Bootstrapping Pions at Large N

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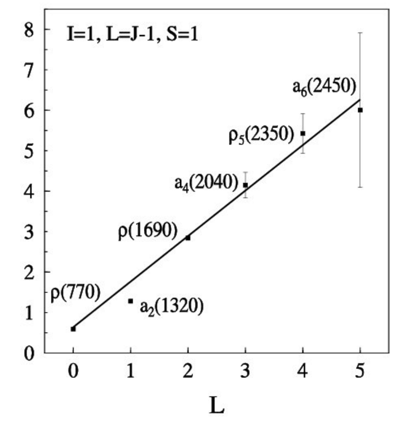

Based on arXiv:2203.11950 with *Jan Albert*

Verlinde^2 Symposium
Amsterdam July 14-15 2022

Happy 60th Erik and Herman!

You have been a great inspiration

Carving out the space of large N confining gauge theories

A confining gauge theory at $N = \infty$ has an infinite tower of stable hadrons. Meromorphic S-matrix.

Consistency of 2-2 scattering imposes constraints on masses, spins and on-shell 3pt couplings.

Carve out this set of data: $\{m_k, J_k; \lambda_{ijk}\}$

Bootstrap equations take the schematic form* $\displaystyle\sum_n \frac{\lambda_n^2}{s - m_n^2} P_{J_n}\left(1 + \frac{2u}{m_n^2}\right) = \sum_n \frac{\lambda_n^2}{u - m_n^2} P_{J_n}\left(1 + \frac{2s}{m_n^2}\right)$

(*): for $2 \to 2$ amplitude $M(s,u)$ of identical massless particles with no t-channel poles and good Regge behavior

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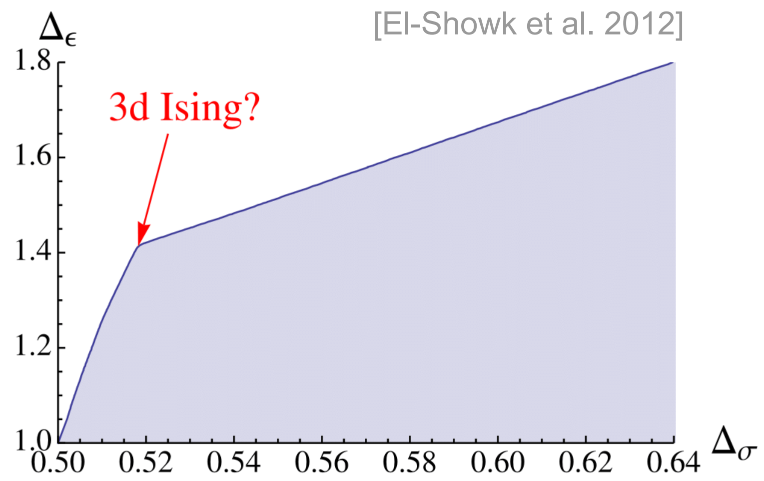
Large N S-matrix Bootstrap:

Carve out large N hadronic data from

- Crossing symmetry
- Unitarity
- Regge boundedness

Does large N QCD sit at a special place?

Conformal Bootstrap:

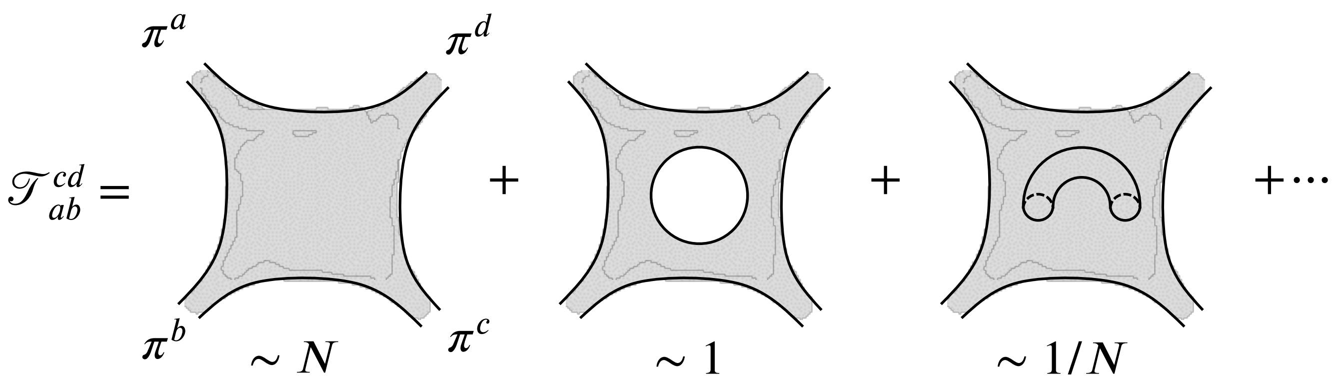


Large N QCD

$D = 4$ $SU(N)$ Yang-Mills with $N_f = 2$ massless quarks in the `t Hooft limit of fixed Λ_{QCD} .

A theory of glueballs, mesons and (heavy) baryons.

Pions π^a $=$ Goldstone bosons of $SU(2)_L \times SU(2)_R \to SU(2)_{\text{diag}}$

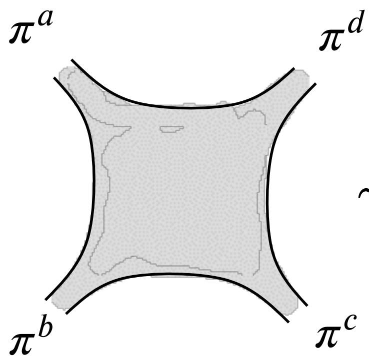


Reminiscent of string theory (`t Hooft), but we won't make any such dynamical assumption.

We'll take a new stab at this classic problem.

Modern theory space perspective & new EFT bootstrap methods ideally suited for this problem.

Pion scattering at large N



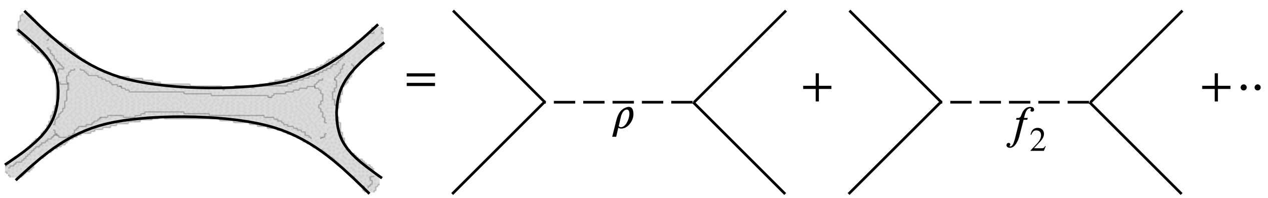
$$\sim \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^c\sigma^d\right)$$

$$2\,\mathcal{T}_{ab}^{cd} = \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^c\sigma^d\right)\ M(s,t) + \mathrm{Tr}\left(\sigma^a\sigma^b\sigma^d\sigma^c\right)\ M(s,u) + \mathrm{Tr}\left(\sigma^a\sigma^c\sigma^b\sigma^d\right)\ M(t,u)$$

Basic amplitude: $M(s,u)$

Crossing symmetry: $M(s,u) = M(u,s)$

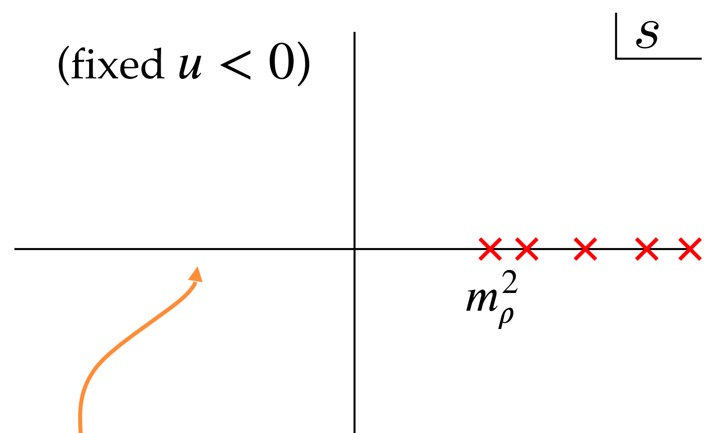
Analytic structure: $M(s,u) = \sum$ mesons poles $=$ meromorphic function



Isospin channels: $\pi\pi:\ I = 1\times 1 = 0+1+2$

$q\bar{q}:\ I = 1/2\times 1/2 = 0+1$

Isospin-two channel $M^{(2)}\left(s\,|\,t,u\right) = 2M(t,u)$



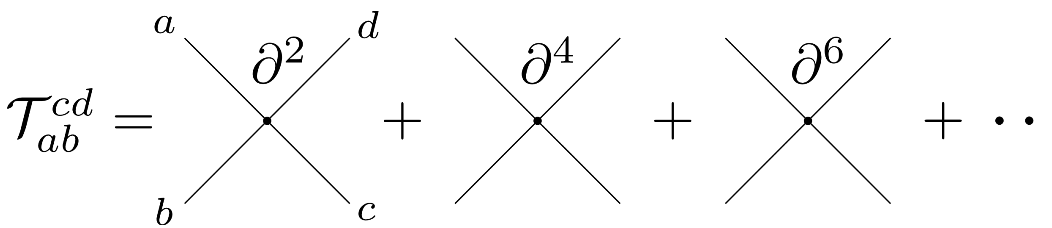
Zweig's rule: No $t-$ channel poles

Effective Field Theory

At low energies ($E < M = m_\rho$), we can use EFT, the standard chiral Lagrangian for $U(x) = e^{\frac{i}{f_\pi}\sigma^a \pi^a(x)}$

$$\mathscr{L}_{\rm Ch} = -\frac{f_\pi^2}{4}\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big) + \frac{\ell_1}{4}\Big[\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial^\mu U\Big)\Big]^2 + \frac{\ell_2}{4}\mathrm{Tr}\Big(\partial_\mu U^\dagger \partial_\nu U\Big)\mathrm{Tr}\Big(\partial^\mu U^\dagger \partial^\nu U\Big) + \cdots$$

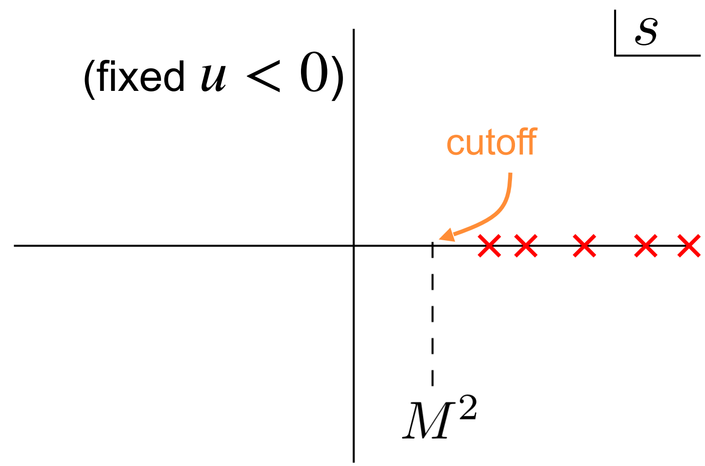
At large N, EFT pion amplitude is the result of tree-level integrating out the heavy exchanged mesons.



$$M_{\rm low}(s,u) = \sum_{n=1}^{\infty}\sum_{\ell=0}^{[n/2]} g_{n,\ell}\left(s^{n-\ell}u^{\ell} + u^{n-\ell}s^{\ell}\right)$$

$$= g_{1,0}(s+u) + g_{2,0}\left(s^2+u^2\right) + 2g_{2,1}su + \cdots$$

$$g_{1,0} \sim \frac{1}{f_\pi^2} \sim \frac{1}{N} \qquad g_{2,0}, g_{2,1} \sim \frac{\ell_1}{f_\pi^4}, \frac{\ell_2}{f_\pi^4} \sim \frac{1}{N}$$



Goal: derive bounds for these low-energy coefficients.

Three Assumptions

Crossing symmetry: $M(s,u) = M(u,s)$

Unitarity: $S = \mathbf{1} + iM \qquad SS^\dagger = 1$

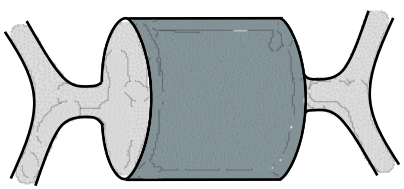
spectral density

$$\mathrm{Im}\, M(s,u) = \sum_J \rho_J(s)\, P_J\left(1 + \frac{2u}{s}\right)$$

Gegenbauer polynomials

$$2 \geq \rho_J(s) \geq 0 \qquad (s > 0)$$

Regge behavior:



- **At finite N:** Controlled by the pomeron (1st glueball).

$$M(s,u) \sim s^{2-\delta}$$

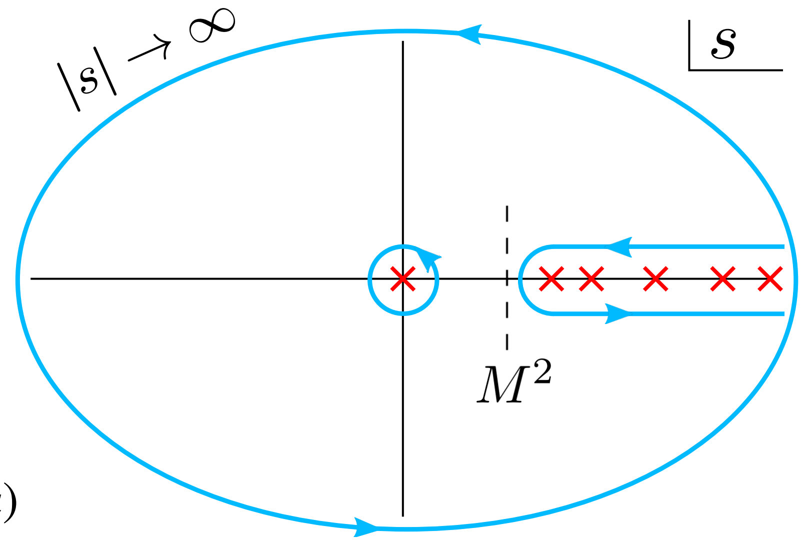
- **At large N:** Controlled by the rho meson, which has intercept ~ 0.5

$$\lim_{|s|\to\infty} \frac{M(s,u)}{s} = 0 \qquad \lim_{|s|\to\infty} \frac{M(s,-s-u)}{s} = 0 \qquad (\text{fixed } u < 0)$$

Dispersion Relations

By the Regge behavior,

$$\frac{1}{2\pi i}\oint_{\infty} ds' \frac{M(s',u)}{s'^{k+1}} = 0 \qquad k = 1,2,\ldots$$



Deforming the contour,

$$\frac{1}{2\pi i}\oint_{0} ds' \frac{M_{\text{low}}(s',u)}{s'^{k+1}} = \frac{1}{\pi}\int_{M^2}^{\infty} ds' \frac{\text{Im}\, M_{\text{high}}(s',u)}{s'^{k+1}}$$

$$= \frac{1}{\pi}\sum_J n_J^{(D)} \int_{M^2}^{\infty} \frac{dm^2}{m^2}\, m^{4-D} \underbrace{\rho_J(m^2)}_{\geq 0} \frac{P_J\left(1+\frac{2u}{m^2}\right)}{m^{2k}} = \left\langle \frac{P_J\left(1+\frac{2u}{m^2}\right)}{m^{2k}} \right\rangle$$

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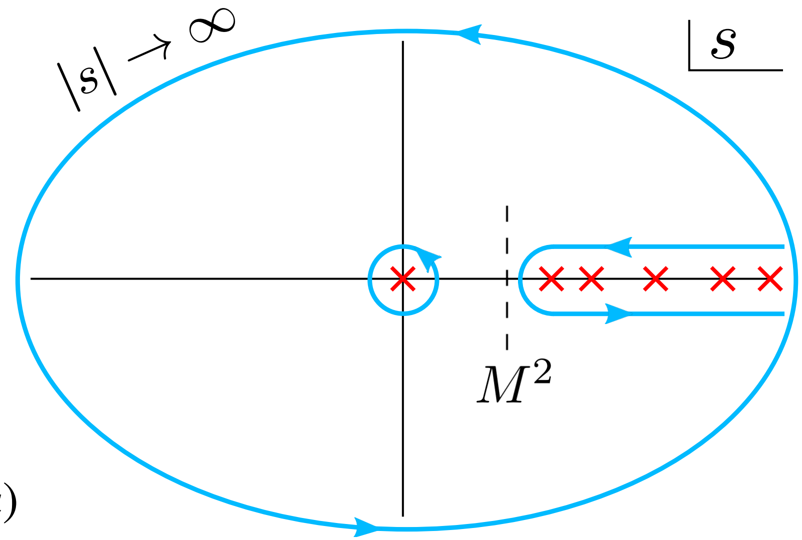

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Expanding around $u \sim 0$,

$$k = 1: \qquad g_{1,0} + 2g_{2,1}u + g_{3,1}u^2 + \ldots = \left\langle \frac{P_J(1)}{m^2} + 2\frac{P_J{}'(1)}{m^4}u + 2\frac{P_J{}''(1)}{m^6}u^2 + \ldots \right\rangle$$

$$k = 2: \qquad g_{2,0} + g_{3,1}u + \ldots = \left\langle \frac{P_J(1)}{m^4} + 2\frac{P_J{}'(1)}{m^6}u + \ldots \right\rangle$$

Sum Rules and Null Constraints

Sum rules: $$g_{1,0} = \left\langle \frac{1}{m^2} \right\rangle, \quad g_{2,0} = \left\langle \frac{1}{m^4} \right\rangle, \quad 2g_{2,1} = \left\langle \frac{J(J+1)}{m^4} \right\rangle, \ \cdots$$

(Due to low-energy crossing symmetry)

$$g_{3,1} = \left\langle 2\frac{P_J{}'(1)}{m^6} \right\rangle = \left\langle 2\frac{P_J{}''(1)}{m^6} \right\rangle$$

[Caron-Huot & Van Duong 2021, Tolley, Wang & Zhou 2021]

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[Caron-Huot & Van Duong 2021, Tolley, Wang & Zhou 2021]

Null constraints: $\mathcal{X}_{3,1}\left(m^2, J\right) = \frac{P_J{}'(1)}{m^6} - \frac{P_J{}''(1)}{m^6}$ $\quad \left\langle \mathcal{X}_{3,1}\left(m^2, J\right) \right\rangle = 0$

Additional set of dispersion relations

$$\frac{1}{2\pi i} \oint_{\infty} ds' \frac{M(s', -s'-u)}{s'^{k+1}} = 0$$

→ New set of null constraints

$$\left\langle \mathcal{Y}_{n,k}\left(m^2, J\right) \right\rangle = 0$$

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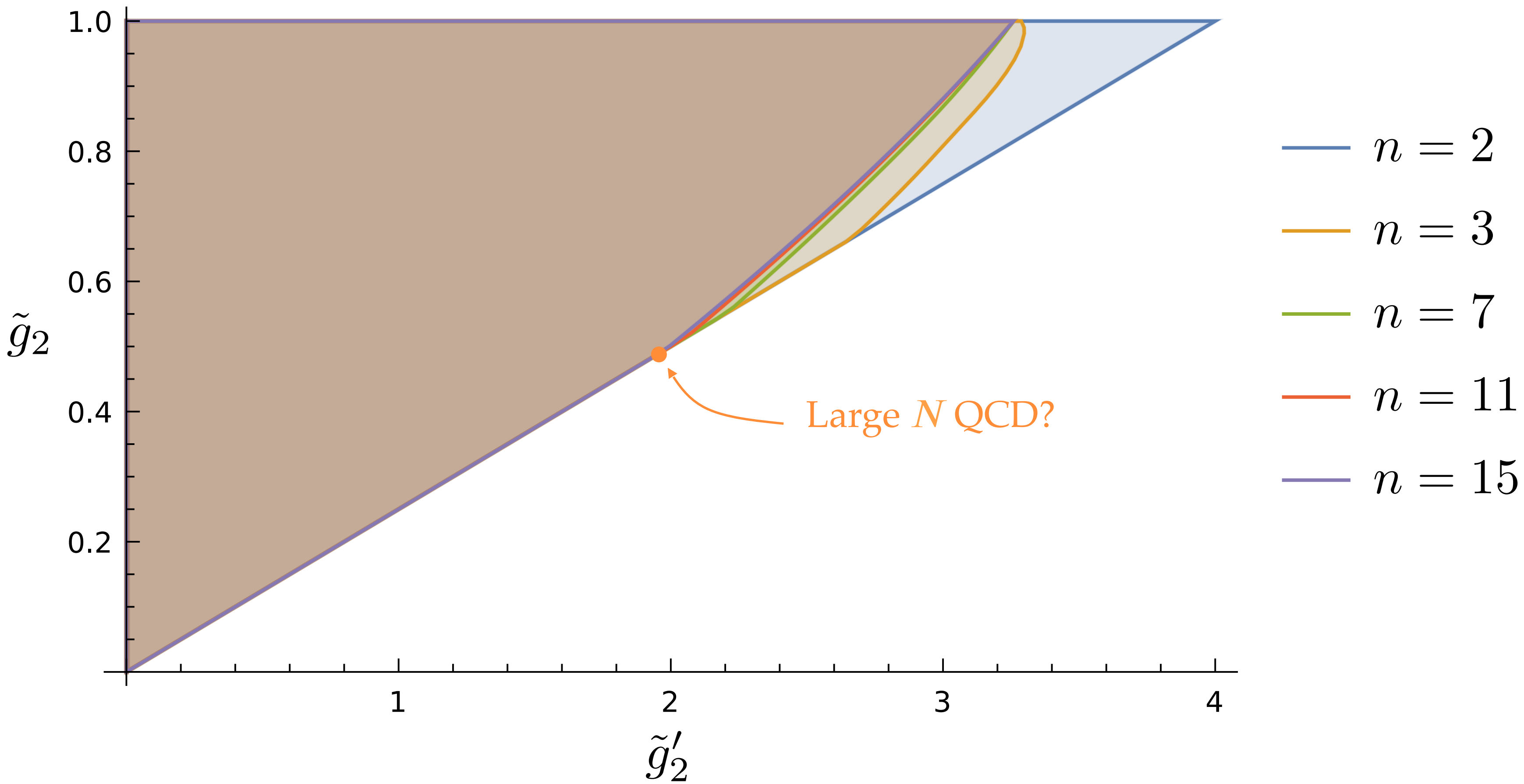
Two-sided bounds: By unitarity, $\langle \ldots \geq 0 \rangle \Rightarrow \langle \ldots \rangle \geq 0.$ $\quad g_{i,j} \geq 0$ [Pham & Truong 1985]

$$0 \leq \tilde{g}_2 \equiv \frac{g_{2,0} M^2}{g_{1,0}} = \frac{\left\langle \frac{M^4}{m^4} \right\rangle}{\left\langle \frac{M^2}{m^2} \right\rangle} \leq 1$$

$m \geq M$

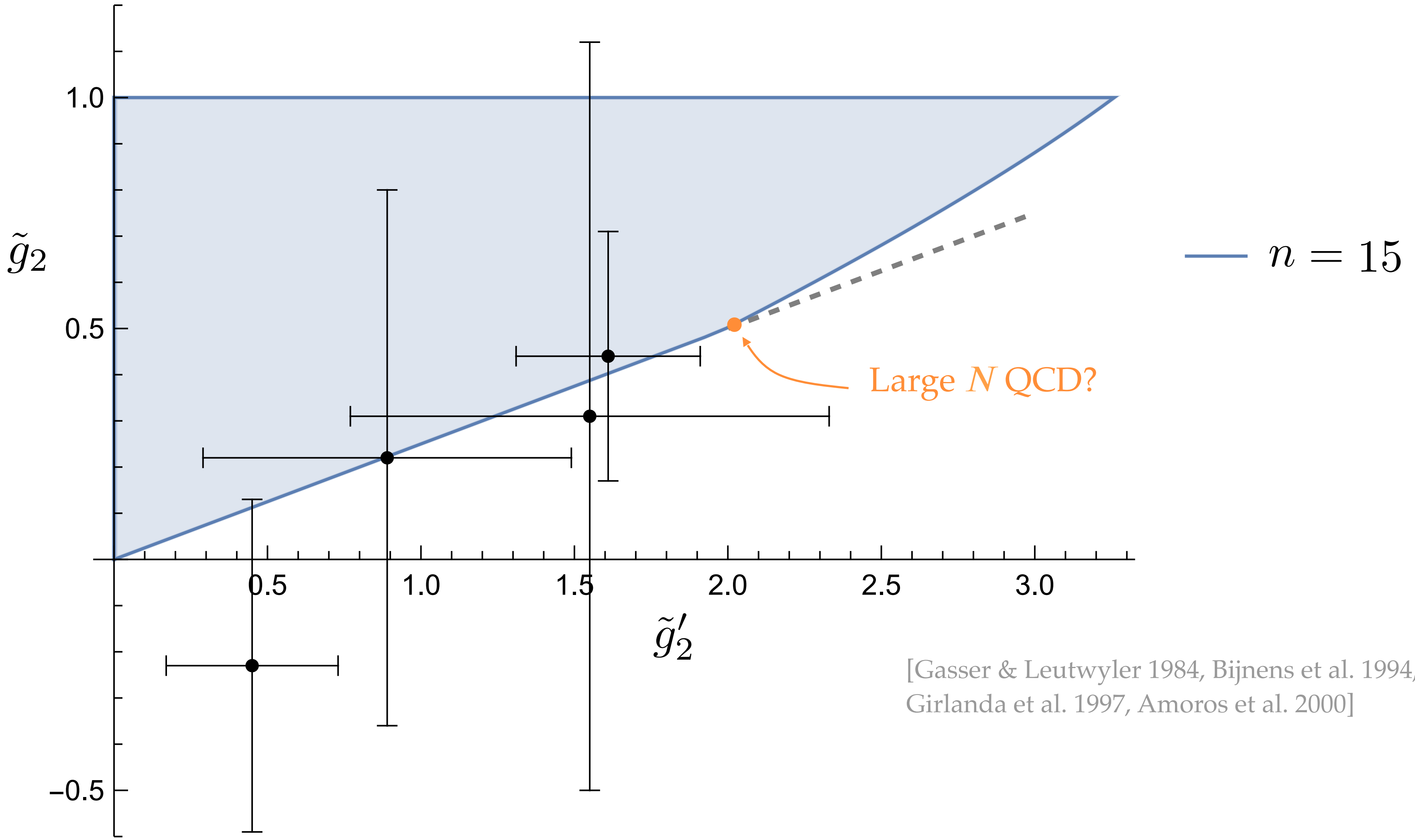
$$0 \leq \tilde{g}_2' \equiv \frac{2 g_{2,1} M^2}{g_{1,0}} \leq \quad ?$$

Exclusion plot



Allowed region in the space of two-derivative couplings.

Healthy theories must lie in the colored region.



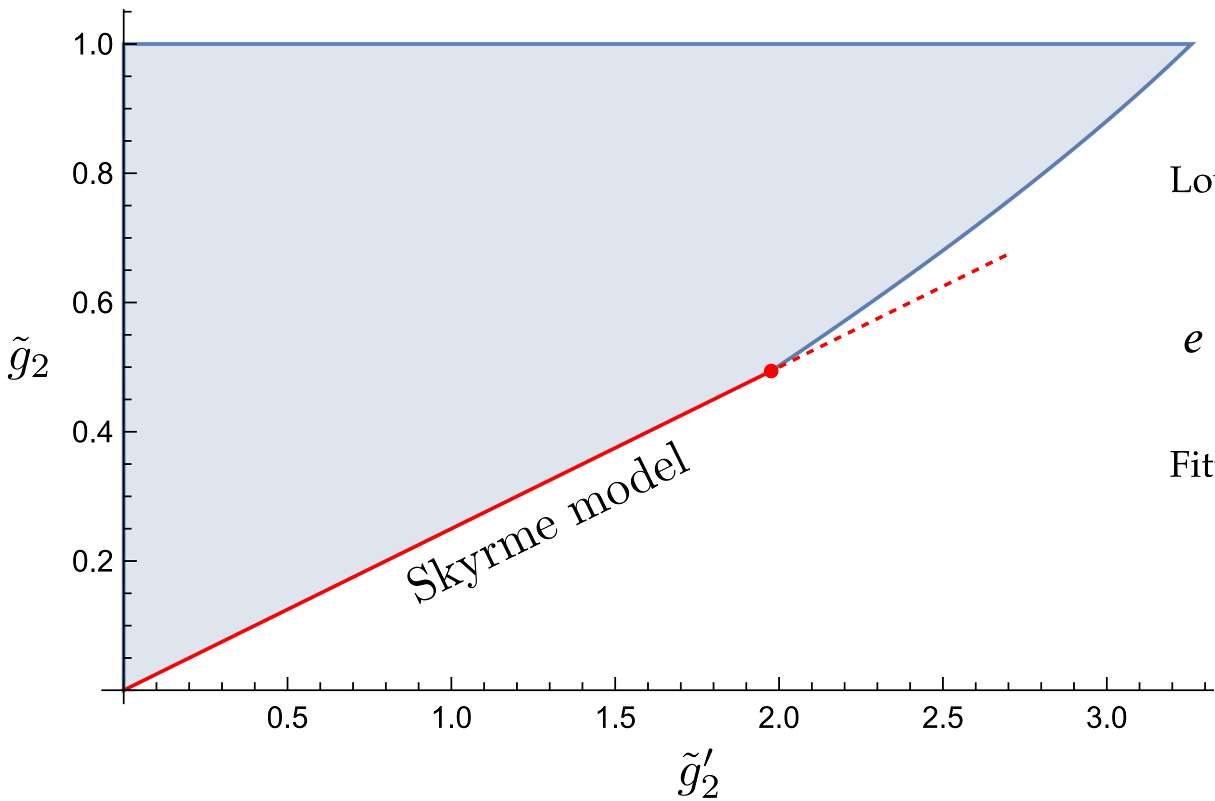
Comparison of the region allowed by unitarity to experiment.

Skyrme model

Used to describe baryons as solitons of the chiral Lagrangian.

$$\mathscr{L}_{\mathrm{Skyr}} = -\frac{f_\pi^2}{4}\mathrm{Tr}\left(\partial_\mu U^\dagger \partial^\mu U\right) + \frac{1}{32e^2}\mathrm{Tr}\left(\left[U^\dagger\partial_\mu U, U^\dagger\partial_\nu U\right]\left[U^\dagger\partial^\mu U, U^\dagger\partial^\nu U\right]\right)$$

Particular choice: $-\ell_1 = \ell_2 = \frac{1}{4e^2}$, $\qquad \tilde{g}_2' = 4\,\tilde{g}_2 = \frac{1}{e^2}\frac{M^2}{f_\pi^2}$



Lower bound on the coupling:

$$e\ f_\pi \geq \sqrt{\frac{M^2}{4\ \tilde{g}_2^{(\mathrm{kink})}}} \simeq 551\ \mathrm{MeV}$$

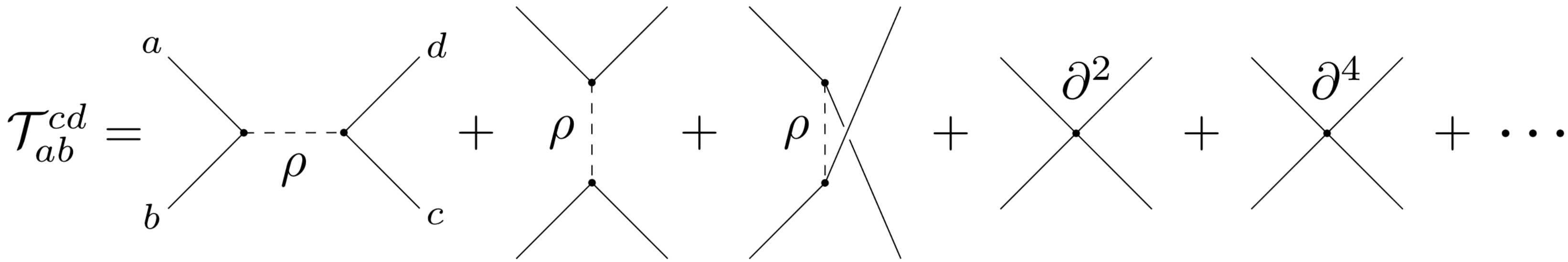
Fitting the nucleon and Δ mass:

$$e\ f_\pi \simeq 352\ \mathrm{MeV}$$

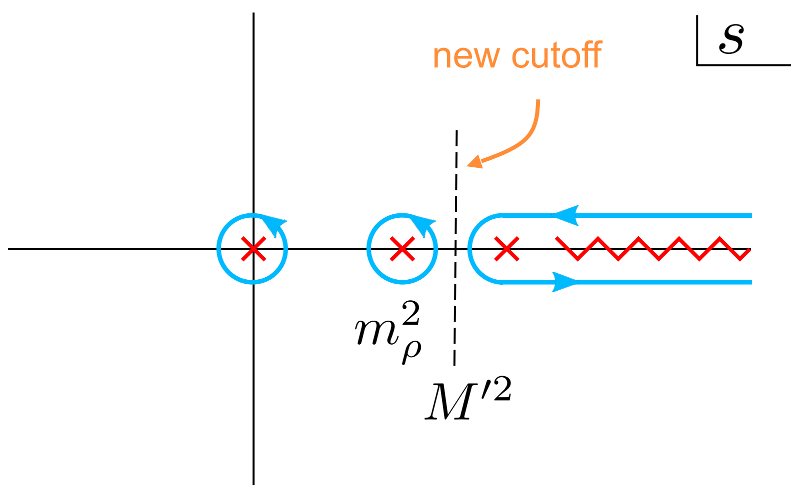
[Adkins, Nappi & Witten 1983]

Including the rho meson

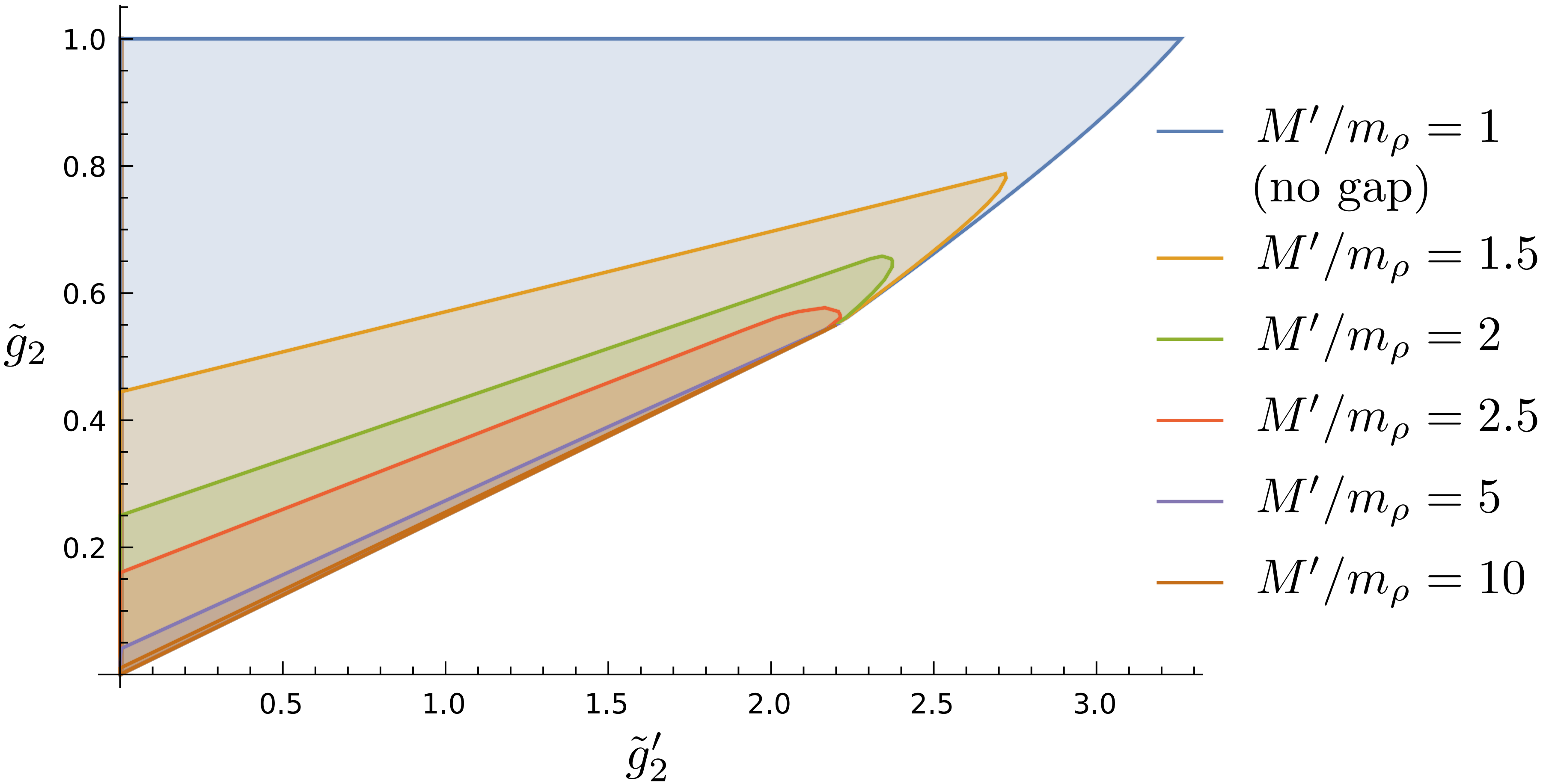
New EFT: We account for the ρ_μ^a, an isospin triplet of spin $J = 1$ and mass m_ρ.



$$M_{\text{low}}^{(\rho)}(s,u) = \frac{1}{2} g_{\pi\pi\rho}^2 \left(\frac{m_\rho^2 + 2u}{m_\rho^2 - s} + \frac{m_\rho^2 + 2s}{m_\rho^2 - u} \right) + \sum_{n=0}^{\infty} \sum_{\ell=0}^{[n/2]} \hat{g}_{n,\ell} \left(s^{n-\ell} u^{\ell} + u^{n-\ell} s^{\ell} \right)$$

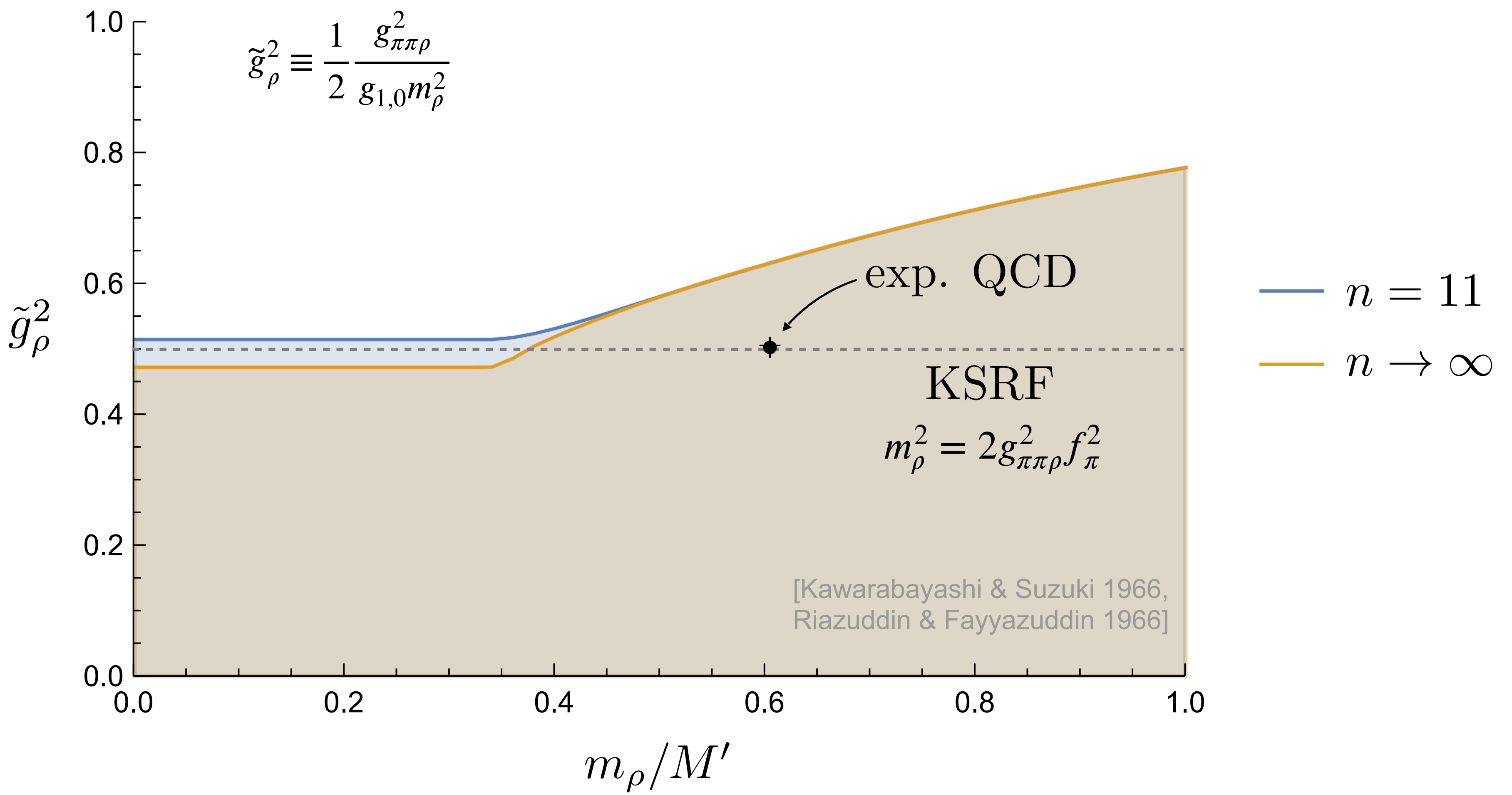


New exclusion plot



Allowed region in the space of two-derivative couplings, as a function of the gap above the rho meson.

Rho coupling plot



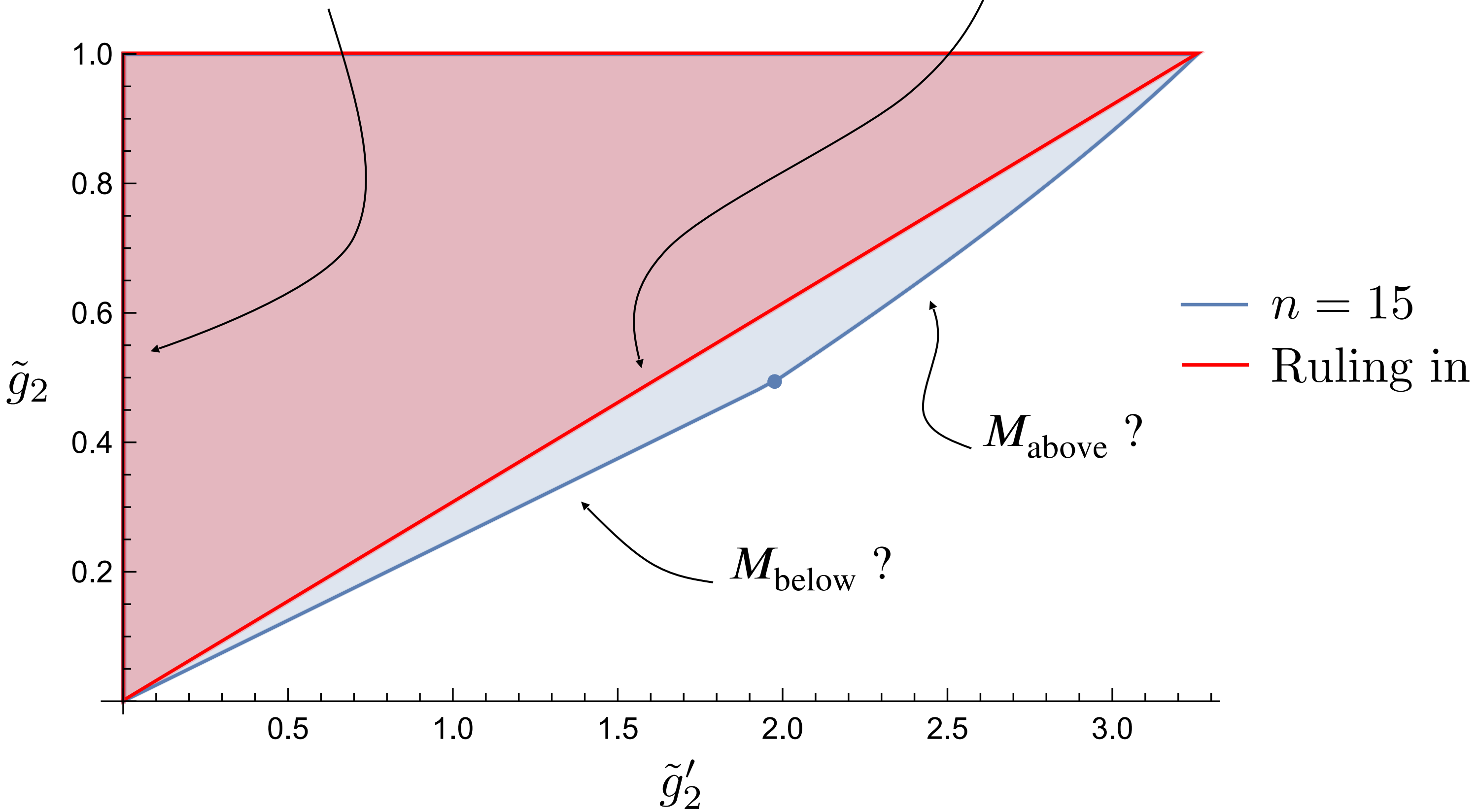
Upper bound on the rho coupling as a function of the gap above the rho.

Analytically ruling in

[Caron-Huot & Van Duong 2021]

$$M_{\text{spin}-0} = \frac{m^2}{m^2 - s} + \frac{m^2}{m^2 - u}$$

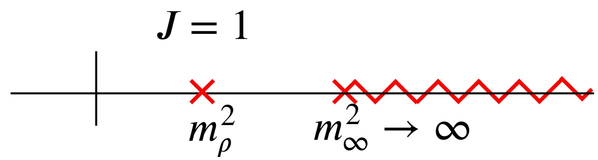
$$M_{su-\text{pole}} = \frac{m^4}{(m^2 - s)(m^2 - u)} - \alpha_0 M_{\text{spin}-0}$$



Simple solutions to crossing turn out to saturate (some of) the bounds.

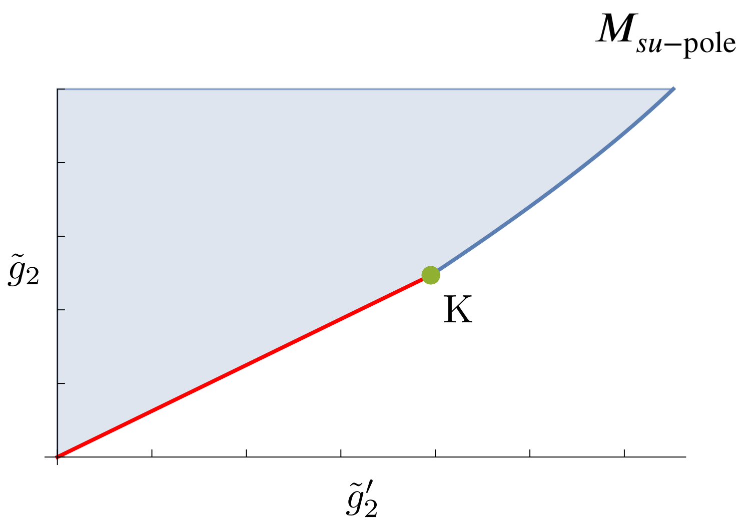
Analytically ruling in

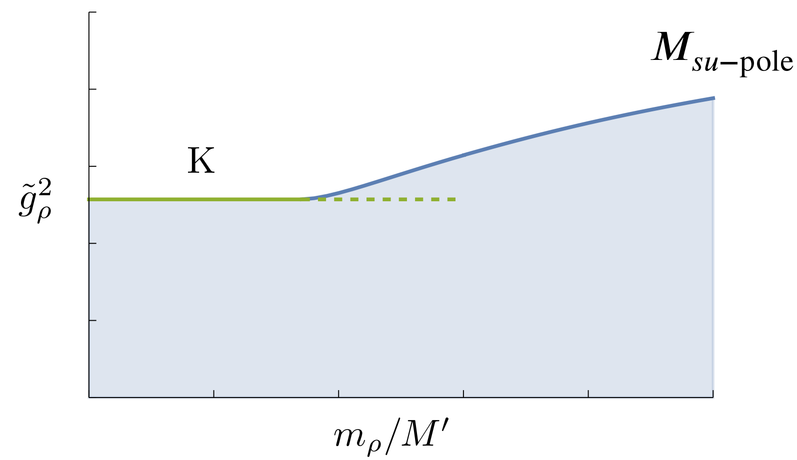
Below the kink: UV completion of the rho meson.



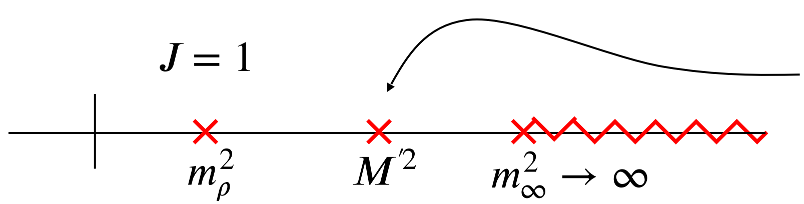
$$M^{(\mathrm{UV})}_{\mathrm{spin}-1} = \frac{m_\rho^2 + 2u}{m_\rho^2 - s}\left(\frac{m_\infty^2}{m_\infty^2 - u}\right) + \frac{m_\rho^2 + 2s}{m_\rho^2 - u}\left(\frac{m_\infty^2}{m_\infty^2 - s}\right)$$

Above the kink: Matching the plots:





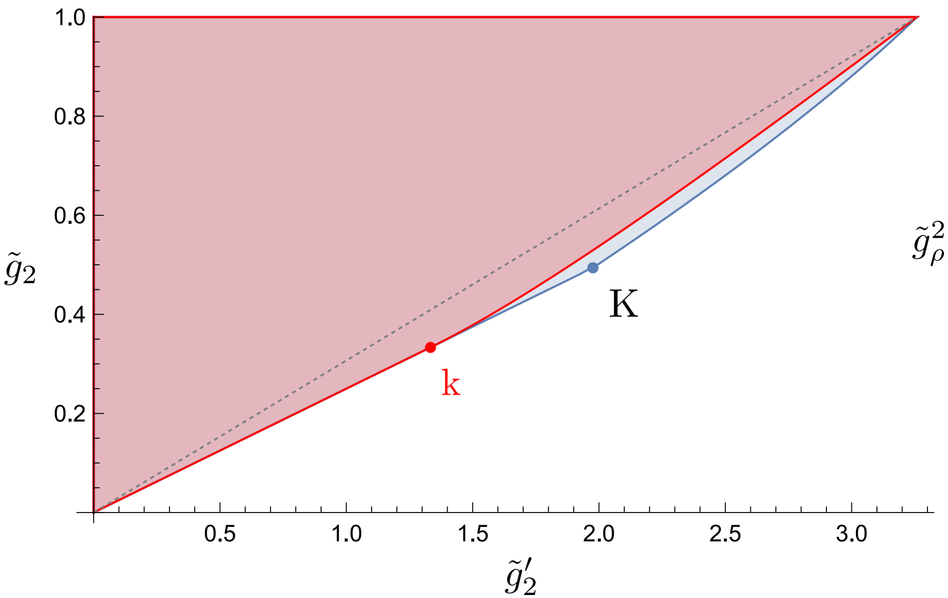
The same (blue) bounds are reproduced with the refined spectrum:

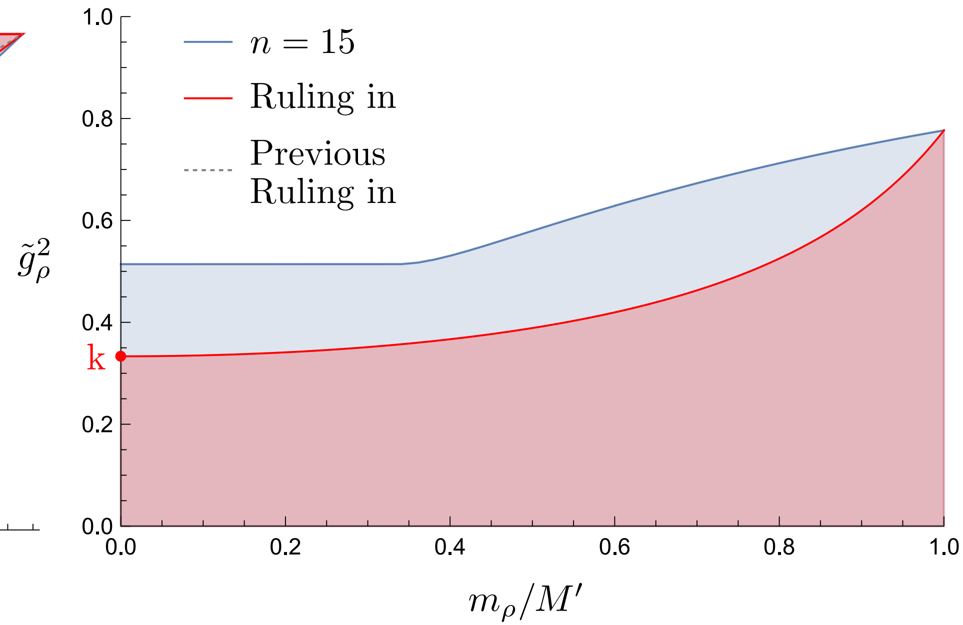


$$\frac{M'^4}{(M'^2 - s)(M'^2 - u)} - \text{spin 1} - \text{spin 0}$$

Analytically ruling in

Our best candidate: $M_{\text{guess}} = M_{su-\text{pole}}\left(m_{\text{hs}}\right) - \alpha_1 M^{(\text{UV})}_{\text{spin}-1}\left(m_{hs}\right) + \alpha_1 M^{(\text{UV})}_{\text{spin}-1}\left(m_\rho\right)$





Further ruling in achieved by M_{guess}.

A possible fix: $M^{(\text{UV})}_{\text{spin}-1} \rightarrow M^{(\text{UV})}_{\text{spin}-1} - M_{(?)}\left(m_\infty\right)$ such that it only changes $g_{1,0}$.

Outlook

Summary:

- A new stab at a very old problem
- New methods yield new bounds
- Exclusion plots show interesting features. Does QCD lie there?
- Intriguing connections with ``vodoo QCD''

In progress:

- Mixed amplitudes with external rhos and pions
- Form factors and anomalies
- Glueball scattering

Future:

- Which assumptions are needed to corner large N QCD or other theories of interest?
- Relation with worldsheet bootstrap?
- $D = 3$, susy, …

Happy Birthday!