}{(r - 2Mr)^{1/2}} + b_6 \cdot \frac{dr}{r^6 (r - 2Mr)^{1/2}}. \end{aligned} \quad (54)$$

This means that the right-hand side of (53) can be written as a general relativity term, plus a quantum correction. We will now integrate both sides of (54). This calculation was done using a Mathematica script.

$$\ln(\rho) = 2i \arcsin \left(\sqrt{\frac{r}{2M}} \right) + \frac{b_6 (r^2 - 2Mr)^{1/2}}{693M^6 r^6} (63M^5 + 35M^4 r + 20M^3 r^2 + 12M^2 r^3 + 8Mr^4 + 8r^5). \quad (55)$$

³In this source $A(r)$ and $B(r)$ have been given in Schwarzschild coordinates. $A_{\text{SS,PPN}}(r)$ and $B_{\text{SS,PPN}}(r)$ given here, are in isotropic coordinates.

We see that this equation is similar to (88) in Appendix A, but with an extra quantum correction. This equation is now much more complex, because of the polynomial in r in the quantum term. Our task is now to find a new coordinate transformation relating r to ρ . Our Ansatz is that r could have the form

$$r = \rho \left(1 + \frac{M}{2\rho} + \frac{\gamma_{3/2}}{\rho\sqrt{\rho}} + \frac{\gamma_2}{\rho^2} + \frac{\gamma_{5/2}}{\rho^2\sqrt{\rho}} + \frac{\gamma_3}{\rho^3} + \frac{\gamma_{7/2}}{\rho^3\sqrt{\rho}} + \frac{\gamma_4}{\rho^4} + \frac{\gamma_{9/2}}{\rho^4\sqrt{\rho}} + \frac{\gamma_5}{\rho^5} + \frac{\gamma_{11/2}}{\rho^5\sqrt{\rho}} + \frac{\gamma_6}{\rho^6} \right)^2, \quad (56)$$

where γ_n is the coefficient for the $\frac{1}{n}$ th power of ρ in the Ansatz. This Ansatz is based on (47), and adds on terms up to an order $\frac{1}{\tau^6}$. The coefficients γ_n should still be determined by substituting the Ansatz into (55). By doing this, we can investigate at which post-Newtonian order the free parameters b_6 appears. We have not been able to find a solution for the coefficients γ_n yet, so this is something that should be done in further research.

5 Tests of a Modified Schwarzschild Metric

Now, we can combine the modified Schwarzschild metric, explained in chapter 4, with the tests of spacetime geometry in chapter 3. Our goal with this, is to find an upper bound for the free parameters a_6 and b_6 . These are the leading terms in the quantum corrections in (40) and (41), so the sub-leading parameters a_7 and b_7 will be neglected.

First, we will analyze the perihelion shift of three systems: the moon and the earth, Mercury and the sun, and the star S2 and Sagittarius A*. We will calculate the higher-order general relativity terms, and the quantum corrections, to the shift of the azimuthal angle ϕ . For each system, we must find approximations for the conserved energy and the conserved angular momentum. With this we can find and compare the values found for the bounds on the free parameters. The second test of spacetime geometry we will look at, is the orbit of photons around the black hole M87. The modified Schwarzschild metric causes the addition of a small perturbation to the radius of the photon ring as described in section 3.3. This perturbation will give a new calculation of the bound on the free parameters.

5.1 Perihelion Shift

The equation found for the perihelion shift in section 3.1 only contains the leading order general relativity terms. In this section, we will systematically expand the approximation for the perihelion shift containing the higher-order general relativity corrections, and the quantum gravity corrections. The last term will depend on the coupling constants a_6 and b_6 .

$$\Delta\phi = 2\pi + \phi_{\text{GR}} + \phi_{\text{GR,HO}} + \phi_{\text{QG}} \quad (57)$$

The first two terms of (57) are the same as in (35). The third term $\phi_{\text{GR,HO}}$ gives the higher-order general relativity corrections, and the fourth term ϕ_{QG} gives the quantum gravity corrections. The derivation of these terms follows approximately the same steps as in section 3.1. The modified effective potential from (46) is defined as

$$V_{\text{eff}} = -\frac{M}{r} + \frac{l^2}{2r^2} - \frac{Ml^2}{r^3} + \frac{\alpha}{r^6}, \quad (58)$$

$$\alpha = \frac{1}{2}b_6 + e^2(a_6 - b_6) \approx \frac{1}{2}b_6. \quad (59)$$

This approximation can be done because usually $e \ll 1$. This potential is substituted into (24); in the denominator, terms with $O(r^{-7})$ are neglected. This makes it possible to separate the denominator into comparable terms as in the Schwarzschild case, with the addition of the quantum gravity term containing α

$$\begin{aligned} 2(\mathcal{E} - V_{\text{eff}}(r))^{-1/2} &= 2\left(\frac{1}{2}e^2 - \frac{1}{2} + \frac{M}{r} - \frac{l^2}{2r^2} + \frac{Ml^2}{r^3} - \frac{\alpha}{r^6}\right)^{-1/2} \\ &\approx \left(e^2 - \left(1 - \frac{2M}{r} + \frac{2\alpha}{r^6}\right)\left(1 + \frac{l^2}{r^2}\right)\right)^{1/2}. \end{aligned}$$

This is substituted into (24) to obtain an equation similar to (60). The difference now is that the expansions that were earlier done up to $O(r^{-3})$ now has to include all contributions up to $O(r^{-6})$ to find the contribution of the higher-order general relativity terms, and subsequently the extra quantum gravity terms

$$\begin{aligned}
\frac{d\phi}{dr} &= \pm \frac{l}{r^2} \left(1 - \frac{2GM}{c^2 r} + \frac{2\alpha}{r^6} \right)^{-1/2} \left(c^2 e^2 \left(1 - \frac{2GM}{c^2 r} + \frac{2\alpha}{r^6} \right)^{-1} - c^2 \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2} \\
&\approx \pm \frac{l}{r^2} \left(1 + \frac{GM}{c^2 r} - \frac{3G^2 M^2}{2c^4 r^2} + \frac{5G^3 M^3}{2c^6 r^3} - \frac{35G^4 M^4}{8c^8 r^4} + \frac{63G^5 M^5}{8c^{10} r^5} - \frac{1}{r^6} \left(\frac{231G^6 M^6}{c^{12}} - \alpha \right) \right) \\
&\quad \cdot \left(c^2 e^2 \left(1 - \frac{2GM}{c^2 r} + \frac{4G^2 M^2}{c^4 r^2} \right) - c^2 \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2} \\
&= \pm \frac{l}{r} P(r) \cdot \left(c^2 e^2 p(r) - c^2 \left(1 + \frac{l^2}{c^2 r^2} \right) \right)^{-1/2}. \tag{60}
\end{aligned}$$

$$P(r) = 1 + \frac{GM}{c^2 r} - \frac{3G^2 M^2}{2c^4 r^2} + \frac{5G^3 M^3}{2c^6 r^3} - \frac{35G^4 M^4}{8c^8 r^4} + \frac{63G^5 M^5}{8c^{10} r^5} - \frac{1}{r^6} \left(\frac{231G^6 M^6}{c^{12}} - \alpha \right) \tag{61}$$

$$p(r) = 1 - \frac{2GM}{c^2 r} + \frac{4G^2 M^2}{c^4 r^2} \tag{62}$$

The term $P(r)$ is expanded up to $O(r^{-6})$, while the other term is still cut off at $O(r^{-3})$. In the other term, we only take the leading order terms of $p(r)$. This is done so that the polynomial $p(r)$, after taking the coordinate transformation $u = \frac{1}{r}$, still has the same two roots u_1 and u_2 as in (30). This is also done to make the integral still analytically solvable.

Because $p(r)$ is identical to the term in (60), it is once again a polynomial in u (with the coordinate transformation $u = \frac{1}{r}$) with the coefficients a , b and η . This polynomial is split into its two turning points in (30). The difference with (32) is that there are now not two integral terms, but six. $P(u)$ and $p(u)$ will now take the forms

$$\begin{aligned}
P(u) &= 1 + \frac{GM}{c^2} u - \frac{3G^2 M^2}{2c^4} u^2 + \frac{5G^3 M^3}{2c^6} u^3 - \frac{35G^4 M^4}{8c^8} u^4 + \frac{63G^5 M^5}{8c^{10}} u^5 - \frac{231G^6 M^6}{c^{12}} u^6 + \alpha u^6 \\
&= 1 + P^{(1)} u + P^{(2)} u^2 + P^{(3)} u^3 + P^{(4)} u^4 + P^{(5)} u^5 + P^{(6)} u^6 + \alpha u^6, \tag{63}
\end{aligned}$$

$$p(u) = 1 - \frac{2GM}{c^2} u + \frac{4G^2 M^2}{c^4} u^2 \tag{64}$$

The integrals, giving the values for the terms in (57) will now take the form

$$\phi_{\text{GR}} = 2l \cdot a^{-1/2} \left(\int_{u_1}^{u_2} \frac{1 + P^{(1)} u}{\sqrt{(u_1 - u)(u - u_2)}} \right), \tag{65}$$

$$\phi_{\text{GR,HO}} = 2l \cdot a^{-1/2} \left(\int_{u_1}^{u_2} \frac{P^{(2)} u^2 + P^{(3)} u^3 + P^{(4)} u^4 + P^{(5)} u^5 + P^{(6)} u^6}{\sqrt{(u_1 - u)(u - u_2)}} \right), \tag{66}$$

$$\phi_{\text{QG}} = 2l \cdot a^{-1/2} \cdot \alpha \cdot \left(\int_{u_1}^{u_2} \frac{u^6}{\sqrt{(u_1 - u)(u - u_2)}} \right). \tag{67}$$

The term (65) has already been calculated in section 3.1 in (35). The term (66) contains higher order general relativity terms, which in comparison to the earlier calculation now do need to be taken into account. The term (67) containing α will eventually give an approximation for the quantum gravity correction and a bound on $|b_6|$. The

	Moon-Earth	Mercury-Sun	S2-Sagittarius A*
M (kg)	$5.9722 \cdot 10^{24} \pm 6 \cdot 10^{20}$	$1.9884 \cdot 10^{30} \pm 2 \cdot 10^{20}$	$4.25 \cdot 10^6 M_{\odot}$
m (kg)	$7.3458 \cdot 10^{22} \pm 2 \cdot 10^{17}$	$3.3020 \cdot 10^{23} \pm 1 \cdot 10^{21}$	$14 M_{\odot}$
l (m ² s ⁻¹)	$3.9452 \cdot 10^{11}$	$2.7567 \cdot 10^{15}$	$1.3823 \cdot 10^{20}$
e	$5.447 \cdot 10^{-12}$	$1.275 \cdot 10^{-8}$	$2.098 \cdot 10^{-5}$

Table 1: Values for the mass of the large and small body in different systems, and the conserved angular momentum and energy.

integrals in these three equations are analytically solvable and are listed in Appendix B.

To find numerical results for the perihelion shift and all higher order corrections, we must find accurate values for the conserved energy e , the conserved angular momentum l and the masses of the involved bodies in different systems. Accurate measurements of the masses of the Moon, Earth, Mercury and the Sun can be found in [16]. The information about the orbit and mass of star S2 can be found in [11] and [17].

The total energy for all the systems is approximated using the equation

$$e = \frac{E}{mc^2} = \frac{1}{mc^2} \frac{GMm}{2a_{\text{sm}}}, \quad (68)$$

where a_{sm} is the length of the semi-major axis in an elliptical orbit. The semi-major axis is the longest diameter of an ellipse and can be calculated using

$$a_{\text{sm}} = \frac{r_{\text{min}}}{1 - \epsilon_{\text{ecc}}}. \quad (69)$$

In this equation r_{min} is the minimal distance between the large and small body, so when it is located in the perihelion, and ϵ_{ecc} is the eccentricity of the orbit. For the orbit of the moon, $r_{\text{min}} = 3.84748 \cdot 10^8$ m and $\epsilon_{\text{ecc}} = 0.0549006$; for the orbit of Mercury around the sun, $r_{\text{min}} = 4.60012 \cdot 10^{10}$ m and $\epsilon_{\text{ecc}} = 0.2056$; for the S2-Sagittarius A* system, $r_{\text{min}} = 120\text{AU}$ and $\epsilon_{\text{ecc}} = 0.88$. The last quantity we must find is the conserved angular momentum. This can be approximated for all the systems using

$$l = \frac{L}{m} \approx v r_{\text{min}}, \quad (70)$$

with v being the orbital speed of the smaller body around the larger one. Using these values, all the terms of the perihelion shift as described in Appendix C can be converted into numerical results. To find a bound, we can impose the following condition

$$\phi_{\text{QG}} \leq \phi_{\text{SS}}. \quad (71)$$

This equation states that the quantum gravity corrections may never exceed the leading Schwarzschild term of the perihelion shift. If this would be the case, the quantum gravity corrections would affect the measured value of the perihelion shift, and this has been experimentally shown to not be the case. From this, we can find a bound for α ,

$$\alpha \leq \frac{\phi_{\text{SS}}}{\tilde{\phi}_{\text{QG}}}, \quad (72)$$

with

$$\tilde{\phi}_{\text{QG}} = 2l \cdot a^{-1/2} \left(\int_{u_1}^{u_2} \frac{u^6}{\sqrt{(u_1 - u)(u - u_2)}} \right). \quad (73)$$

	Moon-Earth	Mercury-Sun	S2-Sagittarius A*
$\delta\phi_{\text{SS}}$	$2.1 \cdot 10^{-10}$	$4.9 \cdot 10^{-7}$	$3.4 \cdot 10^{-3}$
$\delta\phi_2$	$1.8 \cdot 10^{-21}$	$9.4 \cdot 10^{-15}$	$4.9 \cdot 10^{-7}$
$\delta\phi_3$	$-5.8 \cdot 10^{-32}$	$-6.7 \cdot 10^{-22}$	$-2.5 \cdot 10^{-10}$
$\delta\phi_4$	$2.0 \cdot 10^{-42}$	$5.3 \cdot 10^{-29}$	$1.4 \cdot 10^{-13}$
$\delta\phi_5$	$-7.4 \cdot 10^{-53}$	$-4.4 \cdot 10^{-36}$	$8.6 \cdot 10^{-17}$
$\delta\phi_6$	$2.8 \cdot 10^{-63}$	$3.8 \cdot 10^{-43}$	$5.3 \cdot 10^{-20}$
α	$2.5 \cdot 10^{-50}$	$2.6 \cdot 10^{-63}$	$6.0 \cdot 10^{-80}$
$ b_6 \leq$	$1.7 \cdot 10^{40}$	$3.8 \cdot 10^{56}$	$1.2 \cdot 10^{77}$
$ \tilde{b}_6 \leq$	$3.9 \cdot 10^{-26}$	$4.1 \cdot 10^{11}$	$2.1 \cdot 10^{56}$

Table 2: Results for the correction terms on the perihelion advance, an upper bound for b_6 and for the dimensionless quantity \tilde{b}_6 , based on three different systems where a perihelion advance occurs.

By taking the approximation $\alpha \approx \frac{1}{2}b_6$, as seen in section 2.3, the absolute value of b_6 can be bounded from above as

$$|b_6| \leq \frac{2\phi_{\text{SS}}}{\tilde{\phi}_{\text{QG}}}. \quad (74)$$

We would also like to convert the bound for $|b_6|$ into a dimensionless quantity, which gives a more sensible comparison of the three cases of perihelion shift. This can be done by taking

$$|\tilde{b}_6| = |b_6| \cdot \left(\frac{GM}{c^2 r_{\min}} \right)^6 \quad (75)$$

The results in table 2 show that the higher-order general relativity terms are much smaller than the leading order Schwarzschild term. Therefore it is justified to neglect them in the calculation of the perihelion shift in the Schwarzschild metric. The strongest bound on b_6 is given by the moon-earth system. Even though the S2-Sagittarius A* system experiences the largest perihelion shift effect, the worst bound is given by this system. This bound gets even worse when we look at the dimensionless quantity \tilde{b}_6 , because the ratio $\frac{M}{r_{\min}}$ is much higher for Sagittarius A* than for the other systems. This points us in the direction that solar system tests will give the most accurate bounds on the free parameters.

5.2 Photon Orbit around a Black Hole

As shown in section 3.3 the Schwarzschild metric admits a stable circular orbit of photons at $r = \frac{3GM}{c^2}$. The modified Schwarzschild metric gives a small quantum gravity correction to this value. In this section we will show how to calculate this correction and what the consequences are for a bound on $|a_6 - b_6|$.

Like earlier, a stable orbit of photons occurs at the minimum of the potential energy, so when $\frac{dV_{\text{eff}}}{dr} = 0$. Now we will use the modified effective potential in (46), which gives a new equation for the radius of the orbit

$$\frac{dV_{\text{eff}}(r)}{dr} = \frac{1}{r^7} (-l^2 r^4 + 3Ml^2 r^3 - 6e^2 (a_6 - b_6)). \quad (76)$$

Setting (76) to zero will give

$$-l^2 r^4 + 3Ml^2 r^3 - 6e^2(a_6 - b_6) = 0. \quad (77)$$

We know that for a Schwarzschild metric, so for $a_6 = b_6 = 0$, the solution to this equation is $r_0 = 3M$. The solution to the new equation (77) will then be equal to r_0 plus a small perturbation, which we call ϵ , given by

$$r = 3M + \epsilon. \quad (78)$$

We will then substitute (78) into (77) to find an equation for ϵ . All terms of order $O(\epsilon^2)$ or higher are neglected. This yields

$$-l^2((3M)^4 + 4 \cdot (3M)^3 \epsilon) + 3Ml^2((3M)^3 + 3 \cdot (3M)^2 \epsilon) - 6e^2(a_6 - b_6) = 0. \quad (79)$$

By rewriting this equation, we find an equation for ϵ that is dependent on $a_6 - b_6$, given by

$$\epsilon = \frac{6}{(3M)^3} \left(\frac{e}{l}\right)^2 (a_6 - b_6). \quad (80)$$

It is interesting to note that this equation for ϵ contains the inverse of the impact parameter $b = \frac{l}{e}$. The definition of the impact parameter is that it is the perpendicular distance from the path of a photon entering the system, to the center of the system, which is the center of the black hole. A photon will only enter a circular orbit when $b = b_{crit}$, the critical impact parameter. For $b < b_{crit}$ the photon will fall into the black hole. This corresponds to the relationship $\epsilon \sim \frac{1}{b}$: for a very small value of b , there is a larger deviation from the usual circular orbit and the quantum gravity effects are magnified. More information on the impact parameter can be found in sections 2.2 and 2.3 of [18].

By substituting (80) into (78), we find an equation for the bounds on the quantum corrections, in terms of the deviation of r from its theoretical value in general relativity.

$$\begin{aligned} |a_6 - b_6| &\leq \frac{(3M)^3}{6} \left(\frac{l}{e}\right)^2 (r - 3M) \\ &\rightarrow \frac{9}{2} \left(\frac{GM}{c^2}\right)^3 \cdot \left(\frac{l}{ce}\right)^2 \left(r - \frac{3GM}{c^2}\right) \end{aligned} \quad (81)$$

The supermassive black hole M87, observed by the Event Horizon Telescope group in 2019, will be used to calculate this bound. In [19], the group reconstructed an image of the event horizon of M87, using a global array of telescopes. Their observations have measured the mass and radius of this black hole, which is necessary to calculate (81). All values for the parameters of M87 mentioned in the next paragraphs, can be found in the table on page 8 of [19]. The photon ring diameter is equal to $d = 42 \pm 3 \mu as$. The distance from earth to M87 is $D = (16.8 \pm 0.8) Mpc$. which means that the radius in meters is $r = 0.5d \cdot \frac{\pi}{648000} \cdot D = 5.3 \cdot 10^{13} m$.

	Moon-Earth	Mercury-Sun	S2-Sagittarius A*
$ a_6 \leq$	$5.2 \cdot 10^{70}$	$5.2 \cdot 10^{70}$	$1.2 \cdot 10^{77}$

Table 3: Approximations for the upper bound on $|a_6|$, based on (83) and Table 2.

Next, we must find a value for the critical impact parameter for a circular orbit, $b_{\text{crit}} = \frac{l}{e}$. This critical impact parameter is dependent on the effective potential of the photons orbiting around M87. If we would calculate this using (15), b would also be dependent on a_6 and b_6 , which is undesirable, because these are the values we want to find a bound for. Therefore we will approximate b by using the Schwarzschild metric. On p. 206 of [8], we find that

$$b_{\text{crit}} = 3\sqrt{3}\frac{GM}{c^2}. \quad (82)$$

Finally, the mass of M87 is $M = (6.5 \pm 0.7) \cdot 10^7 M_{\odot}$. Using these parameters, (81) evaluates to

$$|a_6 - b_6| \leq 5.2 \cdot 10^{70}. \quad (83)$$

It is convenient to also write this as a dimensionless quantity. This is done like in 75 by multiplying by the factor $(\frac{GM}{c^2 r})^6$, where M is the mass of black hole M87 and r is the radius of the photon orbit. This gives the bound

$$\widetilde{|a_6 - b_6|} \leq 1.9 \cdot 10^{54}. \quad (84)$$

The bound on the dimensionless quantity $\widetilde{|a_6 - b_6|}$ is approximately in the same order as the bound on \tilde{b}_6 given by the perihelion shift of S2. This shows that systems with black holes will give bounds in the same order. The bound given by the photon orbit around M87 is still much less strong than the bounds given by solar system tests.

Using the bounds found for b_6 in section 5.1, we can now also approximate a bound for a_6 . These approximations have been given in Table 3. Because we only have a single bound for $|a_6 - b_6|$, the bounds found on $|a_6|$ will not give us a lot of information. If the bound on $|b_6|$ is smaller than the bound on $|a_6 - b_6|$, then the bound on $|a_6|$ will be equal to $|a_6 - b_6|$. In the case that the bound on $|b_6|$ is larger than the bound on $|a_6 - b_6|$, then the bounds on $|a_6|$ and $|b_6|$ are identical.

6 Discussion

In this thesis, we have found upper bounds for the coupling constants in a Schwarzschild metric modified by quantum gravity terms. The following table gives an overview of all results for different tests of spacetime geometry. The bound found for $|a_6|$ for the photon orbit is an approximation based on the bounds for $|b_6|$ in the three perihelion shift tests, this is shown in section 3.3.

	Moon-Earth	Mercury-Sun	S2-Sagittarius A*	Photon Orbit
$ a_6 \leq$	-	-	-	$10^{70} - 10^{77}$
$ b_6 \leq$	$1.7 \cdot 10^{40}$	$3.8 \cdot 10^{56}$	$1.2 \cdot 10^{77}$	-
$ a_6 - b_6 \leq$	-	-	-	$5.2 \cdot 10^{70}$

Table 4: A summary of all results found for the upper bounds on $|a_6|$, $|b_6|$ and $|a_6 - b_6|$.

We conclude that the perihelion shift tests only give information about the bounds for $|b_6|$, and the photon orbit test gives a bound on $|a_6 - b_6|$ and an approximation for a bound on $|a_6|$.

The accuracy of the bounds depends on accurate approximations of the conserved energy and conserved angular momentum of the system. For the perihelion shift systems, the energy of the system was approximated using (68), a Newtonian equation for the energy. For the angular momentum, (70) is also a Newtonian approximation using the minimal distance between the large and small body, and the average orbital speed. These approximations hold, because the Newtonian energy and angular momentum are the leading terms in general relativity. To make these approximations more accurate, one could use the Schwarzschild metric to find new equations for e and l . The most accurate equations are found using the modified Schwarzschild metric, but this gives the problem that e and l are then dependent on a_6 and b_6 . This makes it impossible to find bounds for these coupling constants.

For the photon orbit around a black hole, the Newtonian limit becomes a more inaccurate approximation, because of the high curvature of spacetime around a black hole. This is why we used the value for the critical impact parameter in the Schwarzschild metric. Further research could also look at a quantum gravity correction on the impact parameter.

Important further research is to look at other tests of general relativity and spacetime geometry and find quantum gravity corrections for the values measured in these tests. The two classical tests which were not worked through in this thesis, are the deflection of light by the sun and the photon redshift. Both of these tests will only find bounds for $|a_6 - b_6|$, since they both concern massless particles. A possible test with massive particles could be the Nordtvedt effect. The Nordtvedt effect is a measure for the violation of the strong equivalence principle, which means that the laws of gravity should be independent of the velocity of a test body. Objects that are held together by gravity, like planets, stars and black holes, should all follow the same trajectory in a gravitational field, if they have the same starting conditions. If the Nordtvedt effect would be present, the earth would fall towards the sun with a different acceleration than the moon. This effect is described using PPN parameters. One could use this effect to calculate a new bound on a_6 and b_6 , and also compare this to the PPN parameters.

Something that still needs to be investigated, is at which order in the parameterized post-Newtonian expansion the free parameters a_6 and b_6 appear. Using the bounds for these parameters, we could derive new bounds for the PPN parameters β and γ . To do this, we need to calculate a method to write the modified Schwarzschild metric in isotropic coordinates. This would give an expansion up to $O(U^6)$, or higher. A possible method to do this was already given at the end of section 4.3. Research on gravitational waves has already looked at higher-order PPN expansions. [3] discusses a formula for the gravitational waveform, which is used to find the form of gravitational waves. This formula has been expanded up to an order 3.5PN beyond the leading term. In [20], expressions for the polarization of gravitational wave are provided up to an order of 5.5PN. Further research could look into these methods and investigate if they provide a method to expand the Schwarzschild metric tensor to higher orders in the PPN expansion.

Appendices

A Schwarzschild Metric in Isotropic Coordinates

The aim of the isotropic coordinates is to write the metric in the form where the spacelike part of the metric is comparable to Euclidean coordinates. We would like to rewrite the metric in the form

$$ds^2 = \tilde{A}(\rho)dt^2 - \tilde{B}(\rho)(d\rho^2 + \rho^2(d\theta^2 + \sin^2(\theta)d\phi^2)), \quad (85)$$

where the whole spacelike part of the metric has the same function $\tilde{B}(r)$ in front of it. To rewrite the metric in this form, we need to calculate a coordinate transformation from the Schwarzschild coordinate r to the isotropic coordinate ρ . Assume that the timelike part of this metric is identical, so $\tilde{A}(\rho) = A(r)$. By comparing (85) and the Schwarzschild metric in (8), we can find for the spacelike part that

$$\begin{aligned} \tilde{B}(\rho)\rho^2 &= r^2, \\ \tilde{B}(\rho)d\rho^2 &= B(r)dr^2. \end{aligned}$$

By dividing the second equation by the first one and substituting the Schwarzschild solution for $B(r)$, we find the following differential equation for r and ρ :

$$\left(\frac{d\rho}{\rho}\right)^2 = \frac{dr^2 B(r)}{r^2}. \quad (86)$$

If we substitute the Schwarzschild solution for $B(r)$ from (10b), and take the square root of the whole equation, we get

$$\frac{d\rho}{\rho} = \frac{dr}{\sqrt{r^2 - 2Mr}}. \quad (87)$$

Now we integrate both sides and add an integration constant. Adding a constant to the right-hand side of the equation is equivalent to putting this constant inside the logarithm, which will make the later calculation clearer.

$$\ln(c_1\rho) = 2i \arcsin\left(\sqrt{\frac{r}{2M}}\right) \quad (88)$$

The arcsin is brought to the other side of the equation to free up r . The sine function of a logarithm expands into a polynomial in $\sqrt{\rho}$.

$$\sqrt{\frac{r}{2M}} = \sin\left(\frac{1}{2}i \ln(c_1\rho)\right) = i\left(-\frac{1}{2\sqrt{c_1\rho}} + \frac{\sqrt{c_1\rho}}{2}\right) \quad (89)$$

By squaring the equation, multiplying by $2M$ and setting the integration constant $c_1 = M/4$, we get back (47), which is a solution to the differential equation in (87).

B Quantum Gravity Perihelion Shift Integrals

The integrals listed in this appendix are the analytical solutions to the integral defined in (65), (66) and (67). These integrals are taken between the turning points u_1 and u_2 , which were defined in (30). The integrals (90) and (91) describe the general relativity terms of the perihelion shift, (92), (93), (94) and (95) describe the higher-order general relativity terms, and (96) describes the last higher-order general relativity term and the quantum gravity correction. All these integrals were analytically calculated using a Mathematica script.

$$\int_{u_1}^{u_2} \frac{1}{\sqrt{(u_1 - u)(u - u_2)}} = 2\pi \quad (90)$$

$$\int_{u_1}^{u_2} \frac{u}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{\pi}{2} (u_1 + u_2) \quad (91)$$

$$\int_{u_1}^{u_2} \frac{u^2}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{\pi}{8} (3u_1^2 + 2u_1u_2 + 3u_2^2) \quad (92)$$

$$\int_{u_1}^{u_2} \frac{u^3}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{\pi}{16} (u_1 + u_2) (5u_1^2 - 2u_1u_2 + 5u_2^2) \quad (93)$$

$$\int_{u_1}^{u_2} \frac{u^4}{\sqrt{(u_1 - u)(u - u_2)}} = \frac{\pi}{128} (35u_1^4 + 20u_1^3u_2 + 18u_1^2u_2^2 + 20u_1u_2^3 + 35u_2^4) \quad (94)$$

$$\begin{aligned} \int_{u_1}^{u_2} \frac{u^5}{\sqrt{(u_1 - u)(u - u_2)}} = & \frac{\pi}{256} (u_1 + u_2) (63u_1^4 - 28u_1^3u_2 + 58u_1^2u_2^2 \\ & - 28u_1u_2^3 + 63u_2^4) \end{aligned} \quad (95)$$

$$\begin{aligned} \int_{u_1}^{u_2} \frac{u^6}{\sqrt{(u_1 - u)(u - u_2)}} = & \frac{\pi}{1024} (231u_1^6 + 126u_1^5u_2 + 105u_1^4u_2^2 \\ & + 100u_1^3u_2^3 + 105u_1^2u_2^4 + 126u_1u_2^5 + 231u_2^6) \end{aligned} \quad (96)$$

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