Galaxy Rotation Curves of a Galactic Mass Distribution

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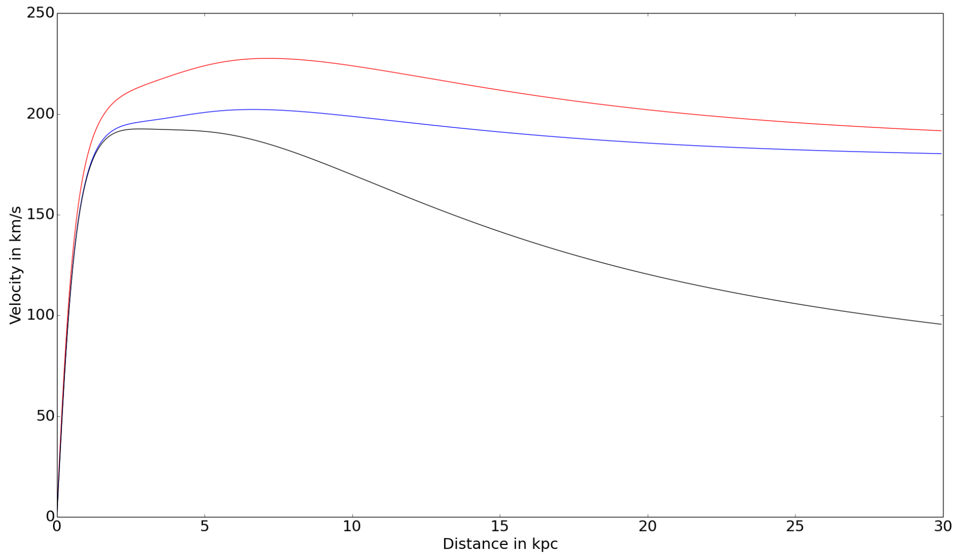

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1 Introduction

A galactic rotation curve describes how the rotation velocity of objects in the galaxy changes as a function of the object's distance to the center. As an example we will consider a central mass like a black hole or a star with an other object moving on a circular path around this central mass. The rotational velocity can in this example be determined from Newton's gravitational equations and the centripetal force:

$$F = \frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v = \sqrt{\frac{GM}{r}} \tag{1}$$

where M is the central mass, m the test mass and r the distance between the central mass and the test mass. The rotational velocity is represented by v. The mass of the galaxy is not distributed in a central point, so we have to consider the actual distribution. The calculation for the galactic distribution requires the integration of Newton's gravitational equations with the entire distribution taken into account. We still assume a circular motion of a test mass around the center of the galaxy. The expected rotation curve would then increase quickly since the considered mass increases rapidly, due to the high density near the center. Based on the visible mass the rotation curve is expected to eventually drop off near the edge of the galaxy like $1/\sqrt{r}$ as in formula (1). This is however not what is measured in astronomy. The rotation curves do not drop off according to Newton's laws but stay flat near the edge of the galaxies. An example of such a measurement for our own galaxy is displayed in figure 1.

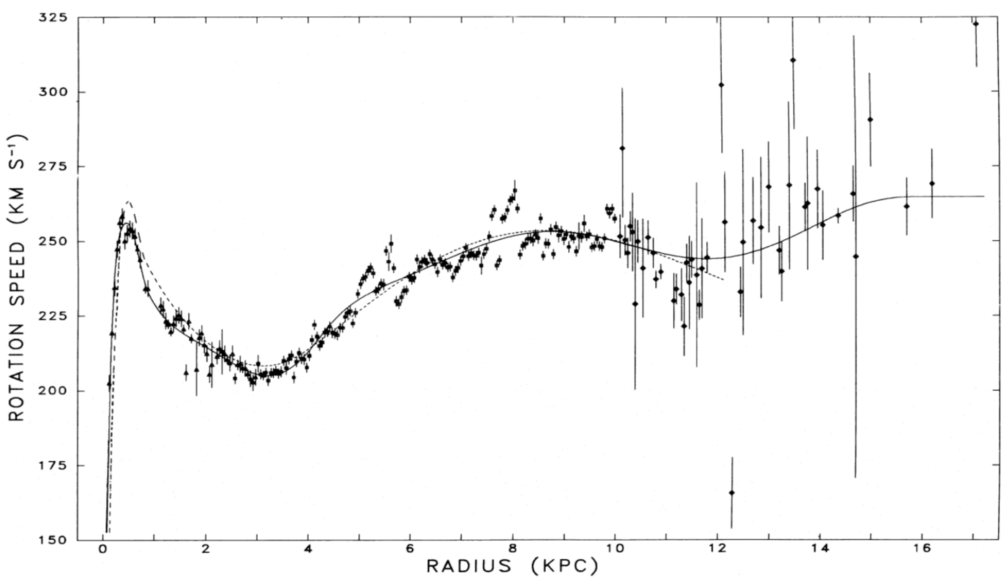

Figure 1: The measured rotation curve from the milky way. The data points are velocity measurements of stars as a function of radial distance to the galactic center. The large error bars are due to the difficulty of measuring stellar gas rather then stars, since the number of stars decrease when the radial distance is increased. [1].

In this thesis we will look at two possible solutions of this discrepancy between Newtonian gravitational theory and the observations. The first idea is that of Modified Newtonian Dynamics. This proposes that our understanding of Newtonian Mechanics is not complete for long distance interactions and proposes a number of changes to Newton's second law. The second solution is the idea of Dark Matter. Instead of altering Newton's laws, it introduces an extra component to the mass distribution of the galaxy, which then changes the shape of the rotation curve.

The goal of the thesis is to develop a method that allows for a quick conversion of a mass distribution to a rotation curve. To achieve this a numerical program has been written that calculates the acceleration as a function of the distance to the center of the galaxy. The rotation curve is then calculated from the acceleration. As an explicit example we will use a cylindrically symmetric distribution that approximates the mass distribution of our galaxy. This automatically implies that we will not be able to predict the wavy pattern in figure 1 since we do not take features such as spiral arms into account.

2 Mass Distribution

The used simplified mass distribution for visible matter in our own galaxy consists of three parts: a thin disk, a thick disk and a bulge[2]. These distributions have cylindrical symmetry. The radial coordinate r is the distance from the center of the galaxy in the plane of the galaxy. The z coordinate is the height perpendicular to the plane of the galaxy.

2.1 The bulge

The mass distribution of the bulge consists of a product of a gaussian and a $r^{-1.8}$ power law. The gaussian has a characteristic length $r_{cut} = 2.1$ kpc while the power law has a characteristic length $r_0 = 0.075$ kpc. The expression for the density is shown in formula (2). The central density $\rho_{b,0} = 9.93 \cdot 10^{10}\ M_\odot\ \text{kpc}^{-3}$ is derived from the assumption that the bulge mass amounts to $M_b = 8.9 \cdot 10^9 M_\odot$, which can be estimated by the mass-to-light ratio. The other parameters are the power of r: $\alpha = 1.8$ and the axis ratio: $q = 0.5$. The axis ratio causes the distribution to fall of faster in the z-direction than the radial direction:

$$\rho_b(r,z) = \frac{\rho_{b,0}}{\left(1 + \frac{\sqrt{r^2+(z/q)^2}}{r_0}\right)^\alpha} e^{-\left(\frac{r^2+(z/q)^2}{r_{cut}^2}\right)} \tag{2}$$

2.2 The disks

The mass distribution for the disk is split into two different distributions to account for a smaller, more dense disk and a bigger, less dense disk. Both mass distributions consist only of an exponential function:

$$\rho_d(r,z) = \frac{\sigma_{d,0}}{2Z_d} e^{-\left(\frac{|z|}{Z_d} + \frac{r}{R_d}\right)} \tag{3}$$

The function is scaled by a characteristic surface density $\sigma_{d,0}$ and is divided by the characteristic thickness Z_d of the disk. The density falls off at different rates in the z and r-direction due to different characteristic lengths. The values for these parameters are given in Table 1.

Disk	$\sigma_{d,0}$ [$10^6 \text{M}_\odot$ kpc^{-2}]	R$_d$ [kpc]	Z$_d$ [kpc]
Thin	816.6	2.6	0.3
Thick	209.5	3.6	0.9

Table 1: Parameters for the thick and thin galactic disks

2.3 Density maps of the mass distribution

The total mass distribution of the galaxy has been plotted as a heatmap, where the axes denote the distance from the center and the color the density. The distribution is displayed from its side in figure 2. The density is highest in the center of the distribution and then falls off when the distance from the center is increased. The density decreases faster in the z-direction than the radial direction. This results in a flat disk in the plane of the galaxy. The distribution has a sharp feature at either $r = 0$ or $z = 0$ because r can only be positive and only the absolute value of z features in formula 3.

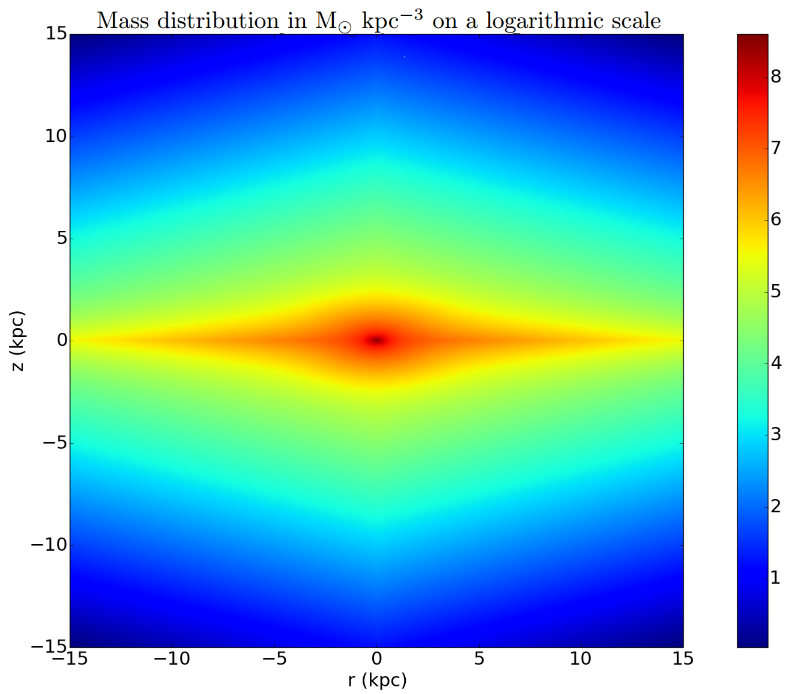

Figure 2: The mass distribution of the galaxy as viewed from the side on a logarithmic density scale.

The top view of the galaxy is shown in figure 3 on a linear density scale and in figure 4 on a logarithmic density scale. Here it is clearly visible that the density is highest in the center of the galaxy and decreases outward. The two parts of the disk are also visible: a sharp disk with a higher density and a diffuse disk with a smaller density.

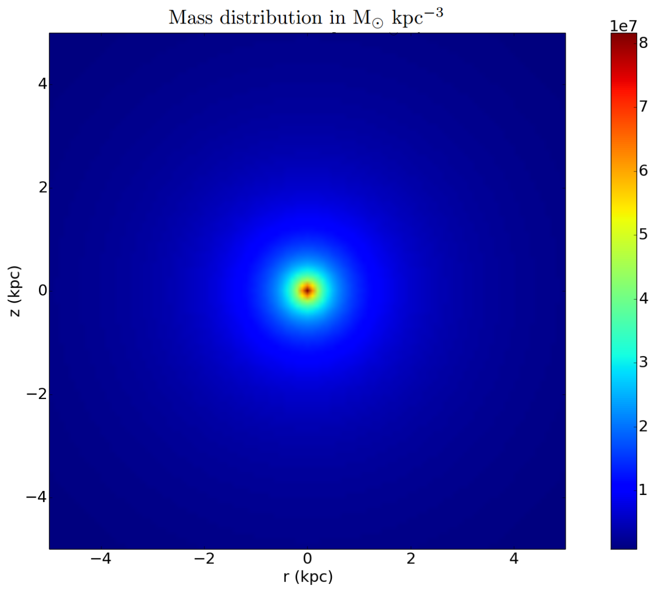

Figure 3: The mass distribution viewed from the top on a linear density scale.

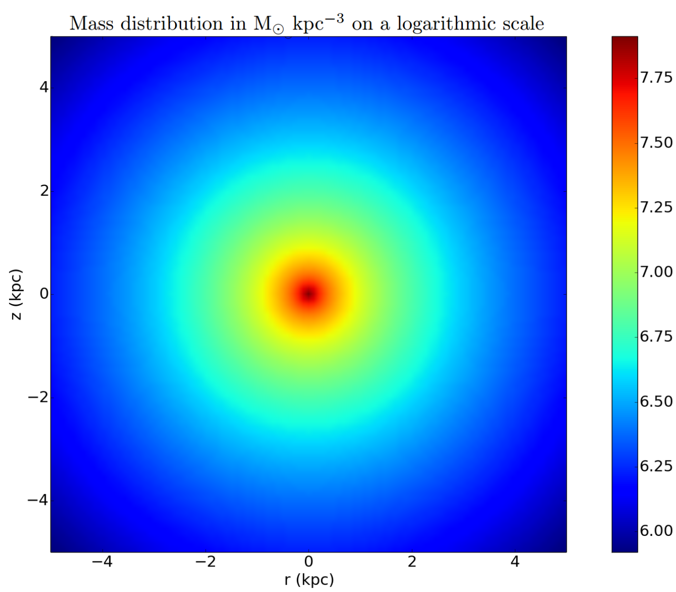

Figure 4: The mass distribution viewed from the top on a logarithmic density scale

3 Gravitational force induced by the galactic mass distribution

To calculate the rotation velocity of a galaxy as a function of the distance to the center, we first need to calculate the force on a test mass in a simplified disk shaped mass distribution. To do this we will first look at a solid uniform disk with surface density σ. The symmetry of the problem is displayed in figure 5. Due to the radial symmetry of the solid uniform disk and the distributions mentioned in Section 2, we can choose our x- and y-axes freely. If we define the x-axis to be the line through the center and the position of the test mass, only the force towards the center remains. Now we only have to consider the distance x_0 from the center.

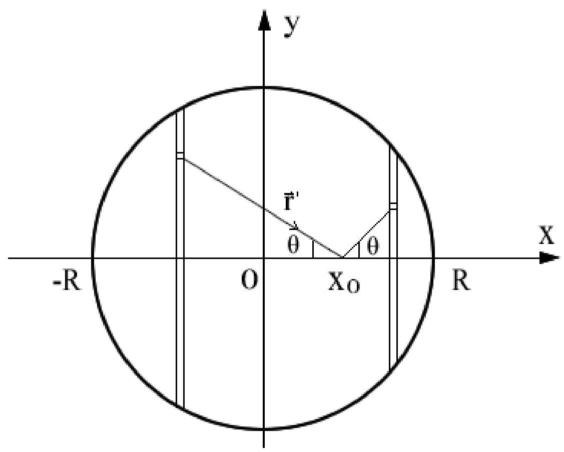

Figure 5: A schematic view of the position of the test mass relative to a mass element in the disk.[3]

The gravitational force experienced by a test mass m sitting at a distance x_0 from the center due to a mass element with size $dxdy$ at a distance r' is given by:

$$d\vec{F} \quad = \quad -\frac{Gm\sigma\ dxdy\ \vec{r}'}{r'^3} \qquad \text{with} \qquad r' = \sqrt{(x_0 - x)^2 + y^2} \tag{4}$$

Due to symmetry the y-component of this infinitesimal contribution always cancels against the contribution from the mass element that is obtained by reflection in the x-axis. The x-component of the force can be written as:

$$dF_x \quad = \quad -\frac{Gm\sigma\ cos\,\theta\ dxdy}{r'^2} \tag{5}$$

$$cos\,\theta \quad = \quad \frac{x_0 - x}{\sqrt{(x_0 - x)^2 + y^2}} \tag{6}$$

This function is integrated to give the total force on a test mass at x_0, resulting in the following formula:

$$F_x \quad = \quad -\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \frac{Gm\sigma\ (x_0 - x)dy}{((x_0 - x)^2 + y^2)^{\frac{3}{2}}} \tag{7}$$

The integral over the y-coordinate can be calculated analytically for a uniform distribution:

$$F_z = -\int_{-R}^{R} \frac{2Gm\sigma\sqrt{R^2 - x^2}\, dx}{(x_0 - x)\sqrt{R^2 + {x_0}^2 - 2x_0 x}} \tag{8}$$

However this analytical solution cannot be used for the mass distribution of the galaxy which is a function of x,y and z. The integral over the solid disk provides some useful insight into the behaviour of the acceleration curve of the galactic mass distribution. The integral over the solid disk has a singularity at the edge of the distribution which leads to an infinite acceleration at the edge of the disk, as displayed in figure 6. This infinity leads to the decision to not integrate the galaxy distributions to infinity but to a finite value instead. These values have been determined to be R=150kpc and Z=15kpc, because in this volume the whole mass of the galaxy is contained.

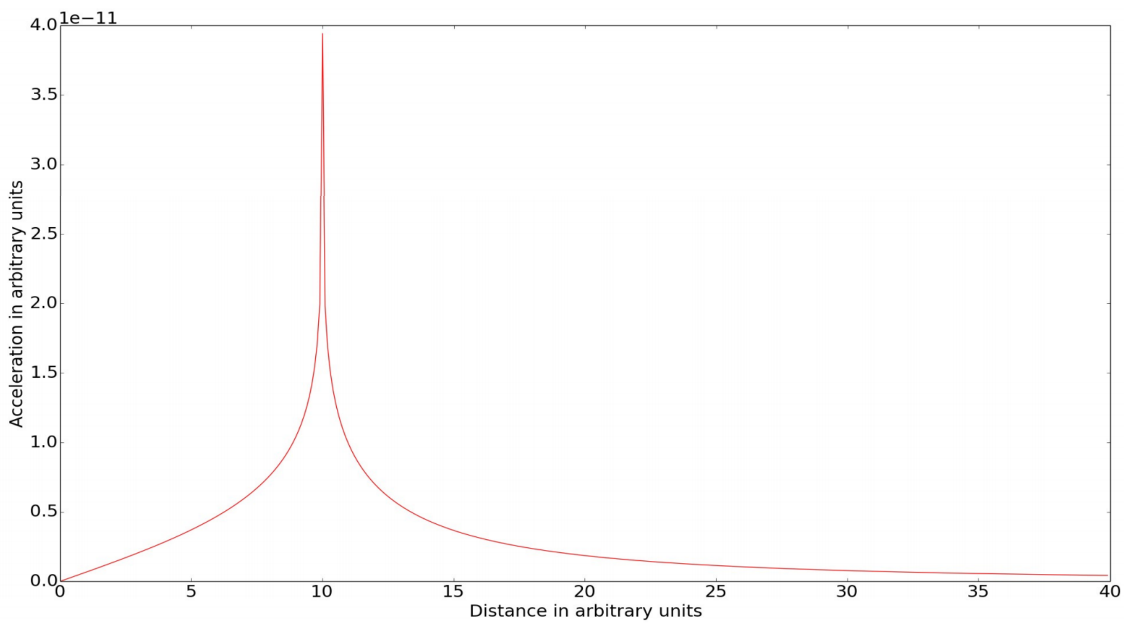

Figure 6: Acceleration of a test mass in a solid uniform disk as a function of the distance to the center. The disk has an arbitrary mass and size. At the edge $x_0 = 10$ a singularity is visible.

To get the galactic acceleration curve from formula 7 we only have to drop the test mass m and replace the surface density σ with an intergral of the volume density over the z-coordinate,

$$\sigma(x, y) = \int_{-Z}^{Z} dz\rho(x, y, z) \tag{9}$$

where $\rho(x, y, z)$ are the distributions mentioned in section 2. We drop the minus sign, which indicates that the acceleration is pointed inward, since we are only interested in its absolute value.

$$a = \left| \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-Z}^{Z} dz \, \frac{G(x_0 - x)}{((x_0 - x)^2 + y^2)^{\frac{3}{2}}} \, \rho(x, y, z) \right| \tag{10}$$

The function for the acceleration has been integrated numerically with an relative error of 10^{-3}. The errors in the input parameters[2], discussed in section 2, are of the order of 10%. The accuracy of the results only depends on the errors in the input parameters since these are larger than the numerical errors. The graph of the acceleration curve of our galaxy is plotted in figure 7. The graph rises quickly and then drops off since the density in the bulge is large compared to that of the disk.

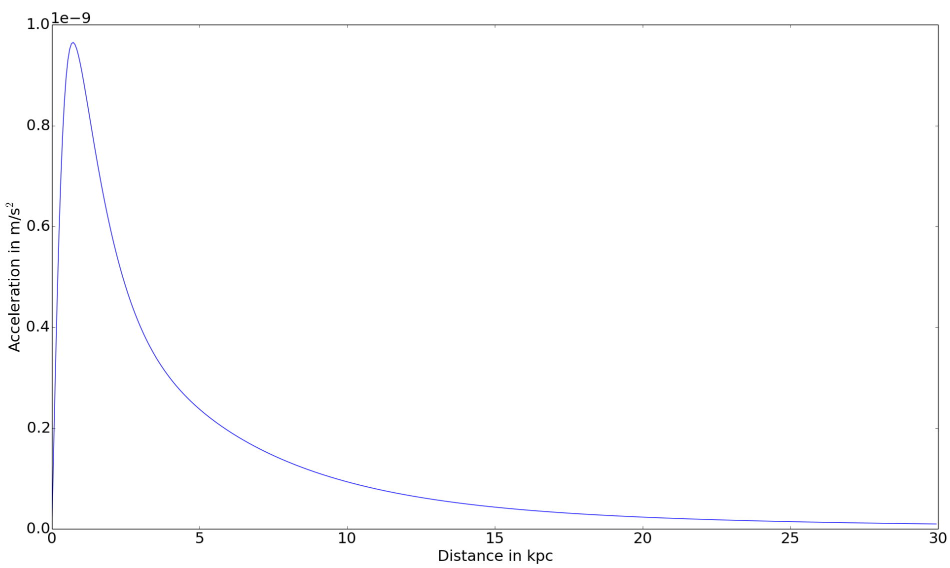

Figure 7: The acceleration curve of the galactic mass distribution as a function of the distance to the galactic center.

The rotation curve that results from this acceleration is plotted in figure 8. The curve again rises quickly due to the high density in the center and then drops off, where we expect $1/\sqrt{r}$ behaviour.

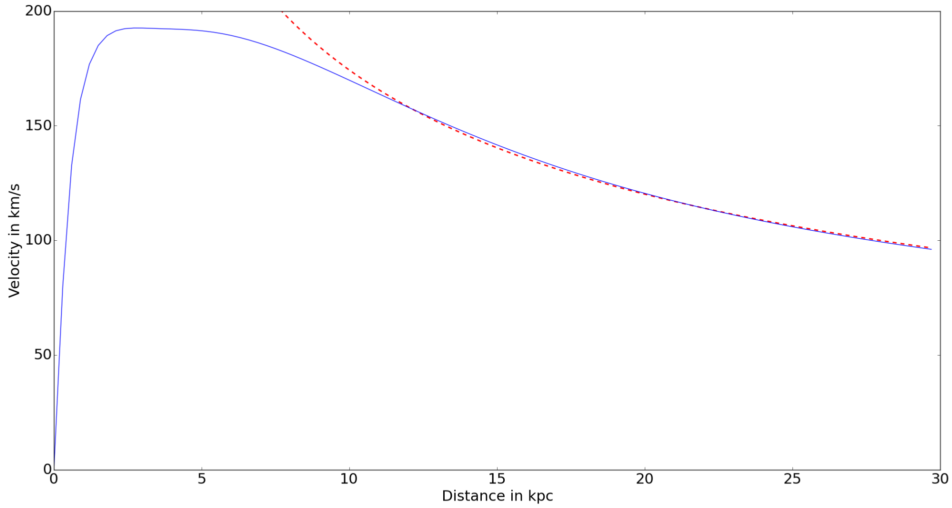

Figure 8: The rotation curve of the galactic mass distribution as a function of the distance to the galactic center. The dotted red line represents $1/\sqrt{r}$ dependence.

Features such as spiral arms were not taken into account in the distribution. This results in a smoother curve when compared to the measurement in figure 1, which displays some wavy patterns.

4 Modified Newtonian Dynamics

Modified Newtonian Dynamics (MOND) was introduced to eliminate the apparent mass discrepancy between observed matter and the matter derived from rotation curves. This theory assumes that our understanding of Newtonian dynamics is incomplete at low accelerations. A number of relativistic theories have been proposed that can be reduced to MOND. MOND consists of two modifications to Newton's second law which are denoted by the Simple and Standard Interpolation functions, see formulas (12) and (13)[4]:

$$\begin{aligned} F &= ma\mu\Big(\frac{a}{a_0}\Big) && (11)\\ \text{Simple: } \mu\Big(\frac{a}{a_0}\Big) &= \Big(1+\frac{a_0}{a}\Big)^{-1} && (12)\\ \text{Standard: } \mu\Big(\frac{a}{a_0}\Big) &= \Big(1+\Big(\frac{a_0}{a}\Big)^2\Big)^{-1/2} && (13) \end{aligned}$$

where a_0 is a fitting parameter. MOND does not predict the value of a_0, we use $a_0 = 1.22 \cdot 10^{-10}$ m/s^2 [4]. The modified version of Newton's second law can be solved for a, which leads to the following expressions for both interpolation functions:

$$\begin{aligned} \text{Simple: } a &= \frac{1}{2}\Big(a_n \pm \sqrt{a_n^2 + 4a_n a_0}\Big) && (14)\\ \text{Standard: } a &= \Big[\frac{1}{2}\Big(a_n^2 \pm \sqrt{a_n^4 + 4a_n^2 a_0^2}\Big)\Big]^{1/2} && (15) \end{aligned}$$

Here a_n is the standard Newtonian acceleration: $a_n = |F|/m$. The modified acceleration curves are displayed in figure 9. To determine how this modification changes the dependence of the acceleration on the distance from the center, we look at the low acceleration (long distance) limit. When $|a_n| \ll a_0$, which happens at a long distance from the center, the terms proportional to a_n^2 can be neglected compared to terms proportional to $a_n a_0$. This results in the following expressions for the new acceleration.

$$\begin{aligned} \text{Simple: } a &\approx \frac{1}{2}\sqrt{4a_n a_0} = \sqrt{a_n a_0} && (16)\\ \text{Standard: } a &\approx \Big[\frac{1}{2}\sqrt{4a_n^2 a_0^2}\Big]^{1/2} = \sqrt{a_n a_0} && (17) \end{aligned}$$

Using the fact that the Newtonian acceleration can be written as:

$$a_n = \frac{C}{r^2} \tag{18}$$

this leads to the following proportionality for both interpolation functions in the low acceleration limit:

$$a = \frac{\sqrt{C a_0}}{r} \tag{19}$$

From this we can conclude that Modified Newtonian Dynamics change Newton's second law in such a way that the acceleration falls off with $1/r$ instead of $1/r^2$, which is visible in figure 9

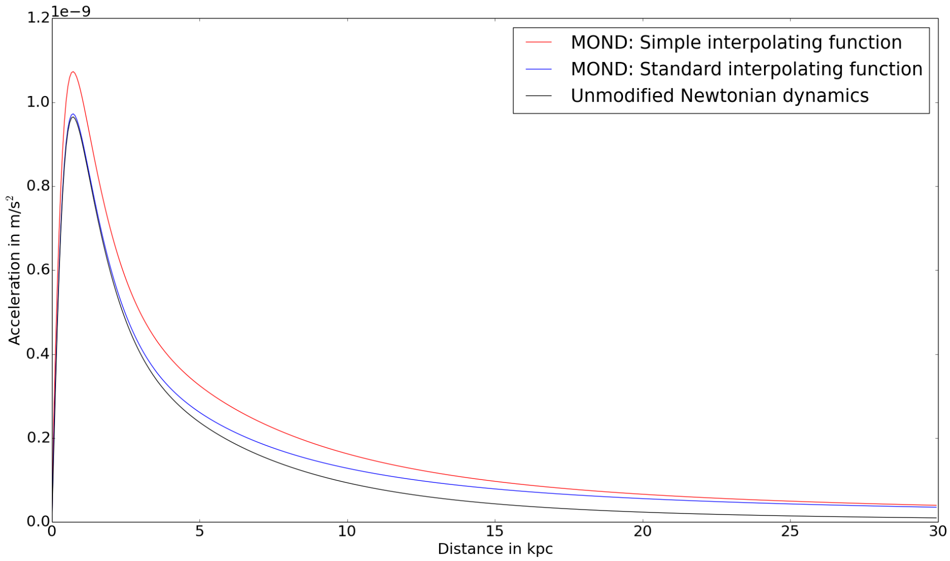

Figure 9: Both acceleration curves for Modified Newtonian Dynamics compared with the unmodified version. In the low acceleration limit both modified versions converge to the same but slower decay rate than the unmodified one.

This change in acceleration has a significant effect on the shape of the rotation curve, as displayed in figure 10. The Modified curve levels off and stays flat, where the Newtonian curve starts to decay. This modification changes the rotation curve to better match to the observed curves. The standard interpolation function induces a smaller change than the simple interpolation function, because for $a > a_0$ the effect of the standard interpolation function is smaller than for the simple one. This means that the modification also has an effect closer to the center of the galaxy for the simple interpolation function, while the standard interpolation function only modifies the curve near the edge of the galaxy.

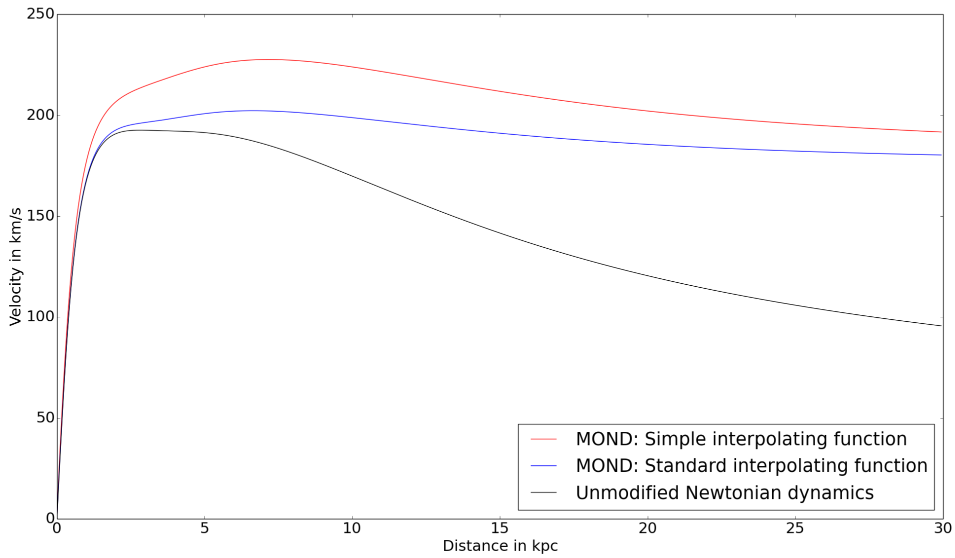

Figure 10: The rotation curves for both the modified and unmodified Newtonian dynamics.

The solution that MOND provides works well for a static system like our own galaxy. MOND cannot completely eliminate the apparent mass discrepancy when applied to a more complex system like a cluster of galaxies[4]. In that case the difference between visible mass and expected mass from measurements can only be reduced.

5 Dark Matter

The idea of Dark Matter does not modify the gravitational laws like MOND, but proposes that there is extra matter that is as yet unobserved. This idea is also supported by gravitational lensing experiments[6]. Dark Matter is not allowed to have electromagnetic or strong interactions, otherwise it would have been observed. It is allowed to have weak and gravitational interactions and seems to behave frictionless. Due to this frictionless behaviour, Dark Matter is predicted by N-body simulations to be distributed in spherically symmetric halos. A number of different distributions have been proposed[7]:

$$\text{NFW: } \rho_{\text{NFW}}(x) = \rho_0 \frac{r_0}{x}\left(1+\frac{x}{r_0}\right)^{-2} \tag{20}$$

$$\text{Einasto: } \rho_{\text{Ein}}(x) = \rho_0 e^{\frac{-2}{\alpha}\left(\left(\frac{x}{r_0}\right)^{\alpha}-1\right)} \tag{21}$$

$$\text{Isothermal: } \rho_{\text{Iso}}(x) = \frac{\rho_0}{1+(x/r_0)^2} \tag{22}$$

$$\text{Burkert: } \rho_{\text{Bur}}(x) = \frac{\rho_0}{(1+x/r_0)\left(1+(x/r_0)^2\right)} \tag{23}$$

$$\text{Moore: } \rho_{\text{Moo}}(x) = \rho_0 \left(\frac{r_0}{x}\right)^{1.16}\left(1+\frac{x}{r_0}\right)^{-1.84} \tag{24}$$

Here x is the distance from the center of the galaxy, r_0 is the characteristic length and ρ_0 is the characteristic density. These parameters are listed in table 2.

DM Profile	α	r_0 [kpc]	ρ_0 [$10^6 M_\odot$ kpc^{-3}]
NFW		24.42	4.82
Einasto	0.17	28.44	0.86
EinastoB	0.11	35.24	0.55
Isothermal		4.38	36.3
Burkert		12.67	18.6
Moore		30.28	2.75

Table 2: The parameters for the Dark Matter density profiles.[7]

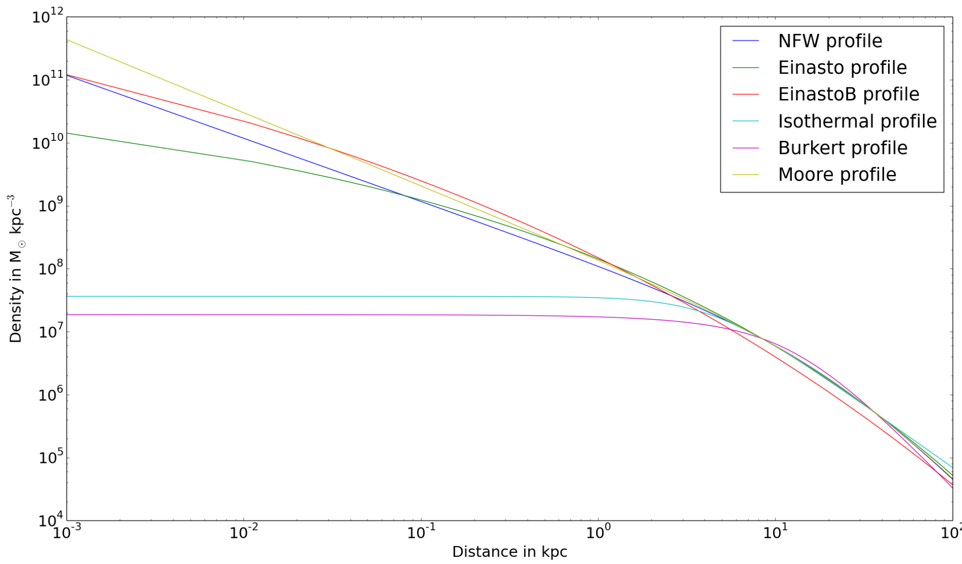

Figure 11: A logarithmic plot of the dark matter profiles

These profiles have two constraints. They are anchored at a distance of 8.33 kpc, which is the distance from the center of our galaxy to the sun. The second constraint is the total predicted mass of the distribution inside a radius of 60 kpc. This mass has been predicted to be M=$4.7 \cdot 10^{11}$ $M_\odot$. The differences between the profiles are made visible in figure 11. The profiles follow a similar shape when the distance from the galactic center is larger than that of the sun. The profiles differ more towards the center of the galaxy. The Burkert and Isothermal profiles become constant and approach their charateristic density, while the NFW and Moore profiles go to infinity. The Einasto profiles approach their characteristic densities scaled by a factor $e^{\frac{2}{\alpha}}$. This profile is parametrized by α, where a smaller α leads to a higher density at the center and a steeper slope. This new additional component to the mass distribution can be taken into account by simply adding a spherical integral over the dark matter profile $\rho_{DM}(x)$ to the component from the visible matter, as discussed in section 3:

$$a = \int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_{-Z}^{Z} dz \; \frac{G(x_0 - x)}{((x_0-x)^2 + y^2)^{\frac{3}{2}}} \, \rho(x,y,z) + \frac{G}{x_0^2} \int_{0}^{x_0} 4\pi x^2 \rho_{DM}(x) dx \tag{25}$$

The effect of this additional term on the acceleration curve is plotted in figure 12. It shows similar graphs as obtained for MOND, the curves fall off with a distinctly slower slope than in the unmodified case.

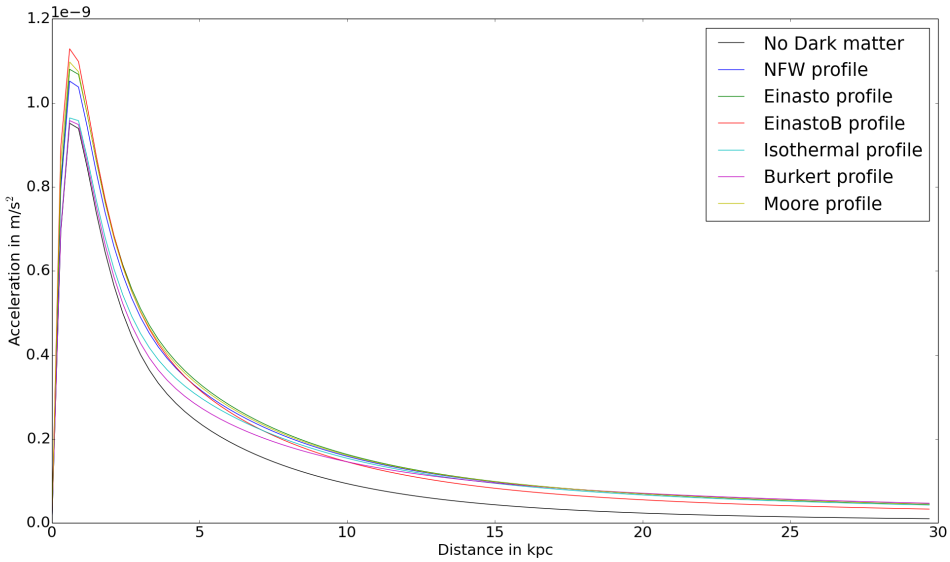

Figure 12: The acceleration curves of the mass distribution with the added dark matter profiles.

The rotation curves that result from these new acceleration curves are plotted in figure 13. The new rotation curves are flatter than the original one when the distance from the galactic center is increased. The different profiles change the rotation curves in different ways dependent on their shape and the total mass added. The EinastoB profile for example falls off faster than the others, which is expected from figure 11. In figure 11 it has the least amount of mass added at a large distance from the galactic center where the correction is most necessary. The Burkert profile does the opposite, it adds the least amount of mass near the center and more towards the edge compared to the other profiles. This is made visible in figure 13 where the rotation curve of this profile does not peak somewhere around 5 to 10 kpc but continues to rise and flattens off near the edge. The Isothermal profile is similar to the Burkert profile since its distibution is also shifted further towards the edge. The difference is that it starts higher than the Burkert profile but falls off quicker. This difference causes the Isothermal profile to have a peak near $\sim$8 kpc and then decrease slightly before flattening off near the edge.

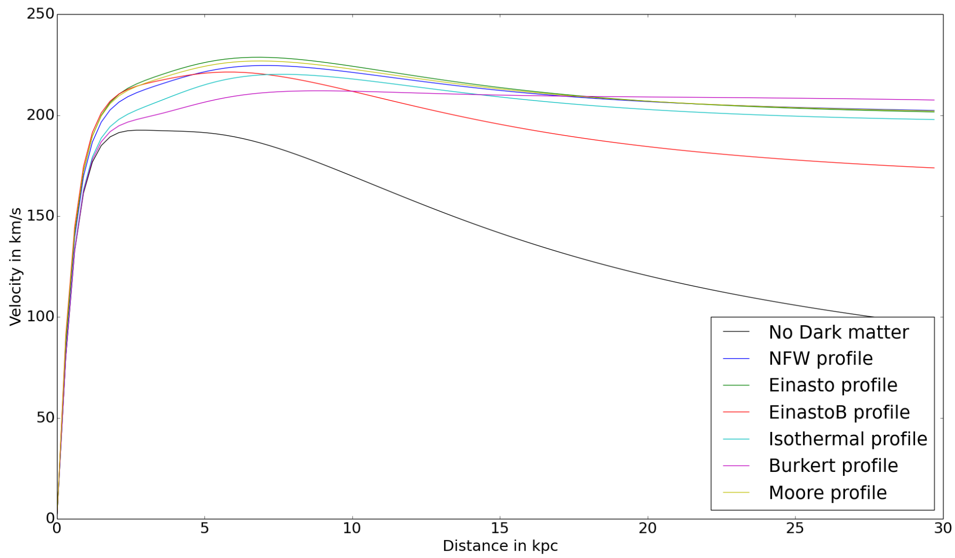

Figure 13: The rotation curves of the mass distribution with the added dark matter profiles.

The new rotation curves better match observed curves like in figure 1. Which of the profiles is most suitable remains unclear and probably also depends on the shape of the visible mass distribution as well, since visible matter accumulates at high dark matter densities and vice versa. Dynamical problems may also be easier to solve with dark matter than with MOND, because dark matter seems to behave frictionless. For example in a collision between galaxies the dark matter distribution will most likely remain undisturbed or very similar to the shape before this collision.

6 Conclusion, Discussion and Outlook

The two modifications of the galaxy rotation curves from both Modified Newtonian Dynamics and the addition of Dark Matter change the curve in the desired way. Modified Newtonian Dynamics changes the r-dependence of the acceleration from r^{-2} to r^{-1} in the long distance limit. This change to Newton's second law results in a rotation curve with a shape that is more similar to the measured curves. However this modification is only a static solution that works on simple systems. When applied to a more complex system the modification does not solve the entire problem of the mass discrepancy and only manages to reduce the difference. The solution that Dark Matter provides is to add unobservable additional matter to the mass distribution. This idea of extra unseen mass is also supported by gravitational lensing experiments[6]. There have been a number of propositions for the shape of the dark matter distribution, since this is unknown. These different propositions all lead to similar rotation curves with minor differences. Some rise quicker and decrease faster, like the EinastoB profile, others rise slower but flatten off instead of decreasing, like the Burkert profile.

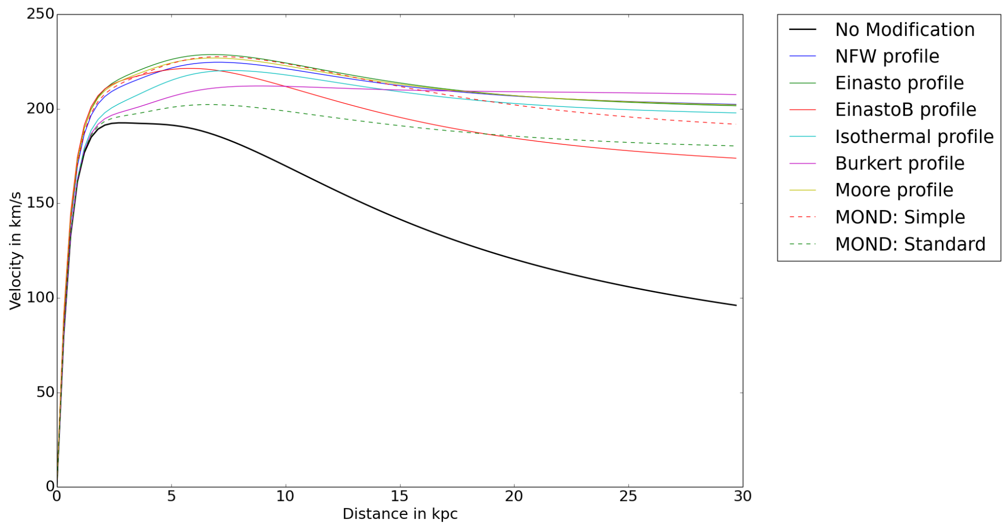

Figure 14: The rotation curves from the dark matter profiles and MOND compared with the unmodified curve.

Dark Matter seems to be a better solution since it is more clear as to where the mass should be added, but the shape of the Dark Matter distribution is unknown and the amount of mass that needs to be added is large compared with the mass of the galaxy itself. By contrast, for MOND the solution is static and the modification should change the curve a long distance away from the center, however this does not solve the problem for more complicated systems. Another idea is to solve the problem with a combination of these two solutions. Dark Matter can be added to complement

MOND, when MOND alone is not sufficient. However the idea of introducing dark matter seems to remove the motivation of MOND, which tries to solve the problem without resorting to adding mass.

The numerical program written for this thesis can be used to quickly evaluate certain mass distributions. When a new mass distribution or a GR-motivated modification of Newton's gravitational law is proposed, the corresponding rotation curve can quickly be evaluated to see if this distribution has the desired shape. This will only work as long as the proposed distributions have a spherical or cylindrical symmetry.

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