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HIGH ENERGY PHYSICS

A curvature-dependent dark energy

AND HOW IT CAN SOLVE THE HUBBLE TENSION

THESIS BSc PHYSICS AND ASTRONOMY

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Abstract

The Hubble tension poses a significant challenge in cosmology, requiring an effective resolution. This thesis introduces a potential solution: the Curvature Dependent Dark Energy (CDDE) model. The CDDE model incorporates a curvature-dependent dark energy term represented as $T_{\mu\nu}^{DE} = C\sqrt{(R/6)}g_{\mu\nu}$ in the Einstein equations. The primary objective of this study is to fit the parameter C within the CDDE model, providing a solution to the Hubble tension. Through careful parameter tuning, we obtain a best-fit value that successfully resolves the tension. The specific value of C determined in the fit is: $C = 1.460^{+0.170}_{-0.182} \times 10^{22} \text{eV}^3$, corresponding with an energy scale of $C^{\frac{1}{3}} = 24.442^{+0.915}_{-1.062} \text{MeV}$ that may hint at an origin in the realm of strong interactions. During the fitting, we determined the necessary value of z_* , representing the redshift at which matter and radiation decouple. Our findings yield a value of $z_* = 1089.59^{+3.22}_{-1.88}$. An intriguing consequence of the CDDE model is the emergence of matter creation, a distinctive feature that sets it apart from conventional models. Furthermore, when comparing the deceleration parameters between the CDDE and Lambda-CDM models, a significant overlap is observed, which adds motivation to further research this model. This overlap is not trivial, as the model by construction only solves the Hubble tension. The presence of such overlap in the deceleration parameters occurs naturally within the model, without requiring further modifications. It implies that the CDDE model can solve the Hubble tension and at the same time still be consistent with the observed acceleration of the universe.

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1 Introduction

Cosmology is a branch of physics that studies the origin, evolution, and structure of the universe as a whole. It deals with understanding the large-scale properties and dynamics of the universe, including its expansion, composition, and ultimate fate. Several key concepts in cosmology include the cosmic microwave background, scale factor (a), redshift (z), Hubble parameter (H), the Hubble tension and dark energy, which will be explained in this introductory chapter.

The scale factor (a) is a fundamental dimensionless quantity and it describes the relative size of the universe at different times. It represents the expansion of the universe, with a value of 1 at the present time. In the future as the universe expands, the scale factor increases, indicating that galaxies and other cosmic structures become more separated over time. If we look back in time, the scale factor decreases meaning that structures were closer together. This way the scale factor can be used as a time scale with a small /large scale factor corresponding with the past/future.

The dimensionless quantity redshift (z) is the increase of wavelength of radiation which means a decrease in frequency and energy. Redshift can occur in three ways, as the Doppler effect, gravitational redshift and cosmological redshift. For cosmology this last one is of course the most important. As space itself expands light waves get stretched out, resulting in a larger wavelength. The relation between redshift and the scale factor is inversely proportional:

$$1 + z = \frac{1}{a} \quad (1)$$

Therefore redshift can be used as an inverse measure of time. The present has a redshift of $z = 0$. A redshift smaller/larger than $z = 0$ corresponds with the future/past.

The Hubble parameter is a time dependent quantity which describes the expansion of the universe. Its units are most of the time expressed in km/s/Mpc. For example, a Hubble constant of 1 km/s/Mpc means that objects with a distance of 1 mega parsec ($\approx 3.09 \times 10^{22}$ meter) to each other move away from each other with a speed of 1 km/s. Mathematically, it is defined as:

$$H(t) = \frac{\dot{a}}{a} \quad (2)$$

with $\dot{a} = \frac{da}{dt}$ being the time derivative of the scale factor.

The Hubble constant is the Hubble parameter today: $H_0 = H(t = t_0)$. The Hubble constant plays an important role in cosmology and is therefore of great importance to measure. However, different measurements lead to different results for the standard cosmological Λ -CDM model. This difference is also known as the Hubble tension. The measurements can be sorted into two groups: late universe and early universe measurements. The early universe measurement uses the cosmic microwave background (CMB). By analysing temperature fluctuations in the CMB and the large-scale structure of the universe, the Hubble constant is calculated. Combining constraints from the Dark Energy Survey (DES), Baryon Acoustic Oscillations (BAO) and Big Bang Nucleosynthesis (BBN), which are early universe measurements, one obtains the value: $H_0^{\text{early}} = 67.4 \pm 1.1$ km/s/Mpc [1, 2]. A combination of multiple late universe measurements result in the value of $H_0^{\text{late}} = 73.3 \pm 0.8$ km/s/Mpc[3]. Inspired by the analysis of [4], we aim to resolve this tension by replacing the cosmological constant by a non-constant curvature-dependent dark energy, giving rise to the model that we will refer to as the Curvature Dependent Dark Energy (CDDE) model.

2 Theory

In this chapter the most important assumptions and mathematical tools will be explained. These will be used for both the Λ -CDM and CDDE model.

2.1 Natural units

In this thesis we will make use of the natural units. Most importantly, we will put the speed of light and the reduced Planck constant to one: $c = \hbar = 1$, by absorbing these constants in the relevant fields and quantities. This will allow us to express dimensionful quantities in units of energy (eV) to some power.

2.2 Flatness

With a good approximation, we can say that the universe is flat (no intrinsic curvature). Observations from the Microwave Anisotropy Probe (WMAP) are consistent with a nearly flat universe [5]. In this thesis, we will therefore simplify all work by assuming $k = 0$ for the intrinsic curvature parameter k .

2.3 The Einstein equations

Einstein's general theory of relativity relates the curvature of the universe to its components. This relation is given by the Einstein equations. The Einstein equations can be written in the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3$$

Here, $R_{\mu\nu}$ represents the Ricci tensor, $g_{\mu\nu}$ the metric tensor, R the Ricci scalar, G the gravitational constant and $T_{\mu\nu}$ the stress-energy tensor. These objects will be explained and derived in the following sections. The left-hand side of the equation corresponds to the curvature of the universe, while the right-hand side corresponds to the components present in the universe.

2.4 FRW-metric

In both the standard cosmological Λ -CDM model and our curvature-dependent dark energy model (CDDE) the Friedmann–Lemaître–Robertson–Walker metric (FRW metric) is used to describe the universe at large scales where there is no preferred direction and location. These characteristics can easily be seen in the FRW metric. The 00 component g_{00} is spatially independent as implied by homogeneity, while the other diagonal components, the ii components g_{ii} ($i \neq t$), have no preferred direction as implied by isotropy. The FRW-metric is:

$$g_{\mu\nu}^{\text{FRW}} = \text{diag}(1, -a^2(t), -a^2(t)r^2, -a^2(t)r^2\sin^2(\theta)) \quad (3)$$

Here we have used spherical coordinates and the flatness property ($k = 0$).

2.5 Christoffel symbols

The Christoffel symbols are calculated using:

$$\Gamma_{\mu\nu}^{\alpha} \equiv \frac{1}{2}g^{\alpha\lambda}(g_{\mu\lambda,\nu} + g_{\nu\lambda,\mu} - g_{\mu\nu,\lambda}) \quad (4)$$

with $g^{\mu\nu}$ being the inverse of the metric $g_{\mu\nu}$ and $g_{\mu\nu,\lambda}$ being the derivative of $g_{\mu\nu}$ with respect to the spacetime coordinate x^λ . Here we make use of the Einstein summation convention. In the Einstein summation convention, whenever an index appears twice in a term (once as a subscript and once as a superscript), it implies summation over all possible values of that index. In the case of cosmology (and in this thesis) these indices are the 4-dimensional space-time indices. The non-zero Christoffel symbols of the FRW metric, again using $k = 0$, are:

$$\begin{aligned}\Gamma_{rr}^t &= a\dot{a} & \Gamma_{\theta\theta}^t &= r^2 a\dot{a} & \Gamma_{\phi\phi}^t &= r^2 \sin^2(\theta) a\dot{a} \\ \Gamma_{tr}^r &= \Gamma_{t\theta}^\theta = \Gamma_{t\phi}^\phi = \frac{\dot{a}}{a} & \Gamma_{\phi\phi}^r &= -r \sin^2(\theta) & \Gamma_{\theta\theta}^r &= -r \\ \Gamma_{r\phi}^\phi &= \frac{1}{r} & \Gamma_{r\theta}^\theta &= \frac{1}{r} & \Gamma_{\phi\phi}^\theta &= -\cos(\theta) \sin(\theta) \\ \Gamma_{\phi\theta}^\phi &= \cot(\theta)\end{aligned}\tag{5}$$

2.6 Riemann tensor and the Ricci scalar and tensor

With the calculated Christoffel symbols we can calculate the Riemann tensor:

$$R_{\nu\alpha\beta}^\mu \equiv \partial_\alpha \Gamma_{\nu\beta}^\mu - \partial_\beta \Gamma_{\nu\alpha}^\mu + \Gamma_{\gamma\alpha}^\mu \Gamma_{\nu\beta}^\gamma - \Gamma_{\gamma\beta}^\mu \Gamma_{\nu\alpha}^\gamma\tag{6}$$

and the Ricci scalar:

$$R \equiv g^{\mu\nu} R_{\mu\nu}\tag{7}$$

Here, the Ricci tensor $R_{\mu\nu}$ is defined as:

$$R_{\mu\nu} \equiv R_{\mu\alpha\nu}^\alpha\tag{8}$$

The 00 component of the Ricci tensor R_{00} for the FRW-metric is:

$$R_{00} = -3\frac{\ddot{a}}{a}\tag{9}$$

The other diagonal terms of the Ricci tensor are:

$$R_{ii} = -\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right) g_{ii}^{FRW}\tag{10}$$

with $i \in \{1, 2, 3\}$. The other non-diagonal terms vanish. With the Ricci tensor it is easy to compute the Ricci scalar:

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\tag{11}$$

2.7 Perfect fluid approximation

In both the Λ -CDM model and our CDDE model, we make use of the perfect fluid approximation. In this approximation effects like heat conductivity and viscosity are neglected. This approximation is motivated by the isotropic and homogeneous properties of the universe at large scales. If there were spatial dependencies in the pressure $p(\vec{x}, t)$ and the energy density $\rho(\vec{x}, t)$, it would contradict the isotropy and homogeneity. Therefore, we assume that the pressure and energy density are spatially independent: $p(\vec{x}, t) = p(t)$ and $\rho(\vec{x}, t) = \rho(t)$. This leads to a stress-energy tensor of the form:

$$T_{\mu\nu}^{PF} = \text{diag} [\rho(t), -p(t)g_{11}^{FRW}, -p(t)g_{22}^{FRW}, -p(t)g_{33}^{FRW}]\tag{12}$$

Here, the energy density $\rho(t)$ represents the total energy density, which is the sum of the (ultrarelativistic) radiation and (non-relativistic) matter energy densities: $\rho(t) = \rho_R(t) + \rho_M(t)$. In the Λ -CDM model, the total pressure $p(t)$ is only dependent on the radiation energy density and has the following dependence: $p(t) = \frac{1}{3}\rho_R(t)$. The presence of the factor $\frac{1}{3}$ and its independence from the matter energy density are related to thermodynamics and statistical physics[6]. For non-relativistic matter, the energy associated with the temperature is negligible compared with the mass of the matter particles and as a consequence: $p_M(t) \ll \rho_M(t)$. In many textbooks and studies, the dark energy term is automatically incorporated into the expression for total pressure. However, in this thesis, we will include the dark energy term as a separate term $\propto g_{\mu\nu}$ in the Einstein equations. Since in the CDDE model the dark-energy contribution to $T_{\mu\nu}$ can not be viewed as an independent fluid component, we will treat it separately. As such, in our set-up the perfect fluid contribution to the stress-energy tensor is given by:

$$T_{\mu\nu}^{\text{PF}} = \text{diag} \left[\rho_R + \rho_M, -\frac{1}{3}\rho_R g_{11}^{\text{FRW}}, -\frac{1}{3}\rho_R g_{22}^{\text{FRW}}, -\frac{1}{3}\rho_R g_{33}^{\text{FRW}} \right] \quad (13)$$

3 Lambda cold dark matter

The Lambda cold dark matter model (Λ -CDM) is the most widely accepted cosmological model due to its success in explaining various observations. The two key features of this model are in its name: the universe has a component called the cosmological constant (Λ) and the cold (non-relativistic) matter is mostly cold dark matter (CDM).

The inclusion of the cosmological constant in the Einstein field equations allows for the possibility of an expanding universe. The growing body of data strongly suggests that the universe is presently experiencing accelerated expansion, which indicates the presence of a positive cosmological constant.

The Einstein's equations of this model can be put in the form:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu}^{\text{eff}} \quad (14)$$

with the effective stress-energy tensor being:

$$T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{\text{PF}} + \frac{\Lambda}{8\pi G}g_{\mu\nu} \quad (15)$$

Here Λ is the cosmological constant and is taken to be positive ($\Lambda > 0$) such that an accelerating expansion is possible. When rewritten like this, we can interpret the cosmological constant $\Lambda/8\pi G$ as a vacuum energy density (often referred to as dark energy density). Notice that this vacuum energy density is constant and is therefore not depending on the expansion of the universe.

The following equations are derived in the appendix (A.1 and A.2). Assuming no intrinsic curvature ($k = 0$), the 00 and ii component of the Einstein equations can respectively be put in the form:

$$H^2 \equiv \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G(\rho_M + \rho_R) + \Lambda}{3} \quad (16)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(2\rho_R + \rho_M) + \frac{\Lambda}{3} \quad (17)$$

with the dot and double dot indicating single and double differentiation with respect to time. Equation 17 shows that with a positive cosmological constant $\Lambda > 0$ an accelerating expansion is possible.

These two equations can be combined into a useful third dependent equation (see appendix A.3):

$$\rho_M + \rho_R + \frac{\dot{a}}{a}(3\rho_M + 4\rho_R) = 0 \quad (18)$$

By dividing both sides of equation 16 by H^2 and subsequently rewriting it, we obtain (see appendix A.1):

$$\Omega_M + \Omega_R + \Omega_\Lambda = 1 \quad (19)$$

where the convention $\Omega_i = \frac{8\pi G}{3H^2}\rho_i$ and $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ is used. It might seem that we do not have enough equations to solve the system, and indeed, we do not. We have three unknowns, $\rho_M(t)$, $\rho_R(t)$ and $H(t)$ but only 2 equations. In this thesis we will only consider the universe during the stage in which matter and radiation are decoupled and therefore do not interact with each other. When this happens equation 18 decouples. The decoupling of matter and radiation happens at the z -value of recombination ($z_* \approx 1100$). The exact value of z_* will be calculated later on.

The decoupling of equation 18 looks as follows:

$$\dot{\rho}_M = -3\frac{\dot{a}}{a}\rho_M, \quad \dot{\rho}_R = -4\frac{\dot{a}}{a}\rho_R \quad (20)$$

Both of these equation can be easily solved using integration and have the solutions:

$$\rho_M = \rho_{M,0}a^{-3}, \quad \rho_R = \rho_{R,0}a^{-4} \quad (21)$$

Or written in term of the redshift z :

$$\rho_M = \rho_{M,0}(z+1)^3, \quad \rho_R = \rho_{R,0}(z+1)^4 \quad (22)$$

Dividing equation 16 by $H_0^2 = \left(\frac{\dot{a}}{a}\right)^2|_{t=t_0}$ and using the equations 21 will result in the following equation:

$$H(z) = H_0 \sqrt{\Omega_{R,0}(z+1)^4 + \Omega_{M,0}(z+1)^3 + \Omega_\Lambda} \quad (23)$$

Here we have used a convention that we will use a lot throughout this thesis:

$$\Omega_i = \frac{8\pi G}{3H^2}\rho_i, \quad \Omega_{i,0} = \Omega_i(t = t_0) \quad (24)$$

These omegas all have values between 0 and 1 and can thus be interpreted as the relative contribution of the specific component in the universe. With this decoupling we have 3 unknown parameters and 3 equations. The only thing necessary now is a set of initial conditions. The benchmark model tells us that at $t = t_0$: $\Omega_{M,0} = 0.334 \pm 0.018$ [7], $\Omega_{R,0} = 7.74_{-0.173}^{+0.178} \times 10^{-5}$ [8] and $\Omega_{\Lambda,0} = 1 - \Omega_M = 0.666 \pm 0.018$, with $\Omega_{\Lambda,0} = 0.666$ corresponding to a positive cosmological constant $\Lambda > 0$.

3.1 Acceleration redshift in the Λ -CDM model

The transition from deceleration to acceleration happens at the turning point called acceleration redshift. This point is defined as the redshift with $\ddot{a} = 0$. Using equation 17 and equating it to zero, gives us the equation: $-\frac{4\pi G}{3}(2\rho_R + \rho_M) + \frac{\Lambda}{3} = 0$. Assuming that the acceleration happened relatively late in the evolution of the universe, which indeed will be the case, we make the approximation of setting the radiation energy density to zero $\rho_R = 0$. The assumption that the radiation energy density in the late universe is negligible is here made because of the fact that radiation falls off much quicker than the matter or dark matter components. This results in the equation: $-\frac{4\pi G}{3}\rho_M + \frac{\Lambda}{3} = 0$.

After rewriting this, we can say that acceleration happened when the matter energy density equaled $\rho_M = \frac{\Lambda}{4\pi G}$. Using the left equation of 21 we find that the acceleration happened at the scale factor:

$$a_{acc} = \left(\frac{4\pi G}{\Lambda} \rho_{M,0} \right)^{\frac{1}{3}} = \left(\frac{\Omega_{M,0}}{2\Omega_{\Lambda,0}} \right)^{\frac{1}{3}} \quad (25)$$

or written in terms of the redshift:

$$z_{acc} = \left(\frac{\Lambda}{4\pi G \rho_{M,0}} \right)^{\frac{1}{3}} - 1 = \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{M,0}} \right)^{\frac{1}{3}} - 1 \quad (26)$$

4 Curvature-dependent dark energy model

To solve the Hubble tension, we introduce a curvature-dependent dark energy term. In addition to the perfect fluid approximation (equation 12) we will add a curvature-dependent stress-energy tensor that looks as follows:

$$T_{\mu\nu}^{\text{DE}} = C \left(\frac{R}{6} \right)^{\frac{1}{2}} g_{\mu\nu}^{\text{FRW}}, \quad C > 0 \quad (27)$$

Here C is a positive constant that has yet to be determined, R is the Ricci scalar and $g_{\mu\nu}^{\text{FRW}}$ denotes the FRW-metric. The constant C needs to be positive for the universe to experience accelerating expansion. The Ricci scalar has units of energy squared ($[R] = \text{eV}^2$), and the curvature dependent stress-energy tensor has units of energy to the power of 4: ($[T_{\mu\nu}^{\text{vac}}] = \text{eV}^4$). Consequently, C has units of energy to the power of 3: ($[C] = \text{eV}^3$). The total (effective) stress-energy tensor is therefore:

$$T_{\mu\nu}^{\text{eff}} = T_{\mu\nu}^{\text{PF}} + T_{\mu\nu}^{\text{DE}} \quad (28)$$

Note that the dark energy stress-energy tensor is not an independent perfect-fluid contribution. It is dependent on the radiation and matter present in the universe because radiation and matter create curvature. The quantity $C\sqrt{R/6}$ should be interpreted as an energy density associated with the presence of matter and radiation, in contrast to a vacuum energy density, which is what the cosmological constant term represents in the Λ -CDM model.

With this stress-energy tensor we again get 2 equations. The 00 and ii component of the Einstein equations can respectively be put in the forms (see appendix B):

$$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{4\pi G}{3} \left[2(\rho_R + \rho_M) + \beta + \sqrt{\beta(\rho_M + \beta)} \right] \quad (29)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[\rho_M + 2\rho_R - \beta - \sqrt{\beta(\rho_M + \beta)} \right] \quad (30)$$

Here we have defined: $\beta = \frac{16\pi G C^2}{3}$. These two equations can be combined into the equation:

$$\dot{\rho}_R + \dot{\rho}_M + \frac{\dot{a}}{a}(3\rho_M + 4\rho_R) = -\frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}}\dot{\rho}_M \quad (31)$$

Just like in the Λ -CDM model we have 2 equations and 3 unknowns, and we will use decoupling to solve the system after recombination. Equation 31 decouples when matter and radiation stop interacting with each other:

$$\dot{\rho}_R = -4\frac{\dot{a}}{a}\rho_R, \quad \dot{\rho}_M \left(1 + \frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \right) = -3\frac{\dot{a}}{a}\rho_M \quad (32)$$

These differential equations are solved by:

$$\rho_R = \rho_{R,0} a^{-4}, \quad \rho_M \left(1 - \frac{2}{1 + \sqrt{\frac{\rho_M + \beta}{\beta}}} \right)^{\frac{1}{4}} = C' a^{-3} \quad (33)$$

The matter and dark energy components are not able to decouple. Matter and dark energy will always interact with each other.

When both sides of Equation 29 are divided by H^2 it can be rewritten as

$$1 = \Omega_M + \Omega_R + \frac{1}{2} \Omega_\beta \left(1 + \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \quad (34)$$

The 3 equations 32 and 34 can be written in terms of $u = \ln(z + 1)$ and Ω_i :

$$\frac{d \ln(H/H_0)}{du} = \frac{3\Omega_M + 4\Omega_R}{2} \quad (35)$$

$$\frac{d\Omega_R}{du} = 4\Omega_R - 2\Omega_R \frac{d \ln(H/H_0)}{du} \quad (36)$$

$$\frac{d\Omega_M}{du} = -2\Omega_M \frac{d \ln(H/H_0)}{du} + \frac{3\Omega_M}{1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}}} \quad (37)$$

We now have the 3 equation to solve the system. The initial conditions H_0 , $\Omega_{M,0}$, $\Omega_{R,0}$ and $\Omega_{\beta,0}$ are the only things we need to find to solve the system. The benchmark model tells us $\Omega_{M,0} = 0.334 \pm 0.018$ [7], $\Omega_{R,0} = 7.74^{+0.178}_{-0.173} \times 10^{-5}$ [8] and $\Omega_{\Lambda,0} = 1 - \Omega_M = 0.666 \pm 0.018$, but these values are found by using the Λ -CDM model. Because we are trying to solve the Hubble tension using a new model, we can not use these values. We have to find these values with the CDDE model.

4.1 Acceleration redshift in the CDDE model

The universe transitioned from deceleration to acceleration at a turning point referred to as the acceleration redshift. This point is defined as the moment when the second derivative of the scale factor with respect to time equals zero ($\ddot{a} = 0$). This happens when equation 30 is equal to zero: $\rho_M + 2\rho_R - \beta - \sqrt{\beta(\rho_M + \beta)} = 0$. Assuming that the acceleration redshift happened relatively late in the evolution of the universe, which indeed will be the case, we again take the radiation energy density to be zero $\rho_R = 0$. This assumption results in the equation: $\rho_M - \beta = \sqrt{\beta(\rho_M + \beta)}$. This is solved by: $\rho_M = 3\beta$, which we will insert into equation 33. This results in:

$$a_{acc}(\beta) = \sqrt[3]{\frac{C'}{3^{3/4}\beta}} \quad (38)$$

or written as the z -value:

$$z_{acc}(\beta) = \sqrt[3]{\frac{3^{3/4}\beta}{C'}} - 1 \quad (39)$$

4.2 Determining z_*

We assume that the decoupling of matter and radiation takes place during recombination. Recombination refers to the moment when the temperature of the universe

sufficiently decreased, allowing electrons and protons to combine and form neutral hydrogen atoms. Before recombination, the plasma state was maintained due to high energy and temperature levels, preventing the long-term existence of neutral atoms. Photons continuously scattered off the free electrons within the plasma. This scattering is called Thomson scattering. During this period, radiation and baryonic (non-dark) matter were heavily interacting and coupled together. However, after recombination, we witness their decoupling.

The moment of recombination will be defined as when the following equation is true: $H_* = \Gamma(z_*)$. When the scattering rate equals or is less than the Hubble parameter, the interactions between photons and electrons weaken significantly. Photons can now travel more freely without constant scattering, leading to the decoupling of photons from the charged particles. H_* is the Hubble constant at recombination, using the early universe measurement:

$$H_* \equiv H(z_*) = H_0^{\text{early}} \sqrt{\Omega_{R,0}(z_* + 1)^4 + \Omega_{M,0}(z_* + 1)^3 + \Omega_\Lambda} \quad (40)$$

Why we use this H_* will be explained in chapter 4.4. $\Gamma(z_*)$ is the scattering rate of Thomson scattering, defined as:

$$\Gamma(z) = n_e(z)\sigma \quad (41)$$

Here n_e is the number density of electrons and σ is the Thomson scattering cross-section ($\sigma = 6.6523 \times 10^{-29} \text{m}^2$ [9]). For determining $n_e(z)$ we will assume that the plasma is neutrally charged particles and contains only electrons, hydrogen, helium and their ions. Helium-4 has a relative contribution to the mass energy density ρ_M of $Y = \frac{\rho_M(^4\text{He})}{\rho_M} = 0.2479 \pm 0.0029$ [10] in the universe. We will denote the number density of a particle i with $n_{i,x}$, with $x=0$ for neutrally charged and $x=1,2$ for the particle having charge $+|e|$, $+2|e|$ in terms of the unit charge $|e|$. Ions are, of course, slightly less massive than their neutrally charged counterparts, but the difference is so small that we often consider them to have comparable mass. The total mass density must therefore be:

$$\rho_M(z) = m_{H0}[n_{H0}(z) + n_{H1}(z)] + m_{He0}[n_{He0}(z) + n_{He1}(z) + n_{He2}(z)] \quad (42)$$

The mass density of helium is:

$$Y\rho_M(z) = m_{He0}[n_{He0}(z) + n_{He1}(z) + n_{He2}(z)] \quad (43)$$

Subtracting these two from each other results in:

$$(1 - Y)\rho_M(z) = m_{H0}[n_{H0}(z) + n_{H1}(z)] \quad (44)$$

In addition the trivial equation of the number density for a neutral plasma reads:

$$n_e(z) = n_{H1}(z) + n_{He1}(z) + 2n_{He2}(z) \quad (45)$$

We also assume the plasma in the universe to be in thermal equilibrium. From this follows the Boltzmann distribution of the particle density of particle i with charge $x|e|$:

$$n_{i,x} = g_{i,x} \left(\frac{m_{i,x} kT(z)}{2\pi} \right)^{\frac{3}{2}} \exp \left(-\frac{m_{i,x}}{kT(z)} \right) \quad (46)$$

here $g_{i,x}$ is the statistical spin weight and $m_{i,x}$ the mass of particle i with charge $x|e|$. The statistical spin weight of an electron g_e is 2. Again we use $m_{i,x+1} \approx m_{i,x}$ to derive:

$$\begin{aligned} \frac{n_{i,x+1}(z)n_e(z)}{n_{i,x}(z)} &= \frac{g_e g_{i,x+1}}{g_{i,x}} \left(\frac{m_{i,x+1} m_e kT(z)}{m_{i,x} 2\pi} \right)^{\frac{3}{2}} \exp \left(\frac{-(m_{i,x+1} + m_e - m_{i,x})}{kT(z)} \right) \\ \implies \frac{n_{i,x+1}(z)n_e(z)}{n_{i,x}(z)} &= \frac{2g_{i,x+1}}{g_{i,x}} \left(\frac{m_e kT(z)}{2\pi} \right)^{\frac{3}{2}} \exp \left(\frac{-Q_{i,x+1}}{kT(z)} \right) \equiv f_{i,x+1}(z) \end{aligned} \quad (47)$$

with $Q_{i,x+1}$ being the ionization energy of the particle i with charge $(x+1)|e|$. This final equation results in:

$$n_{i,x+1}(z) = \frac{n_{i,x}(z)}{n_e(z)} f_{i,x+1}(z) \quad (48)$$

This equation makes it possible to go from the number density of an element to the number density of the next ionized element. With this equation we can rewrite equation 43:

$$\begin{aligned} Y\rho_M(z) &= \frac{m_{He0}n_{He0}(z)}{n_e^2(z)}(n_e^2(z) + f_{He1}(z)n_e(z) + f_{He1}(z)f_{He2}(z)) \equiv \frac{n_{He0}(z)}{n_e^2(z)}P_2(z) \\ \implies n_{He0}(z) &= \frac{n_e^2(z)}{P_2(z)}Y(z)\rho_M(z) \end{aligned} \quad (49)$$

Here we have defined $P_2 = m_{He0}(n_e^2 + f_{He1}n_e + f_{He1}f_{He2})$. Equation 44 can be rewritten as:

$$\begin{aligned} (1 - Y(z))\rho_M(z) &= \frac{m_{H0}n_{H0}(z)}{n_e(z)}(n_e(z) + f_{H1}(z)) \equiv \frac{n_{H0}(z)}{n_e(z)}P_1(z) \\ \implies n_{H0}(z) &= \frac{n_e(z)}{P_1(z)}(1 - Y(z))\rho_M(z) \end{aligned} \quad (50)$$

P_1 is defined as: $P_1 = m_{H0}(n_e + f_{H1})$. Now we use equations 48, 49 and 50 to rewrite equation 45:

$$\begin{aligned} n_e(z) &= \frac{f_{H1}(z)}{n_e(z)}n_{H0}(z) + \left(\frac{f_{He1}(z)}{n_e(z)} + 2\frac{f_{He1}(z)f_{He2}(z)}{n_e^2(z)} \right) n_{He0}(z) \\ \implies n_e(z) &= \frac{f_{H1}(z)}{P_1(z)}(1 - Y(z))\rho_M(z) + \frac{f_{He1}(z)n_e(z) + 2f_{He1}(z)f_{He2}(z)}{P_2(z)}Y(z)\rho_M(z) \end{aligned} \quad (51)$$

This is our final equation for $n_e(z)$. We substitute the expressions for P_1 and P_2 , and employ a numerical approach to solve equation 51 and obtain the positive, real solution. The solution $n_e(z)$ is then applied in equation 41 and is necessary for determining the recombination redshift z_* , which satisfies the equation $H_* = \Gamma(z_*)$. The computation is done using a binary searching algorithm. This results in $z_* = 1088.65^{+0.85}_{-0.78}$. In this computation, we utilize the upper and lower limits of the errors associated with the used parameters.

4.3 Initial conditions

The initial conditions we are looking for are: H_0 , $\Omega_{M,0}$, $\Omega_{R,0}$ and $\Omega_{\beta,0}$. For H_0 , we will take the late universe measurement as it is minimally affected by the Λ -CDM model. We will take $H_0^{\text{late}} = 73.3 \pm 0.8$ km/s/Mpc [4] as the correct Hubble constant.

$\Omega_{R,0}$ is also unaffected by the Λ -CDM model. The main contributions to $\Omega_{R,0}$ are radiation from the cosmic microwave background and neutrinos. This value is calculated using the present day vacuum temperature $T = 2.72548 \pm 0.00057$ K [11], resulting in $\Omega_{R,0} = 7.74^{+0.178}_{-0.173} \times 10^{-5}$ [8].

The measurement of $\Omega_{M,0}$ is affected by Λ -CDM, whereas $\Omega_{\beta,0}$ is not even defined in Λ -CDM. So we will take these two as variables. Formula 34 can express $\Omega_{M,0}$ in terms of $\Omega_{\beta,0}$ as follows:

$$\Omega_{M,0} = \frac{1}{8} \left(-\sqrt{\Omega_{\beta,0}(9\Omega_{\beta,0} - 16\Omega_{R,0} + 16)} - 3\Omega_{\beta,0} - 8\Omega_{R,0} + 8 \right) \quad (52)$$

This means that we only have one parameter left: $\Omega_{\beta,0}$. We will determine the value of $\Omega_{\beta,0}$ through fitting, ensuring that this model resolves the Hubble tension.

4.4 Solving the Hubble tension

We assume that the late universe measurement, $H_0^{\text{late}} = 73.3 \pm 0.8$ km/s/Mpc, is correct and unaffected by the Λ -CDM model. Logically, the Hubble parameter of the CDDE model has to pass through this late universe measurement. The $H_0^{\text{early}} = 67.4 \pm 1.1$ km/s/Mpc is determined using the Λ -CDM model. Consequently, we find a disagreement between both measurements: the Hubble tension. To resolve the Hubble tension, we will employ reverse engineering, projecting the early universe measurement, H_0^{early} , to the redshift of recombination using the Λ -CDM model. This process transforms the Λ -CDM model-affected measurement, H_0^{early} , into an unaffected one at the redshift of recombination, denoted as H_* . This reverse engineering is done using the equation:

$$H_* \equiv H^{\Lambda\text{-CDM}}(z_*) = H_0^{\text{early}} \sqrt{\Omega_{R,0}(z_* + 1)^4 + \Omega_{M,0}(z_* + 1)^3 + \Omega_{\Lambda}} \quad (53)$$

To solve the Hubble tension, we will determine the value of $\Omega_{\beta,0}$ that makes $H(z)$ pass through both H_0^{late} and H_* . By achieving this, both measurements can be correct simultaneously.

The following graph is provided for illustration purposes:

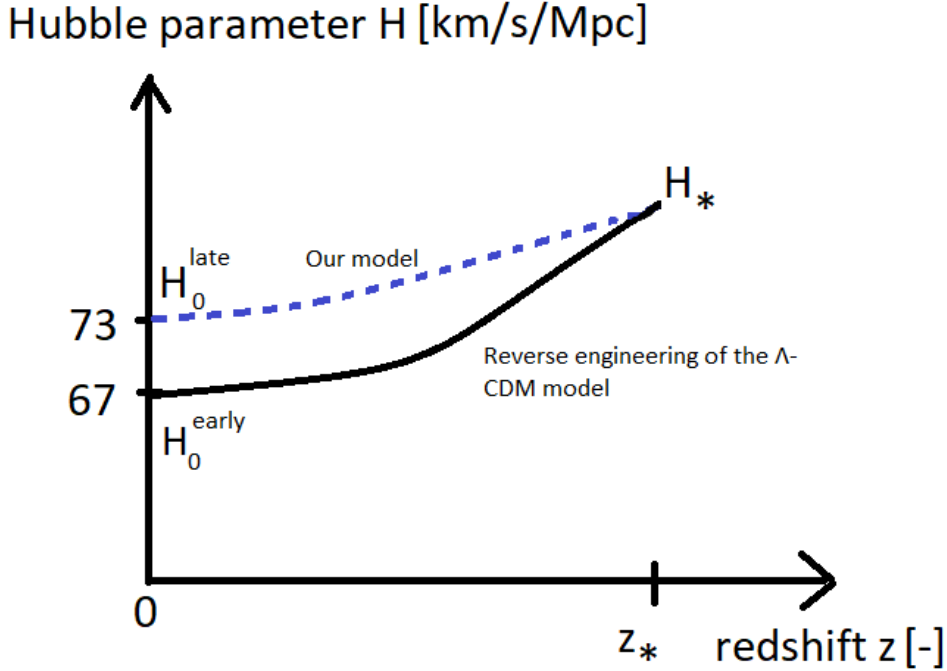


Figure 1: The Hubble parameter is plotted against the redshift. Within our CDDE model we will fit $\Omega_{\beta,0}$ such that the Hubble parameter will pass through both H_0^{late} and H_* (blue dotted line). H_0^{late} represents the Hubble constant obtained from the late universe measurement, while H_* is the Hubble parameter at recombination, which is reverse-engineered using the Λ -CDM early universe measurement (black line).

5 Results

The $\Omega_{\beta,0}$ that is found, while solving the Hubble tension within the CDDE model, is: $\Omega_{\beta,0} = 0.551^{+0.136}_{-0.129}$. This corresponds with $C = 1.460^{+0.170}_{-0.182} \times 10^{22} \text{eV}^3$, resulting in an energy scale of: $C^{\frac{1}{3}} = 24.442^{+0.915}_{-1.062} \text{MeV}$. As the Λ -CDM model falls short in achieving this fit, it is possible that the CDDE model is a successful replacement. When we solve the system of differential equations with the initial conditions we find some interesting results and predictions for the CDDE model

5.1 Components of the universe

The following graph shows us the evolution of the components of the universe against the natural logarithm of $z + 1$:

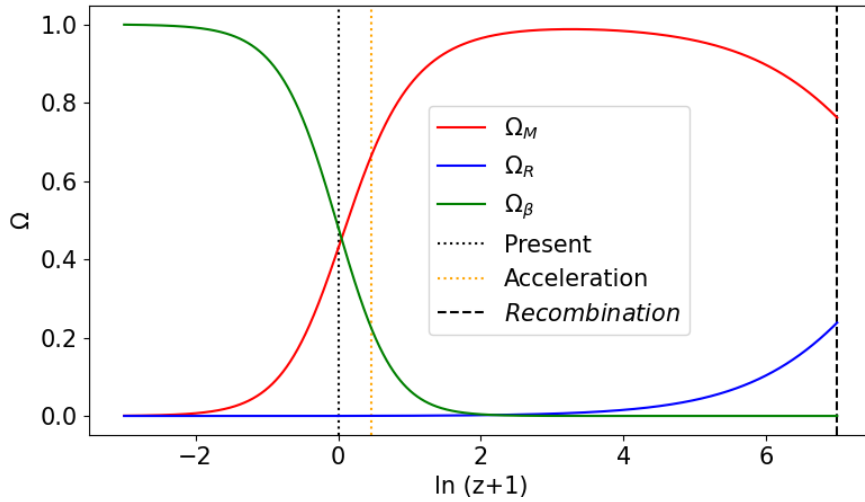


Figure 2: Components of the CDDE model plotted against the natural logarithm of $(z + 1)$. The yellow line 'acceleration' represents the z -value at which the universe began accelerating.

As can be seen, the universe enters a phase of matter domination following recombination. At a certain point, the significance of our dark energy term, denoted as β , begins to emerge. The yellow dotted acceleration line marks the redshift at which the universe initiates its accelerated expansion ($\ddot{a} = 0$). Presently, we live in a universe that is already undergoing acceleration. Looking ahead, as the influence of the matter term keeps decreasing, the dark energy component will become more and more dominant. The use of the logarithmic function $\ln(z + 1)$ on the x-axis allows us to focus our analysis on the present and future while providing a broader perspective on the past. This is done, because over a large range of z -values in the past, minimal changes occurred.

5.2 Creation of matter

When studying the right-hand equation of 33, we encounter something remarkable. In the context of the very late universe, the mass energy density is significantly smaller

compared to the dark energy β , allowing us to take the limit $\rho_M/\beta \rightarrow 0$. By taking the limit $\rho_M/\beta \rightarrow 0$ and performing a Taylor expansion of the equation, we obtain:

$$\rho_M \left(1 - \frac{2}{1 + \sqrt{\frac{\rho_M + \beta}{\beta}}} \right)^{\frac{1}{4}} = C' a^{-3} \xrightarrow{\rho_M/\beta \rightarrow 0} \rho_M \left(\frac{\rho_M}{4\beta} \right)^{\frac{1}{4}} = C' a^{-3} \quad (54)$$

which can be rewritten as: $\rho_M^{\frac{5}{4}} = C'(4\beta)^{\frac{1}{4}} a^{-3}$. Solving this equation for the mass energy density reveals a non-trivial proportionality: $\rho_M \propto a^{-2.4}$. This finding is peculiar because in the Λ -CDM model, we find: $\rho_M \propto a^{-3}$.

The graph below illustrates the relationship between B and $\ln(z+1)$, where B is defined as the power in the formula $\rho_M = \rho_{M,0} a^{-B}$.

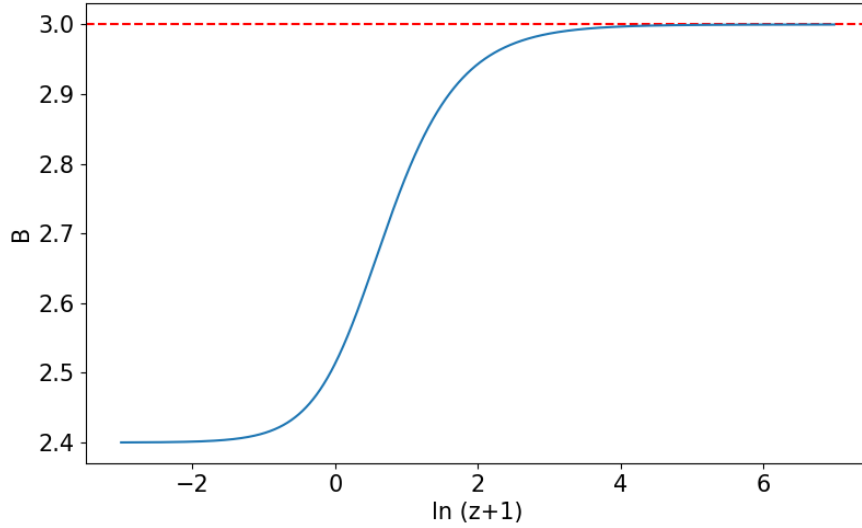


Figure 3: The factor B plotted against $\ln(z+1)$. The factor B begins at 3.0 in the early universe and gradually approaches 2.4 as the universe ages.

The CDDE model predicts a z -dependency of B , while Λ -CDM predicts B to be a constant: 3. This suggests that the CDDE model has a slower matter dilution in the expanding universe than in the Λ -CDM model, implying the presence of matter creation as a consequence of the loss of dark energy density caused by the decrease of R in the expanding universe. The explanation for this matter creation can only be obtained by employing quantum field theory in curved spacetime, which falls outside the scope of this thesis.

5.3 Deviation from Λ -CDM

Comparing the benchmark of the Λ -CDM model (equation 19) to the CDDE model (equation 34), shows a profound difference. The following plot shows the sum of the omegas:

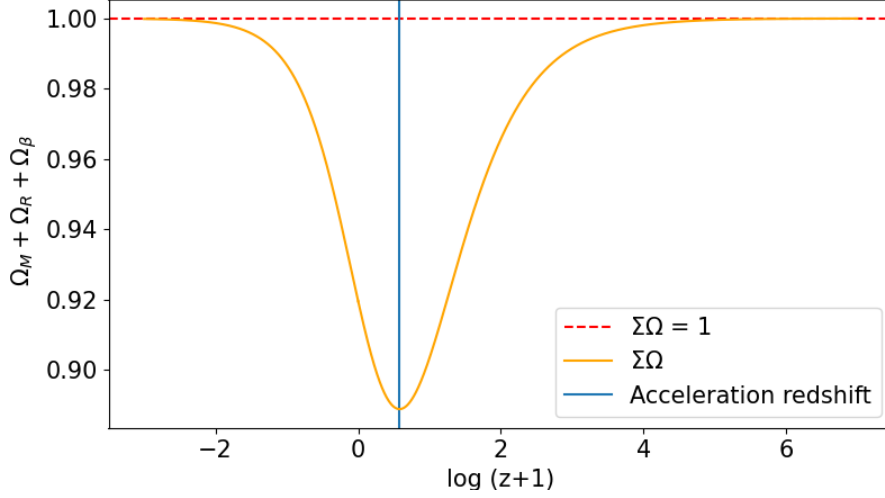


Figure 4: This plot shows on the y-axis the sum of omegas in the CDDE model. The blue acceleration redshift line is the redshift at which the universe initiates its accelerated expansion. The red dotted line shows the benchmark of Λ -CDM in which the sum of omegas is 1.

In this plot we treat β the same way $\Lambda/8\pi G$ gets treated in the Λ -CDM model. Equation 34 can be rewritten as:

$$\Omega_M + \Omega_R + \Omega_\beta = 1 + \frac{1}{2}\Omega_\beta \left(1 - \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \quad (55)$$

It is easy to observe that in the CDDE model, the sum is always lower than 1. In the very late universe ($\beta \gg \rho_M$) the latter term $\frac{1}{2}\Omega_\beta \left(1 - \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \approx 0$ and the sum of the omega becomes 1. Therefore, in the very late universe the curvature dependent dark energy in the CDDE model behaves as a constant vacuum energy, like in the Λ -CDM model.

In the very early universe ($\beta \ll \rho_M$) the latter term can be approximated as follows: $\frac{1}{2}\Omega_\beta \left(1 - \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \approx -\frac{1}{2}\sqrt{\Omega_M\Omega_\beta}$. The quantity is certainly not constant; however, it is exceedingly small, resulting in the sum of the omegas approaching 1. At the acceleration redshift we see the largest deviation from 1.

5.4 Deceleration parameter q

Doing the Taylor series expansion of the scale factor $a(t)$ in the vicinity of $t = t_0$, gives:

$$a(t) = a(t_0) + \left. \frac{da}{dt} \right|_{t=t_0} (t - t_0) + \frac{1}{2} \left. \frac{d^2a}{dt^2} \right|_{t=t_0} (t - t_0)^2 + \dots \quad (56)$$

By utilizing the fact that $a(t_0) = 1$ and considering the first three terms, we obtain the approximation for the scale factor as follows:

$$a(t) \approx 1 + H_0(t - t_0) - \frac{1}{2}q_0 H_0^2(t - t_0)^2 \quad (57)$$

where the deceleration constant q_0 is defined as: $q_0 = -(\ddot{a}/\dot{a}^2)|_{t=t_0}$. Using the CDDE model, we can create a plot of the deceleration parameter $q(z) = -(\ddot{a}(z)a(z)/\dot{a}^2(z))$ against $\ln(z+1)$ for the CDDE model. The plot includes the Λ -CDM model for comparison.

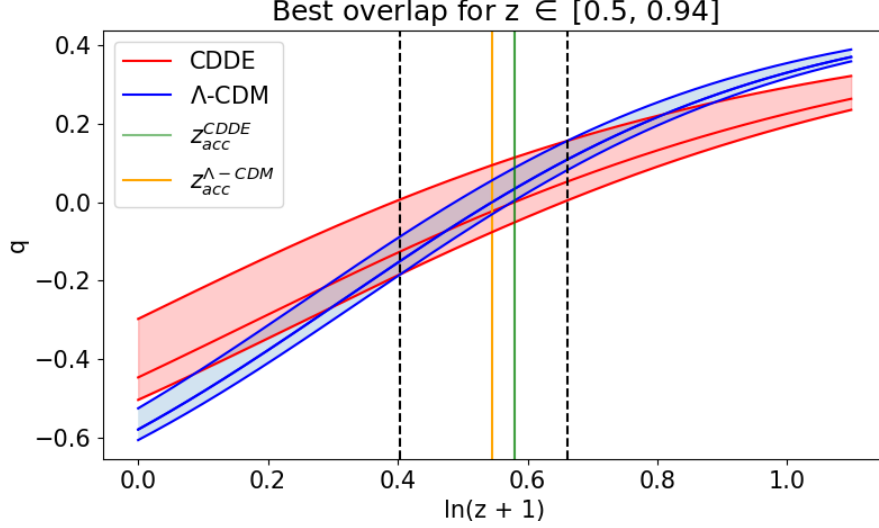


Figure 5: The deceleration parameter $q(z)$ plotted against $\ln(z+1)$ for both the CDDE model (red) and Λ -CDM model (blue). The z_{acc}^{CDDE} and $z_{acc}^{\Lambda-CDM}$ are respectively the acceleration redshifts from the CDDE and Λ -CDM model.

The red bands represent the prediction of the CDDE from the parameter envelope that solves the Hubble tension. Meanwhile the blue bands represent the upper and lower errors from the late universe fit within the Λ -CDM model. The bands for the CDDE and Λ -CDM model come from the upper and lower errors from all parameters in the models. As shown, there is significant overlap between both models. This overlap is not trivial, as the CDDE model only by construction is supposed to solve the Hubble tension. The presence of such overlap in the deceleration parameters occurs naturally within the model, without requiring further modifications. It implies that the CDDE model can solve the Hubble tension and at the same time still be consistent with the observed acceleration of the late universe. This is an encouraging indication for the CDDE model, providing further motivation to continue researching it.

6 Conclusion

The Hubble tension presents a significant challenge in the field of cosmology, in need of a solution. This thesis proposes a potential solution through the Curvature Dependent Dark Energy (CDDE) model, which incorporates a curvature-dependent dark energy term represented as $T_{\mu\nu}^{DE} = C\sqrt{(R/6)}g_{\mu\nu}$.

The primary objective of this study was to accurately determine the parameter C within the CDDE model, aiming to provide a solution to the Hubble tension. Through parameter tuning and analysis, we successfully obtained a best-fit value that resolves the tension. The specific value of C determined in our research is: $C = 1.460^{+0.170}_{-0.182} \times 10^{22} \text{eV}^3$, corresponding with an energy scale of $C^{\frac{1}{3}} = 24.442^{+0.915}_{-1.062} \text{MeV}$.

In the process of parameter fitting, we also investigated the critical value of z_* , which represents the redshift at which matter and radiation decouple in the early universe. Our findings revealed a value of $z_* = 1089.59^{+3.22}_{-1.88}$.

An odd consequence arising from the implementation of the CDDE model is the creation of matter, a distinctive feature that sets it apart from conventional cosmological models. This helped solving the Hubble tension. The CDDE model has 2 ingredients that [4] describes to be necessary to solve the Hubble tension: a non-constant cosmological constant and dilution of matter that does not go with a^{-3} .

Furthermore, when comparing the deceleration parameters between the CDDE and Lambda-CDM models, we observed a significant overlap. This agreement not only reaffirms the viability and compatibility of the CDDE model within the existing cosmological framework but also inspires further investigations into the underlying dynamics and implications of the CDDE model.

7 Discussion

The hypothesis suggesting a direct connection between matter creation and the enigma of dark matter presents an intriguing avenue for future research. In following studies, it would be valuable to investigate the amount of extra matter that is created in the CDDE model compared to the Λ -CDM model and how this creation relates to the amount of dark matter required to describe the universe. Exploring the possibility that matter creation within the CDDE model could contribute to our understanding of dark matter would provide profound insights into the fundamental nature of the universe.

Additionally, I am excited to mention the recent research undertaken by my fellow students and researchers, Thijs van Rossum and Tijmen Melssen. They have embarked on an investigation of Baryon Acoustic Oscillations (BAO) within the CDDE model, specifically focusing on the epoch before recombination. As this epoch lacks one differential equation due to the absence of the decoupling of matter and radiation, their aim is to find this missing equation and further our understanding of this plasma phase.

Although this research is still in its early stages, it holds great promise in enhancing our comprehension of cosmological dynamics during the epoch before recombination. The efforts to uncover the missing equation will contribute valuable insights into our understanding of the CDDE model and its implications for the evolution of the universe.

A Derivations for the Λ -CDM model

A.1 First Friedmann equation

The 00 component of the Einstein equations can be found by inserting equation 3, 9, 12 and 15 into the Einstein equations 14. This will result in:

$$-3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) - \Lambda = 8\pi G(\rho_M + \rho_R) \quad (58)$$

which after simplifying can be brought in the form:

$$H(t)^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G(\rho_M + \rho_R) + \Lambda}{3} \quad (59)$$

Dividing both sides of equation 59 by H^2 results in:

$$1 = \frac{8\pi G}{3H^2} \left(\rho_M + \rho_R + \frac{\Lambda}{8\pi G} \right) \quad (60)$$

which can be rewritten as:

$$\Omega_M + \Omega_R + \Omega_\Lambda = 1 \quad (61)$$

Where the convention $\Omega_i = \frac{8\pi G}{3H^2}\rho_i$ and $\Omega_\Lambda = \frac{\Lambda}{3H^2}$ is used.

A.2 Second Friedman equation

The ii component of the Einstein equation can be found in a similar way by filling in 3, 10, 12 and 15 into the Einstein equations 14. This will result in:

$$-\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2}\right)g_{ii}^{\text{FRW}} + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)g_{ii}^{\text{FRW}} - \Lambda g_{ii}^{\text{FRW}} = -\frac{8\pi G}{3}\rho_R g_{ii}^{\text{FRW}} \quad (62)$$

This can be rewritten and simplified as follows:

$$2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \Lambda - \frac{8\pi G}{3}\rho_R \quad (63)$$

After substituting equation 59 into this equation we get:

$$2\frac{\ddot{a}}{a} + \frac{8\pi G(\rho_R + \rho_M) + \Lambda}{3} = \Lambda - \frac{8\pi G}{3}\rho_R \quad (64)$$

which can be simplified to the second Friedmann equation:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(2\rho_R + \rho_M) + \frac{\Lambda}{3} \quad (65)$$

A.3 The 'third' Friedmann equation

It is useful to rewrite the two Friedmann equations into a new equation. We will start this by multiplying both sides of the first Friedmann equation 59 with a^2 and differentiating both sides with respect to time t . This results in:

$$2\ddot{a}a = 2\dot{a}a\left(\frac{8\pi G(\rho_M + \rho_R) + \Lambda}{3}\right) + a^2\left(\frac{8\pi G(\dot{\rho}_M + \dot{\rho}_R)}{3}\right) \quad (66)$$

Now we will substitute the second Friedmann equation 65 into equation 66:

$$2\dot{a}a\left(-\frac{4\pi G}{3}(2\rho_R + \rho_M) + \frac{\Lambda}{3}\right) = 2\dot{a}a\left(\frac{8\pi G(\rho_M + \rho_R) + \Lambda}{3}\right) + a^2\left(\frac{8\pi G(\dot{\rho}_M + \dot{\rho}_R)}{3}\right) \quad (67)$$

Simplifying this results in the final equation:

$$\dot{\rho}_M + \dot{\rho}_R + \frac{\dot{a}}{a}(3\rho_M + 4\rho_R) = 0 \quad (68)$$

B Derivations for the CDDE model

Using the total stress-energy tensor 28 we will now derive the differential equations that follow from the Einstein equations. Inserting everything in the Einstein equations we will find for the 00 component:

$$-3\frac{\ddot{a}}{a} + 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 8\pi G(\rho_M + \rho_R) + 8\pi G C \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)^{\frac{1}{2}} \quad (69)$$

This can be simplified as:

$$\frac{\dot{a}^2}{a^2} - \frac{8\pi G(\rho_M + \rho_R)}{3} = \frac{8\pi G}{3} C \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)^{\frac{1}{2}} \quad (70)$$

After squaring both sides and rearranging, we end up with:

$$\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} = \left(\frac{3}{8\pi G C} \right)^2 \left(\frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3}(\rho_R + \rho_M) \right)^2 \quad (71)$$

The ii component gives us:

$$\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} \right) g_{ii}^{\text{FRW}} - 3 \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) g_{ii}^{\text{FRW}} = \frac{8\pi G}{3} \rho_R g_{ii}^{\text{FRW}} - 8\pi G C \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)^{\frac{1}{2}} g_{ii}^{\text{FRW}} \quad (72)$$

The g_{ii}^{FRW} can be divided out and after rewriting we will get:

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3} \rho_R = -8\pi G C \left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} \right)^{\frac{1}{2}} \quad (73)$$

The right-hand side of this equation can be replaced by the right-hand side of equation 70 and will result in:

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{8\pi G}{3} \rho_R = -3\frac{\dot{a}^2}{a^2} + 8\pi G(\rho_M + \rho_R) \quad (74)$$

Rewriting this gives:

$$\frac{\dot{a}^2}{a^2} = \frac{\ddot{a}}{a} + \frac{4\pi G}{3}(4\rho_R + 3\rho_M) \quad (75)$$

Substituting equation 75 into equation 73 gives:

$$\left(\frac{\ddot{a}}{a} \right)^2 + \frac{\ddot{a}}{a} \frac{8\pi G}{3}(\rho_M + 2\rho_R - \beta) + \left(\frac{4\pi G}{3} \right)^2 ([\rho_M + 2\rho_R]^2 - \beta[3\rho_M + 4\rho_R]) \quad (76)$$

Here we have defined $\beta = \frac{16\pi G C^2}{3}$. This can easily be solved using the quadratic formula:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[\rho_M + 2\rho_R - \beta \pm \sqrt{(\rho_M + 2\rho_R - \beta)^2 - (\rho_M + 2\rho_R)^2 + \beta(3\rho_M + 4\rho_R)} \right] \quad (77)$$

The part underneath the square root can be much simplified, resulting in:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[\rho_M + 2\rho_R - \beta \pm \sqrt{\beta(\rho_M + \beta)} \right] \quad (78)$$

To determine if the upper or lower sign of the plus-minus is correct we notice that the left-hand side of equation 70 must be positive. This is true because C and the square root term are both positive. From equation 75 we derive that $\frac{\ddot{a}}{a} + \frac{4\pi G}{3}(\rho_M + 2\rho_R)$ must also be positive. This means that $\beta \mp \sqrt{\beta(\rho_M + \beta)} \geq 0$. Because $\sqrt{\beta(\rho_M + \beta)} \geq \beta$ the lower sign is correct. The final result is therefore:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \left[\rho_M + 2\rho_R - \beta - \sqrt{\beta(\rho_M + \beta)} \right] \quad (79)$$

Plugging equation 79 into equation 75 and rewriting yields:

$$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{4\pi G}{3} \left[2(\rho_R + \rho_M) + \beta + \sqrt{\beta(\rho_M + \beta)} \right] \quad (80)$$

Dividing both sides by H^2 and using the convention 24 will result in:

$$1 = \Omega_M + \Omega_R + \frac{1}{2}\Omega_\beta \left(1 + \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \quad (81)$$

Differentiating equation 80 multiplied by a^2 results in:

$$2\ddot{a}a - \frac{8\pi G}{3}\dot{a}a \left[2(\rho_M + \rho_R) + \beta + \sqrt{\beta(\rho_M + \beta)} \right] = \frac{4\pi G}{3}a^2 \left[2(\dot{\rho}_M + \dot{\rho}_R) + \frac{\beta}{2\sqrt{\beta(\rho_M + \beta)}}\dot{\rho}_M \right] \quad (82)$$

Substituting equation 79 into equation 82 and rewriting gives:

$$\dot{\rho}_M + \dot{\rho}_R + \frac{\dot{a}}{a}(3\rho_M + 4\rho_R) = -\frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}}\dot{\rho}_M \quad (83)$$

After recombination this decouples in two equations:

$$\dot{\rho}_R = -4\frac{\dot{a}}{a}\rho_R, \quad \dot{\rho}_M \left(1 + \frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \right) = -3\frac{\dot{a}}{a}\rho_M \quad (84)$$

The right-hand equation of 84 can be analytically solved by integration:

$$\int \frac{d\rho_M}{\rho_M} \left(1 + \frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \right) = \int -3\frac{da}{a} \quad (85)$$

The solution of this is, up to a constant:

$$\ln(\rho_M) + \frac{1}{4} \ln \left(\frac{-1 + \sqrt{\frac{\rho_M + \beta}{\beta}}}{1 + \sqrt{\frac{\rho_M + \beta}{\beta}}} \right) = -3 \ln(a) \quad (86)$$

Now expression 86 will be exponentiated with base e :

$$\rho_M \left(\frac{-1 + \sqrt{\frac{\rho_M + \beta}{\beta}}}{1 + \sqrt{\frac{\rho_M + \beta}{\beta}}} \right)^{\frac{1}{4}} = C' a^{-3} \quad (87)$$

Note here that:

$$C' = \rho_M \left(1 - \frac{2}{1 + \sqrt{\frac{\rho_M}{\beta} + 1}} \right)^{\frac{1}{4}} \bigg|_{a=1} \quad (88)$$

B.1 Transforming the two differential equations

Equations 81 and 83 are the two equations we will use in this thesis. However we will now rewrite them in terms of omegas (equation 24) and $\ln(z + 1)$. We will start by rewriting the equation in terms of z and its derivative. For this we will use equation 1 and the transformation of the time derivative:

$$\frac{d}{dt} = \frac{dz}{dt} \frac{d}{dz} \quad (89)$$

This transformation transforms the time derivative of the scale factor to:

$$\dot{a} = \frac{d}{dt}a = \frac{dz}{dt} \frac{da}{dz} = -\frac{1}{(z+1)^2} \frac{dz}{dt} \quad (90)$$

Using equation 89 and 90 we transform equation 83 in:

$$\frac{dz}{dt} \frac{d\rho_M}{dz} + \frac{dz}{dt} \frac{d\rho_R}{dz} - \frac{1}{1+z} (3\rho_M + 4\rho_R) \frac{dz}{dt} = -\frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \frac{dz}{dt} \frac{d\rho_M}{dz} \quad (91)$$

This can be simplified by removing $\frac{dz}{dt}$ on both sides and rearranging terms:

$$\frac{d\rho_M}{dz} \left(1 + \frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \right) + \frac{d\rho_R}{dz} = \frac{3\rho_M + 4\rho_R}{1+z} \quad (92)$$

To summarize, we have now two final equations:

$$1 = \Omega_M + \Omega_R + \frac{1}{2}\Omega_\beta \left(1 + \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \quad (93)$$

$$\frac{d\rho_M}{dz} \left(1 + \frac{\beta}{4\sqrt{\beta(\rho_M + \beta)}} \right) + \frac{d\rho_R}{dz} = \frac{3\rho_M + 4\rho_R}{z+1} \quad (94)$$

We will now transform these equations in terms of Ω_M , Ω_R , Ω_β and $\ln(z+1)$. We start this by differentiating equation 81 with respect to z :

$$\frac{d}{dz} \left[\Omega_M + \Omega_R + \frac{1}{2}\Omega_\beta \left(1 + \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \right] = 0 \quad (95)$$

This will result in the equation:

$$\left(\Omega_M + \Omega_R + \frac{1}{2}\Omega_\beta \left(1 + \sqrt{1 + \frac{\Omega_M}{\Omega_\beta}} \right) \right) H^2 \frac{d}{dz} H^{-2} + \frac{8\pi G}{3H^2} \left[\frac{d\rho_M}{dz} \left(1 + \frac{1}{4\sqrt{1 + \frac{\rho_M}{\beta}}} \right) + \frac{d\rho_R}{dz} \right] = 0 \quad (96)$$

Using equation 81 and 92 this becomes:

$$H^2 \frac{d}{dz} H^{-2} + \frac{8\pi G}{3H^2} \left(\frac{3\rho_M + 4\rho_R}{1+z} \right) = 0 \quad (97)$$

This is easy to transform to $u = \ln(z+1)$. This relation gives us:

$$\frac{d}{dz} = \frac{du}{dz} \frac{d}{du} = \frac{1}{1+z} \frac{d}{du} \quad (98)$$

Using the convention 24 and equation 98, equation 97 transforms in:

$$-2 \frac{1}{H} \frac{dH}{du} + 3\Omega_M + 4\Omega_R = 0 \quad (99)$$

which can be rewritten using the fact that $\frac{1}{H} \frac{dH}{du} = \frac{d \ln(H/H_0)}{du}$:

$$\frac{d \ln(H/H_0)}{du} = \frac{3\Omega_M + 4\Omega_R}{2} \quad (100)$$

To transform equation 92 we start by differentiating equation 24 with respect to z :

$$\frac{d\Omega_{M,R}}{dz} = \frac{d}{dz} \left(\frac{8\pi G}{3H^2} \rho_{M,R} \right) = \frac{8\pi G}{3H^2} \frac{d\rho_{M,R}}{dz} + \frac{8\pi G}{3} \rho_{M,R} \left(\frac{d}{dz} H^{-2} \right) \quad (101)$$

Here we can again use the identity $\frac{1}{H} \frac{dH}{dz} = \frac{d \ln(H/H_0)}{dz}$:

$$\frac{d\Omega_{M,R}}{dz} = \frac{d}{dz} \left(\frac{8\pi G}{3H^2} \rho_{M,R} \right) = \frac{8\pi G}{3H^2} \frac{d\rho_{M,R}}{dz} - 2\Omega_{M,R} \frac{d \ln(H/H_0)}{dz} \quad (102)$$

Next we will substitute equation 100 into equation 102 using equation 98 and rearrange:

$$\frac{8\pi G}{3H^2} \frac{d\rho_{M,R}}{dz} = \Omega_{M,R} \left(\frac{3\Omega_M + 4\Omega_R}{1+z} \right) + \frac{d\Omega_{M,R}}{dz} \quad (103)$$

Plugging equation 103 into equation 92 yields:

$$\left[\frac{d\Omega_M}{dz} + \Omega_M \left(\frac{3\Omega_M + 4\Omega_R}{1+z} \right) \right] \left(1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}} \right) + \left[\frac{d\Omega_R}{dz} + \Omega_R \left(\frac{3\Omega_M + 4\Omega_R}{1+z} \right) \right] = \left(\frac{3\Omega_M + 4\Omega_R}{1+z} \right) \quad (104)$$

Rewriting this gives:

$$\frac{d\Omega_M}{dz} \left(1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}} \right) + \frac{d\Omega_R}{dz} = \left(\frac{3\Omega_M + 4\Omega_R}{1+z} \right) \left[1 - \Omega_M \left(1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}} \right) - \Omega_R \right] \quad (105)$$

Using equation 98 we can transform this equation in terms of $u = \ln(z+1)$:

$$\frac{d\Omega_M}{du} \left(1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}} \right) + \frac{d\Omega_R}{du} = (3\Omega_M + 4\Omega_R) \left[1 - \Omega_M \left(1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}} \right) - \Omega_R \right] \quad (106)$$

After recombination equation 106 decouples in two equations:

$$\frac{d\Omega_R}{du} = 4\Omega_R - 2\Omega_R \frac{d \ln(H/H_0)}{du} \quad (107)$$

$$\frac{d\Omega_M}{du} = -2\Omega_M \frac{d \ln(H/H_0)}{du} + \frac{3\Omega_M}{1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}}} \quad (108)$$

Now we have transformed the equations 93 and 94 into the equations:

$$\frac{d \ln(H/H_0)}{du} = \frac{3\Omega_M + 4\Omega_R}{2} \quad (109)$$

$$\frac{d\Omega_R}{du} = 4\Omega_R - 2\Omega_R \frac{d \ln(H/H_0)}{du} \quad (110)$$

$$\frac{d\Omega_M}{du} = -2\Omega_M \frac{d \ln(H/H_0)}{du} + \frac{3\Omega_M}{1 + \frac{1}{4\sqrt{1 + \frac{\Omega_M}{\Omega_\beta}}}} \quad (111)$$

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